

EINSTEIN RELATION AND STEADY STATES FOR THE RANDOM CONDUCTANCE MODEL¹

BY NINA GANTERT*, XIAOQIN GUO[†] AND JAN NAGEL*

Technische Universität München and Purdue University[†]*

We consider random walk among i.i.d., uniformly elliptic conductances on \mathbb{Z}^d , and prove the Einstein relation (see Theorem 1). It says that the derivative of the velocity of a biased walk as a function of the bias equals the diffusivity in equilibrium. For fixed bias, we show that there is an invariant measure for the environment seen from the particle. These invariant measures are often called steady states. The Einstein relation follows at least for $d \geq 3$, from an expansion of the steady states as a function of the bias (see Theorem 2), which can be considered our main result. This expansion is proved for $d \geq 3$. In contrast to Guo [*Ann. Probab.* **44** (2016) 324–359], we need not only convergence of the steady states, but an estimate on the rate of convergence (see Theorem 4).

1. Introduction. We consider random walk among i.i.d., uniformly elliptic random conductances. More precisely, let $\mathcal{B}(\mathbb{Z}^d)$ be the set of nonoriented nearest-neighbor bonds in \mathbb{Z}^d , $d \geq 2$ and $\Omega := (0, \infty)^{\mathcal{B}(\mathbb{Z}^d)}$. An element $\omega \in \Omega$ is called an *environment*. For $(x, y) \in \mathcal{B}(\mathbb{Z}^d)$, the weight $\omega(x, y)$ is the *conductance* of the bond (x, y) . For $\omega \in \Omega$, the random walk in the environment ω is the Markov process $(X_n)_{n \geq 1}$ with transition probabilities

$$P_\omega^x(X_{n+1} = y + e | X_n = y) = \frac{\omega(y, y + e)}{\sum_{|e'|=1} \omega(y, y + e')}$$

and $P_\omega^x(X_0 = x) = 1$. In other words, the transition probabilities are proportional to the conductances of the bonds. The distribution P_ω^x of the random walk is called *quenched law*. For a probability measure P on Ω , the averaged measure $\mathbb{P}^x = P \times P_\omega^x := \int P_\omega^x P(d\omega)$ is called the *annealed law*. We will assume throughout this paper that:

(i) the conductances are *uniformly elliptic*, that is, there is a $\kappa > 1$ such that for all $(x, y) \in \mathcal{B}(\mathbb{Z}^d)$

$$\kappa^{-1} \leq \omega(x, y) \leq \kappa,$$

(ii) $(\omega(e))_{e \in \mathcal{B}(\mathbb{Z}^d)}$ are i.i.d. under P .

Received December 2015; revised April 2016.

¹Supported by DFG Grant GA 582/8-1.

MSC2010 subject classifications. Primary 60K37, 60K40; secondary 60J25, 60G10, 82C41.

Key words and phrases. Random conductance model, Einstein relation, steady states.

Now, for $\lambda \in (0, 1)$ and $\ell \in \mathbb{R}^d$, $|\ell| = 1$, define the perturbed environment ω^λ of $\omega \in \Omega$ by

$$\omega^\lambda(x, y) = \omega(x, y)e^{\lambda\ell \cdot (x+y)},$$

where “ \cdot ” denotes the scalar product in \mathbb{R}^d . We denote the quenched and annealed measures of the random walks in the perturbed environment as $P_{\omega, \lambda}^x := P_{\omega^\lambda}^x$ and $\mathbb{P}_\lambda^x := P \times P_{\omega, \lambda}^x$, respectively. If the starting point x is the origin, we will omit the superscript $x = 0$, for instance, we write P_ω instead of P_ω^0 and \mathbb{P} instead of \mathbb{P}^0 . It goes back to [7] that for $\lambda = 0$, the random walk satisfies a functional central limit theorem under \mathbb{P} : it converges under diffusive scaling to a Brownian motion $(B_t)_{t \geq 0}$ with a deterministic, diagonal covariance matrix Σ . For $\lambda > 0$, it was shown in [23] that the random walk is ballistic, that is, there is a deterministic vector $v(\lambda)$ with $v(\lambda) \cdot \ell > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = v(\lambda) \quad \mathbb{P}_\lambda\text{-a.s.}$$

The Einstein relation says that the derivative of the speed with respect to the perturbation relates to the covariance matrix of the unperturbed random walk among random conductances as follows.

THEOREM 1 (Einstein relation).

$$(1) \quad \lim_{\lambda \rightarrow 0} \frac{v(\lambda)}{\lambda} = \Sigma \ell.$$

We remark that Theorem 1 holds true for $d = 1$ as well, but in that case, it can be shown by explicit calculation. Theorem 1 was known for the case when the conductances take only two values and the dimension is at least 3; see [15]. The Einstein relation is conjectured to be true in general for reversible motions which behave diffusively. We refer to [9] for a historical reference and to [24] for further explanations. A weaker form of the Einstein relation holds indeed true under such general assumptions and goes back to [17]. However, (1) was only established in examples: [18] and [19] consider a tagged particle in an exclusion process, [16] and [1] investigate other examples of space–time environments, the paper [15] mentioned above gives the result for particular random walks among random conductances, [10] treats reversible diffusions in random environments and [2] considers biased random walks on Galton–Watson trees. The only result, to our best knowledge, for a nonreversible situation is given in [12]. For results on the steady states in the case of diffusions, we also refer to the recent work [20].

Note that in [12], when the random walk is a martingale, the Einstein relation (1) is a consequence of a more general convergence theorem for the steady states. In the random conductance model, due to the presence of the corrector, to generalize (1) we need finer estimates of the rate of the convergence of the steady states, for which we first introduce some notation.

When $(\theta_z\omega)(x, y) = \omega(z + x, z + y)$ denotes the environment shifted by z , we set $\bar{\omega}_n := \theta_{X_n}\omega, n \geq 0$. The Markov chain $(\bar{\omega}_n)$ is called the *environment seen from the particle* and has generator L acting on bounded functions $f : \Omega \rightarrow \mathbb{R}$ as

$$(Lf)(\omega) = \sum_{y:|y|=1} P_\omega(X_1 = y)(f(\theta_y\omega) - f(\omega)).$$

For $\lambda = 0$, the Markov chain $(\bar{\omega}_n)$ has an invariant measure Q_0 given by

$$(2) \quad \frac{dQ_0}{dP}(\omega) = Z^{-1} \sum_{y:|y|=1} \omega(0, y) \in [1/\kappa^2, \kappa^2],$$

with a normalization constant Z . If $\lambda > 0$ and the law of X_n is given by $P_{\omega,\lambda}$, an invariant measure Q_λ for the Markov chain $\bar{\omega}_n$ can be defined as the \mathbb{P}_λ -a.s. limit

$$(3) \quad Q_\lambda f = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(\bar{\omega}_k),$$

which has an expression in terms of the regeneration times defined in Section 3; see (20). The invariant measures Q_λ are often called “steady states.” The Einstein relation (1) is a consequence of a first-order expansion of Q_λ around $\lambda = 0$; see Theorem 2 below.

To describe the limit, we let \mathcal{H}_{-1} denote the set of all functions $f : \Omega \rightarrow \mathbb{R}$ in $L^2(Q_0)$ such that the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} Q_0 E_\omega \left[\left(\sum_{k=0}^n f(\bar{\omega}_k) \right)^2 \right] =: \sigma^2(f)$$

exists and is finite. For a variational characterization of the space \mathcal{H}_{-1} , we refer to [13, 14]. In the classical paper [13], it is proved that for $f \in \mathcal{H}_{-1}$, the process

$$\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{nt} f(\bar{\omega}_k) \right)_{t \geq 0}$$

converges weakly (under \mathbb{P}_0) to a Brownian motion $(N_t^f)_{t \geq 0}$ with variance $\sigma^2(f)$. For our result, we consider the subspace \mathcal{F} of bounded continuous functions $f : \Omega \rightarrow \mathbb{R}$ depending only on a finite set of conductances, that is, $f(\omega) = \tilde{f}((\omega_e)_{e \in E})$ for a finite set $E \subset \mathcal{B}(\mathbb{Z}^d)$. We remark that it follows from [21] that if $f \in \mathcal{F}$ and $d \geq 3$, then $f - Q_0 f \in \mathcal{H}_{-1}$. Consider the 2-dimensional process

$$\frac{1}{\sqrt{n}} \left(\sum_{k=1}^n f(\bar{\omega}_k) - Q_0 f, \sum_{k=1}^n d(\bar{\omega}_k, 0) \cdot \ell \right),$$

where $d(\omega, x) = E_\omega^x[(X_1 - X_0)]$ is the local drift. If $f - Q_0 f \in \mathcal{H}_{-1}$, this process converges by [13] in distribution under \mathbb{P}_0 to a 2-dimensional normal random variable $(N_1^f, N_1^d \cdot \ell)$. Define then

$$(4) \quad \Lambda f = -\text{Cov}(N_1^f, N_1^d \cdot \ell).$$

The following theorem is our main result.

THEOREM 2. *If $d \geq 3$, we have*

$$(5) \quad \lim_{\lambda \rightarrow 0} \frac{Q_\lambda f - Q_0 f}{\lambda} = \Lambda f$$

for any $f \in \mathcal{F}$.

We remark that it follows by similar arguments as in [12] that $Q_\lambda \Rightarrow Q_0$ for $d \geq 2$. The first-order expansion (5) of the measure Q_λ is obtained in [12] for P that satisfies some ballisticity condition, where a regeneration structure creates enough decorrelation for the environments along the path. In our case where the unperturbed environment P is not ballistic, the first-order expansion (5) is more delicate. When $d \geq 3$, making use of the optimal variance decay for the environment seen from the particle in [5], Theorem 1.1 (for the unperturbed environment) and a 1-dependent regeneration structure (for perturbed environments), we obtain (5). For $d \leq 2$, it is not clear to us whether (5) still holds, since the environment seen from the particle process decorrelates at a slower rate and our argument does not go through.

In order to prove Theorem 2, we will show the following two theorems. To simplify notation, we will write X_t for $X_{\lfloor t \rfloor}$ and similarly for summation limits or other indices which are defined for integer values. In Theorem 3, we center by $Q_0 f$ to point out the relation to Theorem 2, but we remark that $f \in \mathcal{H}_{-1}$ actually implies $Q_0 f = 0$.

THEOREM 3. *For any $t \geq 1$ and $f \in \mathcal{H}_{-1}$, we have*

$$(6) \quad \lim_{\lambda \rightarrow 0} \frac{\frac{\lambda^2}{t} \mathbb{E}_{Q, \lambda} \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k) - Q_0 f}{\lambda} = \Lambda f,$$

where $\mathbb{E}_{Q, \lambda}$ is the expectation with respect to $Q_0 \times P_{\omega, \lambda}$.

THEOREM 4. *If $d \geq 3$, then for any $f \in \mathcal{F}$ there exist constants $C = C(\kappa, d, f)$ and $\lambda_0 = \lambda_0(\kappa, d) > 0$ such that for any $t > 1$ and $\lambda < \lambda_0$,*

$$(7) \quad \left| \frac{\frac{\lambda^2}{t} \mathbb{E}_{Q, \lambda} \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k) - Q_\lambda f}{\lambda} \right| \leq \frac{C}{t^{1/4}}.$$

The paper is organized as follows: In Section 2, we collect some a priori estimates whose proofs are deferred to Section 7. We then define, in Section 3, a regeneration structure which will enable us to prove Theorem 4. As in [10], we here have to take into account how the regeneration times and regeneration distances depend on λ . In Section 4, we prove Theorem 3: the main ingredient is Girsanov transform and the technique of proof is similar to [10]. In Section 5, using the

regeneration structure, we prove (7) for a class of functions that satisfies a nice inequality, namely (32). In Section 6, using the variance estimate of [5, 21], we prove Theorem 4 by verifying (32) for all $f \in \mathcal{F}$ when $d \geq 3$. We also show how the Einstein relation (1) follows, at least in the case $d \geq 3$, from Theorem 2 and give a different argument to show (1) for any d . Finally, in Section 7, we prove the estimates listed in Section 2.

2. A priori estimates. Without loss of generality, assume throughout that $\ell \cdot e_1 = \max_i \{\ell \cdot e_i, -\ell \cdot e_i\}$, where $\{e_i : i = 1, \dots, d\}$ denotes the natural basis for \mathbb{Z}^d . Let $\lambda_1 = \lfloor 1/\lambda \rfloor^{-1}$, such that $1/\lambda_1 \in \mathbb{N}$ and $0 \leq \lambda_1 - \lambda = o(\lambda)$ as $\lambda \rightarrow 0$. For $m \in \mathbb{Z}$ and $L \in \mathbb{N}$, we define the hyperplane

$$H_{m,L} = \{x \in \mathbb{Z}^d \mid x \cdot e_1 = mL/\lambda_1\}$$

and the hitting time

$$T_{m,L} = \inf\{n \geq 0 \mid (X_n - X_0) \in H_{m,L}\}.$$

For $x \in \mathbb{R}^d$, we denote by $|x|$ the 1-norm of x .

The constants appearing in this paper will be allowed to depend on the dimension d and the ellipticity constant κ but we emphasize that they do not depend on λ . In the proofs, we use c, C to denote generic positive constants whose values may change from line to line.

LEMMA 5 (Bounding the probability to go left before going right). *There exist $L_0 \in \mathbb{N}$, $\lambda_0 > 0$, depending only on the ellipticity constant κ and the dimension d , such that*

$$P_{\omega,\lambda}(T_{1,L} < T_{-1,L}) \geq \frac{2}{3}$$

for all $L \geq L_0$, $0 < \lambda \leq \lambda_0$ and for all ω .

In the following, fix L_0 as in Lemma 5 and we write shorter H_m for $H_{m,4L_0}$ and T_m for $T_{m,4L_0}$. For the distance between these hyperplanes, we write

$$(8) \quad L_1 = 4L_0/\lambda_1,$$

tacitly ignoring the dependence on λ .

COROLLARY 6 (Bounding the probability to go far to the left). *Let λ_0 be the same constant as in Lemma 5. For any $\omega \in \Omega$, $n \in \mathbb{N}$ and $\lambda \in (0, \lambda_0)$, we have*

$$P_{\omega,\lambda}(T_{-n/4} < \infty) \leq 2^{-n}.$$

LEMMA 7 (Hitting times of hyperplanes to the right are small with high probability). *There exist positive constants c_1, C_1 such that for any $\omega, n \geq 1$ and $\lambda \in (0, \lambda_0)$ we have*

$$P_{\omega, \lambda} \left(T_n \geq \frac{C_1 n}{\lambda^2} \right) \leq e^{-c_1 n}.$$

LEMMA 8 (Bounds on the moments of the maximum of the walk). *For any $p \geq 1$, there exists a positive constant C_2 , such that for all $\lambda \in (0, \lambda_0)$ and $t \geq 1$ and for any ω ,*

$$E_{\omega, \lambda} \left[\max_{0 \leq s \leq t} |\lambda X_{s/\lambda^2}|^p \right] \leq C_2 t^p.$$

Corollary 6 is an immediate consequence of Lemma 5. The proofs of Lemmas 5, 7 and 8 are given in Section 7. The following parabolic Harnack inequality will be used several times in our paper. Let $B_R = \{x \in \mathbb{Z}^d : |x|_2 \leq R\}$ and $B_R(x) := x + B_R$.

