The Annals of Probability
2017, Vol. 45, No. 3, 1512–1542
DOI: 10.1214/16-AOP1093

A LIMIT THEOREM FOR MOMENTS IN SPACE OF THE INCREMENTS OF BROWNIAN LOCAL TIME

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We prove a limit theorem for moments in space of the increments of Brownian local time. As special cases for the second and third moments, previous results by Chen et al. [*Ann. Prob.* **38** (2010) 396–438] and Rosen [*Stoch. Dyn.* **11** (2011) 5–48], which were later reproven by Hu and Nualart [*Electron. Commun. Probab.* **15** (2010) 396–410] and Rosen [In *Séminaire de Probabilités XLIII* (2011) 95–104 Springer] are included. Furthermore, a conjecture of Rosen for the fourth moment is settled. In comparison to the previous methods of proof, we follow a fundamentally different approach by exclusively working in the space variable of the Brownian local time, which allows to give a unified argument for arbitrary orders. The main ingredients are Perkins' semimartingale decomposition, the Kailath–Segall identity and an asymptotic Ray–Knight theorem by Pitman and Yor.

1. Introduction. Let (L_t^x) be the local time of Brownian motion. In [6], motivated by the form of a Hamiltonian in a certain polymer model, Chen et al. proved that

$$\frac{1}{h^{3/2}}\left(\int_{-\infty}^{\infty}(L_t^{x+h}-L_t^x)^2\,\mathrm{d}x-4ht\right)\xrightarrow{d}\sqrt{\frac{64}{3}\int_{-\infty}^{\infty}(L_t^x)^2\,\mathrm{d}x}\,Z, \tag{1.1}$$

where Z is a standard Gaussian random variable which is independent of $(L_t^x)_{x\in\mathbb{R}}$ and $\xrightarrow{d}$ denotes convergence in distribution. This can be seen as a central limit theorem, as it was also shown in the aforementioned reference that

$$\mathrm{E}\left[h^{-3/2}\int_{-\infty}^{\infty}(L_t^{x+h}-L_t^x)^2\,\mathrm{d}x\right]=4th+\mathcal{O}(h^2).$$

Note that up to a constant, the limit on the right-hand side equals in law $\int_{-\infty}^{\infty}L_t^x\,\mathrm{d}W_x$, where W is a two-sided Brownian motion which is independent of $(L_t^x)_{x\in\mathbb{R}}$. As was pointed out in [6], this integral, when interpreted as a process in t, is known as "Brownian motion in Brownian scenery" and also appears as a limit in several applications, for example, when modelling self-interacting random walks (see [13]) or charged polymers (see [4, 5]).

Received June 2015; revised January 2016.
[1]Supported in part by ERC Grant 277742 Pascal.
MSC2010 subject classifications. 60F05, 60G44, 60H05.
Key words and phrases. Kailath–Segall identity, Brownian local time, central limit theorem, asymptotic Ray–Knight theorem.

The proof of (1.1) was realised by the method of moments, and two different ones were subsequently given by Rosen in [23], using stochastic calculus and Brownian self-intersection local times, and Hu and Nualart in [8], using Malliavin calculus and Pitman and Yor's asymptotic version of the Ray–Knight theorem (see [17]). Later, in [22], Rosen, again using the method of moments, proved a central limit theorem for the third power, which reads

$$\frac{1}{h^2}\int_{-\infty}^{\infty}(L_t^{x+h}-L_t^x)^3\,\mathrm{d}x \xrightarrow{d} \sqrt{192\int_{-\infty}^{\infty}(L_t^x)^3\,\mathrm{d}x}\,Z, \tag{1.2}$$

with Z as above. For this case as well, a Malliavin calculus proof was provided by Hu and Nualart in [9]. Unfortunately, as Rosen mentions in [22], his proof via the method of moments will not work for powers higher than three. It yields, however, the following conjecture.[2]

CONJECTURE 1.1 (Rosen, [22]). *Writing $\Delta_x^h L_t^x = L_t^{x+h} - L_t^x$, it holds that*

$$\begin{aligned}&\frac{1}{h^{5/2}}\biggl(\int_{-\infty}^{\infty}(\Delta_x^h L_t^x)^4\,\mathrm{d}x - 24h\int_{-\infty}^{\infty}(\Delta_x^h L_t^x)^2 L_t^x\,\mathrm{d}x\\ &\qquad + 48h^2\int_{-\infty}^{\infty}(L_t^x)^2\,\mathrm{d}x - \int_{-\infty}^{\infty}(\Delta_x^h L_t^x)L_t^x\,\mathrm{d}x\biggr)\\ &\qquad\xrightarrow{d}\sqrt{\frac{2^9 4!}{5}\int_{-\infty}^{\infty}(L_t^x)^4\,\mathrm{d}x}\,Z,\end{aligned}\tag{1.3}$$

where $Z \sim \mathcal{N}(0,1)$, independent of $(L_t^x)_{x\in\mathbb{R}}$.

Though the techniques are clearly different, all aforementioned proofs approach the problem through the time domain (the variable t). For example, in Rosen's method of moments proofs, intersection local times of the type $\int_{-\infty}^{\infty} L_s^x \widetilde{L}_t^x\,\mathrm{d}x$ are considered, where $\widetilde{L}_t^x$ is the local time of another Brownian motion, independent of the one driving L_t^x, and in the Malliavin calculus approach of Hu and Nualart the quantity $\int_{-\infty}^{\infty}\Delta_x^h L_t^x\,\mathrm{d}x$ is expressed as a stochastic integral indexed by t. In this paper, we take a fundamentally different approach and exclusively work in space (the variable x), which seems to be more natural for the problem at hand. This allows us to prove the following limit theorem for arbitrary integer powers q.

THEOREM 1.2. *For integers $q \ge 2$, it holds that*

$$\frac{1}{h^{\frac{q+1}{2}}}\biggl(\int_{-\infty}^{\infty}(L_t^{x+h}-L_t^x)^q\,\mathrm{d}x + R_{q,h}\biggr) \xrightarrow{d} c_q\sqrt{\int_{-\infty}^{\infty}(L_t^x)^q\,\mathrm{d}x}\,Z, \tag{1.4}$$

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[2]The rightmost integral in the numerator of (1.3) was typeset as $+48h^2\int_{-\infty}^{\infty}(L_t^x)^2 - (\Delta_x^h L_t^x)L_t^x\,\mathrm{d}x$ in the original reference [22]. In the statement of Conjecture 1.1, the author corrected this in a way that seemed the most reasonable at the time of writing.

where Z is a standard Gaussian random variable, independent of $(L_t^x)_{x\in\mathbb{R}}$, the random variable $R_{q,h}$ is given by

$$R_{q,h} = \sum_{k=1}^{\lfloor \frac{q}{2} \rfloor} a_{q,k} \int_{-\infty}^{\infty} (L_t^{x+h} - L_t^x)^{q-2k} \left(4 \int_x^{x+h} L_t^u \, \mathrm{d}u \right)^k \mathrm{d}x$$

and the constants $a_{q,k}$ and c_q are defined as

$$a_{q,k} = \frac{(-1)^k q!}{2^k k!(q-2k)!} \quad \text{and} \quad c_q = \sqrt{\frac{2^{2q+1} q!}{q+1}}. \tag{1.5}$$

It will be shown later that $R_{2,h} = 4ht$ and $R_{3,h} = 0$, so that for $r = 0$, the known central limit theorems (1.1) and (1.2), corresponding to $q = 2$ and $q = 3$, respectively, are included as special cases. Moreover, for $q = 4$, Theorem 1.2 proves Rosen's Conjecture 1.1 (with a slightly different compensator; see Remark 4.2 for details). Starting from $q = 4$, however, the compensator term $R_{q,h}$ becomes random and (1.4) thus no longer states a *central* limit theorem. To remedy the situation, one would have to prove that $R_{q,h}$ can be replaced by its expectation. Unfortunately, we have to leave this problem for further research and again refer to Remark 4.2 for a more detailed discussion of this point.

Our approach allows for generalizations in several directions. For example, as we never touch the time variable t in our proofs, it can be replaced with a suitable stopping time τ (see Theorem 4.4).

The proof of Theorem 1.2 can roughly be sketched as follows. The starting point is the semimartingale decomposition of Browian local time in space, first proven by Perkins in his celebrated paper [16] and subsequently refined by Jeulin in [11]. Decomposing L_t^x into its martingale and finite variation part M_x and V_x, respectively (where we have suppressed the dependence on t for better legibility), some careful stochastic analysis yields that

$$\int_{-\infty}^{\infty} (L_t^{x+h} - L_t^x)^q \, \mathrm{d}x \approx \int_{-\infty}^{\infty} (M_{x+h} - M_x)^q \, \mathrm{d}x.$$

From here, through an iterative application of the Kailath–Segall formula (see [25]), the leading term of the integral on the right-hand side turns out to be a certain iterated integral, whose limit can be obtained by Pitman and Yor's asymptotic Ray–Knight theorem (see [17]). This asymptotic Ray–Knight theorem was also used by Hu and Nualart in [8] and [9] for their Malliavin calculus proofs, with the notable difference that in their case the Dambis–Dubins–Schwarz Brownian motion comes from a time change, while we obtain it through a space change.

The rest of this paper is organized as follows. In Section 2, we introduce some results from the literature and fix the notation which is used throughout this paper. In Section 3, several crucial types of iterated integrals are introduced, including a rather technical analysis of their asymptotics. The proofs of Theorem 1.2 and several generalizations are provided in Section 4.

2. Preliminaries and notation. In the sequel, we will deal with stochastic integrals of the form $\int_{-\infty}^{x} Y_u \,\mathrm{d}M_u$, where $(M_u)_{u\geq-\infty}$ is a continuous martingale. In general, constructing such stochastic integrals is a very delicate task, which, to the best knowledge of the author, has only recently been treated rigorously and in full generality by Basse-O'Connor, Graversen and Pedersen in [1] and [2]. For example, an application of Doob's backward martingale convergence theorem (see, e.g., [7], page 328, Theorem 4.2) shows that two-sided Browian motion can neither be a martingale, nor a local martingale in any filtration, and thus cannot be integrated against in the classical way. To overcome this problem, one can introduce the notion of increment martingale (see [1]).

However, in this paper all processes of the form $(X_u)_{u\geq-\infty}$ vanish at $-\infty$, so that stochastic integration can be defined in the classical way, starting from simple processes and building up to semimartingales as, for example, outlined in the monographs [10, 12, 19–21]. Due to the finite limit at $-\infty$, the usual index interval $[0,\infty)$ can be replaced by $\{-\infty\}\cup\mathbb{R}$ and all tools from standard textbook martingale theory, such as Itô's formula, the Burkholder–Davis–Gundy inequality, etc. remain valid.

