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BSE'S, BSDE'S AND FIXED-POINT PROBLEMS

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In this paper, we introduce a class of backward stochastic equations (BSEs) that extend classical BSDEs and include many interesting examples of generalized BSDEs as well as semimartingale backward equations. We show that a BSE can be translated into a fixed-point problem in a space of random vectors. This makes it possible to employ general fixed-point arguments to establish the existence of a solution. For instance, Banach's contraction mapping theorem can be used to derive general existence and uniqueness results for equations with Lipschitz coefficients, whereas Schauder-type fixed-point arguments can be applied to non-Lipschitz equations. The approach works equally well for multidimensional as for one-dimensional equations and leads to results in several interesting cases such as equations with path-dependent coefficients, anticipating equations, McKean–Vlasov-type equations and equations with coefficients of superlinear growth.

1. Introduction. In this paper, we study backward stochastic equations (BSEs) of the form

$$Y_t + F_t(Y, M) + M_t = \xi + F_T(Y, M) + M_T. \tag{1.1}$$

For a given maturity $T \in \mathbb{R}_+$, a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, a generator F and a terminal condition $\xi \in L^p(\mathcal{F}_T)^d$, a solution to (1.1) consists of a d-dimensional adapted process Y together with a d-dimensional martingale M such that equation (1.1) holds for all $t \in [0, T]$. If $F(Y, M)$ is a finite variation process, (1.1) is a semimartingale backward equation, which as a special case, contains the semimartingale Bellman equation introduced by Chitasvili (1983); see also Mania and Tevzadze (2003) and the references therein. In the case where F is of the form $F_t(Y, M) = \int_0^t f(s, Y, M)\, ds$, BSE (1.1) becomes a generalized backward stochastic differential equation (BSDE),

$$Y_t = \xi + \int_t^T f(s, Y, M)\, ds + M_T - M_t, \tag{1.2}$$

in the spirit of Liang, Lyons and Qian (2011). If in addition, the probability space carries an n-dimensional Brownian motion W and a Poisson random measure N

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on $[0,T]\times(\mathbb{R}^m\setminus\{0\})$ such that every square-integrable martingale M has a unique representation of the form

$$M_t = \int_0^t Z_s^M \, dW_s + \int_0^t \int_{\mathbb{R}^m\setminus\{0\}} U_s^M(x)\tilde{N}(ds,dx) + K_t^M$$

for the compensated Poisson random measure $\tilde{N}$, suitable integrands Z^M and U^M, and a square-integrable martingale K^M strongly orthogonal to W and $\tilde{N}$, one can write equations of the form

$$Y_t = \xi + \int_t^T f(s, Y, Z^M, U^M)\, ds + M_T - M_t. \tag{1.3}$$

This generalizes the jump-diffusion extension of Tang and Li (1994) of the classical BSDEs introduced by Pardoux and Peng (1990) in three directions. First, in Tang and Li (1994) the filtration is generated by the Brownian motion and the Poisson random measure, whereas here it is general; second, at any given time, the driver f in (1.3) can depend on the whole paths of the processes Y, Z^M, U^M and not only on their current values; and finally, f can be a function of Y, Z^M, U^M viewed as random elements instead of just their realizations $Y(\omega)$, $Z^M(\omega)$ and $U^M(\omega)$. As special cases, (1.3) contains BSDEs with drivers that depend on the past or future of Y, Z^M and U^M, such as the time-delayed BSDEs of Delong and Imkeller (2010a, 2010b) or the anticipating BSDEs of Peng and Yang (2009). It also includes mean-field BSDEs as in Buckdahn, Li and Peng (2009), or more generally, McKean–Vlasov-type BSDEs with coefficients depending on the distributions of Y, Z^M and U^M.

Our approach to proving that a BSE has a solution is to translate it into a fixed-point problem for a mapping $G : L^p(\mathcal{F}_T)^d \to L^p(\mathcal{F}_T)^d$. This makes it possible to apply general fixed-point results. For instance, Banach's contraction mapping theorem can be used to derive general existence and uniqueness results for equations with Lipschitz coefficients. In the non-Lipschitz case, one can employ Schauder-type fixed-point arguments. This yields results for equations with coefficients of superlinear growth, but it requires compactness assumptions. By reducing a BSE to a fixed-point problem in $L^p(\mathcal{F}_T)^d$, one eliminates the time-dimension. But one still has to find compact subsets of $L^p(\mathcal{F}_T)^d$. We do that by making use of Sobolev spaces corresponding to infinite-dimensional Gaussian measures.

Our method works equally well for multidimensional as for one-dimensional equations, and in addition to general results for BSEs, it also yields interesting findings for BSDEs. For instance, in Section 3, we obtain existence and uniqueness results for BSDEs with functional drivers depending on the whole processes Y and M. In general, such results require Lipschitz continuity with a small enough Lipschitz constant or, alternatively, a sufficiently short maturity. But in several interesting special cases, it is possible to derive the existence of a unique solution for arbitrary Lipschitz constant and maturity. In Section 4, we use compactness and a theorem by Krasnoselskii (1964), which combines the fixed-point results of Banach and Schauder, to derive existence results for multidimensional BSDEs with functional drivers of superlinear growth. For instance, Corollary 4.7 establishes the existence of solutions to BSDEs with general path-dependent drivers and Corollary 4.10 the existence of a solution to a multidimensional mean-field BSDE with a driver of quadratic growth. The latter complements results by, for example, Tevzadze (2008) and Cheridito and Nam (2015) on multidimensional quadratic BSDEs, which are known to not always have solutions [see, e.g., Peng (1999), or Frei and dos Reis (2011)].

The structure of the paper is as follows. In Section 2, we formally introduce BSEs and relate them to fixed-point problems in $L^p(\mathcal{F}_T)^d$. In Section 3, we derive existence and uniqueness results for various BSEs and BSDEs with general functional Lipschitz coefficients from Banach's contraction mapping theorem. In Section 4, we provide existence results for different non-Lipschitz equations using compactness and Krasnoselskii's fixed-point theorem.

2. BSEs and fixed points in L^p. In this section, we introduce BSEs and show how they can be translated into fixed-point problems in L^p-spaces. We fix a finite time horizon $T \in \mathbb{R}_+$ and let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions. Then all martingales admit a RCLL modification (i.e., right-continuous with left limits). By $|\cdot|$, we denote the Euclidean norm on $\mathbb{R}^d$, and for a d-dimensional random vector X, we define

$$\|X\|_p := \big(\mathbb{E}|X|^p\big)^{1/p} \text{ if } p < \infty \quad \text{and} \quad \|X\|_\infty := \operatorname*{ess\,sup}_{\omega\in\Omega} |X|.$$

For $p \in (1, \infty]$, we set:

- $L^p(\mathcal{F}_t)^d$: all d-dimensional $\mathcal{F}_t$-measurable random vectors X satisfying $\|X\|_p < \infty$.
- $\mathbb{E}_t X := \mathbb{E}[X|\mathcal{F}_t]$.
- $\mathbb{S}^p$: all $\mathbb{R}^d$-valued RCLL adapted processes $(Y_t)_{0\le t\le T}$ satisfying $\|Y\|_{\mathbb{S}^p} := \|\sup_{0\le t\le T} |Y_t|\|_p < \infty$.
- $\mathbb{S}^p_0$: all $Y \in \mathbb{S}^p$ with $Y_0 = 0$.
- $\mathbb{M}^p_0$: all martingales in $\mathbb{S}^p_0$.

A BSE is specified by a generator $F : \mathbb{S}^p \times \mathbb{M}^p_0 \to \mathbb{S}^p_0$ and a terminal condition $\xi \in L^p(\mathcal{F}_T)^d$.

DEFINITION 2.1. A solution to the BSE

$$Y_t + F_t(Y, M) + M_t = \xi + F_T(Y, M) + M_T \tag{2.1}$$

consists of a pair $(Y, M) \in \mathbb{S}^p \times \mathbb{M}^p_0$ such that (2.1) holds for all $t \in [0, T]$.

DEFINITION 2.2. We say F satisfies condition (S) if for all $y \in L^p(\mathcal{F}_0)^d$ and $M \in \mathbb{M}_0^p$, the equation

$$Y_t = y - F_t(Y, M) - M_t \tag{2.2}$$

has a unique solution $Y \in \mathbb{S}^p$.

For a given $V \in L^p(\mathcal{F}_T)^d$, one obtains from Jensen's inequality that $y^V := \mathbb{E}_0 V$ belongs to $L^p(\mathcal{F}_0)^d$ and from Doob's L^p-maximal inequality that $M_t^V := \mathbb{E}_0 V - \mathbb{E}_t V$ is in $\mathbb{M}_0^p$. If F satisfies (S), we denote by Y^V the solution of the equation $Y_t = y^V - F_t(Y, M^V) - M_t^V$.

A BSE depends on the generator F and terminal condition ξ. Provided that F satisfies condition (S), then the pair (F, ξ) also defines a map

$$G : L^p(\mathcal{F}_T)^d \to L^p(\mathcal{F}_T)^d \quad \text{through} \quad V \mapsto \xi + F_T(Y^V, M^V).$$

To relate solutions of the BSE (2.1) to fixed points of G, we define the two mappings:

$$\pi : \mathbb{S}^p \times \mathbb{M}_0^p \to L^p(\mathcal{F}_T)^d \quad \text{and} \quad \phi : L^p(\mathcal{F}_T)^d \to \mathbb{S}^p \times \mathbb{M}_0^p$$

by

$$\pi(Y, M) := Y_0 - M_T \quad \text{and} \quad \phi(V) := (Y^V, M^V).$$

THEOREM 2.3. *Assume F satisfies* (S). *Then the following hold*:

(a) *$V = (\pi \circ \phi)(V)$ for all $V \in L^p(\mathcal{F}_T)^d$. In particular, ϕ is injective.*

(b) *If $V \in L^p(\mathcal{F}_T)^d$ is a fixed point of G, then $\phi(V)$ is a solution of the BSE* (2.1).

(c) *If $(Y, M) \in \mathbb{S}^p \times \mathbb{M}_0^p$ solves the BSE* (2.1), *then $\pi(Y, M)$ is a fixed point of G and $(Y, M) = (\phi \circ \pi)(Y, M)$.*

(d) *V is a unique fixed point of G in $L^p(\mathcal{F}_T)^d$ if and only if $\phi(V)$ is a unique solution of the BSE* (2.1) *in $\mathbb{S}^p \times \mathbb{M}_0^p$.*

PROOF. (a) Is straightforward to check.

(b) If $V \in L^p(\mathcal{F}_T)^d$ is a fixed point of G, then

$$y^V - M_T^V = (\pi \circ \phi)(V) = V = G(V) = \xi + F_T(Y^V, M^V). \tag{2.3}$$

Since Y^V satisfies $Y_t^V = y^V - F_t(Y^V, M^V) - M_t^V$ for all t, (2.3) is equivalent to

$$Y_t^V + F_t(Y^V, M^V) + M_t^V = \xi + F_T(Y^V, M^V) + M_T^V \qquad \text{for all } t,$$

which shows that $\phi(V) = (Y^V, M^V)$ solves the BSE (2.1).

(c) Let $(Y, M) \in \mathbb{S}^p \times \mathbb{M}_0^p$ be a solution of the BSE (2.1). Set $V := \pi(Y, M) = Y_0 - M_T$. Then $y^V = Y_0$ and $M_t^V = M_t$. In particular,

$$Y_t = Y_0 - F_t(Y, M) - M_t = y^V - F_t(Y, M^V) - M_t^V$$

for all t. It follows that $(Y, M) = (Y^V, M^V) = \phi(V) = (\phi \circ \pi)(Y, M)$ and

$$y^V = Y_0^V = \xi + F_T(Y^V, M^V) + M_T^V = G(V) + M_T^V.$$

Since $y^V - M_T^V = V$, this shows that $V = G(V)$.

(d) Follows from (a)–(c). □

In the special case, where F does not depend on Y, condition (S) holds trivially, and it is enough to find a fixed point of the mapping $G_0(V) := G(V) - \mathbb{E}_0 G(V)$ in the subspace

$$L_0^p(\mathcal{F}_T)^d := \{V \in L^p(\mathcal{F}_T)^d : \mathbb{E}_0 V = 0\}.$$

COROLLARY 2.4. *If F does not depend on Y, the following hold*:

(a) *If $V \in L_0^p(\mathcal{F}_T)^d$ is a fixed point of G_0, then the processes $Y_t := \mathbb{E}_0\xi + \mathbb{E}_0 F_T(M) - F_t(M) - M_t$ and $M_t := -\mathbb{E}_t V$ form a solution of the BSE* (2.1) *in $\mathbb{S}^p \times \mathbb{M}_0^p$.*

(b) *If $(Y, M) \in \mathbb{S}^p \times \mathbb{M}_0^p$ solves the BSE* (2.1), *then $-M_T$ is a fixed point of G_0.*

(c) *V is a unique fixed point of G_0 in $L_0^p(\mathcal{F}_T)^d$ if and only if the pair (Y, M) given by $Y_t := \mathbb{E}_0\xi + \mathbb{E}_0 F_T(M) - F_t(M) - M_t$ and $M_t := -\mathbb{E}_t V$ is a unique solution of the BSE* (2.1) *in $\mathbb{S}^p \times \mathbb{M}_0^p$.*

PROOF. (a) If $V = G_0(V)$, then for $\tilde{V} = V + \mathbb{E}_0 G(V)$, one has $M^{\tilde{V}} = M^V$, and, therefore,

$$\tilde{V} = V + \mathbb{E}_0 G(V) = G(V) = \xi + F_T(M^V) = \xi + F_T(M^{\tilde{V}}) = G(\tilde{V}).$$

So it follows from Theorem 2.3 that the pair (Y, M) given by $Y_t := \mathbb{E}_0\xi + \mathbb{E}_0 F_T(M) - F_t(M) - M_t$ and $M_t := -\mathbb{E}_t V$ solves the BSE (2.1).