THEOREM 9 (Parabolic Harnack inequality). *Fix $R \geq \sqrt{d}$. Let $a \in \Omega$ be a configuration of conductances such that $\frac{a(e)}{a(e')} \leq C_V$ for any two bonds e, e' in B_{2R} . Assume that $u : \mathbb{Z}^d \times \mathbb{Z} \rightarrow \mathbb{R}_+$ is a nonnegative function that satisfies the parabolic equation*

$$(PE) \quad u(x, n + 1) = \sum_y P_a^x(X_1 = y) u(y, n) \quad \text{in } B_{2R} \times [0, 4R^2 + 1].$$

Then there exists a constant $C = C(d, C_V)$ such that

$$(PHI) \quad \max_{B_R \times [R, 2R^2]} u \leq C \min_{(x, n) \in B_R \times [3R^2, 4R^2]} [u(x, n) + u(x, n + 1)].$$

We remark that the bound $a(e)/a(e') < C_V$ implies the *volume-doubling condition* (cf. [6], Definition 1.1) and uniform ellipticity. For our exponentially growing conductances ω^λ , we can choose the constant C_V independent of $\lambda \in [0, 1]$, as long as we consider u defined on a subgraph of size C/λ . For the conductances with $a(x, x) > c > 0$, the theorem is [6], Theorem 1.7. In our case $a(x, x) = 0$, (PHI) is obtained by applying [6], Theorem 1.7, to even-step jumps of the random walk; see the remarks below [6], Definition 1.3. The above version of (PHI) can be found in [11], Definition 2.2. Note that when u is not a function of time, that is, $u(x, n) = u(x, m) =: u(x)$ for all $n, m \in \mathbb{Z}$, then u satisfies the *elliptic equation*

$$(EE) \quad u(x) = \sum_y P_a^x(X_1 = y) u(y)$$

and (PHI) becomes the *elliptic Harnack inequality*

$$(EHI) \quad \max_{B_R} u \leq C \min_{B_R} u.$$

3. Regeneration structure. In this section, we will construct a 1-dependent regeneration structure on the random walk path so that the inter-regeneration distances and inter-regeneration times are roughly of order $1/\lambda$ and $1/\lambda^2$, respectively. The regeneration structure will then imply the estimate of Theorem 4. For this, we fix the function $f \in \mathcal{F}$ and allow the constants in this chapter to depend on f .

3.1. *Auxiliary estimates.*

LEMMA 10 (Transversal fluctuations are not too large). *There exists a constant C_3 so that for all $\lambda \in (0, \lambda_0)$*

$$P_{\omega,\lambda}(|X_{T_{3/4}}| \geq C_3/\lambda) \leq 1 - C_3^{-1}.$$

PROOF. By Lemma 7 and Lemma 8, for any $\theta \geq 1$,

$$\begin{aligned} P_{\omega,\lambda}(|X_{T_{3/4}}| \geq \theta/\lambda) &\leq P_{\omega,\lambda}(T_{3/4} \geq C_1/\lambda^2) + P_{\omega,\lambda}\left(\max_{0 \leq s \leq C_1/\lambda^2} |X_s| \geq \theta/\lambda\right) \\ &\leq e^{-c_1} + C_1 C_2/\theta. \end{aligned}$$

Taking θ sufficiently large, the lemma follows. \square

LEMMA 11 (Bounding the exit measure on a hyperplane to the right). *There exists a probability measure $\mu_{\omega,\lambda,1}$ on H_1 , which is independent of $\sigma(\omega(x, y) : x \cdot e_1 \leq L_0/\lambda_1, y \in \mathbb{Z}^d)$, and a constant $c_4 > 0$ such that*

$$P_{\omega,\lambda}(X_{T_1} = \cdot) \geq c_4 \mu_{\omega,\lambda,1}(\cdot).$$

PROOF. We will prove the lemma by showing that, for any $w \in H_1$ and $x = e_1 \cdot 3L_0/\lambda_1$,

$$P_{\omega,\lambda}(X_{T_1} = w) \geq C P_{\omega,\lambda}^x(X_{T_{1/4}} = w | T_{1/4} < T_{-1/4}).$$

Indeed, for any $w \in H_1$,

$$\begin{aligned} P_{\omega,\lambda}(X_{T_1} = w) &\geq \sum_{y:|y-x| < C_3/\lambda} P_{\omega,\lambda}(X_{T_{3/4}} = y) P_{\omega,\lambda}^y(X_{T_{1/4}} = w) \\ &\geq C \sum_{y:|y-x| < C_3/\lambda} P_{\omega,\lambda}(X_{T_{3/4}} = y) P_{\omega,\lambda}^x(X_{T_{1/4}} = w) \\ &\stackrel{\text{Lem. 10}}{\geq} C C_3^{-1} P_{\omega,\lambda}^x(X_{T_{1/4}} = w) \\ &\stackrel{\text{Lem. 5}}{\geq} C P_{\omega,\lambda}^x(X_{T_{1/4}} = w | T_{1/4} < T_{-1/4}), \end{aligned}$$

where in the second inequality we applied the elliptic Harnack inequality (EHI) to the function $u(y) := P_{\omega,\lambda}^y(X_{T_{1/4}} = w)$ in the ball $B_{2C_3/\lambda}(x)$. \square

3.2. *Construction of the regeneration time.* Classical regeneration times are usually defined as times when the random walker crosses a certain hyperplane for the first time and then never comes back. To keep the regeneration times robust for small bias, we allow the path to backtrack a distance of order $1/\lambda$. To decouple the trajectory between these regeneration times, we will then use the “coin trick” by [4].

The starting point is that by Lemma 11, the hitting probability $P_{\omega,\lambda}^x(X_{T_1} = \cdot)$ of the next hyperplane dominates c_4 times a probability measure $\mu_{\omega,\lambda,1}^x$, which is independent of the environment to the left of the hyperplane $H_0^x := x + \frac{L_0}{\lambda_1}e_1 + H_0$. Hence, for $\beta \in (0, c_4)$ the hitting probability can be decomposed as

$$P_{\omega,\lambda}^x(X_{T_1} = \cdot) = \beta \mu_{\omega,\lambda,1}^x(\cdot) + (1 - \beta) \mu_{\omega,\lambda,0}^x(\cdot),$$

where

$$\mu_{\omega,\lambda,0}^x(\cdot) := \frac{P_{\omega,\lambda}^x(X_{T_1} = \cdot) - \beta \mu_{\omega,\lambda,1}^x(\cdot)}{1 - \beta}.$$

By Lemma 11, both $\mu_{\omega,\lambda,1}^x$ and $\mu_{\omega,\lambda,0}^x$ are probability measures on $H_1^x = \{y \in \mathbb{Z}^d \mid (y - x) \cdot e_1 = L_1\}$. Let $(\varepsilon_i)_{i=0}^\infty \in \{0, 1\}^{\mathbb{N}_0}$ be i.i.d. Bernoulli random variables with law q_β :

$$q_\beta(\varepsilon_i = 1) = \beta \quad \text{and} \quad q_\beta(\varepsilon_i = 0) = 1 - \beta.$$

Intuitively, when X_n is at $x \in H_i$ the coin ε_i will determine whether the hitting point of the next hyperplane H_{i+1} is sampled via $\mu_{\omega,\lambda,0}^x$ or $\mu_{\omega,\lambda,1}^x$. Until reaching H_{i+1} , the law of the path will then be the original quenched law, conditioned on the predetermined hitting point.

We now give the formal definition of the regeneration times, for which we first define inductively a path measure given a set of “hitting rules” as described above. Sample the sequence $\varepsilon := (\varepsilon_i)_{i=0}^\infty$ according to the product measure q_β and fix it. Then define $P_{\omega,\lambda,\varepsilon}^x$ on the paths by the following steps:

Step 1. For $x \in \mathbb{Z}^d$, set

$$P_{\omega,\lambda,\varepsilon}^x(X_0 = x) = 1,$$

and for any $\mathcal{O} \in \sigma(X_1, X_2, \dots, X_{T_1})$, put

$$(9) \quad v_{\omega,\lambda,\varepsilon_i}^x(\mathcal{O}) := \sum_y [\varepsilon_i \mu_{\omega,\lambda,1}^x(y) + (1 - \varepsilon_i) \mu_{\omega,\lambda,0}^x(y)] P_{\omega,\lambda}^x(\mathcal{O} \mid X_{T_1} = y).$$

Step 2. Suppose the $P_{\omega,\lambda,\varepsilon}^x$ -law for paths of length $\leq n$ is defined. For any path $(x_i)_{i=0}^{n+1}$ with $x_0 = x$, define

$$P_{\omega,\lambda,\varepsilon}^x(X_{n+1} = x_{n+1}, \dots, X_0 = x_0) \\ := P_{\omega,\lambda,\varepsilon}^x(X_I = x_I, \dots, X_0 = x_0) v_{\omega,\lambda,\varepsilon_J}^{x_I}(X_{n+1-I} = x_{n+1}, \dots, X_1 = x_{I+1}),$$

where

$$J = \max\{j \geq 0 : H_j^{x_0} \cap \{x_i, 0 \leq i \leq n\} \neq \emptyset\}$$

is the rightmost hyperplane visited by $(x_i)_{i=0}^n$ and

$$I = \min\{0 \leq i \leq n : x_i \in H_J^{x_0}\}$$

is the hitting time to the J th level.

Step 3. By induction, the law $P_{\omega,\lambda,\varepsilon}^x$ is well defined for paths of all lengths.

Intuitively, whenever the walker visits new hyperplanes $H_i, i \geq 0$, we make him flip a coin ε_i . If $\varepsilon_i = 0$ (or 1), he then walks following the law $\nu_{\omega,\lambda,0}$ (or $\nu_{\omega,\lambda,1}$) until he reaches the $(i + 1)$ -th hyperplane. The regeneration time τ_1 is defined to be the first time of visiting a new hyperplane H_k such that the outcome ε_{k-1} of the previous coin-tossing is “1” and the path will never backtrack to level H_{k-1} in the future.

Note that a path sampled by $P_{\omega,\lambda,\varepsilon}^x$ is not a Markov chain, but the law of X under

$$\bar{P}_{\omega,\lambda}^x := q\beta \times P_{\omega,\lambda,\varepsilon}^x$$

coincides with $P_{\omega,\lambda}^x$. That is,

$$(10) \quad \bar{P}_{\omega,\lambda}^x(X \in \cdot) = P_{\omega,\lambda}^x(X \in \cdot).$$

We denote by

$$(11) \quad \bar{\mathbb{P}}_\lambda := P \times \bar{P}_{\omega,\lambda}$$

the law of X , averaged over the coins and the environment. Expectations with respect to $\bar{P}_{\omega,\lambda}^x$ and $\bar{\mathbb{P}}_\lambda$ are denoted by $\bar{E}_{\omega,\lambda}^x$ and $\bar{\mathbb{E}}_\lambda$, respectively. Next, for a path $(X_n)_{n \geq 0}$ sampled according to $P_{\omega,\lambda,\varepsilon}$, we will define the regeneration times. To be specific, put $S_0 = 0, M_0 = 0$, and define inductively the times S_k and R_k and the distances M_k by

$$(12) \quad \begin{aligned} S_{k+1} &= \inf\{T_{n+1} : nL_1 \geq M_k \text{ and } \varepsilon_n = 1\}, \\ R_{k+1} &= S_{k+1} + T_{-1/4} \circ \theta_{S_{k+1}}, \\ M_{k+1} &= X_{S_{k+1}} \cdot e_1 + N \circ \theta_{S_{k+1}} \cdot L_1, \quad k \geq 0. \end{aligned}$$

Here, θ_n denotes the time shift of the path, that is, $\theta_n X = (X_{n+i})_{i=0}^\infty$, and

$$(13) \quad N := \inf\{n : nL_1 > (X_i - X_0) \cdot e_1 \text{ for all } i \leq T_{-1/4}\}.$$

We use the convention that $\inf \emptyset = \infty$. Set

$$(14) \quad K := \inf\{k \geq 1 : S_k < \infty, R_k = \infty\},$$

$$(15) \quad \tau_1 := S_K \quad \text{and} \quad \tau_{k+1} = \tau_k + \tau_1 \circ \theta_{\tau_k}.$$

The times $(\tau_k)_{k \geq 1}$ are called *regeneration times*; see Figure 1 for an illustration.

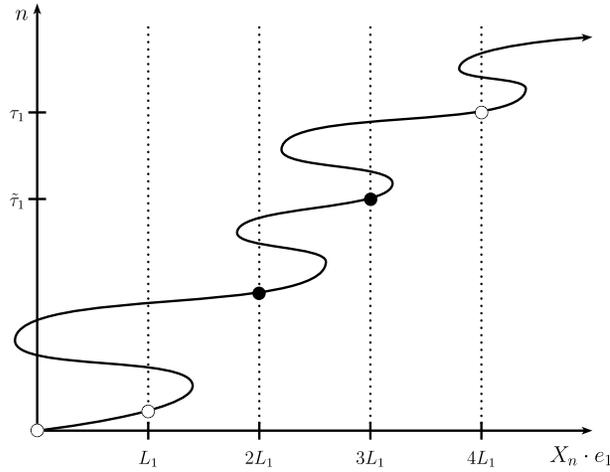


FIG. 1. A sample path of the random walk. A black dot at X_{T_n} represents a successful coin toss $\varepsilon_n = 1$, a white dot corresponds to $\varepsilon_n = 0$. After hitting H_2 , the random walk does not backtrack more than $L_1/4$, but since $\varepsilon_1 = 0$, this is not a regeneration time. Since $\varepsilon_3 = 1$ and after reaching H_4 , the path does not backtrack more than $L_1/4$, we have $\tilde{\tau}_1 = T_3$, $\tau_1 = T_4$. Note that $\varepsilon_3 = 1$ implies that the hitting point X_{T_4} was chosen according to $\mu_{\omega, \lambda, 1}^{X_{T_3}}$.