Let us define several classical spaces of martingales and integrators. We set H_0^2 to be the space of L^2-bounded continuous martingales indexed by $[-\infty,\infty)$ and vanishing at $-\infty$. Given $M\in H_0^2$, the Hilbert space $L^2(M)$ consists of all (equivalence classes of) progressively measurable processes $(Y_x)_{x\geq-\infty}$ such that

$$\|Y\|_M=\mathrm{E}\left[\int_{-\infty}^{\infty} Y_u^2 \,\mathrm{d}\langle M\rangle_u\right]<\infty,$$

where as usual $\langle M\rangle_u$ denotes the quadratic variation of M. A continuous local martingale M belongs to $H_{0,\mathrm{loc}}^2$, if its localized version is an element of H_0^2 and, given a continuous local martingale $M=(M_x)_{x\geq-\infty}$, the space $L_{\mathrm{loc}}^2(M)$ contains all progressively measurable processes Y such that

$$\mathrm{E}\left[\int_{-\infty}^{T_n} Y_u^2 \,\mathrm{d}\langle M\rangle_u\right]<\infty$$

for a sequence (T_n) of stopping times increasing to infinity. We will make frequent use of the following result from stochastic analysis.

THEOREM 2.1 (Stochastic Fubini theorem). *Let $(A,\mathcal{A})$ be a measurable space, μ be a σ-finite, measure on A and denote by $\mathcal{P}$ the predictable σ-algebra on $[-\infty,\infty)\times\Omega$. Furthermore, let $(X_{a,x})_{a\in A,x\in[-\infty,\infty)}$ be a continuous, $\mathcal{A}\otimes\mathcal{P}$-measurable stochastic process indexed by $A\times[-\infty,\infty)$ which is μ-integrable and assume that*

$$\mathrm{E}\left[\int_{-\infty}^{\infty}\int_A X_{a,u}^2\mu(\mathrm{d}a)\,\mathrm{d}\langle M,M\rangle_u^{T_n}\right]<\infty, \tag{2.1}$$

where $(M)_{x\geq-\infty} \in H^2_{0,\mathrm{loc}}$ *and* $(T_n)_{n\geq 1}$ *is a sequence of localizing stopping times for* M. *Then, for all* $x \in [-\infty, \infty)$,

$$\int_{-\infty}^{x} \int_A X_{a,u}\mu(\mathrm{d}a)\,\mathrm{d}M_u \overset{a.s.}{=} \int_A \int_{-\infty}^{x} X_{a,u}\,\mathrm{d}M_u\,\mu(\mathrm{d}a). \tag{2.2}$$

In particular, the double integral on the right-hand side exists.

PROOF. A proof for semimartingales indexed by $[0, \infty]$ can, for example, be found in [18], Chapter IV, Theorem 64. The adaptation to our setting is straightforward. □

As already indicated, we also need (a slightly generalized version of) the asymptotic Ray–Knight theorem by Pitman and Yor (see [17] or the textbook [19], Chapter XIII, Theorem 2.3). Again, adapting the original proof is straightforward.

THEOREM 2.2 (Asymptotic Ray–Knight theorem, [17]). *For* $k \geq 2$ *and* $n \geq 1$ *define a sequence* $(M^n_{1,x}, M^n_{2,x}, \dots, M^n_{k,x})$ *of* k*-tuples of continuous local martingales* $(M^n_{j,x})_{x\geq-\infty} \in H^2_{0,\mathrm{loc}}$ *such that for fixed* $j = 1, \dots, k$ *the limit* $\langle M^n_j, M^n_j\rangle_\infty$ *is either infinite for all* n, *or finite for all* n. *After possibly enlarging the underlying probability space, each* M^n_j *possesses a Dambis–Dubins–Schwarz Brownian motion* β^n_j *and an associated time change* $T^n_j(y)$, *such that*

$$M^n_{j,T^n_j(y)} = \beta^n_{j,y} \tag{2.3}$$

for $y \geq 0$ *and* $1 \leq j \leq k$ (*for a proof, see, e.g.*, [19], Chapter V, Theorem 1.7). *If for* $a \geq 0$ *and* $1 \leq i, j \leq k$ *with* $i \neq j$,

$$\sup_{x\in[-\infty,a]} \left|\langle M^n_i, M^n_j\rangle_x\right| \to 0 \qquad (n \to \infty) \tag{2.4}$$

in probability, then the k*-dimensional process* $\beta^n_y = (\beta^n_{1,y}, \beta^n_{2,y}, \dots, \beta^n_{k,y})_{y\geq 0}$ *converges in distribution to a* k*-dimensional Brownian motion* $(\beta_y)_{y\geq 0}$.

Let us now collect some of the results from Perkins' celebrated paper [16].

THEOREM 2.3 ([16]). *For any* $t > 0$, *there exists a right continuous filtration* $(\mathcal{G}_x)_{x\in\mathbb{R}}$ *such that* $x \mapsto L^x_t$ *is a continuous* $\mathcal{G}_x$*-semimartingale on* $\mathbb{R}$ *with quadratic variation* $4\int_{-\infty}^x L^u_t\,\mathrm{d}u$. *Furthermore, if* $L^x_t = M_{t,x} + \int_{-\infty}^x A_{t,u}\,\mathrm{d}u$ *is its canonical decomposition, then it holds for any* $p \geq 1$ *that*

$$\|L^*_t\|_p < \infty, \qquad \int_{-\infty}^{\infty} |A_{t,x}|\,\mathrm{d}x < \infty \quad \textit{and} \quad \int_{-\infty}^{\infty} \|A_{t,x}\|_p\,\mathrm{d}x < \infty,$$

where $L^*_t = \sup_{x\in\mathbb{R}} |L^x_t|$.

As a corollary of Theorem 2.3 we get the following explicit semimartingale decomposition (see [11], Chapter II, for several extensions).

COROLLARY 2.4. *In the setting of Theorem 2.3, there exists a $\mathcal{G}_x$-Browian motion $(\beta_x)_{x\in\mathbb{R}}$, such that*

$$L_t^x = \begin{cases} L_t^0 + 2\int_0^x \sqrt{L_t^u}\,\mathrm{d}\beta_u + \int_0^x A_{t,u}\,\mathrm{d}u, & \text{if } x \geq 0, \\ L_t^0 - 2\int_x^0 \sqrt{L_t^u}\,\mathrm{d}\beta_u - \int_x^0 A_{t,u}\,\mathrm{d}u, & \text{if } x < 0. \end{cases} \tag{2.5}$$

In fact, as the next lemma shows, the integrability property of the L^p-norms of the finite variation kernel $A_{t,x}$ are also true for the local time itself.

LEMMA 2.5. *For $p \geq 1$ and $a > 0$, it holds that*

$$\int_{-\infty}^{\infty} \|L_t^x\|_p^a \,\mathrm{d}x < \infty. \tag{2.6}$$

PROOF. For $t = 0$, the assertion is trivial. If $t > 0$, we have by scaling that $\|L_t^x\|_p = \sqrt{t}\|L_1^{x/\sqrt{t}}\|$. Thus, through a change of variables in the integral (2.6), we can and do assume without loss of generality that $t = 1$. Furthermore, if B denotes the underlying Brownian motion, we see that $\|L_t^x\|_p = \|L_t^{-x}\|_p$, as $-B$ has the same law as B. Thus,

$$\int_{-\infty}^{\infty} \|L_1^x\|_p^a \,\mathrm{d}x = 2\int_0^{\infty} \|L_1^x\|_p^a \,\mathrm{d}x.$$

In [27], it is proved that

$$\|L_1^x\|_p = \frac{2}{\sqrt{2\pi}}\left(x^{p+1}\int_1^{\infty} \mathrm{e}^{-x^2y^2/2}(y-1)^p \,\mathrm{d}y\right)^{1/p}$$

for $x > 0$, which implies that there exist positive constants α, β and $x_0 > 1$, such that

$$\|L_t^x\|_p^a \leq \alpha \mathrm{e}^{-\beta x^2}$$

for all $x > x_0$. Consequently,

$$\int_0^{\infty} \|L_1^x\|_p^a \,\mathrm{d}x = \int_0^{x_0} \|L_1^x\|_p^a \,\mathrm{d}x + \int_{x_0}^{\infty} \|L_1^x\|_p^a \,\mathrm{d}x < \infty. \qquad \square$$

3. The iterated integrals I_q, J_q and K_q. In this section, we define and study three types of iterated integrals, two of them, namely I_q and J_q, stochastic, the third one deterministic with a random kernel. As already indicated in the Introduction, these integrals will appear later through the Kailath–Segall formula.

DEFINITION 3.1. Let $(M_u)_{u\geq-\infty} \in H^2_{0,\mathrm{loc}}$ and $(Y(u))_{u\geq-\infty} \in L^2_{\mathrm{loc}}(M)$. For $q \geq 0$, and $-\infty \leq x_1 < x_2$, the iterated integrals I_q, J_q and K_q are defined as follows:

$$I_0(x_1, x_2) = 1, \qquad I_{q+1}(x_1, x_2) = \int_{x_1}^{x_2} I_q(u)\,\mathrm{d}M_u, \tag{3.1}$$

$$\begin{aligned} J_0(Y, x_1, x_2) &= Y(x_2), \\ J_{q+1}(Y, x_1, x_2) &= \int_{x_1}^{x_2} J_q(Y, x_1, x_2)\,\mathrm{d}M_u, \end{aligned} \tag{3.2}$$

$$\begin{aligned} K_0(Y, x_1, x_2) &= Y(x_2), \\ K_{q+1}(Y, x_1, x_2) &= \int_{x_1}^{x_2} K_q(Y, x_1, u)\,\mathrm{d}\langle M, M\rangle_u. \end{aligned} \tag{3.3}$$

If $x_1 = -\infty$, we drop it from the arguments, so that $I_q(-\infty, x_2)$ becomes $I_q(x_2)$, $J_q(Y, -\infty, x_2)$ becomes $J_q(Yd, x_2)$ and $K_q(Y, -\infty, x_2)$ becomes $K_q(Y, x_2)$.

As $I_q(x_1, x_2) = J_q(1, x_1, x_2)$, the definition of I_q is redundant. We have included it to improve the readability of subsequent results. Observe that for any integer $q \geq 1$, the integrals I_q and J_q are elements of $H^2_{0,\mathrm{loc}}$. The following is a consequence of the main findings in [25].