(b) If $(Y, M) \in \mathbb{S}^p \times \mathbb{M}_0^p$ solves the BSE (2.1), it follows from Theorem 2.3 that $V := Y_0 - M_T$ is a fixed point of G. So

$$G_0(-M_T) = G_0(Y_0 - M_T) = G(V) - \mathbb{E}_0 G(V) = V - \mathbb{E}_0 V = -M_T^V = -M_T.$$

(c) V is a fixed point of G_0 if and only if $V + \mathbb{E}_0 G(V)$ is a fixed point of G. Therefore, the result follows from part (d) of Theorem 2.3. □

The following lemma provides a sufficient condition for F to satisfy condition (S). For $(Y, M) \in \mathbb{S}^p \times \mathbb{M}_0^p$ and $k \in \mathbb{N}$, define

$$F_t^{(k)}(Y, M) := F_t(Y^{(k,M)}, M),$$

where $Y^{(k,M)}$ is recursively given by

$$Y^{(1,M)} := Y \quad \text{and} \quad Y_t^{(k,M)} := Y_0 - F_t(Y^{(k-1,M)}, M) - M_t, \qquad k \geq 2.$$

LEMMA 2.5. *If for given $y \in L^p(\mathcal{F}_0)^d$ and $M \in \mathbb{M}_0^p$, there exist a number $k \in \mathbb{N}$ and a constant $C < 1$ such that*

$$\|F^{(k)}(Y, M) - F^{(k)}(Y', M)\|_{\mathbb{S}^p} \le C\|Y - Y'\|_{\mathbb{S}^p} \tag{2.4}$$

for all $Y, Y' \in \mathbb{S}^p$ with $Y_0 = Y'_0 = y$, then the SDE (2.2) has a unique solution $Y \in \mathbb{S}^p$.

PROOF. The mapping $Y \mapsto y - F^{(k)}(Y, M) - M$ is a contraction on $\{Y \in \mathbb{S}^p : Y_0 = y\}$. So it follows from Banach's contraction mapping theorem that there exists a unique $Y \in \mathbb{S}^p$ satisfying

$$Y = y - F^{(k)}(Y, M) - M = Y^{(k+1,M)}.$$

This implies

$$\begin{aligned} Y^{(2,M)} &= y - F_t(Y, M) - M_t = y - F_t\big(Y^{(k+1,M)}, M\big) - M_t \\ &= Y^{(k+2,M)} = y - F^{(k)}\big(Y^{(2,M)}, M\big) - M, \end{aligned}$$

from which one deduces $Y = Y^{(2,M)} = y - F(Y, M) - M$. This shows that Y solves the SDE (2.2). If $Y' \in \mathbb{S}^p$ is another solution of (2.2), then $Y' = y - F^{(k)}(Y', M) - M$, and one obtains $Y' = Y$. □

3. Existence and uniqueness of solutions under Lipschitz assumptions. In this section, we consider equations with Lipschitz coefficients and use Banach's contraction mapping theorem to show that they have unique solutions.

3.1. *General existence and uniqueness results.* We start with a result for general Lipschitz BSEs. Let us denote

$$c_2 = \frac{1}{5}, \qquad c_\infty = \frac{1}{4} \quad \text{and} \quad c_p = \frac{p-1}{4p-1} \qquad \text{for } p \in (1, \infty) \setminus \{2\}.$$

Then the following holds.

THEOREM 3.1. *Let $\xi \in L^p(\mathcal{F}_T)^d$ for some $p \in (1, \infty]$. If there exist a number $k \in \mathbb{N}$ and a constant $C < c_p$ such that*

$$\|F^{(k)}(Y, M) - F^{(k)}(Y', M')\|_{\mathbb{S}^p} \le C\big(\|Y - Y'\|_{\mathbb{S}^p} + \|M - M'\|_{\mathbb{S}^p}\big)$$

for all $Y, Y' \in \mathbb{S}^p$ and $M, M' \in \mathbb{M}_0^p$, then the BSE (2.1) has a unique solution (Y, M) in $\mathbb{S}^p \times \mathbb{M}_0^p$.

PROOF. Since $C < 1$, it follows from Lemma 2.5 that F satisfies (S). So by Theorem 2.3, it is enough to prove that G has a unique fixed point in $L^p(\mathcal{F}_T)^d$. This follows from Banach's contraction mapping theorem if we can show that G is a contraction on $L^p(\mathcal{F}_T)^d$. Since for $V \in L^p(\mathcal{F}_T)^d$, Y^V is the unique fixed point of the mapping $Y \mapsto \mathbb{E}_0 V - F(Y, M^V) - M^V$, it follows from the definition of $F^{(k)}$ that $F(Y^V, M^V) = F^{(k)}(Y^V, M^V)$. Hence, one has for all $V, V' \in L^p(\mathcal{F}_T)^d$,

$$\begin{aligned} Y_t^V - Y_t^{V'} &= y^V - y^{V'} - \{F_t^{(k)}(Y^V, M^V) - F_t^{(k)}(Y^{V'}, M^{V'})\} - (M_t^V - M_t^{V'}) \\ &= \mathbb{E}_t(V - V') - \{F_t^{(k)}(Y^V, M^V) - F_t^{(k)}(Y^{V'}, M^{V'})\}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sup_{0 \le t \le T} |Y_t^V - Y_t^{V'}| \\ &\quad \le \sup_{0 \le t \le T} |\mathbb{E}_t(V - V')| + \sup_{0 \le t \le T} |F_t^{(k)}(Y^V, M^V) - F_t^{(k)}(Y^{V'}, M^{V'})|, \end{aligned}$$

and it follows that

$$\begin{aligned} &\|Y^V - Y^{V'}\|_{\mathbb{S}^p} \\ &\quad \le \Big\| \sup_{0 \le t \le T} |\mathbb{E}_t(V - V')| \Big\|_p + \|F^{(k)}(Y^V, M^V) - F^{(k)}(Y^{V'}, M^{V'})\|_{\mathbb{S}^p} \\ &\quad \le \Big\| \sup_{0 \le t \le T} |\mathbb{E}_t(V - V')| \Big\|_p + C(\|Y^V - Y^{V'}\|_{\mathbb{S}^p} + \|M^V - M^{V'}\|_{\mathbb{S}^p}). \end{aligned}$$

In particular,

$$\|Y^V - Y^{V'}\|_{\mathbb{S}^p} \le \frac{1}{1 - C} \Big(\Big\| \sup_{0 \le t \le T} |\mathbb{E}_t(V - V')| \Big\|_p + C \|M^V - M^{V'}\|_{\mathbb{S}^p} \Big)$$

and, therefore,

$$\begin{aligned} &\|G(V) - G(V')\|_p \\ &\quad = \|F_T^{(k)}(Y^V, M^V) - F_T^{(k)}(Y^{V'}, M^{V'})\|_p \\ &\quad \le C(\|Y^V - Y^{V'}\|_{\mathbb{S}^p} + \|M^V - M^{V'}\|_{\mathbb{S}^p}) \\ &\quad \le \frac{C}{1 - C} \Big(\Big\| \sup_{0 \le t \le T} |\mathbb{E}_t(V - V')| \Big\|_p + C \|M^V - M^{V'}\|_{\mathbb{S}^p} \Big) + C \|M^V - M^{V'}\|_{\mathbb{S}^p} \\ &\quad = \frac{C}{1 - C} \Big(\Big\| \sup_{0 \le t \le T} |\mathbb{E}_t(V - V')| \Big\|_p + \|M^V - M^{V'}\|_{\mathbb{S}^p} \Big). \end{aligned}$$

By Doob's L^p-maximal inequality, if we let $C_p = p/(p-1)$ for $p \in (1, \infty)$ and $C_\infty = 1$,

$$\Big\| \sup_{0 \le t \le T} |\mathbb{E}_t(V - V') - \mathbb{E}_0(V - V')| \Big\|_p \le C_p \|V - V' - \mathbb{E}_0(V - V')\|_p$$

and

$$\Big\| \sup_{0 \le t \le T} |\mathbb{E}_t(V - V')| \Big\|_p \le C_p \|V - V'\|_p.$$

Hence,

$$\|M^V - M^{V'}\|_{\mathbb{S}^p} \le \begin{cases} 2\|V - V' - \mathbb{E}_0(V - V')\|_2 \le 2\|V - V'\|_2, & \text{for } p = 2, \\ C_p \|V - V' - \mathbb{E}_0(V - V')\|_p \le 2C_p \|V - V'\|_p, & \text{for } p \neq 2, \end{cases}$$

and

$$\|G(V) - G(V')\|_p \le \begin{cases} \dfrac{4C}{1 - C} \|V - V'\|_2, & \text{for } p = 2, \\ 3C_p \dfrac{C}{1 - C} \|V - V'\|_p, & \text{for } p \neq 2. \end{cases}$$

This shows that G is a contraction. □

REMARK 3.2. One cannot hope to obtain a general existence and uniqueness result like Theorem 3.1 for equations with path-dependent coefficients without the assumption that the Lipschitz constant C is sufficiently small. For instance, if the generator is given by $F_t(Y, M) = atY_0$ for a constant a, the BSE (2.1) takes the form

$$Y_t - a(T - t)Y_0 = \xi + M_T - M_t. \tag{3.1}$$

This is a variant of the equation studied in Example 3.1 of Delong and Imkeller (2010a), who noticed that time-delayed BSDEs with Lipschtitz coefficients are not always well-posed. Obviously, $F(Y, M)$ is Lipschitz in (Y, M). But if one sets $t = 0$ and takes expectation on both sides of (3.1), one obtains $(1 - aT)Y_0 = \mathbb{E}_0\xi$. This shows that for $aT = 1$ and $\mathbb{E}_0\xi \neq 0$, (3.1) cannot have a solution. On the other hand, if $aT = 1$ and $\mathbb{E}_0\xi = 0$ then $Y_t = (1 - t/T)Y_0 + \mathbb{E}_t\xi$ and $M_t = -\mathbb{E}_t\xi$ defines a solution for any initial value $Y_0 \in L^p(\mathcal{F}_0)^d$. So in this case, (3.1) has infinitely many solutions in $\mathbb{S}^p \times \mathbb{M}_0^p$.

If the generator is of integral form $F_t(Y, M) = \int_0^t f(s, Y, M)\, ds$ for a driver,

$$f : [0, T] \times \Omega \times \mathbb{S}^p \times \mathbb{M}_0^p \to \mathbb{R}^d, \tag{3.2}$$

the BSE (2.1) becomes a BSDE of the general form

$$Y_t = \xi + \int_t^T f(s, Y, M)\, ds + M_T - M_t. \tag{3.3}$$

If for a RCLL measurable processe X, one denotes

$$\|X\|_{\mathbb{S}^p_{[0,t]}} := \Big\| \sup_{0 \le s \le t} |X_t| \Big\|_p,$$

the following holds.

PROPOSITION 3.3. *Let $\xi \in L^p(\mathcal{F}_T)^d$ for some $p \in (1,\infty]$. Then the BSDE* (3.3) *has a unique solution $(Y,M) \in \mathbb{S}^p \times \mathbb{M}_0^p$ for every driver of the form* (3.2) *satisfying the following conditions*:

(i) *For all $(Y,M) \in \mathbb{S}^p \times \mathbb{M}_0^p$, $f(\cdot,Y,M)$ is progressively measurable with $\int_0^T \|f(t,0,0)\|_p \, dt < \infty$.*

(ii) *There exist nonnegative constants*

$$C_1 > 0 \quad \text{and} \quad C_2 < \frac{c_p C_1}{e^{C_1 T} - 1}$$

such that

$$\begin{aligned}&\|f(t,Y,M) - f(t,Y',M')\|_p \\ &\quad \le C_1 \|Y - Y_0 + M - (Y' - Y_0' + M')\|_{\mathbb{S}^p_{[0,t]}} + C_2(\|Y_0 - Y_0'\|_p + \|M - M'\|_{\mathbb{S}^p})\end{aligned}$$

for all $(Y,M),(Y',M') \in \mathbb{S}^p \times \mathbb{M}_0^p$.

PROOF. Let $q = p/(p-1) \in [1,\infty)$. It follows from the assumptions that for all $(Y,M) \in \mathbb{S}^p \times \mathbb{M}_0^p$,

$$\begin{aligned}&\left\| \int_0^T |f(t,Y,M)| \, dt \right\|_p \\ &\quad = \sup_{\|X\|_q \le 1} \int_0^T \mathbb{E}[|f(t,Y,M)||X|] \, dt \\ &\quad \le \sup_{\|X\|_q \le 1} \int_0^T \|f(t,Y,M)\|_p \|X\|_q \, dt \\ &\quad = \int_0^T \|f(t,Y,M)\|_p \, dt \\ &\quad \le \int_0^T \|f(t,0,0)\|_p \, dt + TC_1 \|Y - Y_0 + M\|_{\mathbb{S}^p} + TC_2(\|Y_0\|_p + \|M\|_{\mathbb{S}^p}) < \infty.\end{aligned}$$

So $F_t(Y,M) := \int_0^t f(s,Y,M) \, ds$ is a well-defined mapping from $\mathbb{S}^p \times \mathbb{M}_0^p$ to $\mathbb{S}_0^p$ for all $p \in (1,\infty]$.