3.3. *Renewal property of the regenerations.* The regeneration times possess good renewal properties:

1. Set $\tau_0 = 0$. For $k \geq 0$, define

$$\tilde{S}_{k+1} := \inf\{T_n : nL_1 \geq M_k \text{ and } \varepsilon_n = 1\},$$

with M_k as in (12). That is, if we divide the space by the hyperplanes H_k at distance L_1 of each other, \tilde{S}_{k+1} is the hitting time of a hyperplane after the previous maximum M_k is achieved and when the coin toss corresponding to this hyperplane is a success. Note that S_{k+1} is the hitting time of the next hyperplane after \tilde{S}_{k+1} . Let $\tilde{\tau}_k$ be the hitting time to the previous hyperplane of X_{τ_k} . Namely, with K as in (14), set

$$\tilde{\tau}_1 := \tilde{S}_K, \quad \tilde{\tau}_{k+1} := \tau_k + \tilde{\tau}_1 \circ \theta_{\tau_k} \quad (k > 1).$$

Note that at time $\tilde{\tau}_k$ the ε_i -coin toss is a success and, after arriving at X_{τ_k} , the hyperplane of $X_{\tilde{\tau}_k}$ is never visited again. Conditioning on $X_{\tilde{\tau}_k} = x$, the law of X_{τ_k} is $\mu_{\omega, \lambda, 1}^x$, which is independent (under the environment measure P) of $\sigma(\omega(y, z) : y \cdot e_1 \leq x \cdot e_1)$.

Moreover, after time τ_k , the path will never visit $\{y : y \cdot e_1 \leq x \cdot e_1 + 3L_1/4\}$. Therefore, $\tau_{k+1} - \tau_k$ is independent of what happened before $\tilde{\tau}_{k-1}$ and the inter-regeneration times form a 1-dependent sequence.

2. Since $(X_{\tilde{\tau}_{k+1}} - X_{\tau_k})_{k \geq 1}$ are i.i.d. and $(X_{\tau_{k+1}} - X_{\tilde{\tau}_{k+1}}) \cdot e_1 = 1/\lambda_1$, the inter-regeneration distances $((X_{\tau_{k+1}} - X_{\tau_k}) \cdot e_1)_{k \geq 1}$ are i.i.d.

3. From the construction, we see that a regeneration occurs after roughly a geometric number of levels. Thus, we expect $(X_{\tau_{k+1}} - X_{\tau_k}) \cdot e_1 \sim c/\lambda$ and (by Lemma 7) $\tau_{k+1} - \tau_k \sim c/\lambda^2$.

The above properties are made more precise in Lemma 12 below, the proof follows as in [12]. We introduce the σ -field

$$\mathcal{G}_k := \sigma(\tilde{\tau}_k, (X_i)_{i \leq \tilde{\tau}_k}, (\omega(x, y))_{x \cdot e_1 \leq X_{\tilde{\tau}_k} \cdot e_1})$$

and set

$$(16) \quad p_\lambda := E \left[\sum_{y \in H_1} \mu_{\omega, \lambda, 1}(y) P_{\omega, \lambda}^y(T_{-1/4} = \infty) \right].$$

LEMMA 12. *For any appropriate measurable sets B_1, B_2 and any event*

$$B := \{(X_i)_{i \geq 0} \in B_1, (\omega(x, y))_{x \cdot e_1 > -L_1/4} \in B_2\},$$

we have for $k \geq 1$,

$$\bar{\mathbb{P}}_\lambda(B \circ \bar{\theta}_{\tilde{\tau}_k} | \mathcal{G}_k) = E \left[\sum_{y \in H_1} \mu_{\omega, \lambda, 1}(y) \bar{P}_{\omega, \lambda}^y(B \cap \{T_{-1/4} = \infty\}) \right] / p_\lambda,$$

where $\bar{\theta}_n$ is the time-shift defined by

$$B \circ \bar{\theta}_n = \{(X_i)_{i \geq n} \in B_1, (\omega(x, y))_{(x - X_n) \cdot e_1 > -L_1/4} \in B_2\}.$$

We say that a sequence of random variables $(Y_i)_{i \in \mathbb{N}}$ is m -dependent ($m \in \mathbb{N}$) if

$$\sigma(Y_i; 1 \leq i \leq n) \quad \text{and} \quad \sigma(Y_j; j > n + m) \quad \text{are independent, } \forall n \in \mathbb{N}.$$

The law of large numbers and central limit theorem also hold for a stationary m -dependent sequence with finite means and variances; see [3], Theorem 5.2. The following proposition is an immediate consequence of Lemma 12.

PROPOSITION 13. *Under $\bar{\mathbb{P}}_\lambda$, $(X_{\tau_{n+1}} - X_{\tau_n})_{n \geq 1}$ and $(\tau_{n+1} - \tau_n)_{n \geq 1}$ are stationary 1-dependent sequences. Furthermore, for all $n \geq 1$, $(X_{\tau_{n+1}} - X_{\tau_n}, \tau_{n+1} - \tau_n)$ has the following law:*

$$\begin{aligned} & \bar{\mathbb{P}}_\lambda(X_{\tau_{n+1}} - X_{\tau_n} \in \cdot, \tau_{n+1} - \tau_n \in \cdot) \\ &= E \left[\sum_{y \in H_1} \mu_{\omega, \lambda, 1}(y) \bar{P}_{\omega, \lambda}^y(X_{\tau_1} \in \cdot, \tau_1 \in \cdot, T_{-1/4} = \infty) \right] / p_\lambda. \end{aligned}$$

3.4. *Moment estimates of regeneration times.* In this section, we will show that the rescaled inter-regeneration times $\lambda^2(\tau_2 - \tau_1)$ and inter-regeneration distances $\lambda(X_{\tau_2} - X_{\tau_1})$ have finite exponential moments.

THEOREM 14. *There exist constants $c_5, C_5 > 0$ such that for any ω and $\lambda \in (0, \lambda_0)$,*

$$\bar{E}_{\omega, \lambda}[\exp(c_5 \lambda X_{\tau_1} \cdot e_1)] \leq C_5 < \infty.$$

PROOF. First, observe that

$$\begin{aligned} \bar{E}_{\omega, \lambda}[\exp(c \lambda X_{\tau_1} \cdot e_1)] &= \sum_{k \geq 1} \bar{E}_{\omega, \lambda}[\exp(c \lambda X_{S_k} \cdot e_1) \mathbb{1}_{\{S_k < \infty, \tau_1 = S_k\}}] \\ &\leq \sum_{k \geq 1} \bar{E}_{\omega, \lambda}[\exp(c \lambda X_{S_k} \cdot e_1) \mathbb{1}_{\{S_k < \infty\}}]. \end{aligned}$$

Next, by the definition of S_k , when $S_{k+1} < \infty$, we have (recall that $L_1 = 4L_0/\lambda_1$)

$$X_{S_1} \cdot e_1 = G_1 L_1$$

and, recalling (13),

$$(X_{S_{k+1}} - X_{S_k}) \cdot e_1 = (N \circ \theta_{S_k} + G_k) L_1 \quad \text{for } k \geq 1,$$

where $G_k := \inf\{n \geq 1 : \varepsilon_{n+M_k/L_1} = 1\}$ is a geometric random variable with parameter β . Hence, taking $c > 0$ small enough, we can achieve

$$E[e^{cG_k}] \leq \frac{9}{8}.$$

Then, for $k \geq 1$, using the Markov property

$$\begin{aligned} &\bar{E}_{\omega, \lambda}[\exp(c X_{S_{k+1}} \cdot e_1 / L_1) \mathbb{1}_{\{S_{k+1} < \infty\}}] \\ &\leq \frac{9}{8} \bar{E}_{\omega, \lambda}[\exp(c X_{S_k} \cdot e_1 / L_1 + c N \circ \theta_{S_k}) \mathbb{1}_{\{S_k < \infty\}}] \\ &= \frac{9}{8} \sum_x \bar{E}_{\omega, \lambda}[\exp(cx \cdot e_1 / L_1) \mathbb{1}_{\{S_k < \infty, X_{S_k} = x\}}] \bar{E}_{\omega, \lambda}^x[e^{cN} \mathbb{1}_{\{T_{-1/4} < \infty\}}]. \end{aligned}$$

Further, for any ω ,

$$\begin{aligned} &E_{\omega, \lambda}[e^{cN} \mathbb{1}_{\{T_{-1/4} < \infty\}}] \\ &\leq e^c P_{\omega, \lambda}(T_{-1/4} < \infty) + \sum_{n \geq 2} e^{cn} P_{\omega, \lambda}(N = n, T_{-1/4} < \infty) \\ &\leq e^c P_{\omega, \lambda}(T_{-1/4} < \infty) \\ &\quad + \sum_{n \geq 2} \sum_{z \in H_{n-1}} e^{cn} P_{\omega, \lambda}(X_{T_{n-1}} = z) P_{\omega, \lambda}^z(T_{-(n-1)-1/4} < T_1) \\ &\stackrel{\text{Cor. 6}}{\leq} e^c / 2 + \sum_{n \geq 1} (e^c / 16)^n \leq 7/8, \end{aligned}$$

where the last inequality is achieved by taking $c > 0$ sufficiently small. Thus, we conclude that taking $c > 0$ sufficiently small, for $k \geq 1$,

$$\begin{aligned} & \bar{E}_{\omega,\lambda}[\exp(cX_{S_{k+1}} \cdot e_1/L_1)\mathbb{1}_{\{S_{k+1} < \infty\}}] \\ & \leq \frac{63}{64} \bar{E}_{\omega,\lambda}[\exp(cX_{S_k} \cdot e_1/L_1)\mathbb{1}_{\{S_k < \infty\}}]. \end{aligned}$$

Therefore,

$$\begin{aligned} & \bar{E}_{\omega,\lambda}[\exp(cX_{\tau_1} \cdot e_1/L_1)] \\ & \leq \sum_{k \geq 1} \left(\frac{63}{64}\right)^{k-1} \bar{E}_{\omega,\lambda}[\exp(cX_{S_1} \cdot e_1/L_1)\mathbb{1}_{\{S_1 < \infty\}}] \\ & = 64E[e^{cG_1}] \leq 72, \end{aligned}$$

which completes the proof. \square

COROLLARY 15. *There exist constants $c_6, C_6, c_7, C_7 > 0$ such that for any integer $n \geq 0$, any ω and $\lambda \in (0, \lambda_0)$,*

$$(17) \quad \bar{\mathbb{E}}_\lambda[\exp(c_6\lambda(X_{\tau_{n+1}} - X_{\tau_n}) \cdot e_1)] < C_6,$$

$$(18) \quad \bar{\mathbb{E}}_\lambda[\exp(c_7\lambda^2(\tau_{n+1} - \tau_n))] < C_7.$$

PROOF. When $n = 0$, inequality (17) reduces to Theorem 14. For $n \geq 1$, by Proposition 13,

$$\begin{aligned} & \bar{\mathbb{E}}_\lambda[\exp(c_5\lambda(X_{\tau_{n+1}} - X_{\tau_n}) \cdot e_1)] \\ & \leq 2E\left[\sum_{y \in H_1} \mu_{\omega,\lambda,1}(y) \bar{E}_{\omega,\lambda}^y[\exp(c_5\lambda X_{\tau_1} \cdot e_1)\mathbb{1}_{\{T_{-1/4} = \infty\}}]\right] \leq 2C_5, \end{aligned}$$

where we used again Theorem 14 and the fact (see Corollary 6) that $P_{\omega,\lambda}(T_{-1/4} = \infty) \geq 1/2$ for all ω and $\lambda \in (0, \lambda_0)$. This proves (17). To prove (18), it suffices to show that

$$(19) \quad \bar{\mathbb{P}}_\lambda(\tau_{n+1} - \tau_n > Cm/\lambda^2) \leq Ce^{-c_6m}, \quad \forall m \geq 1, n \geq 0.$$

From (17), we get the bound

$$\begin{aligned} & \bar{\mathbb{P}}_\lambda(\tau_{n+1} - \tau_n > C_1m/\lambda^2) \\ & \leq \bar{\mathbb{P}}_\lambda((X_{\tau_{n+1}} - X_{\tau_n}) \cdot e_1 \geq mL_1) + \bar{\mathbb{E}}_\lambda[P_{\omega,\lambda}^{X_{\tau_n}}(T_m > C_1m/\lambda^2)] \\ & \leq C_6e^{-cm} + e^{-c_1m} \end{aligned}$$

by Lemma 7. \square

COROLLARY 16. *Let $\lambda \in (0, \lambda_0)$.*

(a) The speed $v(\lambda)$ satisfies $|v(\lambda)| \in (C\lambda, C'\lambda)$ for positive constants C, C' .

(b) The limit Q_λ in (3) exists \mathbb{P}_λ -almost surely for any bounded continuous f , and it defines an invariant measure for the process $(\bar{\omega}_n)_n$. Moreover, for any $f \in \mathcal{F}$, there exists $\lambda_0 > 0$ such that for $\lambda < \lambda_0$,

$$(20) \quad Q_\lambda f = \bar{\mathbb{E}}_\lambda \left[\sum_{\tau_1 \leq i < \tau_2} f(\bar{\omega}_i) \right] / \bar{\mathbb{E}}_\lambda[\tau_2 - \tau_1].$$

PROOF. (a) Since the inter-regeneration distances and inter-regeneration times are stationary 1-dependent sequences (Proposition 13), and they have exponential moments (Corollary 15), the law of large numbers gives

$$v(\lambda) = \lim_{n \rightarrow \infty} \frac{X_n}{n} = \lim_{n \rightarrow \infty} \frac{X_{\tau_n}}{\tau_n} = \frac{\bar{\mathbb{E}}_\lambda[X_{\tau_2} - X_{\tau_1}]}{\bar{\mathbb{E}}_\lambda[\tau_2 - \tau_1]}.$$

Moreover, by Corollary 15 we have $|v(\lambda)| \geq \frac{1/\lambda}{c/\lambda^2} > C\lambda$. On the other hand,

$$(21) \quad \begin{aligned} E_{\omega, \lambda}[T_1] &\geq \frac{t}{\lambda^2} P_{\omega, \lambda}(T_1 \geq t/\lambda^2) \geq \frac{t}{\lambda^2} P_{\omega, \lambda}\left(\max_{0 \leq s \leq t/\lambda^2} |X_s| \leq L_1\right) \\ &= \frac{t}{\lambda^2} \left(1 - P_{\omega, \lambda}\left(\max_{0 \leq s \leq t/\lambda^2} |X_s| > L_1\right)\right) \\ &\geq \frac{t}{\lambda^2} (1 - Ct) \geq t/2\lambda^2, \end{aligned}$$

for all $\omega \in \Omega$ and $\lambda \in (0, \lambda_0)$ and $t > 0$ a sufficiently small constant (where we used Lemma 8 for the second-last inequality). Then we also have a lower bound for the moment of the inter-regeneration time

$$(22) \quad \bar{\mathbb{E}}_\lambda[\tau_{n+1} - \tau_n] \geq c/\lambda^2 \quad \text{for all } \lambda \in (0, \lambda_0), n \geq 0.$$

So we have $|v(\lambda)| \leq C'\lambda$.