PROPOSITION 3.2. *For $x_1 \geq -\infty$, define the martingale $(M_{x_2})_{x_2 \geq x_1}$ by $M_{x_2} = I_1(x_1, x_2)$. Then, for $q \geq 2$, the* Kailath–Segall identity

$$q I_q(x_1, x_2) = I_{q-1}(x_1, x_2) M_{x_2} - I_{q-2}(x_1, x_2)\langle M, M\rangle_{x_2} \tag{3.4}$$

holds. Furthermore, we have for $q \geq 1$ that

$$q! I_q(x_1, x_2) = \widetilde{H}_q\big(M_{x_2}, \langle M, M\rangle_{x_2}\big) = \sum_{k=0}^{\lfloor \frac{q}{2} \rfloor} a_{q,k} M_{x_2}^{q-2k} \langle M, M\rangle_{x_2}^k. \tag{3.5}$$

Here, $\widetilde{H}_n(x, a) = a^{n/2} H_n(x/\sqrt{a})$, where $H_n(x) = \mathrm{e}^{\frac{x^2}{2}} \frac{\partial^n}{\partial x^n} \mathrm{e}^{-\frac{x^2}{2}}$ denotes the nth Hermite polynomial and the constants $a_{q,k}$ are the Bessel of numbers of the second kind, given by

$$a_{q,k} = \frac{(-1)^k q!}{2^k k!(q-2k)!}. \tag{3.6}$$

PROOF. The identity $H_n(x)' = n H_{n-1}(x)$ and the recursion formula

$$H_n(x) = x H_{n-1}(x) - H_{n-1}(x)' = x H_{n-1}(x) - (n-1) H_{n-2}(x),$$

which are both well known (see, e.g., [26], page 106) and follow inductively from the definition, imply that

$$\widetilde{H}_n(x, a) = x \widetilde{H}_{n-1}(x, a) - (n-1) a \widetilde{H}_{n-2}(x, a) \tag{3.7}$$

and also that

$$\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial t}\right)\widetilde{H}_n(x,a)=0 \quad \text{and} \quad \frac{\partial}{\partial x}\widetilde{H}_n=n\widetilde{H}_{n-1}.$$

Thus (see, e.g., [19], Chapter IV, Proposition 3.8), $\widetilde{H}_n(M_x,\langle M,M\rangle_x)$ is a martingale and satisfies the recursion

$$\widetilde{H}_n(M_x,\langle M,M\rangle_x)=n\int_{-\infty}^{x}\widetilde{H}_{n-1}(M_u,\langle M,M\rangle_u)\,\mathrm{d}M_u,$$

which inductively implies that

$$q!I_q(x_1,x_2)=\widetilde{H}_q(M_{x_2},\langle M,M\rangle_{x_2}). \tag{3.8}$$

Together with (3.7), the Kailath–Segall formula (3.4) follows. Identity (3.5) is a consequence of (3.8) as well, together with the well-known formula (see, e.g., [26], page 106)

$$H_q(x)=\sum_{k=0}^{\lfloor\frac{q}{2}\rfloor}a_{q,k}x^{q-2k},$$

which translates into

$$\widetilde{H}_q(x,a)=\sum_{k=0}^{\lfloor\frac{q}{2}\rfloor}a_{q,k}a^kx^{q-2k}.$$

□

REMARK 3.3. (i) As indicated in the proof, the Bessel numbers $a_{q,k}$ are the coefficients of the Hermite polynomials. The first few values of $a_{q,k}$ for $q,k\geq 1$ (note that $a_{q,0}=1$) are given in Table 1.

The third row for example translates into

$$3!I_3=I_1^3-3I_1\langle M\rangle$$

TABLE 1
First few values of the Bessel numbers $a_{q,k}$

$q\backslash k$	1	2	3	4
1				
2	1			
3	3			
4	6	3		
5	10	15		
6	15	45	15	
7	21	105	105	
8	28	210	420	105
9	36	378	1260	945

and row eighth shows that

$$8!I_8 = I_1^8 - 28I_1^6\langle M\rangle + 210I_1^4\langle M\rangle^2 - 420I_1^2\langle M\rangle^3 + 105\langle M\rangle^4.$$

(ii) The Kailath–Segall identity (3.4) continues to hold verbatim for continuous semimartingales. For general semimartingales, it involves higher-order variations (see [25]).

The following lemma gives an analogue of the Burkholder–Davis–Gundy inequality for the iterated integrals.

LEMMA 3.4 ([3]). *Let $(M_x)_{x\ge -\infty} \in H^2_{0,\mathrm{loc}}$. Then, for the iterated integrals I_q defined with respect to M, we have that*

$$A_{p,q}\big\|\langle M, M\rangle_x^{1/2}\big\|_{pq}^q \le \big\|I_q(x)\big\|_p \le B_{p,q}\big\|\langle M, M\rangle_x^{1/2}\big\|_{pq}^q,$$

where the left-hand side holds for $p > 1$, the right-hand side for $p \ge 1$ and $A_{p,q}$, $B_{p,q}$ denote positive constants depending on p and q.

The proof, which was originally given for martingales indexed by $[0,\infty)$ and continues to work in our framework, uses the Kailath–Segall identity. The constants $A_{p,q}$ and $B_{p,q}$ can be computed explicitly, decay in q and are sharp in a certain sense (none of these facts are needed here, see [3] for details). Unfortunately, the approach cannot be adapted to cover the iterated integrals J_q or K_q, but nevertheless, as the next lemma shows, we can derive rather tight upper bounds which suffice for our purposes.

LEMMA 3.5. *Let J_q and K_q be the iterated integrals as defined above with respect to the local martingale part M_x of the local time L_t^x. Then, for two real numbers $x_1 < x_2$, an integrable continuous process $(X_u)_{u\in[x_1,x_2]}$, $p \ge 1$ and any integer $q \ge 0$, it holds that*

$$\big\|J_q(X, x_1, x_2)\big\|_p \le C_p(x_2 - x_1)^{\frac{q}{2}}\big\|L_t^*\big\|_{2^q p}^{q/2} \sup_{u\in(x_1,x_2)} \|X_u\|_{2^q p} \tag{3.9}$$

and

$$\big\|K_q(X, x_1, x_2)\big\|_p \le (x_2 - x_1)^q\big\|L_t^*\big\|_{2^q p}^q \sup_{u\in(x_1,x_2)} \|X_u\|_{2^q p}, \tag{3.10}$$

where C_p denotes a positive constant and $L_t^* = \sup_{x\in\mathbb{R}} |L_t^x|$.

PROOF. We assume throughout the proof that $p \ge 2$. The general case follows a posteriori as the L^p-norms with respect to a finite measure are increasing in p.

Furthermore, the constant C_p might change from line to line. For $q=0$, both inequalities are trivially satisfied. Inductively, for $q \geq 1$, Burkholder–Davis–Gundy yields that

$$\begin{aligned}\|J_q(X,x_1,x_2)\|_p &= \left\|\int_{x_1}^{x_2} J_{q-1}(X,x_1,u)\,\mathrm{d}M_u\right\|_p \\ &\leq C_p\left\|\left(\int_{x_1}^{x_2} J_{q-1}(X,x_1,u)^2 L_t^u\,\mathrm{d}u\right)^{1/2}\right\|_p.\end{aligned}$$

By Jensen's inequality, the above is less than or equal to

$$C_p(x_2-x_1)^{1/2-1/p}\left(\int_{x_1}^{x_2} \mathrm{E}\big[|J_{q-1}(X,x_1,u)|^p(L_t^u)^{p/2}\big]\,\mathrm{d}u\right)^{1/p}$$

and by Cauchy–Schwarz and the induction hypothesis, this can further be bounded by

$$\begin{aligned}&C_p(x_2-x_1)^{1/2-1/p}\left(\int_{x_1}^{x_2}\|J_{q-1}(X,x_1,u)\|_{2p}^p\|L_t^u\|_p^{p/2}\,\mathrm{d}u\right)^{1/p} \\ &\qquad \leq C_p(x_2-x_1)^{q-2}\|L_t^*\|_p^{1/2}\|L_t^*\|_{2^q p}^{(q-1)/2}\sup_{u\in(x_1,x_2)}\|X_u\|_{2^q p}.\end{aligned}$$

Together with the monotonicity of the L^p norms with respect to a finite measure, this proves inequality (3.9). To show (3.10), apply Jensen's inequality to obtain

$$\begin{aligned}&\|K_q(X,x_1,x_2)\|_p \\ &\quad = \left\|\int_{x_1}^{x_2} K_{q-1}(X,x_1,u)L_t^u\,\mathrm{d}u\right\|_p \\ &\quad \leq (x_2-x_1)^{1-\frac{1}{p}}\left(\int_{x_1}^{x_2}\mathrm{E}\big[|K_{q-1}(X,x_1,u)L_t^u|^p\big]\,\mathrm{d}u\right)^{1/p}\end{aligned}$$

and then again use Cauchy–Schwarz and the induction hypothesis. □

We now turn to the proof of two crucial technical lemmas.

LEMMA 3.6. *For an integer $q \geq 2$, let J_{q-1} be the iterated integral defined with respect to the local martingale part M_x of the local time L_t^x. Furthermore, let $(Y_x)_{x\in\mathbb{R}}$ be a uniformly bounded continuous stochastic process adapted to $(L_t^x)_{x\in\mathbb{R}}$ and define $X_u(v)=\int_u^v Y_x\,\mathrm{d}x$. Then, for any $x_0\in\mathbb{R}$ it holds that*

$$\mathrm{E}\left[\sup_{x\in(-\infty,x_0)}\frac{1}{h^{(q+1)/2}}\left|\int_{-\infty}^{x} J_{q-1}(X_{u-h},u-h,u)L_t^u\,\mathrm{d}u\right|\right]\to 0. \tag{3.11}$$

PROOF. Throughout the proof, C and C_p denote positive constants, the latter depending on p, which might change from line to line. We have that

$$
\begin{aligned}
&\frac{1}{h^{(q+1)/2}} \int_{-\infty}^{x} J_q(X_{u-h}, u-h, u) L_t^u \, \mathrm{d}u \\
&\quad = \frac{1}{h^{(q+1)/2}} \int_{-\infty}^{x} J_{q-1}(X_{u-h}, u-h, u) L_t^{u-h} \, \mathrm{d}u \\
&\qquad + \frac{1}{h^{(q+1)/2}} \int_{-\infty}^{x} J_{q-1}(X_{u-h}, u-h, u) \big(L_t^u - L_t^{u-h}\big) \, \mathrm{d}u.
\end{aligned} \tag{3.12}
$$

In two separate steps, we show that the supremum of each of the two integrals in the sum (3.12) converges to zero in L^1.