For given $Y, Y' \in \mathbb{S}^p$ and $M, M' \in \mathbb{M}_0^p$, set

$$\begin{aligned}\delta &:= \frac{C_2}{C_1}(\|Y_0 - Y_0'\|_p + \|M - M'\|_{\mathbb{S}^p}), \\ H_t^0 &:= H^0 := 2(\|Y - Y'\|_{\mathbb{S}^p} + \|M - M'\|_{\mathbb{S}^p}), \\ H_t^k &:= \|F^{(k)}(Y,M) - F^{(k)}(Y',M')\|_{\mathbb{S}^p_{[0,t]}}.\end{aligned}$$

Then

$$H_t^k \le \int_0^t \|f(s, Y^{(k,M)}, M) - f(s, (Y')^{(k,M')}, M')\|_p \, ds$$
$$\le \int_0^t (C_1 H_s^{k-1} + C_2(\|Y_0 - Y_0'\|_p + \|M - M'\|_{\mathbb{S}^p}))\, ds$$
$$\le C_1 \int_0^t (H_s^{k-1} + \delta)\, ds,$$

and by iteration,

$$H_t^k \le \frac{(C_1 t)^k}{k!} H^0 + \left(C_1 t + \cdots + \frac{(C_1 t)^k}{k!}\right)\delta.$$

In particular,

$$\|F^{(k)}(Y, M) - F^{(k)}(Y', M)\|_{\mathbb{S}^p} \le 2\frac{(C_1 T)^k}{k!}(\|Y - Y'\|_{\mathbb{S}^p} + \|M - M'\|_{\mathbb{S}^p})$$
$$+ (e^{C_1 T} - 1)\frac{C_2}{C_1}(\|Y_0 - Y_0'\|_p + \|M - M'\|_{\mathbb{S}^p}).$$

So for k large enough, there exists a constant $C < c_p$ such that

$$\|F^{(k)}(Y, M) - F^{(k)}(Y', M')\|_{\mathbb{S}^p} \le C(\|Y - Y'\|_{\mathbb{S}^p} + \|M - M'\|_{\mathbb{S}^p}),$$

and the proposition follows from Theorem 3.1. □

REMARK 3.4. The backward stochastic dynamics

$$Y_t = \int_t^T f_0(s, Y_s, L(M)_s)\, ds + \int_t^T f(s, Y_s)\, dB_s - (M_T - M_t)$$

studied by Liang, Lyons and Qian (2011) can be viewed as a BSE with generator

$$F_t(Y, M) = \int_0^t f_0(s, Y_s, L(M)_s)\, ds + \int_0^t f(s, Y_s)\, dB_s.$$

But it also fits into the framework (3.3) if the transformation

$$\tilde{M}_t = \int_0^t f(s, Y_s)\, dB_s, -M_t \quad \text{and}$$
$$\tilde{f}(t, Y, \tilde{M}) = f_0\left(t, Y_t, L\left(\int f(s, Y_s)\, dB_s - \tilde{M}\right)_t\right)$$

is applied. In addition, (3.3) includes BSDEs with drivers depending on the past or future of the processes Y and M, such as the time-delayed BSDEs of Delong and Imkeller (2010a, 2010b) or the anticipating BSDEs of Peng and Yang (2009). Previous existence and uniqueness results like Theorem 3.3 of Liang, Lyons and Qian (2011), Theorem 2.1 of Delong and Imkeller (2010a) or Theorem 2.1 of Delong and Imkeller (2010b), can all be recovered as special cases of Proposition 3.3.

REMARK 3.5. Let $f : [0,T] \times \Omega \times \mathbb{S}^p \times \mathbb{M}_0^p \to \mathbb{R}^d$ be a driver satisfying condition (i) of Proposition 3.3 for some $p \in (1,\infty]$. If there exist nonnegative constants D_1, D_2 such that

$$\|f(t,Y,M) - f(t,Y',M')\|_p \le D_1 \|Y - Y'\|_{\mathbb{S}^p_{[0,t]}} + D_2 \|M - M'\|_{\mathbb{S}^p}$$

for all $Y, Y' \in \mathbb{S}^p$ and $M, M' \in \mathbb{M}_0^p$, then

$$\|f(t,Y,M) - f(t,Y',M')\|_p \le D_1 \|Y - Y_0 + M - (Y' - Y'_0 + M')\|_{\mathbb{S}^p_{[0,t]}} + D_1 \|Y_0 - Y'_0\|_p + (D_1 + D_2)\|M - M'\|_{\mathbb{S}^p}.$$

So the assumptions of Proposition 3.3 only hold if the constants D_1 and D_2 are small enough, or alternatively, the maturity T is sufficiently short. This is in line with Remark 3.2 above [note that (3.1) is a path-dependent BSDE of the form (3.3) with $f(t,Y,M) = aY_0$].

The following corollary gives conditions under which it directly follows from Proposition 3.3 that the BSDE (3.3) has a unique solution for arbitrary Lipschitz constant and maturity. More examples of (3.3) admitting solutions under general Lipschitz assumptions are given in Section 3.2 below.

COROLLARY 3.6. *Let $p \in (1,\infty]$ and consider a terminal condition $\xi \in L^p(\mathcal{F}_T)^d$ together with a driver f of the form (3.2) fulfilling condition (i) of Proposition 3.3 such that $f(t,Y,M) = h(t, Y - Y_0 + M)$ for a mapping $h : [0,T] \times \Omega \times \mathbb{S}_0^p \to \mathbb{R}^d$. If*

$$\|h(t,X) - h(t,X')\|_p \le C \|X - X'\|_{\mathbb{S}^p_{[0,t]}}, \qquad X, X' \in \mathbb{S}_0^p$$

for a constant $C \ge 0$, then the BSDE (3.3) has a unique solution $(Y,M) \in \mathbb{S}^p \times \mathbb{M}_0^p$.

3.2. *Generalized Lipschitz BSDEs based on a Brownian motion and a Poisson random measure.* Let W be an n-dimensional Brownian motion and N an independent Poisson random measure on $[0,T] \times E$ for $E = \mathbb{R}^m \setminus \{0\}$ with an intensity measure of the form $dt\mu(dx)$ for a measure μ over the Borel σ-algebra $\mathcal{B}(E)$ of E satisfying

$$\int_E (1 \wedge |x|^2)\mu(dx) < \infty.$$

Denote by $\tilde{N}$ the compensated random measure $N(dt,dx) - dt\mu(dx)$, and assume that, for $A \in \mathcal{B}(E)$ with $\mu(A) < \infty$, $\tilde{N}([0,t] \times A)$ and W are martingales with respect to $\mathbb{F}$. We need the following spaces of integrands:

(i) $\mathbb{H}^2$: all $\mathbb{R}^{d\times n}$-valued predictable processes Z satisfying

$$\|Z\|_{\mathbb{H}^2} := \left(\int_0^T \mathbb{E}|Z_t|^2\,dt\right)^{1/2} < \infty.$$

(ii) $L^2(\tilde{N})$: all $\mathcal{P}\otimes\mathcal{B}(E)$-measurable mappings $U : [0,T]\times\Omega\times E \to \mathbb{R}^d$ such that

$$\|U\|_{L^2(\tilde{N})} := \left(\int_0^T \int_E \mathbb{E}\big|U_t(x)\big|^2\mu(dx)\,dt\right)^{1/2} < \infty,$$

where $\mathcal{P}$ is the σ-algebra of $\mathbb{F}$-predictable subsets of $[0,T]\times\Omega$.

Any square-integrable $\mathbb{F}$-martingale $M \in \mathbb{M}_0^2$ has a unique representation of the form

$$M_t = \int_0^t Z_s^M\,dW_s + \int_0^t \int_E U_s^M(x)\tilde{N}(ds,dx) + K_t^M \tag{3.4}$$

for a triple $(Z^M, U^M, K^M) \in \mathbb{H}^2 \times L^2(\tilde{N}) \times \mathbb{M}_0^2$ such that K^M is strongly orthogonal to W and $\tilde{N}$ [see, e.g., Jacod (1979)]. This makes it possible to consider BSDEs

$$Y_t = \xi + \int_t^T f\big(s,Y,Z^M,U^M\big)\,ds + M_T - M_t \tag{3.5}$$

for terminal conditions $\xi \in L^2(\mathcal{F}_T)^d$ and drivers

$$f : [0,T]\times\Omega\times\mathbb{S}^2\times\mathbb{H}^2\times L^2(\tilde{N}) \to \mathbb{R}^d. \tag{3.6}$$

In the special case where the filtration $\mathbb{F}$ is generated by W and N, the orthogonal part K^M in the representation (3.4) vanishes [see, e.g., Ikeda and Watanabe (1989)], and as a result, (3.5) can be written as

$$\begin{aligned} Y_t = \xi &+ \int_t^T f\big(s,Y,Z^M,U^M\big)\,ds + \int_t^T Z_s^M\,dW_s \\ &+ \int_t^T \int_E U_s^M(x)\tilde{N}(ds,dx). \end{aligned} \tag{3.7}$$

This generalizes the classical BSDEs of Pardoux and Peng (1990) and Tang and Li (1994), which have drivers that at time s only depend on the realizations $Y_s(\omega)$, $Z_s^M(\omega)$, $U_s^M(\omega)$, to equations with functional drivers that can depend on the full processes Y, Z^M and U^M.

In the rest of this subsection, we consider different specifications of (3.5) with drivers depending on the future, present or past of the processes Y, Z^M and U^M. In all instances, we are able to derive the existence of a unique solution for an arbitrary Lipschitz constant and maturity. In the following proposition, the driver can depend on the present and future of Y, Z^M and U^M, but not on their past—this is ruled out by condition (ii). For its proof, we need the isometry

$$\mathbb{E}|M_t|^2 = \int_0^t \mathbb{E}|Z_s^M|^2\,ds + \int_0^t \int_E \mathbb{E}|U_s^M(x)|^2 \mu(dx)\,ds + \mathbb{E}|K_t^M|^2 \tag{3.8}$$

[see, e.g., Jacod (1979)].

PROPOSITION 3.7. *The BSDE* (3.5) *has a unique solution* $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$ *for every terminal condition* $\xi \in L^2(\mathcal{F}_T)^d$ *and driver*

$$f : [0, T] \times \Omega \times \mathbb{S}^2 \times \mathbb{H}^2 \times L^2(\tilde{N}) \to \mathbb{R}^d$$

satisfying the following two conditions:

(i) *For all* $(Y, Z, U) \in \mathbb{S}^2 \times \mathbb{H}^2 \times L^2(\tilde{N})$, $f(t, Y, Z, U)$ *is progressively measurable with* $\int_0^T \|f(t, 0, 0, 0\|_2\,dt < \infty$.

(ii) *There exists a constant* $C \geq 0$ *such that*

$$\int_t^T \|f(s, Y, Z, U) - f\big(s, Y', Z', U'\big)\|_2\,ds$$
$$\leq C \int_t^T \|Y_s - Y_s'\|_2 + \|Z_s - Z_s'\|_2 + \|U_s - U_s'\|_{L^2(\mathbb{P}\otimes\mu)}\,ds$$

for all $t \in [0, T]$ *and* $(Y, Z, U), (Y', Z', U') \in \mathbb{S}^2 \times \mathbb{H}^2 \times L^2(\tilde{N})$.

PROOF. Choose $\delta > 0$ so that

$$C\sqrt{3\delta(\delta+1)} < \frac{1}{5} \quad \text{and} \quad k := T/\delta \in \mathbb{N}.$$

By (3.8), one has for every $M \in \mathbb{M}_0^2$,

$$\left(\int_0^t \|Z_s^M\|_2 + \|U_s^M\|_{L^2(\mathbb{P}\otimes\mu)}\,ds\right)^2$$
$$\leq t \int_0^t \big(\|Z_s^M\|_2 + \|U_s^M\|_{L^2(\mathbb{P}\otimes\mu)}\big)^2\,ds$$
$$\leq 2t \int_0^t \|Z_s^M\|_2^2 + \|U_s^M\|_{L^2(\mathbb{P}\otimes\mu)}^2\,ds \leq 2t\|M_t\|_2^2.$$

Therefore, one obtains from the assumptions for all $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$

$$\left\|\int_{T-\delta}^T |f\big(s, Y, Z^M, U^M\big)|\,ds\right\|_2$$
$$\leq \int_{T-\delta}^T \|f\big(s, Y, Z^M, U^M\big)\|_2\,ds$$

$$\leq \int_{T-\delta}^{T} \|f(s,0,0,0)\|_2\,ds$$
$$+ C\int_{T-\delta}^{T} \big(\|Y_s\|_2 + \|Z_s^M\|_2 + \|U_s^M\|_{L^2(\mathbb{P}\otimes\mu)}\big)\,ds < \infty,$$

where the first inequality follows from the same argument as in the proof of Proposition 3.3. In particular, for every pair $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$,

$$F_t(Y, M) := \int_0^t f\big(s, Y, Z^M, U^M\big) 1_{[T-\delta,T]}(s)\,ds$$

defines a process in $\mathbb{S}_0^2$. Furthermore, one has

$$\begin{aligned}
&\|F(Y,M) - F\big(Y',M'\big)\|_{\mathbb{S}^2}\\
&\quad\leq \left\| \int_{T-\delta}^{T} \big|f\big(s,Y,Z^M,U^M\big) - f\big(s,Y',Z^{M'},U^{M'}\big)\big|\,ds \right\|_2\\
&\quad\leq \int_{T-\delta}^{T} \|f\big(s,Y,Z^M,U^M\big) - f\big(s,Y',Z^{M'},U^{M'}\big)\|_2\,ds\\
&\quad\leq C\int_{T-\delta}^{T} \|Y_s - Y_s'\|_2 + \|Z_s^M - Z_s^{M'}\|_2 + \|U_s^M - U_s^{M'}\|_{L^2(\mathbb{P}\otimes\mu)}\,ds\\
&\quad\leq C\sqrt{\delta\int_{T-\delta}^{T} \big(\|Y_s - Y_s'\|_2 + \|Z_s^M - Z_s^{M'}\|_2 + \|U_s^M - U_s^{M'}\|_{L^2(\mathbb{P}\otimes\mu)}\big)^2\,ds}\\
&\quad\leq C\sqrt{3\delta\int_{T-\delta}^{T} \|Y_s - Y_s'\|_2^2 + \|Z_s^M - Z_s^{M'}\|_2^2 + \|U_s^M - U_s^{M'}\|_{L^2(\mathbb{P}\otimes\mu)}^2\,ds}\\
&\quad\leq C\sqrt{3\delta^2\|Y - Y'\|_{\mathbb{S}^2}^2 + 3\delta\|M - M'\|_{\mathbb{S}^2}^2}\\
&\quad\leq C\sqrt{3\delta(\delta+1)}\big(\|Y - Y'\|_{\mathbb{S}^2} + \|M - M'\|_{\mathbb{S}^2}\big)
\end{aligned}$$

for all $(Y,M), (Y',M') \in \mathbb{S}^2 \times \mathbb{M}_0^2$. Since $C\sqrt{3\delta(\delta+1)} < 1/5$, one obtains from Theorem 3.1 that the BSDE

$$Y_t = \xi + \int_t^T f\big(s, Y, Z^M, U^M\big) 1_{[T-\delta,T]}(s)\,ds + M_T - M_t$$

has a unique solution $(Y^{(k)}, M^{(k)})$ in $\mathbb{S}^2 \times \mathbb{M}_0^2$. Now, consider the BSDE

$$Y_t = Y_{T-\delta}^{(k)} + \int_t^{T-\delta} f^{(k-1)}\big(s, Y, Z^M, U^M\big) 1_{[T-2\delta,T-\delta]}(s)\,ds + M_{T-\delta} - M_t \tag{3.9}$$

on the time interval $[0, T-\delta]$, where $f^{(k-1)}$ is given by

$$f^{(k-1)}(s,Y,Z,U) := f\big(s, (Y,Z,U)1_{[0,T-\delta)} + \big(Y^{(k)}, Z^{M^{(k)}}, U^{M^{(k)}}\big)1_{[T-\delta,T]}\big).$$