(b) For a ballistic random walk in a uniformly elliptic finitely dependent random environment, recall that the regeneration times (which are different from the regeneration times in our paper) constructed by Comets and Zeitouni in [4] divide both the path and the environment into i.i.d. inter-regeneration pieces. This regeneration structure and the same argument as in the proof of Theorem 3.1 of [25] yields that the annealed law of $\bar{\omega}_n$ converges to an ergodic invariant measure, which we denote by Q_λ , of the sequence $(\bar{\omega}_n)$. Hence,

$$Q_\lambda f = \lim_{n \rightarrow \infty} \bar{\mathbb{E}}_\lambda \left[\frac{1}{n} \sum_{i=0}^n f(\bar{\omega}_i) \right] \quad \text{for any } f \in \mathcal{F}.$$

Suppose $f \in \mathcal{F}$ is $\sigma(\omega(x, \cdot) : |x| \leq K)$ -measurable for some $K > 0$, then by Proposition 13, when $\lambda < 1/(4K)$, the sequence $(\sum_{\tau_k \leq i < \tau_{k+1}} f(\bar{\omega}_i))$, $k \geq 1$, is

1-dependent and stationary. Therefore, by the moment estimates in Corollary 15 and the law of large numbers for 1-dependent stationary sequences, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \bar{\mathbb{E}}_\lambda \left[\frac{1}{n} \sum_{i=0}^n f(\bar{\omega}_i) \right] &= \lim_{k \rightarrow \infty} \bar{\mathbb{E}}_\lambda \left[\frac{1}{\tau_k} \sum_{i=0}^{\tau_k} f(\bar{\omega}_i) \right] \\ &= \bar{\mathbb{E}}_\lambda \left[\sum_{\tau_1 \leq i < \tau_2} f(\bar{\omega}_i) \right] / \bar{\mathbb{E}}_\lambda[\tau_2 - \tau_1]. \end{aligned} \quad \square$$

4. Proof of Theorem 3. Recall the notation $\mathbb{E}_{Q,\lambda}$ in Theorem 3. We start by writing the expectation with respect to the unperturbed measure,

$$\mathbb{E}_{Q,\lambda} \left[\frac{\lambda}{t} \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k) \right] = \mathbb{E}_{Q,0} \left[\frac{\lambda}{t} \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k) \frac{dP_{\omega,\lambda}}{dP_\omega}(X_s; 0 \leq s \leq t/\lambda^2) \right]$$

and first study the Radon–Nikodym derivative

$$(23) \quad G_\omega(\lambda, n) = \frac{dP_{\omega,\lambda}}{dP_\omega}(X_s; 0 \leq s \leq n).$$

4.1. *An expansion of the Radon–Nikodym derivative.* In this subsection, we will derive a formula (25) for the density $G_\omega(\lambda, n)$. For a path (x_0, \dots, x_n) with $x_0 = 0$ we have

$$\begin{aligned} &\frac{P_{\omega,\lambda}((x_0, \dots, x_n))}{P_\omega((x_0, \dots, x_n))} \\ &= \prod_{i=0}^{n-1} e^{\lambda \ell \cdot (x_{i+1} - x_i)} \frac{\sum_{|e|=1} \omega(x_i, x_i + e)}{\sum_{|e|=1} \omega(x_i, x_i + e) e^{\lambda \ell \cdot e}} \\ &= \exp \left\{ \lambda \ell \cdot x_n - \sum_{i=0}^{n-1} \log \left[\frac{\sum_{|e|=1} \omega(x_i, x_i + e) e^{\lambda \ell \cdot e}}{\sum_{|e|=1} \omega(x_i, x_i + e)} \right] \right\}. \end{aligned}$$

Note that, for any $\omega \in \Omega$,

$$\begin{aligned} &\log \left[\frac{\sum_{|e|=1} \omega(0, e) e^{\lambda \ell \cdot e}}{\sum_{|e|=1} \omega(0, e)} \right] \\ &= \log \left[\frac{\sum_{|e|=1} \omega(0, e) (1 + \lambda \ell \cdot e + \frac{(\lambda \ell \cdot e)^2}{2} + o(\lambda^2))}{\sum_{|e|=1} \omega(0, e)} \right] \\ &= \log \left[1 + \lambda E_\omega[X_1 \cdot \ell] + \frac{\lambda^2}{2} E_\omega[(X_1 \cdot \ell)^2] + o(\lambda^2) \right] \\ &= \lambda E_\omega[X_1] \cdot \ell + \frac{\lambda^2}{2} \text{Var}_\omega[X_1 \cdot \ell] + o(\lambda^2), \end{aligned}$$

where in the last inequality we used the expansion $\log(1+x) = x - x^2/2 + o(x^2)$ and we write Var_ω for the variance with respect to P_ω . Then, recalling $d(\omega, x) = E_\omega^x[X_1 - X_0]$, we obtain

$$\frac{P_{\omega,\lambda}((x_0, \dots, x_n))}{P_\omega((x_0, \dots, x_n))} = \exp \left\{ \lambda x_n \cdot \ell - \sum_{i=0}^{n-1} \left(\lambda d(\omega, x_i) \cdot \ell + \frac{\lambda^2}{2} \text{Var}_{\theta_{x_i}\omega}[(X_1 - X_0) \cdot \ell] + o(\lambda^2) \right) \right\}.$$

Hence, writing

$$(24) \quad M_n := \left(X_n - \sum_{i=0}^{n-1} d(\omega, X_i) \right) \cdot \ell, \quad D_\ell(\omega) := \text{Var}_\omega[(X_1 - X_0) \cdot \ell],$$

we conclude that

$$(25) \quad \begin{aligned} G_\omega(\lambda, n) &= \frac{dP_{\omega,\lambda}}{dP_\omega}(X_s; 0 \leq s \leq n) \\ &= \exp \left\{ \lambda M_n - \frac{\lambda^2}{2} \sum_{i=0}^{n-1} D_\ell(\bar{\omega}_i) + n \cdot o(\lambda^2) \right\}. \end{aligned}$$

4.2. *Weak convergence and Girsanov transform.* In this subsection, we will compute the limit of

$$(26) \quad \begin{aligned} &\mathbb{E}_{Q,\lambda} \left[\lambda \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k) \right] \\ &= \mathbb{E}_{Q,0} \left[\lambda \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k) G_\omega(\lambda, t/\lambda^2) \right] \\ &= \mathbb{E}_{Q,0} \left[\lambda \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k) \exp \left\{ \lambda M_{t/\lambda^2} - \frac{\lambda^2}{2} \sum_{i=0}^{t/\lambda^2-1} D_\ell(\bar{\omega}_i) + t \cdot o(1) \right\} \right] \end{aligned}$$

for $f \in \mathcal{H}_{-1}$ and any fixed $t > 0$, as $\lambda \rightarrow 0$. First, we compute the limits of the terms in the expectation (26). Recall that for any $f \in \mathcal{H}_{-1}$, the process $\lambda \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k)$ converges weakly (under $Q_0 \times P_\omega$) to a Brownian motion N_t^f . Furthermore, notice that M_n given in (24) is a P_ω -martingale whose increments are bounded and stationary with respect to $Q_0 \times P_\omega$. Hence, the (joint) martingale CLT yields the joint convergence

$$(27) \quad \lambda \left(\sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k), M_{t/\lambda^2} \right)_{t \geq 0} \xrightarrow[\lambda \rightarrow 0]{} (N_t^f, N_t)_{t \geq 0},$$

in distribution under $Q_0 \times P_\omega$ to a 2-dimensional Brownian motion $(N_t^f, N_t)_{t \geq 0}$. Recall that the process $(\bar{\omega}_n)_{n \geq 0}$ is stationary and ergodic with respect to the initial measure Q_0 defined in (2). By the ergodic theorem, we have

$$\frac{\lambda^2}{2} \sum_{i=0}^{t/\lambda^2-1} D_\ell(\bar{\omega}_i) \xrightarrow{\lambda \rightarrow 0} \frac{t}{2} E_{Q_0}[D_\ell(\omega)]$$

$Q_0 \times P_\omega$ -almost surely and hence also \mathbb{P} -almost surely, where E_{Q_0} denotes the expectation with respect to Q_0 .

Next, we will show that $G_\omega(\lambda, t/\lambda^2)$ is uniformly bounded in $L^p(P_\omega)$, $p \geq 1$. In fact, for any $p \geq 1$ and small enough $\lambda \leq \lambda_0 = \lambda_0(p, t)$, we claim

$$(28) \quad E_\omega[G_\omega(\lambda, t/\lambda^2)^p] \leq e^{p^2 t/2 + 1}.$$

To show (28), note that by the expansion of the Radon–Nikodym derivative in (25),

$$\log G_\omega(p\lambda, t/\lambda^2) - p \log G_\omega(\lambda, t/\lambda^2) = (p^2 - p) \frac{\lambda^2}{2} \sum_{i=0}^{t/\lambda^2-1} D_\ell(\bar{\omega}_i) + C_{p,t} o(1).$$

Since $D_\ell(\omega) = \text{Var}_\omega[(X_1 - X_0) \cdot \ell] \leq 1$ for all ω , we have

$$|\log G_\omega(p\lambda, t/\lambda^2) - p \log G_\omega(\lambda, t/\lambda^2)| \leq p^2 t/2 + 1$$

for all $0 < \lambda \leq \lambda_0(p, t)$, where $\lambda_0(p, t)$ is small enough. Inequality (28) then follows by recalling that $E_\omega[G_\omega(p\lambda, t/\lambda^2)] = 1$.

Finally, we will show that

$$(29) \quad \lim_{\lambda \rightarrow 0} \mathbb{E}_{Q,\lambda} \left[\lambda \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k) \right] = t \text{Cov}(N_1^f, N_1).$$

Note that the uniform integrability of $G_\omega(\lambda, t/\lambda^2)$ yields

$$\mathbb{E}_{Q,0}[G_\omega(\lambda, t/\lambda^2)] \xrightarrow{\lambda \rightarrow 0} E \left[\exp \left\{ N_t - \frac{t}{2} E_{Q_0}[D_\ell(\omega)] \right\} \right]$$

and integrating the density gives $\mathbb{E}_{Q,0}[G_\omega(\lambda, t/\lambda^2)] = 1$ for any $\lambda > 0$, hence we have necessarily $E[N_t^2] = t E_{Q_0}[D_\ell(\omega)]$. Furthermore, since $\lambda \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k)$ is bounded in L^2 and the density $G(\lambda, t/\lambda^2)$ is bounded in L^p for any $p \geq 1$, their product is uniformly integrable, which implies

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \mathbb{E}_{Q,\lambda} \left[\lambda \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k) \right] &= \lim_{\lambda \rightarrow 0} \mathbb{E}_{Q,0} \left[\lambda \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k) G_\omega(\lambda, t/\lambda^2) \right] \\ &\stackrel{(26)}{=} E[N_t^f e^{N_t - \frac{1}{2} E[N_t^2]}]. \end{aligned}$$

By Girsanov’s formula,

$$E[N_t^f e^{N_t - \frac{1}{2} E[N_t^2]}] = E[N_t^f + \text{Cov}(N_t^f, N_t)] = \text{Cov}(N_t^f, N_t),$$

which proves (29).

4.3. *Remarks on the value of Λf .* We have obtained in (29) an expression for the operator Λ in (5)

$$\Lambda f = \text{Cov}(N_1^f, N_1).$$

To complete the proof of Theorem 3, we now show that this coincides with the definition of Λ in (4),

$$\Lambda f = -\text{Cov}(N_1^f, N_1^d \cdot \ell),$$

where $(N_t^d)_{t \geq 0}$ denotes the weak (Gaussian) limit of the process $\lambda \sum_{k=0}^{t/\lambda^2} d(\bar{\omega}_k, 0)$ as $\lambda \rightarrow 0$. In the following, we denote by

$$\Delta X_k := X_{k+1} - X_k,$$

the increments of the random walk. Noting that for any $n \geq 1$, the sequence $(\Delta X_0, \dots, \Delta X_{n-1}, \bar{\omega}_0, \dots, \bar{\omega}_n)$ has the same distribution as $(-\Delta X_{n-1}, \dots, -\Delta X_0, \bar{\omega}_n, \dots, \bar{\omega}_0)$ under $Q_0 \times P_\omega$, we have

$$(30) \quad \mathbb{E}_{Q,0} \left[(X_n - X_0) \cdot \left(\sum_{k=0}^n f(\bar{\omega}_k) \right) \right] = \mathbb{E}_{Q,0} \left[(X_0 - X_n) \cdot \left(\sum_{k=0}^n f(\bar{\omega}_k) \right) \right] = 0.$$

Consequently,

$$\begin{aligned} \text{Cov}(N_1^f, N_1) &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{Q,0} \left[\left(\sum_{k=0}^n f(\bar{\omega}_k) \right) \left(X_n - \sum_{i=0}^n d(\omega, X_i) \right) \cdot \ell \right] \\ &= - \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{Q,0} \left[\left(\sum_{k=0}^n f(\bar{\omega}_k) \right) \left(\sum_{i=0}^n d(\omega, X_i) \cdot \ell \right) \right] \\ &= -\text{Cov}(N_1^f, N_1^d \cdot \ell). \end{aligned}$$

We also remark that replacing the process $\lambda \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k)$ by $\lambda X_{t/\lambda^2}$ in (26), the same argument gives

$$\begin{aligned} (31) \quad \lim_{\lambda \rightarrow 0} \frac{1}{t} \mathbb{E}_{Q,\lambda} [\lambda X_{t/\lambda^2}] &= \text{Cov}(B_1, N_1 \cdot \ell) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{Q,0} \left[X_n \left(X_n - \sum_{i=0}^n d(\omega, X_i) \right) \cdot \ell \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}_{Q,0} [X_n (X_n \cdot \ell)] = \Sigma \ell. \end{aligned}$$

5. A uniform LLN. In this section, we show a quantitative result for the convergence to the steady state of a function g of the environment seen from the particle and the increments of the process, provided that we can control maxima of the sum over g . In the next section, we will show that for functions $f \in \mathcal{F}$, we can control the maxima and can use the following theorem to prove Theorem 4.

THEOREM 17. *Let $g : \Omega \times \{e \in \mathbb{Z}^d : |e| \leq 1\} \rightarrow \mathbb{R}$ be a function such that $g(\cdot, e) \in \mathcal{F}$ for any $|e| \leq 1$. Assume that for all λ smaller than some $\lambda_0 > 0$ and $n \geq 0$,*

$$(32) \quad \left\| \max_{0 \leq m \leq n/\lambda^2} \left| \lambda \sum_{k=0}^m g(\bar{\omega}_k, \Delta X_k) \right| \right\|_{L^{3/2}(Q_0 \times P_{\omega, \lambda})} \leq C e^{c\sqrt{n}}.$$

Then there exists a constant $C_8 = C_8(\kappa, d, g)$ such that for any $t > 1$ and $\lambda \in (0, \lambda_0)$,

$$(33) \quad \left| \frac{\frac{\lambda^2}{t} \mathbb{E}_{Q, \lambda} [\sum_{k=0}^{t/\lambda^2} g(\bar{\omega}_k, \Delta X_k)] - E_{Q_\lambda} [g(\omega, \Delta X_0)]}{\lambda} \right| \leq \frac{C_8}{t^{1/4}}.$$

Note that the sequence $(\bar{\omega}_k, \Delta X_k)_{k \geq 0}$ is stationary under the measure

$$(34) \quad \mathbb{Q}_\lambda := Q_\lambda \times P_{\omega, \lambda}.$$

Let us denote $\xi_n = \sum_{k=0}^n g(\bar{\omega}_k, \Delta X_k)$ and

$$\xi_{m,n}^* = \max_{m \leq j \leq n} \left| \sum_{k=m}^j g(\bar{\omega}_j, \Delta X_j) \right|, \quad \xi_n^* := \xi_{0,n}^*.$$

We first show the following consequence of inequality (32).