Step 1. An iterated application of the stochastic Fubini theorem, justified by Lemma 3.5, yields that

$$
\begin{aligned}
&\int_{-\infty}^{x} J_{q-1}(X_{u_1-h}, u_1-h, u_1) L_t^{u_1-h} \, \mathrm{d}u_1 \\
&\quad = \int_{-\infty}^{x} J_{q-2}(\widetilde{X}_{u_2-h, x-h}, u_2-h, u_2) \, \mathrm{d}M_{u_2},
\end{aligned} \tag{3.13}
$$

where

$$
\widetilde{X}_{u_2-h,x-h}(u_q) = \int_{u_2-h}^{u_q \wedge (x-h)} \int_{u_1}^{u_q} Y_{u_{q+1}} \, \mathrm{d}u_{q+1} L_t^{u_1} \, \mathrm{d}u_1.
$$

Note that for $u_q \in (u_2-h, u_2)$ it holds that

$$
\big|\widetilde{X}_{u_2-h,x-h}(u_q)\big| \leq Ch^2
$$

and thus, by Lemma 3.5,

$$
\big\| J_q(\widetilde{X}_{u_2-h,x-h}, u_2-h, u_2) \big\|_p \leq C_p h^{q/2+2} \| L_t^* \|_{2^q p}^{q/2}. \tag{3.14}
$$

Abbreviating $\widetilde{X}_{u_2-h,x-h}$ by $\widetilde{X}$, we use identity (3.13), Burkholder–Davis–Gundy and the deterministic Fubini theorem to obtain

$$
\begin{aligned}
&\bigg\| \sup_{x \leq x_0} \bigg| \int_{-\infty}^{x} J_{q-1}(X_{u_1-h}, u_1-h, u_1) L_t^{u_1-h} \, \mathrm{d}u_1 \bigg| \bigg\|_2 \\
&\quad = \bigg\| \sup_{x \leq x_0} \bigg| \int_{-\infty}^{x} J_{q-2}(\widetilde{X}, u_2-h, u_2) \, \mathrm{d}M_{u_2} \bigg| \bigg\|_2 \\
&\quad \leq C \bigg\| \bigg(\int_{-\infty}^{\infty} J_{q-2}(\widetilde{X}, u_2-h, u_2)^2 L_t^{u_2} \, \mathrm{d}u_2 \bigg)^{1/2} \bigg\|_2 \\
&\quad \leq C \bigg(\int_{-\infty}^{\infty} \| J_{q-2}(\widetilde{X}, u_2-h, u_2) \|_4^2 \| L_t^{u_2} \|_2 \, \mathrm{d}u_2 \bigg)^{1/2}
\end{aligned} \tag{3.15}
$$

and using (3.14), we can continue to write

$$\leq C h^{(q+2)/2} \| L_t^* \|_{2^q}^{q/2+r} \int_{-\infty}^{\infty} \| L_t^{u_2} \|_2 \, \mathrm{d}u_2. \tag{3.16}$$

Step 2. Before treating the second integral of (3.12), note that

$$|X_u(v)| = \left| \int_u^v Y_x \, \mathrm{d}x \right| \leq C(v-u) \tag{3.17}$$

and thus, by Lemma 3.5,

$$\| J_q(X_{u-h}, u-h, u) \|_p \leq C_p h^{q/2+1} \| L_t^* \|_{2^q p}^{q/2}. \tag{3.18}$$

By Cauchy–Schwarz, it holds that

$$\begin{aligned}
&\left| \int_{-\infty}^{x} J_{q-1}(X_{u-h}, u-h, u) \big(L_t^u - L_t^{u-h} \big) \, \mathrm{d}u \right| \\
&\quad \leq \left(\int_{-\infty}^{\infty} J_{q-1}(X_{u-h}, u-h, u)^2 \, \mathrm{d}u \right)^{1/2} \left(\int_{-\infty}^{\infty} |L_t^u - L_t^{u-h}|^2 \, \mathrm{d}u \right)^{1/2} \\
&\quad = \left(h \int_{-\infty}^{\infty} J_{q-1}(X_{u-h}, u-h, u)^2 \, \mathrm{d}u \right)^{1/2} \\
&\qquad \times \left(\frac{1}{h} \int_{-\infty}^{\infty} |L_t^u - L_t^{u-h}|^2 \, \mathrm{d}u \right)^{1/2}.
\end{aligned}$$

Taking supremum and expectation and applying Cauchy–Schwarz once again yields

$$\begin{aligned}
&\mathrm{E}\left[\sup_{x \leq x_0} \left| \int_{-\infty}^{x} J_{q-1}(X_{u-h}, u-h, u) \big(L_t^u - L_t^{u-h} \big) \, \mathrm{d}u \right| \right] \\
&\quad \leq h^{1/2} \, \mathrm{E}\left[\int_{-\infty}^{\infty} J_{q-1}(X_{u-h}, u-h, u)^2 \, \mathrm{d}u \right]^{1/2} \\
&\qquad \times \mathrm{E}\left[\frac{1}{h} \int_{-\infty}^{\infty} |L_t^u - L_t^{u-h}|^2 \, \mathrm{d}u \right]^{1/2}.
\end{aligned} \tag{3.19}$$

By [14], Theorem 1.1, the second expectation on the right-hand side converges to a bounded quantity, so we are done if we can show that the square root of the first is of order h^{ε} for some $\varepsilon > q/2$. An application of Itô's formula gives

$$\begin{aligned}
&J_{q-1}(X_{u-h}, u-h, u)^2 \\
&\quad = \int_{u-h}^{u} J_{q-1}(X_{u-h}, u-h, v) J_{q-2}(X_{u-h}, u-h, v) \, \mathrm{d}M_v \\
&\qquad + \int_{u-h}^{u} J_{q-2}(X_{u-h}, u-h, v)^2 \, \mathrm{d}\langle M, M \rangle_v.
\end{aligned}$$

Stochastic Fubini and Burkholder–Davis–Gundy yield that

$$\begin{aligned}
&\mathrm{E}\left[\int_{-\infty}^{\infty}\int_{u-h}^{u} J_{q-1}(X,u-h,v)J_{q-2}(X,u-h,v)\,\mathrm{d}M_v\,\mathrm{d}u\right]\\
&\quad=\mathrm{E}\left[\int_{-\infty}^{\infty}\int_{v}^{v+h} J_{q-1}(X,u-h,v)J_{q-2}(X,u-h,v)\,\mathrm{d}u\,\mathrm{d}M_v\right]\\
&\quad\le\mathrm{E}\left[\int_{-\infty}^{\infty}\left(\int_{v}^{v+h} J_{q-1}(X,u-h,v)J_{q-2}(X,u-h,v)\,\mathrm{d}u\right)^2 4L_t^v\,\mathrm{d}v\right]^{1/2},
\end{aligned}$$

and several applications of deterministic Fubini and the Cauchy–Schwarz inequality further bound the preceding by

$$\begin{aligned}
&2h^{3/4}\left(\int_{-\infty}^{\infty}\left(\int_{v}^{v+h}\|J_{q-1}(X,u-h,v)\|_8^4\|J_{q-2}(X,u-h,v)\|_8^4\,\mathrm{d}u\right)^{1/2}\right.\\
&\qquad\left.\times\|L_t^v\|_2\,\mathrm{d}v\right)^{1/2}.
\end{aligned}$$

Together with (3.14), we arrive at

$$\begin{aligned}
&\mathrm{E}\left[\int_{-\infty}^{\infty}\int_{u-h}^{u} J_{q-1}(X,u-h,v)J_{q-2}(X,u-h,v)\,\mathrm{d}M_v\,\mathrm{d}u\right]\\
&\qquad\le Ch^{q+3/2}\|L_t^*\|_{2^{q+2}}^{q-3/2}\left(\int_{-\infty}^{\infty}\|L_t^v\|_2\,\mathrm{d}v\right)^{1/2}.
\end{aligned}\tag{3.20}$$

Similarly, using Fubini, Jensen's inequality and (3.14), we can show that

$$\begin{aligned}
&\mathrm{E}\left[\int_{-\infty}^{\infty}\int_{u-h}^{u} J_{q-2}(X,u-h,v)^2\,\mathrm{d}\langle M,M\rangle_v\,\mathrm{d}u\right]\\
&\qquad\le Ch^{q+1}\|L_t^*\|_{2^{q-1}}^{q-2}\int_{-\infty}^{\infty}\|L_t^v\|_2\,\mathrm{d}v.
\end{aligned}\tag{3.21}$$

Plugging (3.20) and (3.21) back into (3.19) yields that

$$\mathrm{E}\left[\sup_{x\le x_0}\left|\int_{-\infty}^{x} J_{q-1}(X_{r,u-h},u-h,u)\big(L_t^u-L_t^{u-h}\big)\,\mathrm{d}u\right|\right]^{1/2}$$

is of order $h^{(q+1)/2}$, completing the proof. □

LEMMA 3.7. *Let J_{q-1} be defined with respect to the local martingale part M_x of the local time L_t^x and $X_u(v)=\int_u^v Y_x\,\mathrm{d}x$, where for some $\alpha>0$, $(Y_x)_{x\in\mathbb{R}}$ is a uniformly bounded, almost surely α-Hölder continuous stochastic process adapted to $(L_t^x)_{x\in\mathbb{R}}$. Then, for any positive integer $q\ge 2$ and $x\in\mathbb{R}$, it holds that*

$$\begin{aligned}
&\mathrm{E}\left[\left|\frac{1}{h^{q+1}}\int_{-\infty}^{x} J_{q-1}(X_{u-h},u-h,u)^2\,\mathrm{d}\langle M,M\rangle_u\right.\right.\\
&\qquad\left.\left.-\frac{2^{2q+1}}{(q+1)!}\int_{-\infty}^{x}(L_t^u)^q Y_u^2\,\mathrm{d}u\right|\right]\to 0.
\end{aligned}\tag{3.22}$$

PROOF. Throughout the proof, the first argument of J_{q-1} will not change. For better readability, we will drop it and for example write $J_{q-1}(u-h,u)$ instead of $J_{q-1}(X_{u-h},u-h,u)$. As before, C and C_p denote positive constants, the latter depending on p, which might change from line to line. By Itô's formula,

$$\begin{aligned}J_{q-1}(u-h,u)^2 &= \left(\int_{u-h}^{u} J_{q-2}(u-h,v)\,\mathrm{d}M_v\right)^2\\ &= 2\int_{u-h}^{u} J_{q-1}(u-h,v)J_{q-2}(u-h,v)\,\mathrm{d}M_v\\ &\quad + \int_{u-h}^{u} J_{q-2}(u-h,v)^2\,\mathrm{d}\langle M,M\rangle_v.\end{aligned}$$

Recursively, this yields the identity

$$J_{q-1}(u-h,u)^2 = 2\sum_{j=0}^{q-2} K_j(U_{j,q-1,u-h},u-h,u) + K_{q-1}\big(X_{u-h}^2,u-h,u\big),$$

where

$$U_{j,q-1,u-h}(v) = \int_{u-h}^{v} J_{q-1-j}(u-h,v)J_{q-2-j}(u-h,v)\,\mathrm{d}M_v.$$

Let us first show that, for $0\le j\le q-2$,

$$\mathrm{E}\left[\frac{1}{h^{q+1}}\left|\int_{-\infty}^{x} K_j(U_{j,q-1,u-h},u-h,u)\,\mathrm{d}\langle M,M\rangle_u\right|\right]\to 0 \tag{3.23}$$

and then, in a second step, that

$$\begin{aligned}\mathrm{E}\Bigg[\Bigg|&\frac{1}{h^{q+1}}\int_{-\infty}^{x} K_{q-1}\big(X_{u-h}^2,u-h,u\big)\,\mathrm{d}\langle M,M\rangle_u\\ &- \frac{2^{2q+1}}{(q+1)!}\int_{-\infty}^{x}(L_t^u)^q Y_u^2\,\mathrm{d}u\Bigg|\Bigg]\to 0\end{aligned} \tag{3.24}$$

as $h\to 0$.