Then the conditions (i)–(ii) still hold. So (3.9) has a unique solution $(Y^{(k-1)}, M^{(k-1)})$ in $\mathbb{S}^2 \times \mathbb{M}_0^2$ over the time interval $[0, T-\delta]$. Repeating the same argument, one obtains solutions $(Y^{(j)}, M^{(j)})$, $j = 1, \ldots, k$. If one sets $Y_t := Y_t^{(1)}$, $M_t := M_t^{(1)}$ for $0 \le t \le \delta$ and $Y_t := Y_t^{(j)}$, $M_t - M_{(j-1)\delta} := M_t^{(j)} - M_{(j-1)\delta}^{(j)}$ for $(j-1)\delta < t \le j\delta$, $j = 2, \ldots, k$, then $(Z_t^M, U_t^M) = (Z_t^{M^{(j)}}, U_t^{M^{(j)}})$ for $(j-1)\delta < t \le j\delta$. Since this construction is backwards in time and by condition (ii), $f(t, Y, Z^M, U^M)$ cannot depend on the past of the processes Y, Z^M and U^M, the pair (Y, M) forms a unique solution of (3.5) in $\mathbb{S}^2 \times \mathbb{M}_0^2$. □

REMARK 3.8. The assumptions of Proposition 3.7 allow for drivers f such that $f(t, Y, Z, U)$ depends on the future of the processes Y, Z, U in a general $\mathcal{F}_t$-measurable way. This covers BSDEs with anticipating drivers of the form

$$-dY_t = f(t, Y_t, Z_t, \mathbb{E}_t Y_{t+\delta(t)}, \mathbb{E}_t Z_{t+\zeta(t)})\,dt + Z_t\,dW_t, \qquad t \in [0, T]$$
$$(Y_t, Z_t) = (\xi_t, \eta_t), \qquad t \in [T, T+K]$$

or more generally,

$$\begin{aligned} -dY_t &= f(t, Y_t, Z_t, Y_{t+\delta(t)}, Z_{t+\zeta(t)})\,dt + Z_t\,dW_t, \qquad t \in [0, T] \\ (Y_t, Z_t) &= (\xi_t, \eta_t), \qquad t \in [T, T+K] \end{aligned} \tag{3.10}$$

for a Brownian motion $(W_t)_{t\in[0,T]}$, continuous functions $\delta, \zeta : [0, T] \to \mathbb{R}_+$, and stochastic processes $(\xi_t)_{t\in[T,T+K]}$, $(\eta_t)_{t\in[T,T+K]}$. Equations of the form (3.10) were introduced by Peng and Yang (2009) as duals of time-delayed forward SDEs. Their existence and uniqueness result, Theorem 4.2, as well as extensions for equations with jumps, can easily be derived from Proposition 3.7.

As an immediate consequence of Proposition 3.7 one obtains the following result for BSDEs with functional drivers depending on Y_s, Z_s^M and U_s^M.

COROLLARY 3.9. *The BSDE*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s^M, U_s^M)\,ds + M_T - M_t \tag{3.11}$$

has a unique solution $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$ *for every terminal condition* $\xi \in L^2(\mathcal{F}_T)^d$ *and driver*

$$f : [0, T] \times \Omega \times L^2(\mathcal{F}_T)^d \times L^2(\mathcal{F}_T)^{d\times n} \times L^2(\Omega \times E, \mathcal{F}_T \otimes \mathcal{B}(E), \mathbb{P} \otimes \mu; \mathbb{R}^d) \\ \to \mathbb{R}^d$$

satisfying the following two conditions:

(i) *For all* $(Y, Z, U) \in \mathbb{S}^2 \times \mathbb{H}^2 \times L^2(\tilde{N})$, $f(t, Y_t, Z_t, U_t)$ *is progressively measurable with* $\int_0^T \|f(t, 0, 0, 0\|_2\,dt < \infty$.

(ii) *There exists a constant* $C \geq 0$ *such that*

$$\begin{aligned}\|f(t, Y_t, Z_t, U_t) - f(t, Y'_t, Z'_t, U'_t)\|_2 \\ \leq C(\|Y_t - Y'_t\|_2 + \|Z_t - Z'_t\|_2 + \|U_t - U'_t\|_{L^2(\mathbb{P}\times\mu)})\end{aligned}$$

for all $t \in [0, T]$ *and* $(Y, Z, U), (Y', Z', U') \in \mathbb{S}^2 \times \mathbb{H}^2 \times L^2(\tilde{N})$.

Corollary 3.9 can be used in conjunction with Theorem 2.3 to deduce that the following time-delayed BSDE has a unique solution. This extends Theorem 2.3 of Delong and Imkeller (2010a) to the case of multidimensional BSDEs with jumps and functional dependence in the driver. In addition, our integrability condition on the terminal condition is a bit weaker.

PROPOSITION 3.10. *Let* $\xi \in L^2(\mathcal{F}_T)^d$ *and* ν *be a finite Borel measure on* $[0, T]$. *Then the BSDE*

$$Y_t = \xi + \int_t^T \int_{[0,s]} g(s - r, Z^M_{s-r}, U^M_{s-r})\nu(dr)\, ds + M_T - M_t \tag{3.12}$$

has a unique solution $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}^2_0$ *for every mapping*

$$g : [0, T] \times \Omega \times L^2(\mathcal{F}_T)^{d\times n} \times L^2(\Omega \times E, \mathcal{F}_T \otimes \mathcal{B}(E), \mathbb{P} \otimes \mu; \mathbb{R}^d) \to \mathbb{R}^d$$

satisfying the following two conditions:

(i) *For all* $(Z, U) \in \mathbb{H}^2 \times L^2(\tilde{N})$, $g(t, Z_t, U_t)$ *is progressively measurable*, *and* $\int_0^T \|g(t, 0, 0\|_2\, dt < \infty$.

(ii) *There exists a constant* $C \geq 0$ *such that*

$$\|g(t, Z_t, U_t) - g(t, Z'_t, U'_t)\|_2 \leq C(\|Z_t - Z'_t\|_2 + \|U_t - U'_t\|_{L^2(\mathbb{P}\otimes\mu)})$$

for all $t \in [0, T]$ *and* $(Z, U), (Z', U') \in \mathbb{H}^2 \times L^2(\tilde{N})$.

PROOF. The generator corresponding to the BSDE (3.12) is given by

$$F_t(M) = \int_0^t \int_{[0,s]} g(s - r, Z^M_{s-r}, U^M_{s-r})\nu(dr)\, ds.$$

Since it does not depend on Y, it satisfies condition (S). So, by Theorem 2.3, it is enough to show that there exists a unique $V \in L^2(\mathcal{F}_T)^d$ such that

$$V = G(V) = \xi + \int_0^T \int_{[0,s]} g(s - r, Z^{M^V}_{s-r}, U^{M^V}_{s-r})\nu(dr)\, ds. \tag{3.13}$$

From Fubini's theorem and a change of variable, one obtains

$$\int_0^T \int_{[0,s]} g(s - r, Z^{M^V}_{s-r}, U^{M^V}_{s-r})\nu(dr)\, ds = \int_0^T \nu([0, T - s])g(s, Z^{M^V}_s, U^{M^V}_s)\, ds.$$

Since the driver $h(s, Z_s, U_s) = \nu([0, T - s])g(s, Z_s, U_s)$ satisfies the conditions of Corollary 3.9, the BSDE

$$Y_t = \xi + \int_t^T h(s, Z_s^M, U_s^M)\, ds + M_T - M_t$$

has a unique solution in $\mathbb{S}^2 \times \mathbb{M}_0^2$. The associated generator, $\tilde{F}_t(M) = \int_0^T h(s, Z_s^M, U_s^M)\, ds$, does not depend on Y either. So it also satisfies condition (S), and one obtains from Theorem 2.3 that there exists a unique $V \in L^2(\mathcal{F}_T)^d$ satisfying (3.13). This completes the proof. □

As special cases of Corollary 3.9 and Proposition 3.10, one obtains existence and uniqueness results for McKean–Vlasov-type BSDEs with drivers depending on the realizations $Y_s(\omega)$, $Z_s^M(\omega)$, $U_s^M(\omega)$ as well as the distributions $\mathcal{L}(Y_s)$, $\mathcal{L}(Z_s^M)$, $\mathcal{L}(U_s^M)$ of Y_s, Z_s^M and U_s^M. We recall that if $\mathcal{M}(\mathcal{X})$ is the set of all probability measures defined on the Borel σ-algebra of a normed vector space $(\mathcal{X}, \|\cdot\|)$, the p-Wasserstein metric on $\mathcal{M}_p(\mathcal{X}) := \{\eta \in \mathcal{M}(\mathcal{X}) : \int_{\mathcal{X}} \|x\|^p \eta(dx) < \infty\}$ is given by

$$\mathcal{W}_p(\eta, \eta') := \inf\biggl\{\int_{\mathcal{X}\times\mathcal{X}} \|x - x'\|^p \psi(dx, dx') : \\ \psi \in \mathcal{M}_p(\mathcal{X} \times \mathcal{X}) \text{ with marginals } \eta \text{ and } \eta'\biggr\}^{1/p}.$$

The following is a consequence of Corollary 3.9 and generalizes the existence and uniqueness result for mean-field BSDEs of Buckdahn, Li and Peng (2009).

COROLLARY 3.11. *Consider a BSDE of the form*

$$\begin{aligned} Y_t = \xi + \int_t^T f(s, Y_s, Z_s^M, U_s^M, \mathcal{L}(Y_s), \mathcal{L}(Z_s^M), \mathcal{L}(U_s^M))\, ds \\ + M_T - M_t \end{aligned} \tag{3.14}$$

for a terminal condition $\xi \in L^2(\mathcal{F}_T)^d$ *and a driver* f *from* $[0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \times L^2(E, \mathcal{B}(E), \mu; \mathbb{R}^d) \times \mathcal{M}_2(\mathbb{R}^d) \times \mathcal{M}_2(\mathbb{R}^{d\times n}) \times \mathcal{M}_2(L^2(E, \mathcal{B}(E), \mu; \mathbb{R}^d))$ *to* $\mathbb{R}^d$. *Then* (3.14) *has a unique solution* (Y, M) *in* $\mathbb{S}^2 \times \mathbb{M}_0^2$ *if for fixed* $(y, z, u, \eta, \zeta, \kappa)$ *in* $\mathbb{R}^d \times \mathbb{R}^{d\times n} \times L^2(E, \mathcal{B}(E), \mu; \mathbb{R}^d) \times \mathcal{M}_2(\mathbb{R}^d) \times \mathcal{M}_2(\mathbb{R}^{d\times n}) \times \mathcal{M}_2(L^2(E, \mathcal{B}(E), \mu; \mathbb{R}^d))$, $f(\cdot, y, z, u, \eta, \zeta, \kappa)$ *is progressively measurable*, *and the following two conditions hold*:

(i) $\int_0^T \|f(t, 0, 0, 0, \mathcal{L}(0), \mathcal{L}(0)), \mathcal{L}(0)\|_2\, dt < \infty$.

(ii) *There exists a constant* $C \geq 0$ *such that*

$$\begin{aligned} &|f(t, y, z, u, \eta, \zeta, \kappa) - f(t, y', z', u', \eta', \zeta', \kappa')| \\ &\quad \leq C(|y - y'| + |z - z'| + \|u - u'\|_{L^2(\mu)} + \mathcal{W}_2(\eta, \eta') \\ &\qquad + \mathcal{W}_2(\zeta, \zeta') + \mathcal{W}_2(\kappa, \kappa')). \end{aligned}$$

PROOF. It follows from the assumptions that the driver f is progressively measurable in (t,ω) and continuous in $(y,z,u,\eta,\zeta,\kappa)$. Since

$$\mathbb{R}^d \times \mathbb{R}^{d\times n} \times L^2(E,\mathcal{B}(E),\mu;\mathbb{R}^d) \times \mathcal{M}_2(\mathbb{R}^d) \times \mathcal{M}_2(\mathbb{R}^{d\times n}) \\ \times \mathcal{M}_2(L^2(E,\mathcal{B}(E),\mu;\mathbb{R}^d))$$

is a separable metric space, one obtains from Lemma 4.51 of Aliprantis and Border (2006) that f is jointly measurable in all its arguments. This implies that $f(t,Y_t,Z_t,U_t,\mathcal{L}(Y_t),\mathcal{L}(Z_t),\mathcal{L}(U_t))$ is progressively measurable for every triple $(Y,Z,U) \in \mathbb{S}^2 \times \mathbb{H}^2 \times U \in L^2(\tilde{N})$. It follows that condition (i) of Corollary 3.9 holds, and it just remains to show that

$$\|f(t,Y_t,Z_t,U_t,\mathcal{L}(Y_t),\mathcal{L}(Z_t),\mathcal{L}(U_t)) - f(t,Y'_t,Z'_t,U'_t,\mathcal{L}(Y'_t),\mathcal{L}(Z'_t),\mathcal{L}(U'_t)\|_2 \\ \le D(\|Y_t - Y'_t\|_2 + \|Z_t - Z'_t\|_2 + \|U_t - U'_t\|_{L^2(\mathbb{P}\times\mu)})$$

for some constant D. But this is a consequence of condition (ii) since one has

$$\mathcal{W}_2^2(\mathcal{L}(Y_t),\mathcal{L}(Y'_t)) \le \int_{\mathbb{R}^d\times\mathbb{R}^d} |y-y'|^2 \mathcal{L}(Y_t,Y'_t)(dy,dy') = \|Y_t - Y'_t\|_2^2,$$

and analogously,

$$\mathcal{W}_2^2(\mathcal{L}(Z_t),\mathcal{L}(Z'_t)) \le \|Z_t - Z'_t\|_2^2,$$
$$\mathcal{W}_2^2(\mathcal{L}(U_t),\mathcal{L}(U_t)) \le \|U_t - U'_t\|_{L^2(\mathbb{P}\times\mu)}^2. \qquad \square$$

Using the same arguments as in the proof of Corollary 3.11, one obtains from Proposition 3.10 the following result for time-delayed McKean–Vlasov-type BSDEs.