LEMMA 18. *Assume that (32) holds. Then there exists a constant $C > 0$ such that*

$$\bar{\mathbb{E}}_{Q, \lambda} [(\lambda \xi_{\tau_2}^*)^{4/3}] \leq C, \quad \forall \lambda \in (0, \lambda_0).$$

PROOF. Let $\bar{\mathbb{P}}_{Q, \lambda} = Q_0 \times \bar{P}_{\omega, \lambda}$. Since $(\xi_n^*)_{n \geq 0}$ is monotonically increasing, by Minkowski's and Hölder's inequalities,

$$\begin{aligned} \|\lambda \xi_{\tau_2}^*\|_{L^{4/3}(\bar{\mathbb{P}}_{Q, \lambda})} &\leq \left\| \sum_{n=0}^{\infty} \lambda \xi_{n/\lambda^2, (n+1)/\lambda^2}^* \mathbb{1}_{\{\tau_2 \geq n/\lambda^2\}} \right\|_{L^{4/3}(\bar{\mathbb{P}}_{Q, \lambda})} \\ &\leq \sum_{n=0}^{\infty} \|\lambda \xi_{n/\lambda^2, (n+1)/\lambda^2}^* \mathbb{1}_{\{\tau_2 \geq n/\lambda^2\}}\|_{L^{4/3}(\bar{\mathbb{P}}_{Q, \lambda})} \\ &\leq \sum_{n=0}^{\infty} \|\lambda \xi_{n/\lambda^2, (n+1)/\lambda^2}^*\|_{L^{3/2}(\bar{\mathbb{P}}_{Q, \lambda})} \bar{\mathbb{P}}_{Q, \lambda}(\tau_2 \geq n/\lambda^2)^{1/12}. \end{aligned}$$

By Corollary 15 and the fact that dQ_0/dP is bounded [cf. (2)], we have

$$\bar{\mathbb{P}}_{Q, \lambda}(\tau_2 \geq n/\lambda^2) \leq C \bar{\mathbb{P}}_\lambda(\tau_2 \geq n/\lambda^2) \leq C e^{-cn} \quad \forall n \geq 0.$$

Then the lemma follows by applying inequality (32). \square

An immediate consequence of (22) and Lemma 18 is

$$(35) \quad |E_{Q_\lambda}[g(\omega, \Delta X_0)]| \stackrel{(20)}{=} \left| \frac{\bar{\mathbb{E}}_\lambda[\xi_{\tau_2} - \xi_{\tau_1}]}{\bar{\mathbb{E}}_\lambda[\tau_2 - \tau_1]} \right| \leq \frac{\bar{\mathbb{E}}_{Q,\lambda}[\xi_{\tau_1}^* + \xi_{\tau_2}^*]}{c/\lambda^2} \leq C\lambda.$$

PROOF OF THEOREM 17. Note that by (35), inequality (32) still holds if g is replaced by $g - E_{Q_\lambda}[g(\omega, \Delta X_0)]$. Thus without loss of generality, we assume that

$$E_{Q_\lambda}[g(\omega, \Delta X_0)] = 0.$$

1. First, we will show that

$$(36) \quad \bar{\mathbb{E}}_{Q,\lambda} \left[\max_{1 \leq k \leq n} |\lambda \xi_{\tau_k}|^{4/3} \right] \leq Cn, \quad \forall n \geq 1.$$

Indeed, for $m \geq 0$, let $N_m = \sum_{i=\tau_m}^{\tau_{m+1}-1} g(\bar{\omega}_i, \Delta X_i)$. Then for any $m \geq 1$ by similar arguments as in the proof of (20),

$$\bar{\mathbb{E}}_\lambda[N_m] = E_{Q_\lambda}[g(\omega, \Delta X_0)] \bar{\mathbb{E}}_\lambda[\tau_2 - \tau_1] = 0.$$

Hence, $(\sum_{m=0}^n N_m)_{n \geq 1}$ is a sequence with 1-dependent zero-mean increments under the measure $P \times P_{\omega,\lambda}$, and hence by (2) also under $Q_0 \times P_{\omega,\lambda}$. Moreover, by Lemma 18, we have

$$\bar{\mathbb{E}}_{Q,\lambda}[|\lambda N_m|^{4/3}] = \bar{\mathbb{E}}_{Q,\lambda}[|\lambda N_1|^{4/3}] \leq 2\bar{\mathbb{E}}_{Q,\lambda}[|\lambda \xi_{\tau_2}^*|^{4/3}] \leq C, \quad \forall m \geq 0.$$

By von Bahr–Esseen’s inequality [26], Theorem 2, we have

$$\bar{\mathbb{E}}_{Q,\lambda} \left[\left| \sum_{m \leq n \text{ is odd}} \lambda N_m \right|^{4/3} \right] \leq 2 \sum_{m \leq n \text{ is odd}} \bar{\mathbb{E}}_{Q,\lambda}[|\lambda N_m|^{4/3}] \leq Cn$$

and then by Doob’s L^p -martingale inequality,

$$\bar{\mathbb{E}}_{Q,\lambda} \left[\max_{k \leq n} \left| \sum_{m \leq k \text{ is odd}} \lambda N_m \right|^{4/3} \right] \leq C \bar{\mathbb{E}}_{Q,\lambda} \left[\left| \sum_{m \leq n \text{ is odd}} \lambda N_m \right|^{4/3} \right] \leq Cn.$$

Similarly, we have

$$\bar{\mathbb{E}}_{Q,\lambda} \left[\max_{k \leq n} \left| \sum_{m \leq k \text{ is even}} \lambda N_m \right|^{4/3} \right] \leq Cn.$$

Combining these inequalities, we conclude that

$$\bar{\mathbb{E}}_{Q,\lambda} \left[\max_{k \leq n} \left| \sum_{m=0}^k \lambda N_m \right|^{4/3} \right] \leq Cn,$$

which is exactly (36).

2. Next, for $t > 0$ fixed, we let $r = r(\lambda, t) \in \mathbb{N}$ be the integer that satisfies

$$\bar{\mathbb{E}}_\lambda[\tau_{r-1}] \leq 4t/\lambda^2 < \bar{\mathbb{E}}_\lambda[\tau_r].$$

We will show that

$$(37) \quad \bar{\mathbb{P}}_\lambda(t/\lambda^2 \geq \tau_r) \leq Ce^{-ct}.$$

Note that by Corollary 15 and (22), we have

$$cr \leq t < Cr.$$

Since by Corollary 15 the sequences $(\lambda^2\tau_1)_{\lambda \in (0, \lambda_0)}$ and $(\lambda^2(\tau_2 - \tau_1))_{\lambda \in (0, \lambda_0)}$ are uniformly integrable [with respect to the measures $\bar{\mathbb{P}}_\lambda, \lambda \in (0, \lambda_0)$], by the lower bound (22), there exists a constant $M > 0$ such that for all $n \geq 0$ and $\lambda \in (0, \lambda_0)$,

$$\bar{\mathbb{E}}_\lambda[M \wedge \lambda^2(\tau_{n+1} - \tau_n)] \geq \frac{1}{2}\bar{\mathbb{E}}_\lambda[\lambda^2(\tau_{n+1} - \tau_n)].$$

We set

$$\bar{\tau}_n = \sum_{k=0}^{n-1} (\tau_{k+1} - \tau_k) \wedge (M/\lambda^2).$$

Then, noting that $t/\lambda^2 - \bar{\mathbb{E}}_\lambda[\bar{\tau}_r] \leq t/\lambda^2 - \frac{1}{2}\bar{\mathbb{E}}_\lambda[\tau_r] < -t/\lambda^2$, we have

$$\begin{aligned} \bar{\mathbb{P}}_\lambda(t/\lambda^2 \geq \tau_r) &\leq \bar{\mathbb{P}}_\lambda(t/\lambda^2 - \bar{\mathbb{E}}_\lambda[\bar{\tau}_r] \geq \bar{\tau}_r - \bar{\mathbb{E}}_\lambda[\bar{\tau}_r]) \\ &\leq \bar{\mathbb{P}}_\lambda(\lambda^2|\bar{\tau}_r - \bar{\mathbb{E}}_\lambda[\bar{\tau}_r]| \geq t) \\ &\leq C \exp\left(-\frac{ct^2}{M^2r}\right) \leq Ce^{-ct}, \end{aligned}$$

where in the third inequality we applied Azuma–Hoeffding’s inequality to $\lambda^2(\bar{\tau}_r - \bar{\mathbb{E}}_\lambda[\bar{\tau}_r])$, which is a sum of bounded 1-dependent increments. Estimate (37) is proved.

3. To conclude the proof of Theorem 17, we will show

$$(38) \quad \mathbb{E}_{Q,\lambda}[|\xi_{t/\lambda^2}|] \leq Ct^{3/4}/\lambda, \quad \forall t \geq 1,$$

as this implies (33). Now, by (37) and (32),

$$(39) \quad \bar{\mathbb{E}}_{Q,\lambda}[|\xi_{t/\lambda^2}| \mathbb{1}_{\{t/\lambda^2 \geq \tau_r\}}] \leq \|\xi_{t/\lambda^2}^*\|_{L^{3/2}(Q_0 \times P_{\omega,\lambda})} \bar{\mathbb{P}}_\lambda(t/\lambda^2 \geq \tau_r)^{1/3} \leq C/\lambda.$$

On the other hand,

$$\begin{aligned} &\bar{\mathbb{E}}_{Q,\lambda}[\xi_{t/\lambda^2} \mathbb{1}_{\{t/\lambda^2 < \tau_r\}}] \\ &\leq \bar{\mathbb{E}}_{Q,\lambda}\left[\max_{1 \leq k \leq r} |\xi_{\tau_k}| \right] + \sum_{k=0}^{r-1} \bar{\mathbb{E}}_{Q,\lambda}[\xi_{\tau_k, \tau_{k+1}}^* \mathbb{1}_{\{\tau_k \leq t/\lambda^2 < \tau_{k+1}\}}] \end{aligned}$$

$$\begin{aligned}
 &\stackrel{(36)}{\leq} \frac{Cr^{3/4}}{\lambda} + \sum_{k=0}^{r-1} \|\xi_{\tau_k, \tau_{k+1}}^*\|_{L^{4/3}(\bar{\mathbb{P}}_{Q,\lambda})} \bar{\mathbb{P}}_{Q,\lambda}(\tau_k \leq t/\lambda^2 < \tau_{k+1})^{1/4} \\
 &\leq \frac{Ct^{3/4}}{\lambda} + \frac{C}{\lambda} \sum_{k=0}^{r-1} \bar{\mathbb{P}}_{Q,\lambda}(\tau_k \leq t/\lambda^2 < \tau_{k+1})^{1/4},
 \end{aligned}$$

where we used Lemma 18 and $r \leq Ct$ in the last step. Since by Hölder’s inequality,

$$\begin{aligned}
 \sum_{k=0}^{r-1} \bar{\mathbb{P}}_{Q,\lambda}(\tau_k \leq t/\lambda^2 < \tau_{k+1})^{1/4} &\leq r^{3/4} \left(\sum_{k=0}^{r-1} \bar{\mathbb{P}}_{Q,\lambda}(\tau_k \leq t/\lambda^2 < \tau_{k+1}) \right)^{1/4} \\
 &\leq Ct^{3/4},
 \end{aligned}$$

we conclude that

$$(40) \quad \bar{\mathbb{E}}_{Q,\lambda}[|\xi_{t/\lambda^2}| \mathbb{1}_{\{t/\lambda^2 < \tau_r\}}] \leq Ct^{3/4}/\lambda.$$

The combination of (39) and (40) yields (38). \square

6. Proofs of Theorem 4 and the Einstein relation.

6.1. *Proof of Theorem 4.* For $f \in \mathcal{F}$, let $g(\omega, x) := f(\omega)$. By Theorem 17, we only need to show that (32) holds for g in dimension $d \geq 3$. Note that now $\xi_n = \sum_{k=0}^n f(\bar{\omega}_k)$ and recall that $\xi_{m,n}^* = \max_{m \leq j \leq n} |\sum_{k=m}^j f(\bar{\omega}_k)|$. In fact, we will obtain an estimate stronger than (32).

THEOREM 19. *Let $d \geq 3$. Let $f \in \mathcal{F}$ be a function that satisfies $Q_0 f = 0$. Then for all $\lambda \in [0, \lambda_0)$ and $n \geq 1$,*

$$\|\lambda \xi_{n/\lambda^2, (n+1)/\lambda^2}^*\|_{L^{3/2}(Q_0 \times P_{\omega,\lambda})} \leq Cn^{6d+14}.$$

Our proof contains several steps.

1. First, we will obtain a moment estimate under the unperturbed measure

$$(41) \quad \|\xi_n^*\|_{L^2(Q_0 \times P_\omega)} \leq C\sqrt{n} \quad \forall n \in \mathbb{N}.$$

To prove inequality (41), recall the following theorem.

THEOREM 20 ([5, 21]). *If $d \geq 3$ and $f : \Omega \rightarrow \mathbb{R}$ is an $L^2(Q_0)$ local function with $Q_0 f = 0$, then for all $n \in \mathbb{N}$,*

$$E_{Q_0}[E_\omega[f(\bar{\omega}_n)]^2] \leq C_f n^{-d/2}.$$

Moreover, $(\sqrt{n}\xi_{nt})_{t \geq 0}$ converges weakly to a Brownian motion.

Noting that $f(\bar{\omega}_n)$ is a stationary sequence under $Q_0 \times P_\omega$, by the maximal inequality for stationary sequences in [22], Theorem 1, we have

$$\|\xi_n^*\|_{L^2(Q_0 \times P_\omega)} \leq C\sqrt{n}(\|f\|_{L^2(Q_0)} + \delta_f),$$

where $\delta_f = \sum_{m=1}^\infty m^{-3/2} \|E_\omega[\xi_m]\|_{L^2(Q_0)}$. We only need to show that $\delta_f < \infty$. When $d \geq 3$, using the Cauchy–Schwarz inequality,

$$\begin{aligned} \|E_\omega[\xi_m]\|_{L^2(Q_0)}^2 &= E_{Q_0} \left[\left(\sum_{j=1}^m E_\omega[f(\bar{\omega}_j)] \right)^2 \right] \\ &\leq E_{Q_0} \left[\sum_{j=1}^m \frac{1}{j^{d/4}} \cdot \sum_{j=1}^m j^{d/4} (E_\omega[f(\bar{\omega}_j)])^2 \right] \\ &\stackrel{\text{Thm. 20}}{\leq} C_f \left(\sum_{j=1}^m j^{-d/4} \right)^2 \\ &\stackrel{d \geq 3}{\leq} C_f m^{1/2}. \end{aligned}$$

Hence, $\delta_f \leq C_f \sum_{m=1}^\infty m^{-3/2+1/4} < \infty$ for $d \geq 3$ and (41) is proved.