Step 1. It holds that $|X_u(v)|\le C(v-u)$, and thus, by Lemma 3.5,

$$\|J_q(u-h,u)\|_p \le C_p h^{q/2+1}\|L_t^*\|_{2^q p}^{q/2}. \tag{3.25}$$

Together with Burkholder–Davis–Gundy and Jensen's inequality, this implies for $p\ge 2$ that

$$\begin{aligned}&\|U_{j,q-1,u_1-h}(u_1)\|_p\\ &\quad\le C_p\left\|\left(\int_{u_1-h}^{u_1} J_{q-1-j}(u_1-h,u_2)^2 J_{q-2-j}(u_1-h,u_2)^2 L_t^{u_2}\,\mathrm{d}u_2\right)^{1/2}\right\|_p\\ &\quad\le C_p h^{\frac12-\frac1p}\end{aligned}$$

$$\times\left(\int_{u_1-h}^{u_1}\mathrm{E}\big[\big|J_{q-1-j}(u_1-h,u_2)J_{q-2-j}(u_1-h,u_2)\big|^p(L_t^{u_2})^{p/2}\big]\,\mathrm{d}u_2\right)^{1/p} \tag{3.26}$$
$$\le C_p h^{\frac{1}{2}-\frac{1}{p}}\|L_t^*\|_p^{1/2}$$
$$\times\left(\int_{u_1-h}^{u_1}\|J_{q-1-j}(u_1-h,u_2)\|_{4p}^p\|J_{q-2-j}(u_1-h,u_2)\|_{4p}^p\,\mathrm{d}u_2\right)^{1/p}$$
$$\le C_p h^{q-j+1}\|L_t^*\|_{2^{q+1-j}p}^{q-j-1}.$$

Using this result, Jensen's inequality and Lemma 3.5 gives

$$\mathrm{E}\bigg[\bigg|\int_{-\infty}^{x}K_j(U_{j,q-1,u-h},u-h,u)\,\mathrm{d}\langle M,M\rangle_u\bigg|\bigg]$$
$$=4\int_{-\infty}^{x}\mathrm{E}\big[\big|K_j(U_{j,q-1,u-h},u-h,u)L_t^u\big|\big]\,\mathrm{d}u \tag{3.27}$$
$$\le C_p h^{q+1}\|L_t^*\|_{2^{q+2}}^{q+1}\int_{-\infty}^{x}\|L_t^u\|_2\,\mathrm{d}u,$$

which shows that the expectation appearing in (3.23) is bounded. Before continuing to show that it actually vanishes in the limit, let us informally describe the technique we are going to use. It is an important observation that if instead of one of the local times $L_t^{u_j}$ inside of K_j or $U_{j,q-1,u_1-h}$ we would have encountered a difference of the form $L_t^{u_j}-L_t^{u_j-a}$ in the above calculations, where $|a|<h$, then the corresponding norm of L_t^*, appearing as a factor in the bound on the right-hand side of (3.27) would instead be the norm of the increment and could thus be bounded by h^ε for any $\varepsilon\in(0,1/2)$, increasing the order of the right-hand side of (3.27) to $h^{q+1+\varepsilon}$. Mutatis mutandis, the same argument is valid for the Hölder continuous process Y_u. By linearity of the (stochastic) integral, this reasoning allows us to replace $L_t^{u_j}$ by $L_t^{u_j-a}+(L_t^{u_j}-L_t^{u_j-a})$ and Y_{u_j} by $Y_{u_j-a}+(Y_{u_j}-Y_{u_j-a})$ at any place in the calculations above, at the cost of introducing a negligible summand. In what follows, we will make frequent use of this fact, in order to nudge the processes occurring in the iterations of K_j back in space and make them adapted to the Brownian motion driving the stochastic integral inside $U_{j,q-1,u_1-h}$. This enables us to iteratively apply the stochastic Fubini theorem and bring this Brownian motion to the very outside, effectively replacing the stochastic differential by a deterministic one and increasing the order of convergence by a factor of $h^{1/2}$, which suffices to conclude that (3.22) holds. Restating these arguments in a more rigorous way, we claim that for $0\le j\le q-1$, the L^1-norms of the integrals

$$\int_{-\infty}^{x}K_j(U_{j,q,u-h},u-h,u)\,\mathrm{d}\langle M,M\rangle_u$$
$$=4\int_{-\infty}^{x}K_j(U_{j,q,u-h},u-h,u)L_t^u\,\mathrm{d}u \tag{3.28}$$

are of order $o(h^{q+1-\varepsilon})$ for any $\varepsilon \in (0, 1/2)$. To prove this claim for $j = 0$, note that

$$
\begin{aligned}
&\int_{-\infty}^{x} K_0(U_{0,q-1,u_1-h}, u_1 - h, u_1) L_t^{u_1} \,\mathrm{d}u_1 \\
&\quad = \int_{-\infty}^{x} U_{0,q-1,u_1-h}(u_1) L_t^{u_1} \,\mathrm{d}u_1 \\
&\quad = \int_{-\infty}^{x} U_{0,q-1,u_1-h}(u_1) L_t^{u_1-h} \,\mathrm{d}u_1 \\
&\qquad + \int_{-\infty}^{x} U_{0,q-1,u_1-h}(u_1) \big(L_t^{u_1} - L_t^{u_1-h}\big) \,\mathrm{d}u_1.
\end{aligned} \tag{3.29}
$$

For the first integral on the right-hand side of (3.29), stochastic Fubini yields

$$
\begin{aligned}
&\int_{-\infty}^{x} U_{0,q-1,u_1-h}(u_1) L_t^{u_1-h} \,\mathrm{d}u_1 \\
&\quad = \int_{-\infty}^{x} \int_{u_1-h}^{u_1} J_{q-1}(u_1 - h, u_2) J_{q-2}(u_1 - h, u_2) \,\mathrm{d}M_{u_2} L_t^{u_1-h} \,\mathrm{d}u_1 \\
&\quad = \int_{-\infty}^{x} \int_{u_2}^{(u_2+h)\wedge x} J_{q-1}(u_1 - h, u_2) J_{q-2}(u_1 - h, u_2) L_t^{u_1-h} \,\mathrm{d}u_1 \,\mathrm{d}M_{u_2}.
\end{aligned}
$$

A straightforward application of the Burkholder–Davis–Gundy, Cauchy–Schwarz and Jensen inequalities, as well as Lemma 3.5, thus yields so that by Burkholder–Davis–Gundy and deterministic Fubini,

$$
\begin{aligned}
&\mathrm{E}\bigg[\bigg|\int_{-\infty}^{x} U_{0,q-1,u_1-h}(u_1) L_t^{u_1-h} \,\mathrm{d}u_1\bigg|\bigg] \\
&\quad = \mathrm{E}\bigg[\bigg|\int_{-\infty}^{x} \int_{u_2}^{(u_2+h)\wedge x} J_q(u_1 - h, u_2) J_{q-1}(u_1 - h, u_2) L_t^{u_1-h} \,\mathrm{d}u_1 \,\mathrm{d}M_{u_2}\bigg|\bigg] \\
&\quad \le C_p h^{2q+3/2} \|L_t^*\|_{2^{q+1}}^{2q+3/2} \bigg(\int_{-\infty}^{x} \|L_t^{u_2}\|_2 \,\mathrm{d}u_2\bigg)^{1/2}.
\end{aligned}
$$

To treat the second integral on the right-hand side of (3.29), use Cauchy–Schwarz to get

$$
\begin{aligned}
&\mathrm{E}\bigg[\bigg|\int_{-\infty}^{x} U_{0,q-1,u_1-h}(u_1) \big(L_t^{u_1} - L_t^{u_1-h}\big) \,\mathrm{d}u_1\bigg|\bigg] \\
&\quad \le \mathrm{E}\bigg[h \int_{-\infty}^{x} \big(U_{0,q-1,u_1-h}(u_1)\big)^2 \,\mathrm{d}u_1\bigg]^{1/2} \\
&\qquad \times \mathrm{E}\bigg[\frac{1}{h} \int_{-\infty}^{x} \big(L_t^{u_1} - L_t^{u_1-h}\big)^2 \,\mathrm{d}u_1\bigg]^{1/2}.
\end{aligned} \tag{3.30}
$$

By [14], Theorem 1.1, the second expectation converges to a bounded quantity. Itô's formula, applied to the integrand of the first, gives

$$\begin{aligned}
&U_{0,q-1,u_1-h}(u_1)^2 \\
&\quad = \left(\int_{u_1-h}^{u_1} J_{q-1}(u_1-h,u_2) J_{q-2}(u_1-h,u_2)\,\mathrm{d}M_{u_2}\right)^2 \\
&\quad = 2\int_{u_1-h}^{u_1} U_{0,q-1,u_1-h}(u_2) J_{q-1}(u_1-h,u_2) J_{q-2}(u_1-h,u_2)\,\mathrm{d}M_{u_2} \\
&\qquad + \int_{u_1-h}^{u_1} J_{q-1}(u_1-h,u_2)^2 J_{q-2}(u_1-h,u_2)^2\,\mathrm{d}\langle M,M\rangle_{u_2} \\
&\quad = V_1 + V_2.
\end{aligned}$$

By stochastic Fubini, we obtain that

$$\begin{aligned}
\mathrm{E}\left[\int_{-\infty}^{x} V_1\,\mathrm{d}u_1\right] = 2\,\mathrm{E}\Bigg[\int_{-\infty}^{x}\int_{u_2}^{u_2+h} & U_{0,q-1,u_1-h}(u_2) \\
& \times J_{q-1}(u_1-h,u_2) J_{q-2}(u_1-h,u_2)\,\mathrm{d}u_1\,\mathrm{d}M_{u_2}\Bigg].
\end{aligned}$$

Thus, after straightforward application of Burkholder–Davis–Gundy and Jensen's inequality, the estimates (3.25) and (3.26) yield the bound

$$\mathrm{E}\left[\int_{-\infty}^{x} V_1\,\mathrm{d}u_1\right] \le Ch^{2q+5/2}\|L_t^*\|_{2^{q+5}}^{2(q+r-2)}\left(\int_{-\infty}^{x}\|L_t^{u_2}\|_2\,\mathrm{d}u_2\right)^{1/2}.$$

The same arguments also work for $\mathrm{E}[\int_{-\infty}^{x}(-V_1)\,\mathrm{d}u_1]$, so that we obtain

$$\mathrm{E}\left[\left|\int_{-\infty}^{x} V_1\,\mathrm{d}u_1\right|\right] \le Ch^{2q+5/2}\|L_t^*\|_{2^{q+5}}^{2(q+r-2)}\left(\int_{-\infty}^{x}\|L_t^{u_2}\|_2\,\mathrm{d}u_2\right)^{1/2}.$$

Similarly,

$$\mathrm{E}\left[\left|\int_{-\infty}^{x} V_2\,\mathrm{d}u_1\right|\right] \le Ch^{2q+2}\|L_t^*\|_{2^{q+2}}^{2q+4r-3}\int_{-\infty}^{x}\|L_t^{u_2}\|_2\,\mathrm{d}u_2.$$