COROLLARY 3.12. *Consider a BSDE of the form*

$$Y_t = \xi + \int_t^T \int_0^s g(s-r, Z^M_{s-r}, U^M_{s-r}, \mathcal{L}(Z^M_{s-r}), \mathcal{L}(U^M_{s-r}))\nu(dr)\,ds \\ + M_T - M_t \tag{3.15}$$

for a terminal condition $\xi \in L^2(\mathcal{F}_T)^d$, *a finite Borel measure* ν *on* $[0,T]$ *and a mapping*

$$g : [0,T] \times \Omega \times \mathbb{R}^{d\times n} \times L^2(E,\mathcal{B}(E),\mu;\mathbb{R}^d) \times \mathcal{M}_2(\mathbb{R}^{d\times n}) \\ \times \mathcal{M}_2(L^2(E,\mathcal{B}(E),\mu;\mathbb{R}^d)) \to \mathbb{R}^d.$$

Then (3.15) *has a unique solution* (Y,M) *in* $\mathbb{S}^2 \times \mathbb{M}_0^2$ *if for fixed*

$$(z,u,\zeta,\kappa) \in \mathbb{R}^{d\times n} \times L^2(E,\mathcal{B}(E),\mu;\mathbb{R}^d) \times \mathcal{M}_2(\mathbb{R}^{d\times n}) \\ \times \mathcal{M}_2(L^2(E,\mathcal{B}(E),\mu;\mathbb{R}^d)),$$

$g(\cdot,z,u,\zeta,\kappa)$ *is progressively measurable, and the following two conditions hold*:

(i) $\int_0^T \|g(t,0,0,\mathcal{L}(0)),\mathcal{L}(0)\|_2\,dt < \infty$.

(ii) *There exists a constant* $C \ge 0$ *such that*

$$\begin{aligned}&\left|g(t,z,u,\zeta,\kappa) - g(t,z',u',\zeta',\kappa')\right|\\&\quad\le C\big(|z-z'| + \|u-u'\|_{L^2(\mu)} + \mathcal{W}_2(\zeta,\zeta') + \mathcal{W}_2(\kappa,\kappa')\big).\end{aligned}$$

4. Existence of solutions to non-Lipschitz equations. In this section, we use compactness assumptions to derive existence results for different BSEs and BSDEs with non-Lipschitz coefficients. To find compact sets in the space $L^2(\mathcal{F}_T)^d$, we assume in all of Section 4 that the sample space Ω is an infinite-dimensional separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$. We fix a complete orthonormal system e_j, $j \in \mathbb{N}$, of Ω together with positive numbers λ_j, $j \in \mathbb{N}$ satisfying $\sum_{j\in\mathbb{N}} \lambda_j < \infty$. Then $Qe_j := \lambda_j e_j$ defines a positive self-adjoint trace class operator $Q : \Omega \to \Omega$. The mean zero Gaussian measure $\mathbb{P}$ with covariance Q is the unique probability measure on the Borel σ-algebra $\mathcal{B}(\Omega)$ of Ω under which the functions $\phi_j(\omega) = \langle\omega, e_j\rangle$, $j \in \mathbb{N}$, are independent normal random variables with mean zero and variance λ_j, $j \in \mathbb{N}$; see Da Prato (2006) for details. The map $e_j \mapsto \phi_j/\sqrt{\lambda_j}$ has a unique continuous linear extension $W : \Omega \to L^2(\Omega)$, called white noise mapping. It is an isometry between Ω and the closed subspace of $L^2(\Omega)$ generated by ϕ_j, $j \in \mathbb{N}$.

To define the Sobolev space $W^{1,2}(\Omega)$ in $L^2(\Omega)$, let $\mathcal{E}(\Omega)$ be the linear span of all real and imaginary parts of functions of the form $\omega \mapsto e^{i\langle\omega,\eta\rangle}$ for some $\eta \in \Omega$. For $\varphi \in \mathcal{E}(\Omega)$, we denote by $D_j\varphi$ the derivative of φ in the direction of e_j:

$$D_j\varphi(\omega) = \lim_{\varepsilon\to 0} \frac{\varphi(\omega + \varepsilon e_j) - \varphi(\omega)}{\varepsilon}.$$

The mapping $D : \mathcal{E}(\Omega) \subseteq L^2(\Omega) \to L^2(\Omega;\Omega)$, $\varphi \mapsto D\varphi := \sum_{j\in\mathbb{N}} D_j\varphi e_j$ is closable. We maintain the notation D for the closure of D and denote its domain by $W^{1,2}(\Omega)$. Endowed with the inner product

$$\langle\varphi,\psi\rangle_{W^{1,2}} := \mathbb{E}\big(\varphi\psi + \langle D\varphi, D\psi\rangle\big),$$

the Sobolev space $W^{1,2}(\Omega)$ becomes a Hilbert space. For $\varphi \in L^2(\Omega)^d$ and $\psi \in W^{1,2}(\Omega)^d$, we set

$$\|\varphi\|_2^2 := \sum_{i=1}^d \mathbb{E}\varphi_i^2, \qquad \|D\psi\|_2^2 := \sum_{i=1}^d \mathbb{E}\langle D\psi_i, D\psi_i\rangle$$

and $\|\psi\|_{W^{1,2}}^2 := \|\psi\|_2^2 + \|D\psi\|_2^2$. Theorem 10.25 of Da Prato (2006) shows that every $\varphi \in W^{1,2}(\Omega)^d$ satisfies the Poincaré inequality:

$$\mathbb{E}|\varphi - \mathbb{E}\varphi|^2 \le \lambda\|D\varphi\|_2^2 \qquad \text{for } \lambda := \max_j \lambda_j. \tag{4.1}$$

Moreover, by Theorem 10.16 of Da Prato (2006), every bounded set in $W^{1,2}(\Omega)^d$ is relatively compact in $L^2(\Omega)^d$.

We say a function $\varphi : \Omega \to \mathbb{R}^d$ is ω-Lipschitz with constant $L \ge 0$ if

$$|\varphi(\omega) - \varphi(\omega')| \le L\|\omega - \omega'\| \qquad \text{for all } \omega, \omega' \in \Omega.$$

It follows from Proposition 10.11 of Da Prato (2006) that every ω-Lipschitz function $\varphi : \Omega \to \mathbb{R}^d$ with constant L belongs to $W^{1,2}(\Omega)^d$ with $\|D\varphi\|_2 \le L$. In particular, one obtains that for given numbers $K, L \ge 0$, the set of all ω-Lipschitz $\varphi : \Omega \to \mathbb{R}^d$ with constant L satisfying $|\mathbb{E}\varphi| \le K$ is compact in $L^2(\Omega)^d$. Moreover, the following holds.

LEMMA 4.1. *Let $h : l^1 \to \mathbb{R}^d$ be a mapping satisfying $|h(x) - h(y)| \le K\|x - y\|_1$ for some constant $K \ge 0$. Then for any $x \in l^2$,*

$$\varphi = h\big(\sqrt{\lambda_j} x_j W(e_j), j \in \mathbb{N}\big)$$

is an ω-Lipschitz random variable with constant $K\|x\|_2$.

PROOF. One has

$$|\varphi(\omega) - \varphi(\omega')| \le K\big\|x_j\langle\omega - \omega', e_j\rangle, j \in \mathbb{N}\big\|_1 \le K\|x\|_2\|\omega - \omega'\|. \qquad \square$$

REMARK 4.2. The assumptions on Ω in this section are not restrictive for the purpose of studying BSEs and BSDEs. For instance, they allow for probability spaces rich enough to support an n-dimensional Brownian motion together with an independent Poisson random measure on $[0, T] \times \mathbb{R}^m \setminus \{0\}$. For an explicit construction, one can, for example, choose Ω to be of the form $\Omega = L^2([0, T]; \mathbb{R}^n) \oplus l^2$, where $L^2([0, T]; \mathbb{R}^n)$ is the space of square-integrable measurable functions from $[0, T]$ to $\mathbb{R}^n$ and l^2 the space of square-summable sequences. The inner product on $L^2([0, T]; \mathbb{R}^n) \oplus l^2$ is given by

$$\langle (h, x), (h', x') \rangle = \int_0^T h(s) \cdot h'(s)\, ds + \sum_{j \in \mathbb{N}} x_j x'_j,$$

where $\cdot$ denotes the standard scalar product on $\mathbb{R}^n$. Let $\mathbb{P}$ be a mean zero Gaussian measure corresponding to a positive self-adjoint trace class operator given by $Qe_j = \lambda_j e_j$ for a complete orthonormal system (e_j) of Ω and positive numbers (λ_j) satisfying $\sum_{j \in \mathbb{N}} \lambda_j < \infty$. If $W : \Omega \to L^2(\Omega)$ is the corresponding white noise mapping, b_i denotes the ith unit vector in $\mathbb{R}^n$ and (c_j) is a complete orthonormal system in l^2, then $W^i_t := W(b_i 1_{[0,t]}, 0)$ defines an n-dimensional Brownian motion independent of the sequence $\zeta_j := W(0, c_j)$ of independent standard normals. For a given σ-finite measure μ on the Borel σ-algebra of $\mathbb{R}^m \setminus \{0\}$, a Poisson random measure N on $[0, T] \times \mathbb{R}^m \setminus \{0\}$ with intensity measure $dt\mu(dx)$ can be realized as a function of ζ_j, $j \in \mathbb{N}$. Alternatively, N can be realized with only ζ_{2j-1}, $j \in \mathbb{N}$, and ζ_{2j}, $j \in \mathbb{N}$, can be used to model additional noise.

4.1. *Non-Lipschitz BSEs and BSDEs with path-dependent generators.* Denote by $\mathcal{F}$ the completion of the Borel σ-algebra $\mathcal{B}(\Omega)$ with respect to $\mathbb{P}$, and let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be a general filtration satisfying the usual conditions. The following theorem provides a general existence result for non-Lipschitz BSEs. It uses the theorem of Krasnoselskii (1964), which combines the fixed-point results of Banach and Schauder; for a textbook treatment see, for example, Smart (1974).

THEOREM 4.3. *Let $\xi \in L^2(\mathcal{F}_T)^d$ and assume F is of the form $F = F^1 + F^2$ for mappings $F^1, F^2 : \mathbb{S}^2 \times \mathbb{M}_0^2 \to \mathbb{S}_0^2$. Then the BSE (2.1) has a solution $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$ if there exist constants $C < 1$ and $R_1, R_2, R_3 \geq 0$ such that the following hold*:

(i) $\|F(Y,M) - F(Y',M)\|_{\mathbb{S}^2} \leq C\|Y - Y'\|_{\mathbb{S}^2}$ *and $F(Y,M) \in \mathbb{S}_0^2$ is continuous in $M \in \mathbb{M}_0^2$.*

(ii) $\|F_T^1(Y,M) - F_T^1(Y',M')\|_2 \leq C\sqrt{\|Y_0 - Y_0'\|_2^2 + \|M - M'\|_{\mathbb{S}^2}^2/4}$.

(iii) *For all $(Y,M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$ satisfying $\sqrt{\|Y_0\|_2^2 + \|M\|_{\mathbb{S}^2}^2/4} \leq R_1$, one has $F_T^2(Y,M) \in W^{1,2}(\Omega)^d$ with $\|F_T^2(Y,M)\|_2 \leq R_2$ and $\|DF_T^2(Y,M)\|_2 \leq R_3$.*

(iv) $\|\xi\|_2 + \|F_T^1(0,0)\|_2 + CR_1 + R_2 \leq R_1$.

PROOF. By Lemma 2.5, it follows from condition (i) that F satisfies (S). So by Theorem 2.3, it is enough to show that the mapping $V \mapsto G(V) = \xi + F_T(Y^V, M^V)$ has a fixed point in $L^2(\mathcal{F}_T)^d$. To do that, we define $\mathcal{C} := \{V \in L^2(\mathcal{F}_T)^d : \|V\|_2 \leq R_1\}$, $G^1(V) := \xi + F_T^1(Y^V, M^V)$, $G^2(V) := F_T^2(Y^V, M^V)$ and show the following: (1) G^1 is a contraction on $L^2(\mathcal{F}_T)^d$; (2) G^2 is continuous with respect to $\|\cdot\|_2$; (3) G^2 maps $\mathcal{C}$ into a compact subset of $L^2(\mathcal{F}_T)^d$; and (4) $G^1(V) + G^2(V') \in \mathcal{C}$ for all $V, V' \in \mathcal{C}$. Then it follows from Krasnoselskii's theorem that G has a fixed point.

Step 1: $G^1 : L^2(\mathcal{F}_T)^d \to L^2(\mathcal{F}_T)^d$ is a contraction with respect to $\|\cdot\|_2$: It follows from (ii) that

$$\begin{aligned}\|G^1(V) - G^1(V')\|_2^2 &= \|F_T^1(Y^V, M^V) - F_T^1(Y^{V'}, M^{V'})\|_2^2 \\ &\leq C^2\left(\|Y_0^V - Y_0^{V'}\|_2^2 + \frac{1}{4}\|M^V - M^{V'}\|_{\mathbb{S}^2}^2\right).\end{aligned}$$

By Doob's L^2-maximal inequality, one has

$$\|M^V - M^{V'}\|_{\mathbb{S}^2} \leq 2\|M_T^V - M_T^{V'}\|_2.$$

Therefore,

$$\begin{aligned}\|G^1(V) - G^1(V')\|_2^2 &\leq C^2(\|\mathbb{E}_0(V - V')\|_2^2 + \|M_T^V - M_T^{V'}\|_2^2) \\ &\leq C^2\|V - V'\|_2^2,\end{aligned}$$

which shows that G^1 is a contraction.