2. Next, we will show that

$$(42) \quad \|\lambda \xi_{1/\lambda^2}^*\|_{L^{5/3}(Q_0 \times P_{\omega,\lambda})} \leq C.$$

Recalling the definition of the Radon–Nikodym derivative $G_\omega(\lambda, t)$ in (23),

$$\begin{aligned} \|\lambda \xi_{1/\lambda^2}^*\|_{L^{5/3}(Q_0 \times P_{\omega,\lambda})}^{5/3} &= \mathbb{E}_{Q,0} [G_\omega(\lambda, 1/\lambda^2) (\lambda \xi_{1/\lambda^2}^*)^{5/3}] \\ &\leq \mathbb{E}_{Q,0} [G_\omega(\lambda, 1/\lambda^2)^6]^{1/6} \mathbb{E}_{Q,0} [(\lambda \xi_{1/\lambda^2}^*)^2]^{5/6} \\ &\stackrel{(28), (41)}{\leq} C, \end{aligned}$$

which proves (42).

3. For any fixed $\lambda \in (0, \lambda_0)$, let

$$(43) \quad h(\omega) = h(\lambda, \omega) = E_{\omega,\lambda} [(\lambda \xi_{1/\lambda^2}^*)^{3/2}].$$

Let $\square_n = \{x \in \mathbb{Z}^d : \|x\|_\infty \leq n \lfloor 1/\lambda \rfloor\}$ denote the box of side-length $2n \lfloor 1/\lambda \rfloor$ and set

$$A_h(\omega) := \frac{1}{|\square_1|} \sum_{x \in \square_1} h(\theta_x \omega)$$

to be an average of h in the box \square_1 . We will show that for any integer $n \geq 0$,

$$(44) \quad \mathbb{E}_{Q,\lambda} [h(\bar{\omega}_{n/\lambda^2})] \leq C \mathbb{E}_{Q,\lambda} [A_h(\bar{\omega}_{(n+1)/\lambda^2}) + A_h(\bar{\omega}_{1+(n+1)/\lambda^2})].$$

To this end, we only need to prove that

$$\mathbb{E}_{Q,\lambda}[h(\bar{\omega}_{n/\lambda^2})] \leq C E_{Q,\lambda}[h(\theta_x \bar{\omega}_{(n+1)/\lambda^2}) + h(\theta_x \bar{\omega}_{1+(n+1)/\lambda^2})], \quad \forall x \in \square_1.$$

Further, noting that

$$E_{\omega,\lambda}[h(\bar{\omega}_n)] = \sum_y P_{\omega,\lambda}(X_n = y)h(\theta_y \omega),$$

it suffices to show that for any $y \in \mathbb{Z}^d$ and $x \in \square_1$,

$$(45) \quad \begin{aligned} &P_{\omega,\lambda}(X_{n/\lambda^2} = y) \\ &\leq C [P_{\omega,\lambda}(X_{(n+1)/\lambda^2} = y + x) + P_{\omega,\lambda}(X_{1+(n+1)/\lambda^2} = y + x)]. \end{aligned}$$

Indeed, for $k \in \mathbb{Z}$ and $y \in \mathbb{Z}^d$, set

$$u(y, k) = P_{\omega,\lambda}^y(X_k = 0).$$

Then, by reversibility and uniform ellipticity,

$$P_{\omega,\lambda}(X_k = y) = \frac{\sum_{e'} \omega^\lambda(y, y + e')}{\sum_{e'} \omega^\lambda(0, e')} u(y, k) \asymp e^{2\lambda y \cdot \ell} u(y, k),$$

where $A \asymp B$ means $cB \leq A \leq CB$ for some constants $c, C > 0$. Thus (45) is equivalent to

$$u(y, n/\lambda^2) \leq C [u(y + x, (n + 1)/\lambda^2) + u(y + x, 1 + (n + 1)/\lambda^2)]$$

for all $x \in \square_1, y \in \mathbb{Z}^d$, which follows by (PHI) and the fact that $u(\cdot, \cdot)$ satisfies the parabolic equation (PE) for the environment ω^λ in $B_{2\sqrt{d}/\lambda}(y) \times [(n - 2)/\lambda^2, (n + 2)/\lambda^2]$. Our proof of (44) is complete.

4. Note that by the Markov property,

$$\|\lambda \xi_{n/\lambda^2, (n+1)/\lambda^2}^*\|_{L^{3/2}(Q_0 \times P_{\omega,\lambda})}^{3/2} = \mathbb{E}_{Q,\lambda}[h(\bar{\omega}_{n/\lambda^2})].$$

Thus by (44), to prove Theorem 19, it suffices that $\mathbb{E}_{Q,\lambda}[A_h(\bar{\omega}_{n/\lambda^2})] \leq Cn^{9d+21}$, which by the bounds of dQ_0/dP in (2) is equivalent to

$$(46) \quad \mathbb{E}_\lambda[A_h(\bar{\omega}_{n/\lambda^2})] \leq Cn^{9d+21}, \quad \forall n \in \mathbb{N}.$$

5. We say that a box $\square_1(x) := x + \square_1$ centered at $x \in \mathbb{Z}^d$ is k -good (with respect to the environment ω) if

$$A_h(\theta_z \omega) \leq k^{9d+18} \quad \text{for all } z \in \square_1(x).$$

Otherwise, we say that $\square_1(x)$ is k -bad. We claim that

$$(47) \quad P(\square_1 \text{ is } k\text{-bad}) \leq Ck^{-10d-20}.$$

Indeed, observing that $\sum_{y \in \square_2} h(\theta_y \omega) / |\square_2| \geq 2^{-d} A_h(\theta_z \omega)$ for all $z \in \square_1$, we have

$$\begin{aligned} P(\square_1 \text{ is } k\text{-bad}) &\leq P\left(\sum_{y \in \square_2} h(\theta_y \omega) / |\square_2| \geq 2^{-d} k^{9d+18}\right) \\ &\leq CE \left[\left(\frac{1}{|\square_2|} \sum_{y \in \square_2} h(\theta_y \omega) \right)^{10/9} \right] / k^{10d+20} \\ &\leq CE \left[\frac{1}{|\square_2|} \sum_{y \in \square_2} h(\theta_y \omega)^{10/9} \right] / k^{10d+20} \\ &= CE [h^{10/9}] / k^{10d+20} \stackrel{(43)}{\leq} \mathbb{E}_\lambda [(\lambda \xi_{1/\lambda^2}^*)^{5/3}] / k^{10d+20}, \end{aligned}$$

where we used the translation-invariance of the measure P in the second to last equality. Display (47) then follows by (42) and (2).

6. Finally, we will prove (46). Clearly,

$$\begin{aligned} \mathbb{E}_\lambda [A_h(\bar{\omega}_{n/\lambda^2})] &\leq \sum_{k=0}^\infty (k+1)^{9d+18} \mathbb{P}_\lambda (A_h(\bar{\omega}_{n/\lambda^2}) \in [k^{9d+18}, (k+1)^{9d+18})) \\ &\leq 1 + C \sum_{k=1}^\infty k^{9d+18} \mathbb{P}_\lambda (A_h(\bar{\omega}_{n/\lambda^2}) \geq k^{9d+18}). \end{aligned}$$

Further, for each $k \in \mathbb{N}$, we can decompose the box \square_k into k^d boxes $(\square^{(i)})_{1 \leq i \leq k^d}$ of side-length $2 \lfloor 1/\lambda \rfloor$. Hence,

$$\begin{aligned} \mathbb{P}_\lambda (A_h(\bar{\omega}_{n/\lambda^2}) \geq k^{9d+18}) &\leq \mathbb{P}_\lambda \left(\max_{0 \leq s \leq n/\lambda^2} |X_s| \geq k/\lambda \right) + P(\text{one of the } k^d \text{ boxes } \square^{(i)} \text{ is } k\text{-bad}) \\ &\stackrel{\text{Lem. 8, (47)}}{\leq} C \frac{n^{9d+20}}{k^{9d+20}} + C k^d k^{-10d-20}. \end{aligned}$$

Therefore, for any $n \in \mathbb{N}$,

$$\mathbb{E}_\lambda [A_h(\bar{\omega}_{n/\lambda^2})] \leq C \sum_{k=1}^\infty \frac{n^{9d+20}}{k^2} \leq C n^{9d+20}.$$

Inequality (46) is proved.

Our proof of Theorem 19 is now complete.

6.2. *Proof of Theorem 1.* We note that by the ergodic theorem we can write the velocity as

$$v(\lambda) = \lim_{n \rightarrow \infty} \frac{X_n}{n} = Q_\lambda [d(\omega^\lambda, 0)] = E_{Q_\lambda} [X_1], \quad \mathbb{P}_\lambda\text{-a.s.}$$

Let $g(\omega, e) = e \cdot e_i$ for a fixed unit vector e_i . Then, by Lemma 8, (32) holds for this choice of g and by Theorem 17 we have

$$\left| \frac{\frac{\lambda^2}{t} \mathbb{E}_{Q,\lambda}[X_{t/\lambda^2} \cdot e_i] - E_{Q,\lambda}[X_1 \cdot e_i]}{\lambda} \right| \leq \frac{C}{t^{1/4}}.$$

The collection of these inequalities for $i = 1, \dots, d$, together with (31), yields Theorem 1.

6.3. *Einstein relation as a corollary of Theorem 2.* We remark that Theorem 2 can be considered a more general statement than the Einstein relation (1). Indeed, since

$$\begin{aligned} v(\lambda) &= E_{Q_\lambda} E_{\omega,\lambda}[X_1] \stackrel{(25)}{=} E_{Q_\lambda} E_\omega[e^{\lambda M_1 + O(\lambda^2)} X_1] \\ &= E_{Q_\lambda} E_\omega[(1 + \lambda M_1) X_1] + o(\lambda) \\ &= Q_\lambda[d(\omega, 0)] + \lambda E_{Q_\lambda} E_\omega[M_1 X_1] + o(\lambda), \end{aligned}$$

we have by Theorem 2, applied to the collection of local functions $d(\omega, 0) \cdot e_i$ for $i = 1, \dots, d$,

$$\lim_{\lambda \rightarrow 0} \frac{v(\lambda)}{\lambda} = -\text{Cov}(N_1^d, N_1^d \cdot \ell) + \mathbb{E}_{Q,0}[M_1 X_1].$$

By the ergodic theorem and the fact that $\Delta X_k - d(\bar{\omega}_k, 0)$ are P_ω -martingale differences,

$$\begin{aligned} &\text{Cov}(B_1 - N_1^d, N_1) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{Q,0} \left[\left(X_n - \sum_{k=0}^{n-1} d(\bar{\omega}_k, 0) \right) \left(X_n - \sum_{k=0}^{n-1} d(\bar{\omega}_k, 0) \right) \cdot \ell \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{Q,0} \left[\sum_{k=0}^{n-1} (\Delta X_k - d(\bar{\omega}_k)) (\Delta X_k - d(\bar{\omega}_k)) \cdot \ell \right] \\ &= \mathbb{E}_{Q,0}[(X_1 - d(\omega, 0))(X_1 - d(\omega, 0)) \cdot \ell] \\ &= \mathbb{E}_{Q,0}[X_1 M_1]. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{v(\lambda)}{\lambda} &= -\text{Cov}(N_1^d, N_1^d \cdot \ell) + \text{Cov}(B_1 - N_1^d, N_1) \\ &= -E[N_1^d(N_1^d \cdot \ell)] + E[(B_1 - N_1^d)(B_1 - N_1^d) \cdot \ell] \\ &= E[B_1(B_1 \cdot \ell)] = \Sigma \ell, \end{aligned}$$

since by (30), $E[B_1 N_1^d] = 0$. Note that we proved Theorem 2 only for $d \geq 3$. Hence, it does not cover the case $d = 2$ of Theorem 1.

7. Proof of the a priori estimates.

7.1. *Proof of Lemma 5.* We will prove Lemma 5 by contradiction. Let $u(x) = P_{\omega,\lambda}^x(T_{1,L} < T_{-1,L})$ and assume that for some ω we have

$$u(0) < \frac{2}{3}.$$

Recall that $T_{m,L}$ is the hitting time of $\{z \in \mathbb{Z}^d \mid z \cdot e_1 = nL/\lambda_1\}$ with $\lambda_1 = \lfloor 1/\lambda \rfloor^{-1}$. For a set $G \subset \mathbb{R}^d$, define its discrete boundary as

$$\partial G = \{x \in \mathbb{Z}^d \cap G : x \sim y \text{ for some } y \in \mathbb{Z}^d \setminus G\}.$$

For a function $h : G \rightarrow \mathbb{R}$, denote its Dirichlet energy on G as

$$\mathcal{E}(h, G) = \sum_{x,y \in G, x \sim y} \omega^\lambda(x, y)(h(x) - h(y))^2.$$

We let $S_G = S_G(u, \omega)$ be the set of functions $v : G \rightarrow \mathbb{R}$ such that $v = u$ on ∂G . Since u solves the elliptic equation (EE) in $\{x \in \mathbb{Z}^d : |x \cdot e_1| \leq L/\lambda_1\}$, by Dirichlet’s principle (see [8]), we have

$$\mathcal{E}(u, G) = \min_{v \in S_G} \mathcal{E}(v, G) \quad \text{for any } G \subset \{x \in \mathbb{Z}^d : |x \cdot e_1| \leq L/\lambda_1\}.$$

We first find a lower bound for the Dirichlet energy of u on the set $\Pi_0 = [-\frac{L}{\lambda_1}, \frac{L}{\lambda_1}] \times [-\frac{1}{\lambda_1}, \frac{1}{\lambda_1}]^{d-1}$ by ignoring all edges which are not in the direction of e_1 . The conductance of such an edge in Π_0 connecting (x_1, \bar{x}) with $(x_1 + 1, \bar{x})$, where we write $x = (x_1, \bar{x})$ with $\bar{x} \in \mathbb{Z}^{d-1}$, is bounded from below by $c_\kappa e^{2\lambda\ell_1 x_1}$. A lower bound for the energy is then

$$\begin{aligned} \mathcal{E}(u, \Pi_0) &\geq \sum_{\|\bar{x}\| \leq 1/\lambda_1} \sum_{i=0}^{L/\lambda_1-1} c_\kappa e^{2\lambda\ell_1 i} (u(i+1, \bar{x}) - u(i, \bar{x}))^2 \\ &\geq C \sum_{\|\bar{x}\| \leq 1/\lambda_1} \left(\sum_{i=0}^{L/\lambda_1-1} u(i+1, \bar{x}) - u(i, \bar{x}) \right)^2 \left(\sum_{i=0}^{L/\lambda_1-1} e^{-2\lambda\ell_1 i} \right)^{-1} \\ &\geq C \sum_{\|\bar{x}\| \leq 1/\lambda_1} (1 - u(0, \bar{x}))^2 (1 - e^{-2\lambda\ell_1}), \end{aligned}$$

where we used the Cauchy–Schwarz inequality in the second inequality. Note that $(1 - e^{-2\lambda\ell_1}) > c\lambda$ when $\lambda < \lambda_0$ for some small enough $\lambda_0 > 0$. Since u satisfies (EE) on $[-\frac{L}{\lambda_1}, \frac{L}{\lambda_1}] \times [-\frac{2}{\lambda_1}, \frac{2}{\lambda_1}]^{d-1}$, the elliptic Harnack inequality yields $1 - u(0, \bar{x}) \geq C(1 - u(0))$ whenever $|x| \leq 1/\lambda_1$. Thus,

$$(48) \quad \mathcal{E}(u, \Pi_0) \geq C\lambda \cdot \lambda^{-(d-1)} (1 - u(0))^2 \geq C\lambda^{2-d}$$

by our assumption.