Plugged back into (3.30), we see that

$$\mathrm{E}\left[\left|\int_{-\infty}^{x} U_{0,q-1,u_1-h}(u_1)\big(L_t^{u_1}-L_t^{u_1-h}\big)\,\mathrm{d}u_1\right|\right]$$

is of order $\mathcal{O}(h^{q+3/2})$, concluding the proof for $j=0$. To obtain the asymptotic order of (3.28) for $j\ge 1$, we write

$$\begin{aligned}
&\int_{u_1-h}^{u_j} U_{j,q-1,u_1-h}(u_{j+1}) L_t^{u_{j+1}}\,\mathrm{d}u_{j+1} \\
&\qquad = \int_{u_1-h}^{u_j} U_{j,q-1,u_1-h}(u_{j+1}) L_t^{u_{l+1}-u_j+u_1-h}\,\mathrm{d}u_{j+1} + R_h,
\end{aligned} \tag{3.31}$$

where

$$R_h = \int_{u_1-h}^{u_j} U_{j,q-1,u_1-h}(u_{j+1})\big(L_t^{u_{j+1}} - L_t^{u_{l+1}-u_j+u_1-h}\big)\,\mathrm{d}u_{j+1}.$$

Note that R_h, when plugged back into the integral (3.28), introduces a negligible summand [see (3.27) and the arguments afterward]. The stochastic process

$$u_{j+2} \mapsto \left(\int_{u_{j+2}}^{u_j} L_t^{u_{j+1}-u_j+u_1-h}\,\mathrm{d}u_{j+1}\right), \qquad u_1-h \le u_{j+2} \le u_j$$

is by construction adapted to $(\mathcal{F}_{u_{j+2}})_{u_1-h\le u_{j+2}\le u_j}$ as $u_{j+1}-u_j+u_1-h \le u_1-h$, where $(\mathcal{F}_x)_{x\in\mathbb{R}}$ denotes the filtration of the underlying probability space. Therefore, we can apply stochastic Fubini and get

$$\begin{aligned}
&\int_{u_1-h}^{u_j} U_{j,q-1,u_1-h}(u_{j+1}) L_t^{u_{j+1}-u_j+u_1-h}\,\mathrm{d}u_{j+1}\\
&\quad= \int_{u_1-h}^{u_j}\int_{u_1-h}^{u_{j+1}} J_{q-1-j}(u_1-h,u_{j+2})\\
&\qquad \times J_{q-2-j}(u_1-h,u_{j+2})\,\mathrm{d}M_{u_{j+2}} L_t^{u_{j+1}-u_j+u_1-h}\,\mathrm{d}u_{j+1}\\
&\quad= \int_{u_1-h}^{u_j}\int_{u_{j+2}}^{u_j} L_t^{u_{j+1}-u_j+u_1-h}\,\mathrm{d}u_{j+1} J_{q-1-j}(u_1-h,u_{j+2})\\
&\qquad \times J_{q-2-j}(u_1-h,u_{j+2})\,\mathrm{d}M_{u_{j+2}}.
\end{aligned}$$

Iterating this procedure of interchanging a deterministic and a stochastic integral at the cost of introducing negligible terms, we obtain that

$$\begin{aligned}
&K_j(U_{j,q-1,u_1-h},u_1-h,u_1)\\
&\quad= \int_{u_1-h}^{u_1}\cdots\int_{u_1-h}^{u_j} U_{j,q-1,u_1-h}(u_{j+1}) L_t^{u_{j+1}}\,\mathrm{d}u_{j+1}\cdots L_t^{u_1}\,\mathrm{d}u_2\\
&\quad= \int_{u_1-h}^{u_1}\int_{u_{j+2}}^{u_1}\int_{u_{j+2}}^{u_2}\cdots\int_{u_{j+2}}^{u_j} L_t^{u_{j+1}-u_j+u_1-h}\,\mathrm{d}u_{j+1}\cdots\\
&\qquad \times L_t^{u_3-u_2+u_1-h}\,\mathrm{d}u_{j_3} L_t^{u_2-u_1+u_1-h}\,\mathrm{d}u_{j_2} J_{q-1-j}(u_1-h,u_{j+2})\\
&\qquad \times J_{q-2-j}(u_1-h,u_{j+2})\,\mathrm{d}M_{u_{j+2}} + R_h,
\end{aligned}$$

where $\mathrm{E}[|R_h|] = o(h^{q+1})$. Consequently, by another application of stochastic Fubini,

$$\begin{aligned}
&\int_{-\infty}^{x} K_j(U_{j,q-1,u_1},u_1-h,u_1) L_t^{u_1}\,\mathrm{d}u_1\\
&\quad= \int_{-\infty}^{x}\int_{u_1-h}^{u_1}\int_{u_{j+2}}^{u_1}\int_{u_{j+2}}^{u_2}\cdots\int_{u_{j+2}}^{u_j} L_t^{u_{j+1}-u_j+u_1-h}\,\mathrm{d}u_{j+1}\cdots
\end{aligned}$$

$$
\begin{aligned}
&\times L_t^{u_3-u_2+u_1-h}\,\mathrm{d}u_{j_3}L_t^{u_2-u_1+u_1-h}\,\mathrm{d}u_{j_2}J_{q-1-j}(u_1-h,u_{j+2})\\
&\times J_{q-2-j}(u_1-h,u_{j+2})\,\mathrm{d}M_{u_{j+2}}L_t^{u_1}\,\mathrm{d}u_1+R_h \qquad (3.32)\\
&=\int_{-\infty}^{x}\int_{u_{j+2}}^{(u_{j+2}+h)\wedge x}\int_{u_{j+2}}^{u_1}\int_{u_{j+2}}^{u_2}\cdots\int_{u_{j+2}}^{u_j}L_t^{u_{j+1}-u_j+u_1-h}\,\mathrm{d}u_{j+1}\cdots\\
&\times L_t^{u_3-u_2+u_1-h}\,\mathrm{d}u_{j_3}L_t^{u_2-u_1+u_1-h}\,\mathrm{d}u_{j_2}J_{q-1-j}(u_1-h,u_{j+2})\\
&\times J_{q-2-j}(u_1-h,u_{j+2})L_t^{u_1}\,\mathrm{d}u_1\,\mathrm{d}M_{u_{j+2}}+\widetilde{R}_h,
\end{aligned}
$$

where again, $\mathrm{E}[|\widetilde{R}_h|]=o(h^{q+1})$. From here, a tedious but straightforward application of Burkholder–Davis–Gundy, Jensen's inequality and Lemma 3.5, in the same way as we have done to treat the case $q=0$, yields that the L^1-norm of the iterated integral (3.32) is of order $o(h^{q+1+\varepsilon})$ for any $\varepsilon\in(0,1/2)$.

Step 2. As $|X_{r,u}(v)|\le C(v-u)$, Lemma 3.5 implies that

$$
\|K_{q-1}(X^2_{r,u-h},u-h,u)\|_2\le Ch^{q+1}\|L_t^*\|_{2^q}^{q-1}
$$

and thus, by Fubini,

$$
\begin{aligned}
&\mathrm{E}\biggl[\biggl|\int_{-\infty}^{x}K_{q-1}(X^2_{r,u-h},u-h,u)\,\mathrm{d}\langle M,M\rangle_u\biggr|\biggr]\\
&\quad=4\int_{-\infty}^{x}\mathrm{E}\bigl[|K_{q-1}(X^2_{r,u-h},u-h,u)L_t^u|\bigr]\,\mathrm{d}u\\
&\quad\le 4\int_{-\infty}^{x}\|K_{q-1}(X^2_{r,u-h},u-h,u)\|_2\|L_t^u\|_2\,\mathrm{d}u\\
&\quad\le Ch^{q+1}\|L_t^*\|_{2^q}^{q-1}\int_{-\infty}^{x}\|L_t^u\|_2\,\mathrm{d}u.
\end{aligned}
$$

Reasoning as in Step 1, we see that replacing one of the local times inside K_q or $X_{r,u-h}$ by a difference $L_t^u-L_t^{u-a}$ such that $|a|<h$ has the effect of replacing one power of $\|L_t^*\|$ by h^ε, and consequently introducing an additional, negligible summand. To exhibit the asymptotic behaviour of

$$
\begin{aligned}
&K_{q-1}(X^2_{r,u_1-h},u_1-h,u_1)\\
&\quad=\int_{u_1-h}^{u_1}\cdots\int_{u_1-h}^{u_{q-2}}\int_{u_1-h}^{u_{q-1}}\\
&\qquad\times\biggl(\int_{u_1-h}^{u_q}Y_{u_{q+1}}\,\mathrm{d}u_{q+1}\biggr)^2\mathrm{d}\langle M,M\rangle_{u_q}\,\mathrm{d}\langle M,M\rangle_{u_{q-1}}\cdots\mathrm{d}\langle M,M\rangle_{u_2} \qquad (3.33)\\
&\quad=4^{q-1}\int_{u_1-h}^{u_1}\cdots\int_{u_1-h}^{u_{q-2}}\int_{u_1-h}^{u_{q-1}}\biggl(\int_{u_1-h}^{u_q}Y_{u_{q+1}}\,\mathrm{d}u_{q+1}\biggr)^2\\
&\qquad\times L_t^{u_q}\,\mathrm{d}u_qL_t^{u_{q-1}}\,\mathrm{d}u_{q-1}\cdots L_t^{u_2}\,\mathrm{d}u_2,
\end{aligned}
$$

note that for two real numbers a and b, it holds that

$$a^r - b^r = \sum_{k=1}^{r} a^{r-k}(a-b)b^{k-1}.$$

Thus, setting $a = Y_{u_{q+1}}$, $b = Y_{u_1}$ and exploiting the Hölder continuity of Y, we can replace the innermost integral $\int_{u_1-h}^{u_q} Y_{u_{q+1}} \,\mathrm{d}u_{q+1}$ in (3.33) by $Y_{u_1}(u_{q+1} - u_1 + h)$, at the cost of introducing negligible summands. In formulas, up to negligible summands, the right-hand side of (3.33) is equal to

$$4^{q-1} Y_{u_1}^2 \int_{u_1-h}^{u_1} \cdots \int_{u_1-h}^{u_{q-2}} \int_{u_1-h}^{u_{q-1}} (u_{q+1} - u_1 + h)^2 L_t^{u_q} \,\mathrm{d}u_q L_t^{u_{q-1}} \,\mathrm{d}u_{q-1} \cdots L_t^{u_2} \,\mathrm{d}u_2.$$

Repeating this procedure, iteratively replacing $L_t^{u_q}$, $L_t^{u_{q-1}}$, etc. and evaluating the resulting purely deterministic integral, we see that

$$K_{q-1}(X_{u_1-h}^2, u_1 - h, u_1) = \frac{2^{2q-1}}{(q+1)!} h^{q+1} (L_t^{u_1})^{q-1} Y_{u_1}^2 + o(h^{q+1}).$$

Consequently,

$$\frac{1}{h^{q+1}} \int_{-\infty}^{x} K_{q-1}(X_{u-h}^2, u-h, u) \,\mathrm{d}\langle M, M\rangle_u = \frac{2^{2q+1}}{(q+1)!} \int_{-\infty}^{x} (L_t^u)^q Y_u^2 \,\mathrm{d}u + o(1),$$

completing the proof. $\square$

4. Main result. We now have all necessary tools at our disposal to prove Theorem 1.2 stated in the Introduction. We will proceed in two steps. First, in the forthcoming Theorem 4.1, we will prove a limit theorem for a certain iterated integral, which, in the proof of Theorem 1.2, will turn out to be the leading term when applying the Kailath–Segall identity.