Step 2: $G^2 : L^2(\mathcal{F}_T)^d \to L^2(\mathcal{F}_T)^d$ is continuous with respect to $\|\cdot\|_2$: By Doob's L^2-maximal inequality, $V \mapsto M^V$ is a continuous mapping from $L^2(\mathcal{F}_T)^d$ to $\mathbb{M}_0^2$. Moreover, since

$$Y_t^V = \hat{M}_t^V - F_t(Y^V, M^V) \qquad \text{for } \hat{M}_t^V := \mathbb{E}_t V = \mathbb{E}_0 V - M_t^V,$$

one obtains from the first part of condition (i) that

$$\begin{aligned}\|Y^V - Y^{V'}\|_{\mathbb{S}^2} &\leq \|\hat{M}^V - \hat{M}^{V'}\|_{\mathbb{S}^2} + \|F(Y^V, M^V) - F(Y^{V'}, M^{V'})\|_{\mathbb{S}^2} \\ &\leq 2\|V - V'\|_2 + \|F(Y^V, M^V) - F(Y^V, M^{V'})\|_{\mathbb{S}^2} \\ &\quad + \|F(Y^V, M^{V'}) - F(Y^{V'}, M^{V'})\|_{\mathbb{S}^2} \\ &\leq 2\|V - V'\|_2 + \|F(Y^V, M^V) - F(Y^V, M^{V'})\|_{\mathbb{S}^2} \\ &\quad + C\|Y^V - Y^{V'}\|_{\mathbb{S}^2}.\end{aligned}$$

Therefore,

$$(1 - C)\|Y^V - Y^{V'}\|_{\mathbb{S}^2} \leq 2\|V - V'\|_2 + \|F(Y^V, M^V) - F(Y^V, M^{V'})\|_{\mathbb{S}^2},$$

and it follows from the second part of (i) that $V \mapsto Y^V$ is continuous from $L^2(\mathcal{F}_T)^d$ to $\mathbb{S}^2$. Since $F^2 = F - F^1$, one obtains from (i) and (ii) that $(Y^V, M^V) \mapsto F_T^2(Y^V, M^V)$ is continuous from $\mathbb{S}^2 \times \mathbb{M}_0^2$ to $L^2(\mathcal{F}_T)^d$. This proves the continuity of G^2.

Step 3: $G^2(\mathcal{C})$ is contained in a compact subset of $L^2(\mathcal{F}_T)^d$: For $V \in \mathcal{C}$, one has

$$\|Y_0^V\|_2^2 + \frac{1}{4}\|M^V\|_{\mathbb{S}^2}^2 \leq \|\mathbb{E}_0 V\|_2^2 + \|M_T^V\|_2^2 = \|V\|_2^2 \leq R_1^2. \tag{4.2}$$

So it follows from (iii) that $F_T^2(Y^V, M^V)$ is in $W^{1,2}(\Omega)^d$ with $\|F_T^2(Y^V, M^V)\|_2 \leq R_2$ and $\|DF_T^2(Y^V, M^V)\|_2 \leq R_3$. Since bounded subsets of $W^{1,2}(\Omega)^d$ are relatively compact in $L^2(\Omega)^d$, this shows that $G^2(\mathcal{C})$ is contained in a compact subset of $L^2(\mathcal{F}_T)^d$.

Step 4: $G^1(V) + G^2(V') \in \mathcal{C}$ for all $V, V' \in \mathcal{C}$: If $V \in \mathcal{C}$, one obtains from (4.2) that $\|Y_0^V\|_2^2 + \|M^V\|_{\mathbb{S}^2}^2/4 \leq R_1^2$. So it follows from (ii) that

$$\begin{aligned}\|G^1(V)\|_2 &\leq \|\xi\|_2 + \|F_T^1(Y^V, M^V)\|_2 \\ &\leq \|\xi\|_2 + \|F_T^1(0,0)\|_2 + C(\|Y_0^V\|_2^2 + \|M^V\|_{\mathbb{S}^2}^2/4)^{1/2} \\ &\leq \|\xi\|_2 + \|F_T^1(0,0)\|_2 + CR_1.\end{aligned}$$

By (iii), one has $\|G^2(V')\|_2 \leq R_2$. Therefore, one obtains from (iv) that $\|G^1(V) + G^2(V')\|_2 \leq R_1$.

So Krasnoselskii's theorem applies, and one can conclude that G has a fixed point in $L^2(\mathcal{F}_T)^d$. □

Assumption (i) of Theorem 4.3 is needed to ensure that condition (S) holds and $F^2_T(Y,M)$ is continuous in (Y,M). In the following special case, it is not needed.

PROPOSITION 4.4. *Let $\xi \in L^2(\mathcal{F}_T)^d$ and assume F is of the form $F(Y,M) = F^1(Y_0,M) + F^2(Y_0,M)$ for mappings $F^1, F^2 : L^2(\mathcal{F}_0)^d \times \mathbb{M}^2_0 \to \mathbb{S}^2_0$. Then the BSE* (2.1) *has a solution $(Y,M) \in \mathbb{S}^2 \times \mathbb{M}^2_0$ if there exist a constant $C < 1$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\limsup_{x\to\infty} \frac{\rho(x)}{x} < 1 - C \tag{4.3}$$

such that the following two conditions hold:

(i) $\|F^1_T(Y_0,M) - F^1_T(Y'_0,M')\|_2 \le C\sqrt{\|Y_0 - Y'_0\|_2^2 + \|M - M'\|^2_{\mathbb{S}^2}/4}$.

(ii) $F^2_T : L^2(\mathcal{F}_0)^d \times \mathbb{M}^2_0 \to L^2(\mathcal{F}_T)^d$ *is continuous and takes values in* $W^{1,2}(\Omega)^d$ *with*

$$\left|\mathbb{E}F^2_T(Y_0,M)\right|^2 + \lambda\left\|DF^2_T(Y_0,M)\right\|_2^2 \le \rho^2\left(\sqrt{\|Y_0\|_2^2 + \|M\|^2_{\mathbb{S}^2}/4}\right).$$

PROOF. Since F only depends on Y_0 and M, condition (S) holds trivially. By Theorem 2.3, the proposition follows if we can show that $V \mapsto G(V) = \xi + F_T(Y^V_0, M^V)$ has a fixed point in $L^2(\mathcal{F}_T)^d$. To do that, we fix a constant $R_1 \ge 0$ and define $\mathcal{C}$, G^1 and G^2 as in the proof of Theorem 4.3. Then one obtains from (i) like in the proof of Theorem 4.3 that G^1 is a contraction on $L^2(\mathcal{F}_T)^d$. Condition (ii) implies that G^2 is continuous with respect to $\|\cdot\|_2$, and since

$$\rho^2\left(\sqrt{\|Y^V_0\|_2^2 + \|M^V\|^2_{\mathbb{S}^2}/4}\right) \le \rho^2\left(\sqrt{\|Y^V_0\|_2^2 + \|M^V_T\|_2^2}\right) = \rho^2(\|V\|_2),$$

that $G^2(\mathcal{C})$ is relatively compact in $L^2(\mathcal{F}_T)^d$. Due to (4.3), one has

$$\|\xi\|_2 + \left\|F^1_T(0,0)\right\|_2 + CR_1 + \rho(R_1) \le R_1$$

if R_1 is chosen large enough. Then for $V, V' \in \mathcal{C}$,

$$\begin{aligned}\left\|G^1(V)\right\|_2 &\le \|\xi\|_2 + \left\|F^1_T(Y^V_0, M^V)\right\|_2\\ &\le \|\xi\|_2 + \left\|F^1_T(0,0)\right\|_2 + C\left(\|Y^V_0\|_2^2 + \|M^V_T\|_2^2\right)^{1/2}\\ &\le \|\xi\|_2 + \left\|F^1_T(0,0)\right\|_2 + CR_1\end{aligned}$$

and, by Poincaré's inequality,

$$\begin{aligned}\|G^2(V')\|_2^2 &\le |\mathbb{E}F^2_T(Y^{V'}_0, M^{V'})|^2 + \lambda\|DF^2_T(Y^{V'}_0, M^{V'})\|_2^2\\ &\le \rho^2\left(\sqrt{\|Y^{V'}_0\|_2^2 + \|M^{V'}\|^2_{\mathbb{S}^2}/4}\right)\\ &\le \rho^2\left(\sqrt{\|Y^{V'}_0\|_2^2 + \|M^{V'}_T\|_2^2}\right) = \rho^2(\|V'\|_2).\end{aligned}$$

Therefore,

$$\|G^1(V) + G^2(V')\|_2 \le \|\xi\|_2 + \|F_T^1(0,0)\|_2 + CR_1 + \rho(R_1) \le R_1,$$

and it follows from Krasnoselskii's theorem that G has a fixed point in $L^2(\mathcal{F}_T)^d$. □

As a consequence of Proposition 4.4, one obtains an existence result for BSDEs

$$Y_t = \xi + \int_t^T f(s, Y_0, M)\, ds + M_T - M_t \tag{4.4}$$

with drivers f depending on Y_0 and the whole martingale M.

COROLLARY 4.5. *Let $\xi \in L^2(\mathcal{F}_T)^d$ and assume f to be of the form $f = f^1 + f^2$ for mappings $f^1, f^2 : [0,T] \times \Omega \times L^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2 \to \mathbb{R}^d$. Then the BSDE (4.4) has a solution $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$ if there exist a constant $C < T^{-1}$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\limsup_{x \to \infty} \frac{\rho(x)}{x} < 1 - CT$$

such that the following two conditions hold:

(i) *For all $(Y_0, M) \in L^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2$, $f^1(\cdot, Y_0, M)$ is progressively measurable with $\int_0^T |f^1(t, 0, 0)|\, dt \in L^2(\mathcal{F}_T)$, and*

$$\|f^1(t, Y_0, M) - f^1(t, Y_0', M')\|_2 \le C\sqrt{\|Y_0 - Y_0'\|_2^2 + \|M - M'\|_{\mathbb{S}^2}^2/4}.$$

(ii) *For all $(Y_0, M) \in L^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2$, $f^2(\cdot, Y_0, M)$ is progressively measurable with $\int_0^T |f^2(t, Y_0, M)|\, dt \in L^2(\mathcal{F}_T)$, and*

$$J(Y_0, M) := \int_0^T f^2(t, Y_0, M)\, dt$$

defines a continuous mapping $J : L^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2 \to L^2(\mathcal{F}_T)^d$ with values in $W^{1,2}(\Omega)^d$ such that

$$|\mathbb{E}J(Y_0, M)|^2 + \lambda \|DJ(Y_0, M)\|_2^2 \le \rho^2\left(\sqrt{\|Y_0\|_2^2 + \|M_T\|_{\mathbb{S}^2}^2/4}\right).$$

PROOF. It follows from the assumptions that for all Y_0 and M, $F_t^i(Y_0, M) = \int_0^t f^i(s, Y_0, M)\, ds$ belongs to $\mathbb{S}_0^2$ for $i = 1, 2$ and

$$\mathbb{E}|F_T^1(Y_0, M) - F_T^1(Y', M')|^2 \le C^2 T^2 (\|Y_0 - Y_0'\|_2^2 + \|M - M'\|_{\mathbb{S}^2}^2/4).$$

So the conditions of Proposition 4.4 hold with CT instead of C, and the corollary follows. □

If F does not depend on Y, the assumptions of Theorem 4.3 can be relaxed further, and one obtains the following.

THEOREM 4.6. *Let $\xi \in L^2(\mathcal{F}_T)^d$ and assume F is of the form $F(Y, M) = F^1(M) + F^2(M)$ for mappings $F^1, F^2 : \mathbb{M}_0^2 \to \mathbb{S}_0^2$. Then the BSE* (2.1) *has a solution $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$ if there exist a constant $C < 1/2$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\limsup_{x\to\infty} \frac{\rho(x)}{x} < \frac{1/2 - C}{\sqrt{\lambda}} \tag{4.5}$$

such that the following two conditions hold:

(i) $\|F_T^1(M) - \mathbb{E}_0 F_T^1(M) - (F_T^1(M') - \mathbb{E}_0 F_T^1(M'))\|_2 \le C\|M - M'\|_{\mathbb{S}^2}$.

(ii) $F_T^2 : \mathbb{M}_0^2 \to L^2(\mathcal{F}_T)^d$ *is continuous and takes values in* $W^{1,2}(\Omega)^d$ *with* $\|DF_T^2(M)\|_2 \le \rho(\|M\|_{\mathbb{S}^2})$.