Next, we will derive an upper bound for the energy. Define the function

$$\bar{u}(x) = \frac{e^{2L} - e^{-2\lambda_1 x_1}}{e^{2L} - e^{-2L}},$$

the potential of a nonrandom biased random walk. Then $\bar{u}(x) = 0$ when $x_1 = -L/\lambda_1$ and $\bar{u}(x) = 1$ when $x_1 = L/\lambda_1$, that is, \bar{u} satisfies in the strip $-L/\lambda_1 \leq e_1 \cdot x \leq L/\lambda_1$ the same boundary conditions as u . Let $R_1 = R_1(\lambda) = \frac{L}{\lambda_1} \lfloor e^{L/d} \rfloor$ and $R_2 = 2R_1$. We will calculate energies on shifted sets $\Pi_1(y) = y + [-\frac{L}{\lambda_1}, \frac{L}{\lambda_1}] \times [-R_1, R_1]^{d-1}$ and $\Pi_2(y) = y + [-\frac{L}{\lambda_1}, \frac{L}{\lambda_1}] \times [-R_2, R_2]^{d-1}$. For L, λ fixed, let

$$E_{L,\lambda} = \sup_{y \in \mathbb{Z}^d: y_1=0} \mathcal{E}(u, \Pi_1(y)) < \infty$$

and choose $y = y(L, \lambda) \in \mathbb{Z}^d$ such that

$$\mathcal{E}(u, \Pi_0) \leq \mathcal{E}(u, \Pi_1(0)) \leq \mathcal{E}(u, \Pi_1(y))$$

and

$$\mathcal{E}(u, \Pi_1(y)) + \mathcal{E}(\bar{u}, \Pi_1(y)) > E_{L,\lambda},$$

which is possible, since $\mathcal{E}(\bar{u}, \Pi_1(y))$ has for all λ, L a positive lower bound independent of y . From now on, we fix such a y and write Π_1 for $\Pi_1(y)$ and Π_2 for $\Pi_2(y)$. We will show that for some positive constants c_1, c_2 independent of L, λ ,

$$(49) \quad \mathcal{E}(u, \Pi_1) \leq c_1 \lambda \cdot \lambda^{-(d-1)} e^{-c_2 L},$$

which contradicts (48) if L is large enough, since $\mathcal{E}(u, \Pi_0) \leq \mathcal{E}(u, \Pi_1)$.

To show (49), set

$$v(x) = (1 - d(x))\bar{u}(x) + d(x)u(x) \quad \text{for } x \in \Pi_2,$$

where $d(x) = \text{dist}(x, \Pi_1)/R_1 \leq 1$. Note that $v = \bar{u}$ in Π_1 and $v = u$ on $\partial\Pi_2$. By Dirichlet's principle, $\mathcal{E}(u, \Pi_2) \leq \mathcal{E}(v, \Pi_2)$. For $x, y \in \Pi_2$ and $y \sim x$

$$\begin{aligned} v(x) - v(y) &= (1 - d(x))(\bar{u}(x) - \bar{u}(y)) + d(x)(u(x) - u(y)) \\ &\quad + (d(x) - d(y))(u(y) - \bar{u}(y)). \end{aligned}$$

Hence, observing $|d(x) - d(y)| \leq R_1^{-1} \mathbb{1}_{x,y \notin \Pi_1}$ for $x \sim y$, by Jensen's inequality,

$$\begin{aligned} (v(x) - v(y))^2 &\leq (1 - d(x))(\bar{u}(x) - \bar{u}(y))^2 + d(x)(u(x) - u(y))^2 \\ &\quad + R_1^{-2} \mathbb{1}_{x,y \notin \Pi_1} (\bar{u}(y) - u(y))^2 \\ &\quad + 2R_1^{-1} \mathbb{1}_{x,y \notin \Pi_1} |\bar{u}(y) - u(y)| [(1 - d(x))|\bar{u}(x) - \bar{u}(y)| \\ &\quad + d(x)|u(x) - u(y)|]. \end{aligned}$$

Multiplying both sides by $\omega^\lambda(x, y)$ and summing over all $x, y \in \Pi_2$, this yields

$$\mathcal{E}(v, \Pi_2) \leq \mathcal{E}(\bar{u}, \Pi_2) + \mathcal{E}(u, \Pi_2 \setminus \Pi_1^+)$$

$$\begin{aligned} \text{(a)} \quad & + R_1^{-2} \sum_{y \in \Pi_2 \setminus \Pi_1, x \sim y} \omega^\lambda(x, y) (u(y) - \bar{u}(y))^2 \\ & + 2R_1^{-1} \sum_{y \in \Pi_2 \setminus \Pi_1, x \sim y} \omega^\lambda(x, y) |\bar{u}(y) - u(y)| [|\bar{u}(x) - \bar{u}(y)| \\ \text{(b)} \quad & + |u(x) - u(y)|], \end{aligned}$$

where $\Pi_1^+ := \Pi_1 \setminus \partial \Pi_1$. We will find upper bounds for the sums (a) and (b). Starting with the first one,

$$\begin{aligned} \text{(a)} \leq & 2R_1^{-2} \sum_{y \in \Pi_2 \setminus \Pi_1, x \sim y} \omega^\lambda(x, y) (u(y) - 1)^2 \\ & + 2R_1^{-2} \sum_{y \in \Pi_2 \setminus \Pi_1, x \sim y} \omega^\lambda(x, y) (\bar{u}(y) - 1)^2 \end{aligned}$$

with

$$(u(y) - 1)^2 = \left(\sum_{i=y_1}^{L/\lambda_1} u(i, \bar{y}) - u(i + 1, \bar{y}) \right)^2 \leq 2 \frac{L}{\lambda_1} \sum_{i=y_1}^{L/\lambda_1} (u(i, \bar{y}) - u(i + 1, \bar{y}))^2$$

by Cauchy–Schwarz’s inequality. Bounding $(\bar{u}(y) - 1)^2$ analogously, we get

$$\begin{aligned} \text{(a)} \leq & 2R_1^{-2} \frac{L}{\lambda_1} \left(\sum_{y \in \Pi_2 \setminus \Pi_1, x \sim y} \omega^\lambda(x, y) \sum_{i=y_1}^{L/\lambda_1} (u(i, \bar{y}) - u(i + 1, \bar{y}))^2 \right. \\ & \left. + (\bar{u}(i, \bar{y}) - \bar{u}(i + 1, \bar{y}))^2 \right) \\ \leq & cR_1^{-2} \frac{L}{\lambda_1} \left(\sum_{y \in \Pi_2 \setminus \Pi_1} \sum_{i=y_1}^{L/\lambda_1} \omega^\lambda((i, \bar{y}), (i + 1, \bar{y})) [(u(i, \bar{y}) - u(i + 1, \bar{y}))^2 \right. \\ & \left. + (\bar{u}(i, \bar{y}) - \bar{u}(i + 1, \bar{y}))^2] \right) \\ \leq & cR_1^{-2} \left(\frac{L}{\lambda_1} \right)^2 (\mathcal{E}(u, \Pi_2 \setminus \Pi_1^+) + \mathcal{E}(\bar{u}, \Pi_2 \setminus \Pi_1^+)) \\ \leq & cR_1^{-1} \frac{L}{\lambda_1} (\mathcal{E}(u, \Pi_2 \setminus \Pi_1^+) + \mathcal{E}(\bar{u}, \Pi_2 \setminus \Pi_1^+)). \end{aligned}$$

For the summand (b), Hölder’s inequality yields

$$\begin{aligned}
 \text{(b)} &\leq 2R_1^{-1} \left(\sum_{y \in \Pi_2 \setminus \Pi_1, x \sim y} \omega^\lambda(x, y) (\bar{u}(y) - u(y))^2 \right)^{1/2} \\
 &\quad \times \left(\sum_{y \in \Pi_2 \setminus \Pi_1, x \sim y} \omega^\lambda(x, y) (u(x) - u(y))^2 \right. \\
 &\quad \left. + \sum_{y \in \Pi_2 \setminus \Pi_1, x \sim y} \omega^\lambda(x, y) (\bar{u}(x) - \bar{u}(y))^2 \right)^{1/2}.
 \end{aligned}$$

The first sum can be bounded as we did for (a) by $C(L/\lambda_1)^2(\mathcal{E}(u, \Pi_2 \setminus \Pi_1^+) + \mathcal{E}(\bar{u}, \Pi_2 \setminus \Pi_1^+))$ such that

$$\text{(b)} \leq cR_1^{-1} \frac{L}{\lambda_1} (\mathcal{E}(u, \Pi_2 \setminus \Pi_1^+) + \mathcal{E}(\bar{u}, \Pi_2 \setminus \Pi_1^+)).$$

Collecting the upper bounds for (a), (b) and rearranging, using $\mathcal{E}(u, \Pi_2) - \mathcal{E}(u, \Pi_2 \setminus \Pi_1^+) = \mathcal{E}(u, \Pi_1)$ and $\mathcal{E}(u, \Pi_2) \leq \mathcal{E}(v, \Pi_2)$, we arrive at

$$(50) \quad \mathcal{E}(u, \Pi_1) \leq \mathcal{E}(\bar{u}, \Pi_2) + cR_1^{-1} \frac{L}{\lambda_1} (\mathcal{E}(u, \Pi_2 \setminus \Pi_1^+) + \mathcal{E}(\bar{u}, \Pi_2)).$$

Next, we estimate the energy of \bar{u} . Since $\bar{u}(x) = \bar{u}(y)$ if $x_1 = y_1$, we have

$$\begin{aligned}
 \mathcal{E}(\bar{u}, \Pi_2) &= \sum_{\|\bar{x}\| \leq R_2} \sum_{x_1 = -L/\lambda_1}^{L/\lambda_1 - 1} \omega^\lambda((x_1, \bar{x}), (x_1 + 1, \bar{x})) \\
 &\quad \times \left(\frac{e^{2L} - e^{-2\lambda_1 x_1}}{e^{2L} - e^{-2L}} - \frac{e^{2L} - e^{-2\lambda_1(x_1+1)}}{e^{2L} - e^{-2L}} \right)^2 \\
 &\leq cR_2^{d-1} (e^{2L} - e^{-2L})^{-2} \sum_{x_1 = -L/\lambda_1}^{L/\lambda_1 - 1} e^{2\lambda_1 x_1} (e^{-2\lambda_1 x_1} - e^{-2\lambda_1(x_1+1)})^2,
 \end{aligned}$$

where we used that $\omega^\lambda(x, y) \leq \kappa e^{\lambda \ell \cdot (x+y)} \leq \kappa e^{2\lambda_1 x_1}$. Now simple calculations give the upper bound (recall that $R_1 = \frac{L}{\lambda_1} \lfloor e^{L/d} \rfloor$)

$$\begin{aligned}
 \mathcal{E}(\bar{u}, \Pi_2) &\leq cR_1^{d-1} (e^{2L} - e^{-2L})^{-2} \sum_{x_1 = -L/\lambda_1}^{L/\lambda_1 - 1} e^{-2\lambda_1 x_1} (1 - e^{-2\lambda_1})^2 \\
 &\leq cR_1^{d-1} \lambda (e^{2L} - e^{-2L})^{-2} \sum_{x_1 = -L/\lambda_1}^{L/\lambda_1 - 1} \lambda e^{-2\lambda_1 x_1} \\
 &\leq cR_1^{d-1} \lambda (e^{2L} - e^{-2L})^{-1} \\
 &\leq c \left(\frac{L}{\lambda} \right)^{d-1} \lambda e^{-L}.
 \end{aligned}$$

Since Π_1 and Π_2 are of comparable size, and the energy over a box with the size of Π_1 is bounded by $E_{L,\lambda}$, we have

$$\mathcal{E}(u, \Pi_2 \setminus \Pi_1^+) \leq 3^{d-1} E_{L,\lambda} \leq 3^{d-1} (\mathcal{E}(\bar{u}, \Pi_1) + \mathcal{E}(u, \Pi_1)).$$

Using this inequality, the definition of R_1 and (50), we obtain

$$\mathcal{E}(u, \Pi_1) \leq \mathcal{E}(\bar{u}, \Pi_2) + ce^{-cL} (\mathcal{E}(\bar{u}, \Pi_2) + \mathcal{E}(u, \Pi_1)).$$

Therefore,

$$\mathcal{E}(u, \Pi_1) \leq \frac{1 + ce^{-cL}}{1 - ce^{-cL}} \mathcal{E}(\bar{u}, \Pi_2) \leq C \left(\frac{L}{\lambda}\right)^{d-1} \lambda e^{-L}.$$

For L large enough, this implies (49) which then contradicts the lower bound (48).