THEOREM 4.1. *Let M_x be the local martingale part of Brownian local time L_t^x and I_q be the iterated integrals with respect to M_x. Then, for any integer $q \geq 2$ it holds that*

$$\frac{q!}{h^{(q+1)/2}} \int_{-\infty}^{\infty} I_q(x, x+h) \,\mathrm{d}x \to c_q \sqrt{\int_{-\infty}^{\infty} (L_t^x)^q \,\mathrm{d}x}\, Z, \tag{4.1}$$

where $Z \sim \mathcal{N}(0,1)$, independent of $(L_t^x)_{x\in\mathbb{R}}$ and the constant c_q is given by

$$c_q = \frac{2^{2q+1} q!}{q+1}.$$

PROOF. By definition, we have that

$$\int_{-\infty}^{\infty} I_q(x, x+h) \,\mathrm{d}x = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 1_{(x,x+h)}(u) I_{q-1}(x, u) \,\mathrm{d}M_u \,\mathrm{d}x. \tag{4.2}$$

If we set

$$\phi(x,u) = 1_{(x,x+h)}(u) I_{q-1}(x,u),$$

then, by the deterministic Fubini theorem, Cauchy–Schwarz, Jensen's inequality and Lemma 3.5 [recall that $I_q(x,y) = J_q(1,x,y)$],

$$\begin{aligned}
&\mathrm{E}\Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,u)^2 \,\mathrm{d}x \,\mathrm{d}\langle M, M\rangle_u\Big] \\
&\quad = 4\,\mathrm{E}\Big[\int_{-\infty}^{\infty}\int_{u-h}^{u} I_{q-1}(x,u)^2 \,\mathrm{d}x\, L_t^u \,\mathrm{d}u\Big] \\
&\quad \le 4\int_{-\infty}^{\infty}\Big\|\int_{u-h}^{u} I_{q-1}(x,u)^2 \,\mathrm{d}x\Big\|_2 \|L_t^u\|_2 \,\mathrm{d}u \\
&\quad \le 4\int_{-\infty}^{\infty}\Big\|\int_{u-h}^{u} I_{q-1}(x,u)^2 \,\mathrm{d}x\Big\|_2 \|L_t^u\|_2 \,\mathrm{d}u \\
&\quad \le C\int_{-\infty}^{\infty} \|L_t^u\|_2 \,\mathrm{d}u < \infty.
\end{aligned}$$

This shows that we can apply the stochastic Fubini theorem to the right-hand side of (4.2) and also that

$$\int_{-\infty}^{\cdot}\int_{-\infty}^{\infty} \phi(x,u)\,\mathrm{d}x\,\mathrm{d}M_u$$

is a square integrable martingale (which in particular possesses a limit). Therefore, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} I_q(x,x+h)\,\mathrm{d}x &= \int_{-\infty}^{\infty}\int_{u_1-h}^{u_1} I_{q-1}(x,u_1)\,\mathrm{d}M_{u_1}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\int_{u_1-h}^{u_1}\int_{x}^{u_1} I_{q-2}(x,u_2)\,\mathrm{d}M_{u_2}\,\mathrm{d}x\,\mathrm{d}M_{u_1}.
\end{aligned}$$

Iterating this procedure another $q-1$-times, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} I_q(x,x+h)\,\mathrm{d}x &= \int_{-\infty}^{\infty}\int_{u_1-h}^{u_1}\cdots\int_{u_1-h}^{u_{q-1}}\int_{u_1-h}^{u_q} \mathrm{d}x\,\mathrm{d}M_{u_q}\cdots\mathrm{d}M_{u_2}\,\mathrm{d}M_{u_1} \\
&= \int_{-\infty}^{\infty} J_{q-1}(X_u, u-h, u)\,\mathrm{d}M_u,
\end{aligned}$$

where

$$X_u(v) = v - u + h.$$

By Lemma 3.5, the process $(\widetilde{M}_x^h)_{x\ge -\infty}$, defined by

$$\widetilde{M}_x^h = \frac{q!}{h^{(q+1)/2}}\int_{-\infty}^{x} J_{q-1}(X_u, u-h, u)\,\mathrm{d}M_u,$$

is a L^p-bounded, uniformly integrable martingale for any $h > 0$, which vanishes at $-\infty$. Moreover, Lemma 3.6 yields that for $x_0 > -\infty$

$$\sup_{x\in(-\infty,x_0]} |\langle \widetilde{M}^h, M\rangle_x| \to 0$$

and Lemma 3.7 shows that for $x \in \mathbb{R} \cup \{-\infty, \infty\}$,

$$\langle \widetilde{M}^h, \widetilde{M}^h\rangle_x \to c_q^2 \int_{-\infty}^{x} (L_t^u)^q \,\mathrm{d}u,$$

where both convergences hold in L^1 [and we define $\int_{-\infty}^{-\infty} f(u)\,\mathrm{d}u = 0$]. Consequently, the asymptotic Ray–Knight Theorem 2.2 implies that

$$\widetilde{M}_x^h \xrightarrow{d} c_q \sqrt{\int_{-\infty}^{x} (L_t^x)^q \,\mathrm{d}x}\, Z \tag{4.3}$$

for $h \to 0$, where $Z \sim \mathcal{N}(0, 1)$, independent of $(M_x)_{x\in\mathbb{R}}$ [and thus also of $(L_t^x)_{x\in\mathbb{R}}$ and the underlying Brownian motion]. Indeed, if β and β^h denote the Dambis–Dubins–Schwarz Brownian motions of M and $\widetilde{M}^h$, respectively, Theorem 2.2 yields that

$$(\beta, \beta^h, \langle \widetilde{M}^h, \widetilde{M}^h\rangle) \xrightarrow{d} \left(\beta, \widetilde{\beta}, c_q^2 \int_{-\infty}^{\cdot} (L_t^u)^q \,\mathrm{d}u\right),$$

where $\widetilde{\beta}$ is a standard Brownian motion which is independent of β. Consequently,

$$\widetilde{M}_x^h = \beta^h_{\langle \widetilde{M}^h, \widetilde{M}^h\rangle_x} \xrightarrow{d} \widetilde{\beta}_{c_q^2 \int_{-\infty}^{x} (L_t^u)^q \,\mathrm{d}u}.$$

Letting x tend to infinity completes the proof. $\square$

We now turn to the proof of Theorem 1.2 from the Introduction.

PROOF OF THEOREM 1.2. Let $L_t^x = M_x + V_x$ be the canonical semimartingale decomposition of Brownian local time (we suppress the dependence of the fixed parameter t for brevity) and I_q the iterated integrals with respect to the local martingale M_x. Throughout the proof, we use the shorthand notation $I_q^h(x) = I_q(x, x+h)$, $\Delta_x^h L_t^x = L_t^{x+h} - L_t^x$ and $\Delta_x^h V_{t,x} = V_{t,x+h} - V_{t,x}$. By the binomial theorem and the fact that $M_{x+h} - M_x = I_1^h(x)$, we get that

$$(\Delta_x^h L_t^x)^q = (I_1^h(x) + \Delta_x^h V_{t,x})^q = \sum_{k=0}^{q} \binom{q}{k} (I_1^h(x))^{q-k} (\Delta_x^h V_{t,x})^k. \tag{4.4}$$

As by Burkholder–Davis–Gundy,

$$\begin{aligned}\|I_1^h(x)^{q-2k}\|_p &\le C_p \left\| \left(4 \int_x^{x+h} L_t^u \,\mathrm{d}u\right)^{1/2} \right\|_{(q-2k)p}^{q-2k} \\ &\le C_p h^{(q-2k)/2} \|L_t^*\|_p^{(q-2k)/2}\end{aligned}$$

and, writing $\Delta_x^h V_x = \int_x^{x+h} A_u \,\mathrm{d}u$,

$$\left\| \left(\frac{\Delta_x^h V_x}{h} \right)^k \right\|_p \leq \left\| \left(\frac{1}{h} \int_x^{x+h} |A_u| \,\mathrm{d}u \right)^k \right\|_p \xrightarrow{h \to 0} \|A_u\|_{kp}^k$$

we see that

$$h^{-(q+k)/2} \,\mathrm{E}\left[\int_{-\infty}^{\infty} |I_1^h(x)|^{q-k} |\Delta_x^h V_x|^k \,\mathrm{d}x \right] < \infty.$$

Thus, all those summands in the sum on the right-hand side of (4.4) for which $k > 1$ do not contribute to the limit. To be more precise, it holds that

$$\begin{aligned} &\int_{-\infty}^{\infty} (\Delta_x^h L_t^x)^q \,\mathrm{d}x \\ &\quad = \int_{-\infty}^{\infty} (I_1^h(x))^q \,\mathrm{d}x + q \int_{-\infty}^{\infty} (I_1^h(x))^{q-1} (\Delta_x^h V_x) \,\mathrm{d}x + R_{1,h}, \end{aligned} \tag{4.5}$$

where $R_{1,h}/h^{(q+1)/2}$ converges to zero in L^p for $h \to 0$. The Kailath–Segall identity (3.4) and another application of the binomial theorem yields

$$\begin{aligned} &q! I_q^h(x) - (I_1^h(x))^q \\ &\quad = \sum_{k=1}^{\lfloor \frac{q}{2} \rfloor} a_{q,k} (I_1^h(x))^{q-2k} \left(4 \int_x^{x+h} L_t^u \,\mathrm{d}u \right)^k \\ &\quad = \sum_{k=1}^{\lfloor \frac{q}{2} \rfloor} a_{q,k} (\Delta_x^h L_t^x - \Delta_x^h V_{t,x})^{q-2k} \left(4 \int_x^{x+h} L_t^u \,\mathrm{d}u \right)^k \\ &\quad = \delta_{\lfloor \frac{q}{2} \rfloor, \frac{q}{2}} a_{\frac{q}{2},k} \left(4 \int_x^{x+h} L_t^u \,\mathrm{d}u \right)^{q/2} \\ &\qquad + \sum_{k=1}^{\lfloor \frac{q-1}{2} \rfloor} \sum_{j=0}^{q-2k} (-1)^j a_{q,k} \binom{q-2k}{j} \\ &\qquad \times (\Delta_x^h L_t^x)^{q-2k-j} (\Delta_x^h V_{t,x})^j \left(4 \int_x^{x+h} L_t^u \,\mathrm{d}u \right)^k, \end{aligned} \tag{4.6}$$