PROOF. By Corollary 2.4, it is enough to show that the mapping

$$V \mapsto G_0(V) = \xi - \mathbb{E}_0\xi + F_T(M^V) - \mathbb{E}_0 F_T(M^V)$$

has a fixed point in $L_0^2(\mathcal{F}_T)^d$. For a given constant $R \ge 0$, define

$$\begin{aligned} \mathcal{C} &:= \{V \in L_0^2(\mathcal{F}_T)^d : \|V\|_2 \le R\}, \\ G_0^1(V) &:= \xi - \mathbb{E}_0\xi + F_T^1(M^V) - \mathbb{E}_0 F_T^1(M^V) \quad \text{and} \\ G_0^2(V) &:= F_T^2(M^V) - \mathbb{E}_0 F_T^2(M^V). \end{aligned}$$

By (i) and Doob's L^2-maximal inequality, one has

$$\begin{aligned} \|G_0^1(V) - G_0^1(V')\|_2 &\le \|F_T^1(M^V) - \mathbb{E}_0 F_T^1(M^V) - (F_T^1(M^{V'}) - \mathbb{E}_0 F_T^1(M^{V'}))\|_2 \\ &\le C\|M^V - M^{V'}\|_{\mathbb{S}^2} \le 2C\|M_T^V - M_T^{V'}\|_2 \le 2C\|V - V'\|_2. \end{aligned}$$

So G_0^1 is a contraction on $L_0^2(\mathcal{F}_T)^d$. Moreover, it follows from (ii) that $G_0^2 : L_0^2(\mathcal{F}_T)^d \to L_0^2(\mathcal{F}_T)^d$ is continuous and $G_0^2(\mathcal{C})$ is relatively compact in $L_0^2(\mathcal{F}_T)^d$. Finally, let $V, V' \in \mathcal{C}$. Then

$$\|G_0^1(V)\|_2 \le \|\xi - \mathbb{E}_0\xi\|_2 + \|F_T^1(0) - \mathbb{E}_0 F_T^1(0)\|_2 + 2CR$$

and

$$\begin{aligned} \|G_0^2(V')\|_2 &= \|F_T^2(M^{V'}) - \mathbb{E}_0 F_T^2(M^{V'})\|_2 \\ &\le \sqrt{\lambda}\|DF_T^2(M^{V'})\|_2 \le \sqrt{\lambda}\rho(\|M^{V'}\|_{\mathbb{S}^2}) \le \sqrt{\lambda}\rho(2R). \end{aligned}$$

By (4.5), one has $G_0^1(V) + G_0^2(V') \in \mathcal{C}$ for R large enough. So it follows like in the proof of Theorem 4.3 from Krasnoselskii's theorem that $G_0 = G_0^1 + G_0^2$ has a fixed point in $L_0^2(\mathcal{F}_T)^d$. □

COROLLARY 4.7. *A BSDE of the form*

$$Y_t = \xi + \int_t^T \big(f^1(s, M) + f^2(s, M)\big)\, ds + M_T - M_t$$

for a terminal condition $\xi \in L^2(\mathcal{F}_T)^d$ *and mappings* $f^1, f^2 : [0, T] \times \Omega \times \mathbb{M}_0^2 \to \mathbb{R}^d$ *has a solution* $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$ *if there exist a constant* $C < (2T)^{-1}$ *and a nondecreasing function* $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying*

$$\limsup_{x\to\infty} \frac{\rho(x)}{x} < \frac{1/2 - CT}{\sqrt{\lambda}}$$

such that the following two conditions hold:

(i) *For all* $M \in \mathbb{M}_0^2$, $f^1(\cdot, M)$ *is progressively measurable with* $\int_0^T |f^1(t, 0)|\, dt \in L^2(\mathcal{F}_T)$, *and*

$$\|f^1(t, M) - f^1(t, M')\|_2 \le C\|M - M'\|_{\mathbb{S}^2}.$$

(ii) *For all* $M \in \mathbb{M}_0^2$, $f^2(\cdot, M)$ *is progressively measurable with* $\int_0^T |f^2(t, M)|\, dt \in L^2(\mathcal{F}_T)$, *and* $J(M) := \int_0^T f^2(t, M)\, dt$ *defines a continuous map* $J : \mathbb{M}_0^2 \to L^2(\mathcal{F}_T)^d$ *such that for all* $M \in M_0^2$, $J(M)$ *is* ω-*Lipschitz with constant* $\rho(\|M\|_{\mathbb{S}^2})$.

PROOF. As in Corollary 4.5, it follows from the assumptions that $F_t^i(M) = \int_0^t f^i(s, M)\, ds$ is in $\mathbb{S}_0^2$ for $i = 1, 2$ and all $M \in \mathbb{M}_0^2$. Moreover,

$$\mathbb{E}\big|F_T^1(M) - F_T^1(M')\big|^2 \le C^2T^2\|M - M'\|_{\mathbb{S}^2}^2,$$

and since $\int_0^T f^2(s, M)\, ds$ is ω-Lipschitz with constant $\rho(\|M\|_{\mathbb{S}^2})$, one has $\|DF_T^2(M)\|_2 \le \rho(\|M\|_{\mathbb{S}^2})$. So the conditions of Theorem 4.6 hold with CT instead of C, and the corollary follows as a consequence. □

REMARK 4.8. As a special case of Corollary 4.7, one obtains that the BSDE

$$Y_t = \xi + \int_t^T f(s, M)\, ds + M_T - M_t$$

has a solution for every terminal condition $\xi \in L^2(\mathcal{F}_T)^d$ and driver f satisfying condition (ii) of Corollary 4.7. This provides an existence result for multidimensional BSDEs with drivers exhibiting general dependence on the whole process M. In contrast to the BSDE results in Section 3, here the driver is not required to be Lipschitz in M. On the other hand, it is supposed to satisfy the ω-Lipschitzness assumption contained in condition (ii) of Corollary 4.7.

4.2. *Non-Lipschitz BSDEs based on a Brownian motion and a Poisson random measure.* We now focus on BSDEs with non-Lipschtiz coefficients that depend on an n-dimensional Brownian motion W and an independent Poisson random measure N on $[0,T] \times E$, where $E = \mathbb{R}^m \setminus \{0\}$, with an intensity measure of the form $dt\mu(dx)$ for a measure μ over the Borel σ-algebra $\mathcal{B}(E)$ of E satisfying

$$\int_E (1 \wedge |x|^2)\mu(dx) < \infty$$

(see Remark 4.2 above for a construction of W and N in the case where $\mathbb{P}$ is a mean zero Gaussian measure on the infinite-dimensional separable Hilbert space Ω).

As in Section 4.1, we denote by $\mathcal{F}$ the completed Borel σ-algebra on Ω and let $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ be a filtration satisfying the usual conditions. Let $\tilde{N}$ be the compensated random measure $N(dt, dx) - dt\mu(dx)$, and assume that, for $A \in \mathcal{B}(E)$ with $\mu(A) < \infty$, $\tilde{N}([0,t] \times A)$ and W are martingales with respect to $\mathbb{F}$. The next proposition gives an existence result for BSDEs with functional drivers of the form

$$Y_t = \xi + \int_t^T f(s, Z_s^M, U_s^M)\, ds + M_T - M_t. \tag{4.6}$$

PROPOSITION 4.9. *Let $\xi \in L^2(\mathcal{F}_T)^d$ and assume the driver is of the form $f = f^1 + f^2$ for mappings*

$$f^1, f^2 : [0,T] \times \Omega \times L^2(\mathcal{F}_T)^{d\times n} \times L^2(\Omega \times E, \mathcal{F}_T \otimes \mathcal{B}(E), \mathbb{P} \otimes \mu)^d \to \mathbb{R}^d.$$

Then the BSDE (4.6) has a solution $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$ if there exist a constant $C \ge 0$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $M, M' \in \mathbb{M}_0^2$, the following two conditions hold:

(i) *$f^1(t, Z_t^M, U_t^M)$ is progressively measurable with $\int_0^T \|f^1(t,0,0)\|_2\, dt < \infty$, and*

$$\begin{aligned}\|f^1(t, Z_t^M, U_t^M) - f^1(t, Z_t^{M'}, U_t^{M'})\|_2 \\ \le C(\|Z_t^M - Z_t^{M'}\|_2 + \|U_t^M - U_t^{M'}\|_{L^2(\mathbb{P}\otimes\mu)}).\end{aligned}$$

(ii) *$f^2(t, Z_t^M, U_t^M)$ is progressively measurable with $\int_0^T \|f^2(t,0,0)\|_2\, dt < \infty$, and*

$$\begin{aligned}&\left\| \int_0^T |f^2(t, Z_t^M, U_t^M) - f^2(t, Z_t^{M'}, U_t^{M'})|\, dt \right\|_2 \\ &\quad \le \rho(\|Z^M\|_{\mathbb{H}^2} + \|Z^{M'}\|_{\mathbb{H}^2} + \|U^M\|_{L^2(\tilde{N})} + \|U^{M'}\|_{L^2(\tilde{N})}) \\ &\qquad \times (\|Z^M - Z^{M'}\|_{\mathbb{H}^2} + \|U^M - U^{M'}\|_{L^2(\tilde{N})}),\end{aligned}$$

and $f^2(t, Z_t^M, U_t^M)$ is ω-Lipschitz with constant $C(1 + \|Z_t^M\|_2 + \|U_t^M\|_{L^2(\mathbb{P}\otimes\mu)})$.

PROOF. Choose $\delta > 0$ so that

$$\sqrt{2\delta}C(1+\sqrt{\lambda}) < \frac{1}{2} \quad \text{and} \quad k := T/\delta \in \mathbb{N}.$$

Set $F_t^i(M) = \int_0^t f^i(s, Z_s^M, U_s^M) 1_{[T-\delta,T]}(s)\, ds$. It follows from the assumptions that $F^i(M) \in \mathbb{S}_0^2$ for $i = 1, 2$ and all $M \in \mathbb{M}_0^2$. Moreover,

$$\begin{aligned}
&\| F_T^1(M) - \mathbb{E}_0 F_T^1(M) - \big(F_T^1(M') - \mathbb{E}_0 F_T^1(M')\big)\|_2^2 \\
&\quad \le \| F_T^1(M) - F_T^1(M')\|_2^2 \\
&\quad \le 2\delta C^2 \int_{T-\delta}^T \big(\| Z_s^M - Z_s^{M'}\|_2^2 + \| U_s^M - U_s^{M'}\|_{L^2(\mathbb{P}\otimes\mu)}^2\big)\, ds \\
&\quad \le 2\delta C^2 \| M - M'\|_{\mathbb{S}^2}^2.
\end{aligned}$$

From condition (ii), one obtains that $M \in \mathbb{M}_0^2 \mapsto F_T^2(M) \in L^2(\mathcal{F}_T)^d$ is continuous, and

$$\begin{aligned}
&\left| \int_{T-\delta}^T f^2\big(s, Z_s^M, U_s^M\big)(\omega) - f^2\big(s, Z_s^M, U_s^M\big)(\omega')\, ds \right| \\
&\quad \le \int_{T-\delta}^T \big| f^2\big(s, Z_s^M, U_s^M\big)(\omega) - f^2\big(s, Z_s^M, U_s^M\big)(\omega')\big|\, ds \\
&\quad \le C\left(\int_{T-\delta}^T \big(1 + \| Z_s^M\|_2 + \| U_s^M\|_{L^2(\mathbb{P}\otimes\mu)}\big)\, ds\right) \|\omega - \omega'\| \\
&\quad \le \left(\delta C + \sqrt{\delta} C \sqrt{\int_{T-\delta}^T 2\big(\| Z_s^M\|_2^2 + \| U_s^M\|_{L^2(\mathbb{P}\otimes\mu)}^2\big)\, ds}\right) \|\omega - \omega'\| \\
&\quad \le \big(\delta C + \sqrt{2\delta} C \| M\|_{\mathbb{S}^2}\big) \|\omega - \omega'\|.
\end{aligned}$$

It follows that for all $M \in \mathbb{M}_0^2$, $F_T^2(M)$ is in $W^{1,2}(\Omega)^d$ with $\| D F_T^2(M)\|_2 \le \delta C + \sqrt{2\delta} C \| M\|_{\mathbb{S}^2}$. So the conditions of Theorem 4.6 hold with $\sqrt{2\delta}C$ instead of C and $\rho(x) = \delta C + \sqrt{2\delta} C x$. Therefore,

$$Y_t = \xi + \int_t^T \big(f^1\big(s, Z_s^M, U_s^M\big) + f^2\big(s, Z_s^M, U_s^M\big)\big) 1_{[T-\delta,T]}(s)\, ds + M_T - M_t$$

has a solution $(Y^{(k)}, M^{(k)}) \in \mathbb{S}^2 \times \mathbb{M}_0^2$. From the same argument, one obtains that, for $t \le T - \delta$,

$$Y_t = Y_{T-\delta}^{(k)} + \int_t^{T-\delta} \big(f^1(s, Z_s) + f^2(s, Z_s)\big) 1_{[T-2\delta, T-\delta]}(s)\, ds + M_{T-\delta} - M_t,$$

has a solution $(Y^{(k-1)}, Z^{(k-1)}) \in \mathbb{S}^2 \times \mathbb{M}_0^2$. Iterating this procedure, one obtains $(Y^{(j)}, Z^{(j)})$, $j = 1, \dots, k$. Now, define

$$Y_t := Y_t^{(1)}, \quad M_t := M_t^{(1)} \qquad \text{for } 0 \le t \le \delta \quad \text{and}$$

$$Y_t := Y_t^{(j)}, \quad M_t - M_{(j-1)\delta} := M_t^{(j)} - M_{(j-1)\delta}^{(j)}$$

for $(j-1)\delta < t \le j\delta$, $j = 2, \dots, k$. Then $(Z_t^M, U_t^M) = (Z_t^{M^{(j)}}, U_t^{M^{(j)}})$ for $(j-1)\delta < t \le j\delta$. So (Y, M) is a solution of (4.6) in $\mathbb{S}^2 \times \mathbb{M}_0^2$. $\square$

As a consequence of Proposition 4.9, one obtains the following existence result for multidimensional mean-field BSDEs with drivers of quadratic growth and square integrable terminal conditions. While there exist general existence and uniqueness results for one-dimensional BSDEs with drivers of quadratic growth [see, e.g., Kobylanski (2000), Briand and Hu (2006, 2008) or Delbaen, Hu and Richou (2011)], multidimensional quadratic BSDEs do not always admit solutions [see Peng (1999), or Frei and dos Reis (2011)]. An existence and uniqueness result for multidimensional BSDEs with general drivers of quadratic growth was given by Tevzadze (2008). But it only holds for terminal conditions with small L^∞-norm. Other results, such as the ones in Cheridito and Nam (2015), require the driver to have special structure.

COROLLARY 4.10. *Let $\xi \in L^2(\mathcal{F}_T)^d$ and assume the driver is of the form*

$$f(t, Z_t, U_t) = \tilde{\mathbb{E}} a(t, Z_t, \tilde{Z}_t, U_t, \tilde{U}_t) + B\big(t, \mathbb{E} b(t, Z_t, U_t)\big)$$

for mappings

$$a : [0, T] \times \Omega \times \big(\mathbb{R}^{d\times n}\big)^2 \times \big(L^2(\mu)\big)^2 \to \mathbb{R}^d,$$
$$b : [0, T] \times \Omega \times \mathbb{R}^{d\times n} \times L^2(\mu) \to \mathbb{R}^l, \quad \textit{and}$$
$$B : [0, T] \times \Omega \times \mathbb{R}^l \to \mathbb{R}^d,$$

where $(\tilde{Z}_t, \tilde{U}_t)$ is a copy of (Z_t, U_t) living on a separate probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and $\tilde{\mathbb{E}} a(t, Z_t, \tilde{Z}_t, U_t, \tilde{U}_t)$ means $\int_{\tilde{\Omega}} a(t, Z_t, \tilde{Z}_t, U_t, \tilde{U}_t)\, d\tilde{\mathbb{P}}$.