7.2. *Proof of Lemma 7.* Let $\tilde{T}_1 = T_1 \wedge T_{-1}$. We will begin by estimating an upper bound for $P_{\omega,\lambda}(\tilde{T}_1 > cn/\lambda^2)$ for all $n \in \mathbb{N}$ and $\lambda > 0$ small enough. Note that this quantity depends only on the environments between the hyperplanes H_{-1} and H_1 , thus we let $a \in \Omega$ be a modified environment such that $a(x, y) = \omega^\lambda(x, y)$ for any $x, y \in \{z \in \mathbb{Z}^d : z \cdot e_1 \in (-L_1, L_1)\}$ [recall the definition of L_1 in (8)] and a satisfies $\frac{a(e)}{a(e')} < C$ for any bonds e, e' in \mathbb{Z}^d . Clearly,

$$P_{\omega,\lambda}(\tilde{T}_1 \leq (4L_1)^2) = P_a(\tilde{T}_1 \leq (4L_1)^2) \geq P_a(e_1 \cdot X_{(4L_1)^2} > L_1).$$

By the heat-kernel estimate in [11], Theorem 3.1(i), there exist positive constants c_1, c_2 such that

$$P_a(X_n = y) \geq \frac{c_1 m(y)}{V(0, \sqrt{n})} e^{-c_2 |y|^2/n},$$

whenever $|y| \leq n$ and $|y| + n$ is even, where $m(y) = \sum_{z \sim y} a(y, z)$ and $V(x, r) = \sum_{z: |x-z| \leq r} m(z)$. Define the set

$$A = \left\{ z \in \mathbb{Z}^d \mid \frac{4L_0}{\lambda_1} \leq e_1 \cdot z \leq \frac{8L_0}{\lambda_1}, |e_i \cdot z| \leq \frac{4L_0}{\lambda_1} \text{ for } i = 2, \dots, d \right\},$$

then $A \subset B_{4L_1}$ and $e_1 \cdot z \geq L_1$ for all $z \in A$. Set $n = (4L_1)^2$ and let A' be the set of $y \in A$ such that $|y| + (4L_1)^2$ is even, then we get

$$\begin{aligned} P_a(\tilde{T}_1 \leq (4L_1)^2) &\geq \sum_{y \in A'} P_a(X_{(4L_1)^2} = y) \\ &\geq C \sum_{y \in A'} \frac{m(y)}{V(0, 4L_1)} e^{-c_2 |y|^2 / (16L/\lambda)^2} \\ &\geq C \sum_{y \in A'} \frac{e^{\lambda \ell \cdot y}}{\kappa^2 \sum_{z: |z| \leq 4L_1} e^{\lambda \ell \cdot z}} e^{-c_2} > C. \end{aligned}$$

This shows that there are positive constants $c, \delta > 0$ such that

$$(51) \quad P_{\omega, \lambda} \left(\tilde{T}_1 \leq \frac{c}{\lambda^2} \right) > \delta \quad \text{for all } \omega \in \Omega.$$

Then, by the Markov property and (51) we get for any $m \geq 1$,

$$P_{\omega, \lambda} \left(\tilde{T}_1 > \frac{mc}{\lambda^2} \right) \leq \sup_{-L_1 \leq x, e_1 \leq L_1} P_{\omega, \lambda}^x \left(\tilde{T}_1 > \frac{c}{\lambda^2} \right)^m \leq (1 - \delta)^m.$$

If we set $t_0 = 0$ and define recursively $t_{i+1} = t_i + \tilde{T}_1 \circ \theta_{t_i}$, where θ_n denotes the time shift of the trajectory (recall that T_m is defined relative to the starting position), the exponential tail of \tilde{T}_1 implies by the exponential Markov inequality

$$(52) \quad P_{\omega, \lambda} \left(t_n > \frac{Cn}{\lambda^2} \right) \leq e^{-2n}$$

for some C sufficiently large. We define the one-dimensional process $Y_n = (X_{t_n} \cdot e_1) / L_1$, which indicated the subsequent hyperplanes visited by $(X_n)_n$, then by Lemma 5, Y_n jumps to the right with probability at least $\frac{2}{3}$ and then

$$(53) \quad P_{\omega, \lambda} \left(\sup_{k \leq Cn} Y_k < n \right) \leq e^{-2n}$$

for C sufficiently large. If we combine (52) and (53), we obtain

$$P_{\omega, \lambda} \left(T_n > \frac{Cn}{\lambda^2} \right) \leq P_{\omega, \lambda} \left(\sup_{k \leq Cn} Y_k < n \right) + P_{\omega, \lambda} \left(t_n > \frac{Cn}{\lambda^2} \right) \leq 2e^{-2n}.$$

7.3. Proof of Lemma 8. We first find a uniform lower bound for the probability $P_{\omega, \lambda}(D_{1/\lambda} \geq \frac{r}{\lambda^2})$ for $r \leq 1$ small enough, where $D_{1/\lambda}$ is the exit time from the ball $B_{1/\lambda}$ of radius $\frac{1}{\lambda}$ in the 1-norm. Note that $P_{\omega, \lambda}(D_{1/\lambda} \geq \frac{r}{\lambda^2})$ depends only on the environments inside the box. Hence, as in Section 7.2, we let $a \in \Omega$ be a modified environment such that $a(e) = \omega(e)$ for any bond e in $B_{2/\lambda}$, and $a(e')/a(e'') < C$ for all bonds e', e'' in \mathbb{Z}^d . Then $P_{\omega, \lambda}(D_{1/\lambda} \leq \frac{4r}{\lambda^2}) = P_a(D_{1/\lambda} \leq \frac{4r}{\lambda^2})$, and

$$\begin{aligned} P_a \left(D_{1/\lambda} \leq \frac{4r}{\lambda^2} \right) &\leq P_a \left(|X_{4r/\lambda^2}| \geq \frac{1}{2\lambda} \right) \\ &\quad + P_a \left(|X_{4r/\lambda^2}| < \frac{1}{2\lambda}, D_{1/\lambda} \leq \frac{4r}{\lambda^2} \right) \\ &\leq P_a \left(|X_{4r/\lambda^2}| \geq \frac{1}{2\lambda} \right) \\ &\quad + E_a \left[P_a^{X_{D_{1/\lambda}}} \left(|X_{4r/\lambda^2 - D_{1/\lambda}}| \geq \frac{1}{2\lambda} \right) \mathbb{1}_{\{D_{1/\lambda} \leq \frac{4r}{\lambda^2}\}} \right]. \end{aligned}$$

By the heat kernel upper bound in [11], Theorem 3.1(i), we get for $1/2\lambda_1 \leq k \leq 4r/\lambda_1^2$

$$\begin{aligned} P_a\left(|X_k| \geq \frac{1}{2\lambda_1}\right) &= \sum_{m=1/2\lambda_1}^k \sum_{|x|=m} P_a(X_k = x) \\ &\leq C \sum_{m=1/2\lambda_1}^k \sum_{|x|=m} \frac{m(x)}{V(0, \sqrt{k})} e^{-\bar{c}_2|x|^2/k} \\ &\leq Ck^{-d/2} \sum_{m=1/2\lambda_1}^k m^{d-1} e^{-\bar{c}_2m^2/k}. \end{aligned}$$

Comparing this sum with an integral and using $k \leq 4r/\lambda_1^2$, the last line is bounded by $Ce^{-c/r}$ which is smaller than $\frac{1}{4}$ for r small enough. This yields $P_{\omega,\lambda}(D_{1/\lambda} \leq \frac{4r}{\lambda^2}) \leq \frac{1}{2}$ and so

$$E_{\omega,\lambda}[e^{-\lambda^2 D_{1/\lambda}}] \leq 1 - \delta$$

for some $\delta > 0$ when $r, \lambda > 0$ are small enough. Now we can proceed similarly to the proof of Lemma 4.5 in [10]:

$$\begin{aligned} E_{\omega,\lambda}\left[\max_{0 \leq s \leq t} |\lambda X_{s/\lambda^2}|^p\right] &= \int_0^\infty py^{p-1} P_{\omega,\lambda}\left(\max_{0 \leq s \leq t} |\lambda X_{s/\lambda^2}|^p \geq y\right) dy \\ &= \int_0^\infty py^{p-1} P_{\omega,\lambda}\left(D_{y/\lambda^p} \leq \frac{t}{\lambda^2}\right) dy \\ &\leq e^t \int_0^\infty py^{p-1} E_{\omega,\lambda}[e^{-\lambda^2 D_{y/\lambda^p}}] dy. \end{aligned}$$

The exit time of the ball of radius $\frac{y}{\lambda^2}$ can be bounded as

$$D_{y/\lambda^2} \geq D_{1/\lambda^2} + D_{1/\lambda^2} \circ \theta_{D_{1/\lambda^2}} + \dots + D_{1/\lambda^2} \circ \theta_{D_{(\lfloor y \rfloor - 1)/\lambda^2}}.$$

The Markov property and the inequality $E_{\omega,\lambda}[e^{-\lambda^2 D_{1/\lambda}}] \leq 1 - \delta$ imply then for $t \leq 1$

$$E_{\omega,\lambda}\left[\max_{0 \leq s \leq t} |\lambda X_{s/\lambda^2}|^p\right] \leq e^t \int_0^\infty py^{p-1} (1 - \delta)^{\lfloor y \rfloor} dy \leq C,$$

with C depending only on p , the bounds for the conductances and the dimension. The Markov property implies then for $t \geq 1$

$$\begin{aligned} E_{\omega,\lambda}\left[\max_{0 \leq s \leq t} |\lambda X_{s/\lambda^2}|^p\right] &\leq E_{\omega,\lambda}\left[\left(\sum_{k=1}^{\lceil t \rceil} \max_{k-1 \leq s \leq k} |\lambda(X_{s/\lambda^2} - X_{(k-1)/\lambda^2})|\right)^p\right] \\ &\leq \lceil t \rceil^{p-1} \sum_{k=1}^{\lceil t \rceil} E_{\omega,\lambda}\left[\max_{k-1 \leq s \leq k} |\lambda(X_{s/\lambda^2} - X_{(k-1)/\lambda^2})|^p\right] \\ &\leq C \cdot t^p, \end{aligned}$$

which is equivalent to the claimed inequality.

Acknowledgments. This paper owes a lot to discussions with Pierre Mathieu. In particular, X. Guo and J. Nagel thank Pierre for several invitations to Marseille, and for pointing us to the space \mathcal{H}_{-1} and to reference [21].

We also thank the anonymous referee for a very careful reading of the first version, resulting in many improvements.

REFERENCES

- [1] AVENA, L., DOS SANTOS, R. S. and VÖLLERING, F. (2013). Transient random walk in symmetric exclusion: Limit theorems and an Einstein relation. *ALEA Lat. Am. J. Probab. Math. Stat.* **10** 693–709. [MR3108811](#)
- [2] BEN AROUS, G., HU, Y., OLLA, S. and ZEITOUNI, O. (2013). Einstein relation for biased random walk on Galton–Watson trees. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 698–721. [MR3112431](#)
- [3] BILLINGSLEY, P. (1956). The invariance principle for dependent random variables. *Trans. Amer. Math. Soc.* **83** 250–268. [MR0090923](#)
- [4] COMETS, F. and ZEITOUNI, O. (2004). A law of large numbers for random walks in random mixing environments. *Ann. Probab.* **32** 880–914. [MR2039946](#)
- [5] DE BUYER, P. and MOURRAT, J.-C. (2015). Diffusive decay of the environment viewed by the particle. *Electron. Commun. Probab.* **20** no. 23, 12. [MR3320411](#)
- [6] DELMOTTE, T. (1999). Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Mat. Iberoam.* **15** 181–232. [MR1681641](#)
- [7] DE MASI, A., FERRARI, P. A., GOLDSTEIN, S. and WICK, W. D. (1989). An invariance principle for reversible Markov processes. Applications to random motions in random environments. *J. Stat. Phys.* **55** 787–855. [MR1003538](#)
- [8] DOYLE, P. G. and SNELL, J. L. (1984). *Random Walks and Electric Networks*. Mathematical Association of America, Washington, DC. [MR0920811](#)
- [9] EINSTEIN, A. (1956). *Investigations on the Theory of the Brownian Movement*. Dover Publications, New York. [MR0077443](#)
- [10] GANTERT, N., MATHIEU, P. and PIATNITSKI, A. (2012). Einstein relation for reversible diffusions in a random environment. *Comm. Pure Appl. Math.* **65** 187–228. [MR2855544](#)
- [11] GRIGOR’YAN, A. and TELCS, A. (2002). Harnack inequalities and sub-Gaussian estimates for random walks. *Math. Ann.* **324** 521–556. [MR1938457](#)
- [12] GUO, X. (2016). Einstein relation for random walks in random environment. *Ann. Probab.* **44** 324–359. [MR3456340](#)
- [13] KIPNIS, C. and VARADHAN, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.* **104** 1–19. [MR0834478](#)
- [14] KOMOROWSKI, T., LANDIM, C. and OLLA, S. (2012). *Fluctuations in Markov Processes: Time Symmetry and Martingale Approximation*. Springer, Heidelberg. [MR2952852](#)
- [15] KOMOROWSKI, T. and OLLA, S. (2005). Einstein relation for random walks in random environments. *Stochastic Process. Appl.* **115** 1279–1301. [MR2152375](#)
- [16] KOMOROWSKI, T. and OLLA, S. (2005). On mobility and Einstein relation for tracers in time-mixing random environments. *J. Stat. Phys.* **118** 407–435. [MR2123642](#)
- [17] LEBOWITZ, J. L. and ROST, H. (1994). The Einstein relation for the displacement of a test particle in a random environment. *Stochastic Process. Appl.* **54** 183–196. [MR1307334](#)
- [18] LOULAKIS, M. (2002). Einstein relation for a tagged particle in simple exclusion processes. *Comm. Math. Phys.* **229** 347–367. [MR1923179](#)

- [19] LOULAKIS, M. (2005). Mobility and Einstein relation for a tagged particle in asymmetric mean zero random walk with simple exclusion. *Ann. Inst. Henri Poincaré Probab. Stat.* **41** 237–254. [MR2124642](#)
- [20] MATHIEU, P. and PIATNITSKI, A. (2016). Steady states, fluctuation-dissipation theorems and homogenization for diffusions in a random environment with finite range of dependence. Preprint. Available at [arXiv:1601.02944](#) [math.PR].
- [21] MOURRAT, J.-C. (2011). Variance decay for functionals of the environment viewed by the particle. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** 294–327. [MR2779406](#)
- [22] PELIGRAD, M., UTEV, S. and WU, W. B. (2007). A maximal \mathbb{L}_p -inequality for stationary sequences and its applications. *Proc. Amer. Math. Soc.* **135** 541–550 (electronic). [MR2255301](#)
- [23] SHEN, L. (2002). Asymptotic properties of certain anisotropic walks in random media. *Ann. Appl. Probab.* **12** 477–510. [MR1910636](#)
- [24] SPOHN, H. (1989). Scaling limits for stochastic particle systems. In *IXth International Congress on Mathematical Physics (Swansea, 1988)* 272–275. Hilger, Bristol. [MR1033775](#)
- [25] SZNITMAN, A.-S. and ZERNER, M. (1999). A law of large numbers for random walks in random environment. *Ann. Probab.* **27** 1851–1869. [MR1742891](#)
- [26] VON BAHR, B. and ESSEEN, C.-G. (1965). Inequalities for the r th absolute moment of a sum of random variables, $1 \leq r \leq 2$. *Ann. Math. Stat.* **36** 299–303. [MR0170407](#)

N. GANTERT
J. NAGEL
FAKULTÄT FÜR MATHEMATIK
TECHNISCHE UNIVERSITÄT MÜNCHEN
85748 GARCHING BEI MÜNCHEN
GERMANY
E-MAIL: gantert@ma.tum.de
jan.nagel@ma.tum.de

X. GUO
DEPARTMENT OF MATHEMATICS
PURDUE UNIVERSITY
WEST LAFAYETTE, INDIANA 47907
USA
E-MAIL: guo297@purdue.edu