Then the BSDE (4.6) has a solution $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$ if there exists a constant $C \ge 0$ such that for all $z, \tilde{z}, z', \tilde{z}' \in \mathbb{R}^{d\times n}$, $u, \tilde{u}, u', \tilde{u}' \in L^2(\mu)$ and $x, x' \in \mathbb{R}^k$, $a(\cdot, z, \tilde{z}, u, \tilde{u})$, $b(\cdot, z, u)$ and $B(\cdot, x)$ are progressively measurable and the following hold:

(i) $a(\cdot, 0, 0, 0, 0) \in \mathbb{H}^2$ *and*

$$\begin{aligned}&\big|a(t, z, \tilde{z}, u, \tilde{u}) - a\big(t, z', \tilde{z}', u', \tilde{u}'\big)\big| \\ &\quad \le C\big(|z - z'| + |\tilde{z} - \tilde{z}'| + \|u - u'\|_{L^2(\mu)} + \|\tilde{u} - \tilde{u}'\|_{L^2(\mu)}\big)\end{aligned}$$

(ii) $|b(t, 0, 0)|, |B(t, 0)| \le C$ *and at least one of the following two conditions is satisfied*:

(a) *For any given* $t \in [0,T], x, x' \in \mathbb{R}^l, z, z' \in \mathbb{R}^{d\times n}$, *and* $u, u' \in L^2(\mu)$, $B(t,x)$ *is* ω*-Lipschitz with constant* $C(1+\sqrt{|x|})$, *and*

$$\begin{aligned}
&|b(t,z,u) - b(t,z',u')| \\
&\quad \le C(1+|z|+|z'|+\|u\|_{L^2(\mu)}+\|u'\|_{L^2(\mu)}) \\
&\qquad \times (|z-z'|+\|u-u'\|_{L^2(\mu)}), \\
&|B(t,x) - B(t,x')| \le C|x-x'|.
\end{aligned}$$

(b) *For any given* $t \in [0,T], x, x' \in \mathbb{R}^l, z, z' \in \mathbb{R}^{d\times n}$, *and* $u, u' \in L^2(\mu)$, $B(t,x)$ *is* ω*-Lipschitz with constant* $C(1+|x|)$, *and*

$$\begin{aligned}
|b(t,z,u) - b(t,z',u')| &\le C(|z-z'|+\|u-u'\|_{L^2(\mu)}), \\
|B(t,x) - B(t,x')| &\le C(1+|x|+|x'|)|x-x'|.
\end{aligned}$$

PROOF. It is enough to show that

$$f^1(t,Z_t,U_t) := \tilde{\mathbb{E}} a(t,Z_t,\tilde{Z}_t,U_t,\tilde{U}_t) \quad \text{and} \quad f^2(t,Z_t,U_t) := B(t,\mathbb{E} b(t,Z_t,U_t))$$

satisfy the conditions of Proposition 4.9. As in the proof of Corollary 3.11, one can deduce from Lemma 4.51 of Aliprantis and Border (2006) that $f^i(t,Z_t,U_t)$ is progressively measurable and satisfies $\int_0^T \|f^i(t,0,0)\|_2\,dt < \infty$ for $i = 1,2$ and all $Z \in \mathbb{H}^2$ and $U \in L^2(\tilde{N})$.

Now consider $Z, Z' \in \mathbb{H}^2$, $U, U' \in L^2(\tilde{N})$, and let $(\tilde{Z},\tilde{U},\tilde{Z}',\tilde{U}')$ be a copy of (Z,U,Z',U') on $\tilde{\Omega}$. Then, for fixed $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}|\tilde{\mathbb{E}} a(t,Z_t,\tilde{Z}_t,U_t,\tilde{U}_t) - \tilde{\mathbb{E}} a(t,Z'_t,\tilde{Z}'_t,U'_t,\tilde{U}'_t)|^2 \\
&\quad \le \mathbb{E}\tilde{\mathbb{E}}|a(t,Z_t,\tilde{Z}_t,U_t,\tilde{U}_t) - a(t,Z'_t,\tilde{Z}'_t,U'_t,\tilde{U}'_t)|^2 \\
&\quad \le 4C^2(\mathbb{E}|Z_t - Z'_t|^2 + \tilde{\mathbb{E}}|\tilde{Z}_t - \tilde{Z}'_t|^2 + \mathbb{E}\|U_t - U'_t\|^2_{L^2(\mu)} + \tilde{\mathbb{E}}\|\tilde{U}_t - \tilde{U}'_t\|^2_{L^2(\mu)}) \\
&\quad = 8C^2(\|Z_t - Z'_t\|_2^2 + \|U_t - U'_t\|^2_{L^2(\mathbb{P}\otimes\mu)}).
\end{aligned}$$

On the other hand, if condition (ii)(a) holds, then

$$\begin{aligned}
&\left\| \int_0^T |B(t,\mathbb{E} b(t,Z_t,U_t)) - B(t,\mathbb{E} b(t,Z'_t,U'_t))|\,dt \right\|_2 \\
&\quad \le C\int_0^T |\mathbb{E} b(t,Z_t,U_t) - \mathbb{E} b(t,Z'_t,U'_t)|\,dt \\
&\quad \le C\mathbb{E}\int_0^T |b(t,Z_t,U_t) - b(t,Z'_t,U'_t)|\,dt \\
&\quad \le C^2\mathbb{E}\int_0^T (1+|Z_t|+|Z'_t|+\|U_t\|_{L^2(\mu)}+\|U'_t\|_{L^2(\mu)})
\end{aligned}$$

$$
\begin{aligned}
&\times \bigl(|Z_t - Z_t'| + \|U_t - U_t'\|_{L^2(\mu)}\bigr)\,dt \\
&\le C^2 \sqrt{\mathbb{E}\int_0^T \bigl(1 + |Z_t| + |Z_t'| + \|U_t\|_{L^2(\mu)} + \|U_t'\|_{L^2(\mu)}\bigr)^2\,dt} \\
&\quad \times \sqrt{\mathbb{E}\int_0^T \bigl(|Z_t - Z_t'| + \|U_t - U_t'\|_{L^2(\mu)}\bigr)^2\,dt} \\
&\le C^2\sqrt{10}\sqrt{T + \|Z\|_{\mathbb{H}^2}^2 + \|Z'\|_{\mathbb{H}^2}^2 + \|U\|_{L^2(\tilde{N})}^2 + \|U'\|_{L^2(\tilde{N})}^2} \\
&\quad \times \sqrt{\|Z - Z'\|_{\mathbb{H}^2}^2 + \|U - U'\|_{L^2(\tilde{N})}^2} \\
&\le C^2\sqrt{10}\bigl(\sqrt{T} + \|Z\|_{\mathbb{H}^2} + \|Z'\|_{\mathbb{H}^2} + \|U\|_{L^2(\tilde{N})} + \|U'\|_{L^2(\tilde{N})}\bigr) \\
&\quad \times \bigl(\|Z - Z'\|_{\mathbb{H}^2} + \|U - U'\|_{L^2(\tilde{N})}\bigr).
\end{aligned}
$$

Moreover, $B(t, \mathbb{E}b(t, Z_t, U_t))$ is ω-Lipschitz with constant $C(1 + \sqrt{|\mathbb{E}b(t, Z_t, U_t)|})$, and

$$
\begin{aligned}
\bigl|\mathbb{E}b(t, Z_t, U_t)\bigr| &\le \mathbb{E}\bigl|b(t, Z_t, U_t)\bigr| \\
&\le C\mathbb{E}\bigl(1 + |Z_t| + \|U_t\|_{L^2(\mu)}\bigr)\bigl(|Z_t| + \|U_t\|_{L^2(\mu)}\bigr) \\
&\le C\bigl(1 + \|Z_t\|_2 + \|U_t\|_{L^2(\mathbb{P}\otimes\mu)}\bigr)\bigl(\|Z_t\|_2 + \|U_t\|_{L^2(\mathbb{P}\otimes\mu)}\bigr),
\end{aligned}
$$

from which one obtains that $B(t, \mathbb{E}b(t, Z_t, U_t))$ is ω-Lipschitz with constant

$$
C\bigl(1 + \sqrt{C}\bigl(1 + \|Z_t\|_2 + \|U_t\|_{L^2(\mathbb{P}\otimes\mu)}\bigr)\bigr).
$$

Similarly, if condition (ii)(b) holds, one has

$$
\begin{aligned}
&\bigl|B\bigl(t, \mathbb{E}b(t, Z_t, U_t)\bigr) - B\bigl(t, \mathbb{E}b\bigl(t, Z_t', U_t'\bigr)\bigr)\bigr| \\
&\quad \le C\bigl(1 + \bigl|\mathbb{E}b(t, Z_t, U_t)\bigr| + \bigl|\mathbb{E}b\bigl(t, Z_t', U_t'\bigr)\bigr|\bigr)\bigl|\mathbb{E}b(t, Z_t, U_t) - \mathbb{E}b\bigl(t, Z_t', U_t'\bigr)\bigr| \\
&\quad \le C\bigl(1 + \mathbb{E}\bigl|b(t, Z_t, U_t)\bigr| + \mathbb{E}\bigl|b\bigl(t, Z_t', U_t'\bigr)\bigr|\bigr)\mathbb{E}\bigl|b(t, Z_t, U_t) - b\bigl(t, Z_t', U_t'\bigr)\bigr| \\
&\quad \le C^2\bigl(1 + 2\mathbb{E}\bigl|b(t, 0, 0)\bigr| + C\mathbb{E}\bigl(|Z_t| + |Z_t'| + \|U_t\|_{L^2(\mu)} + \|U_t'\|_{L^2(\mu)}\bigr)\bigr) \\
&\qquad \times \mathbb{E}\bigl(|Z_t - Z_t'| + \|U_t - U_t'\|_{L^2(\mu)}\bigr).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left\|\int_0^T \bigl|B\bigl(t, \mathbb{E}b(t, Z_t, U_t)\bigr) - B\bigl(t, \mathbb{E}b\bigl(t, Z_t', U_t'\bigr)\bigr)\bigr|\,dt\right\|_2 \\
&\qquad \le C^2\sqrt{\int_0^T \bigl(1 + 2C + C\mathbb{E}\bigl(|Z_t| + |Z_t'| + \|U_t\|_{L^2(\mu)} + \|U_t'\|_{L^2(\mu)}\bigr)\bigr)^2\,dt}
\end{aligned}
$$

$$\times \sqrt{\int_0^T (\mathbb{E}|Z_t - Z'_t| + \mathbb{E}\|U_t - U'_t\|_{L^2(\mu)})^2\, dt}$$

$$\leq C^3 \sqrt{\int_0^T 6(C^{-2} + 4 + \|Z_t\|_2^2 + \|Z'_t\|_2^2 + \|U_t\|^2_{L^2(\mathbb{P}\otimes\mu)} + \|U'_t\|^2_{L^2(\mathbb{P}\otimes\mu)})\, dt}$$

$$\times \sqrt{\int_0^T 2(\|Z_t - Z'_t\|_2^2 + \|U_t - U'_t\|^2_{L^2(\mathbb{P}\otimes\mu)})\, dt}$$

$$\leq C^3\sqrt{12}\big(\sqrt{T(C^{-2}+4)} + \|Z\|_{\mathbb{H}^2} + \|Z'\|_{\mathbb{H}^2} + \|U\|_{L^2(\tilde{N})} + \|U'\|_{L^2(\tilde{N})}\big)$$

$$\times \big(\|Z - Z'\|_{\mathbb{H}^2} + \|U - U'\|_{L^2(\tilde{N})}\big).$$

Moreover, $B(t, \mathbb{E}b(s, Z_t, U_t))$ is ω-Lipschitz with constant $C(1 + |\mathbb{E}b(t, Z_t, U_t)|)$. So since

$$\big|\mathbb{E}b(t, Z_t, U_t)\big| \leq \mathbb{E}\big|b(t, Z_t, U_t)\big| \leq C\big(1 + \mathbb{E}\big(|Z_t| + \|U_t\|_{L^2(\mu)}\big)\big)$$
$$\leq C\big(1 + \|Z_t\|_2 + \|U_t\|_{\mathbb{P}\otimes L^2(\mu)}\big),$$

$B(t, \mathbb{E}b(t, Z_t, U_t))$ is ω-Lipschitz with constant $C(1 + C(1 + \|Z_t\|_2 + \|U_t\|_{\mathbb{P}\otimes L^2(\mu)}))$. This shows that the conditions of Proposition 4.9 hold, and the corollary follows. □

EXAMPLE 4.11. A simple example of a driver satisfying the conditions of Corollary 4.10 is given by

$$f(Z_t) = f^1(Z_t) + f^2(Z_t),$$

for a Lipschitz function $f^1 : \mathbb{R}^{d\times n} \to \mathbb{R}^d$ and a mapping $f^2 : L^2(\mathcal{F}_T)^{d\times n} \to \mathbb{R}^d$ of the form

$$f^2(Z_t) := \alpha + \mathbb{E}\big(Z_t|Z_t|\big)\beta$$

with constant vectors $\alpha \in \mathbb{R}^{d\times 1}$ and $\beta \in \mathbb{R}^{n\times 1}$. In particular, if W is an n-dimensional Brownian motion generating the filtration $\mathbb{F}$, the BSDE

$$Y_t = \xi + \int_t^T f(Z_s)\, ds + \int_t^T Z_s\, dW_s$$

has a solution $(Y, Z) \in \mathbb{S}^2 \times \mathbb{H}^2$ for every terminal condition $\xi \in L^2(\mathcal{F}_T)^d$.

Since f^2 has quadratic growth, the contraction mapping principle used by Buckdahn, Li and Peng (2009) cannot be applied here. Also, if $d > 1$ and f^2 were a function with quadratic growth of the realizations $Z_t(\omega)$, the existence of a global solution could not be guaranteed; see Frei and dos Reis (2011) for a counterexample.

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