where $\delta_{\lfloor \frac{q}{2} \rfloor, \frac{q}{2}} = 1$ if q is even and zero otherwise. By the Hölder continuity property of the Brownian local time, $\Delta_x^h L_t^x < h^\varepsilon$ for any $\varepsilon \in (0, 1/2)$. Therefore, a similar argument as above shows that if we first multiply (4.6) by $(L_t^x)^r$ and then integrate on both sides, all summands in the double sum for which $\frac{q-2k-j}{2} + k + j > \frac{q+1}{2}$, i.e. for which $j > 1$ do not contribute to the limit. In formulas, we have

$$
\begin{aligned}
&q!\int_{-\infty}^{\infty} I_q^h(x)\,\mathrm{d}x - \int_{-\infty}^{\infty} (I_1^h(x))^q\,\mathrm{d}x \\
&\quad= \sum_{k=1}^{\lfloor \frac{q}{2}\rfloor} a_{q,k}\int_{-\infty}^{\infty} (\Delta_x^h L_t^x)^{q-2k}\left(4\int_x^{x+h} L_t^u\,\mathrm{d}u\right)^k \mathrm{d}x \\
&\qquad - \sum_{k=1}^{\lfloor \frac{q-1}{2}\rfloor} a_{q,k}(q-2k) \\
&\qquad \times \int_{-\infty}^{\infty} (\Delta_x^h L_t^x)^{q-2k-1}\Delta_x^h V_x\left(4\int_x^{x+h} L_t^u\,\mathrm{d}u\right)^k \mathrm{d}x + R_{2,h},
\end{aligned}
\tag{4.7}
$$

where $R_{2,h}/h^{(q+1)/2}$ converges to zero in L^p for $h \to 0$. Analogously, we derive that

$$
\begin{aligned}
&q!\int_{-\infty}^{\infty} I_{q-1}^h(x)(\Delta_x^h V_x)\,\mathrm{d}x - q\int_{-\infty}^{\infty} (I_1^h(x))^{q-1}\Delta_x^h V_x (L_t^x)^r\,\mathrm{d}x \\
&\quad= q\sum_{k=1}^{\lfloor \frac{q-1}{2}\rfloor} a_{q-1,k}\int_{-\infty}^{\infty} (\Delta_x^h L_t^x)^{q-1-2k}\Delta_x^h V_x\left(4\int_x^{x+h} L_t^u\right)^k \mathrm{d}x \\
&\qquad + R_{3,h},
\end{aligned}
\tag{4.8}
$$

where $R_{3,h}/h^{(q+1)/2}$ converges to zero in L^p for $h \to 0$. If we now plug (4.7) and (4.8) into (4.5) and exploit the identity $qa_{q-1,k} - (q-2k)a_{q,k} = 0$ for $q \ge 2$ and $1 \le k \le \lfloor \frac{q-1}{2}\rfloor$ (which can be shown by straightforward induction), we obtain that

$$
\begin{aligned}
\int_{-\infty}^{\infty} (\Delta_x^h L_t^x)^q\,\mathrm{d}x = q!\int_{-\infty}^{\infty} I_q^h(x)\,\mathrm{d}x &\\
&- \sum_{k=1}^{\lfloor \frac{q}{2}\rfloor} a_{q,k}\int_{-\infty}^{\infty} (\Delta_x^h L_t^x)^{q-2k}\left(4\int_x^{x+h} L_t^u\,\mathrm{d}u\right)^k \mathrm{d}x \\
&+ q!\int_{-\infty}^{\infty} I_{q-1}^h(x)\Delta_x^h V_x\,\mathrm{d}x \\
&+ \widetilde{R}_h,
\end{aligned}
$$

where $\widetilde{R}_h/h^{(q+1)/2}$ converges to zero in L^p for $h \to 0$. By Theorem 4.1, the proof is complete if we can show that

$$
\frac{1}{h^{(q+1)/2}}\int_{-\infty}^{\infty} I_{q-1}^h(x)\Delta_x^h V_x\,\mathrm{d}x \xrightarrow{d} 0.
\tag{4.9}
$$

For $V_x = \int_{-\infty}^{x} A_u \,\mathrm{d}u$, we write

$$
\begin{aligned}
&\frac{1}{h^{(q+1)/2}} \int_{-\infty}^{\infty} I_{q-1}^h(x) \Delta_x^h V_x \,\mathrm{d}x \\
&\quad = \frac{1}{h^{(q-1)/2}} \int_{-\infty}^{\infty} I_{q-1}^h(x) A_x \,\mathrm{d}x \\
&\qquad + \int_{-\infty}^{\infty} \frac{I_{q-1}^h(x)}{h^{(q-1)/2}} \biggl(\frac{1}{h} \int_x^{x+h} A_u \,\mathrm{d}u - A_x \biggr) \mathrm{d}x.
\end{aligned} \tag{4.10}
$$

For the first integral on the right-hand side of (4.10), stochastic Fubini yields that

$$
\begin{aligned}
\frac{1}{h^{(q-1)/2}} \int_{-\infty}^{\infty} I_{q-1}^h(x) A_{t,x} \,\mathrm{d}x &= \frac{1}{h^{(q-1)/2}} \int_{-\infty}^{\infty} \int_x^{x+h} I_{q-2}(x,u) \,\mathrm{d}M_u A_x \,\mathrm{d}x \\
&= \frac{1}{h^{(q-1)/2}} \int_{-\infty}^{\infty} \int_{u-h}^{u} I_{q-2}(x,u) A_x \,\mathrm{d}x \,\mathrm{d}M_u.
\end{aligned}
$$

Thus, by Burkholder–Davis–Gundy and Jensen's inequality,

$$
\begin{aligned}
&\biggl\| \frac{1}{h^{(q-1)/2}} \int_{-\infty}^{\infty} \int_{u-h}^{u} I_{q-2}(x,u) A_x \,\mathrm{d}x \,\mathrm{d}M_u \biggr\|_2 \\
&\quad \le C \biggl\| \biggl(\int_{-\infty}^{\infty} \biggl(\frac{1}{h^{(q-1)/2}} \int_{u-h}^{u} I_{q-2}(x,u) A_x \,\mathrm{d}x \biggr)^2 L_t^u \,\mathrm{d}u \biggr)^{1/2} \biggr\|_2 \\
&\quad \le C \biggl(\int_{-\infty}^{\infty} \mathrm{E}\biggl[\biggl(\frac{1}{h^{(q-1)/2}} \int_{u-h}^{u} I_{q-2}(x,u) A_x \,\mathrm{d}x \biggr)^2 L_t^u \biggr] \mathrm{d}u \biggr)^{1/2} \\
&\quad \le C \biggl(\frac{1}{h^{q-1}} \int_{-\infty}^{\infty} \biggl\| \int_{u-h}^{u} I_{q-2}(x,u) A_x \,\mathrm{d}x \biggr\|_4^2 \| L_t^u \|_2 \,\mathrm{d}u \biggr)^{1/2}.
\end{aligned} \tag{4.11}
$$

By Cauchy–Schwarz, Jensen and Lemma 3.5, it follows that

$$
\begin{aligned}
&\biggl\| \int_{u-h}^{u} I_{q-2}(x,u) A_x \,\mathrm{d}x \biggr\|_4 \\
&\quad \le \biggl\| \biggl(\int_{u-h}^{u} I_{q-2}(x,u)^2 \,\mathrm{d}x \biggr)^{1/2} \biggl(\int_{u-h}^{u} A_x^2 \,\mathrm{d}x \biggr)^{1/2} \biggr\|_4 \\
&\quad \le \biggl\| \int_{u-h}^{u} I_{q-2}(x,u)^2 \,\mathrm{d}x \biggr\|_8^{1/2} \biggl\| \int_{u-h}^{u} A_x^2 \,\mathrm{d}x \biggr\|_8^{1/2} \\
&\quad \le \biggl(h^7 \int_{u-h}^{u} \| I_{q-2}(x,u) \|_{16}^{16} \,\mathrm{d}x \biggr)^{1/16} \biggl\| \int_{u-h}^{u} A_x^2 \,\mathrm{d}x \biggr\|_8^{1/2} \\
&\quad \le C h^{(q-1)/2} \| L_t^* \|_{2^{q+2}}^{(q-2)/2} \biggl\| \int_{u-h}^{u} A_x^2 \,\mathrm{d}x \biggr\|_8^{1/2},
\end{aligned}
$$

which, plugged into (4.11), yields

$$\left\| \frac{1}{h^{(q-1)/2}} \int_{-\infty}^{\infty} \int_{u-h}^{u} I_{q-2}(x,u) A_x \, \mathrm{d}x \, \mathrm{d}M_u \right\|_2$$
$$\leq C \| L_t^* \|_{2q+2}^{(q-2)/2} \left(\int_{-\infty}^{\infty} \left\| \int_{u-h}^{u} A_x^2 \, \mathrm{d}x \right\|_8 \| L_t^u \|_2 \, \mathrm{d}u \right)^{1/2}$$

and by the Vitali convergence theorem (see, e.g., [24], page 133), the integral on the right-hand side converges to zero.

Let us turn to the second integral on the right-hand side of (4.10). By deterministic Fubini,

$$\mathrm{E}\left[\left| \int_{-\infty}^{\infty} \frac{I_{q-1}^h(x)}{h^{(q-1)/2}} \left(\frac{1}{h} \int_x^{x+h} A_u \, \mathrm{d}u - A_x \right) \mathrm{d}x \right| \right]$$
$$\leq \int_{-\infty}^{\infty} \mathrm{E}\left[\left| \frac{I_{q-1}^h(x)}{h^{(q-1)/2}} \left(\frac{1}{h} \int_x^{x+h} A_u \, \mathrm{d}u - A_x \right) \right| \right] \tag{4.12}$$
$$\leq \int_{-\infty}^{\infty} \left\| \frac{I_{q-1}^h(x)}{h^{(q-1)/2}} \right\|_2 \left\| \frac{1}{h} \int_x^{x+h} A_u \, \mathrm{d}u - A_x \right\|_4 \mathrm{d}x.$$

Lemma 3.4 gives that

$$\left\| \frac{I_{q-1}^h(x)}{h^{(q-1)2}} \right\|_2 \leq \left\| \frac{1}{h} \int_x^{x+h} \langle M, M \rangle_u \right\|_{2(q-1)}^{(q-1)/2} = 4^{(q-1)/2} \left\| \frac{1}{h} \int_x^{x+h} L_t^u \, \mathrm{d}u \right\|_{2(q-1)}^{(q-1)/2}$$

and the right-hand side converges to $4^{(q-2)/2} \| L_t^x \|_{2(q-1)}^{(q-1)/2}$. As by Lemma 2.5, it holds that

$$\int_{-\infty}^{\infty} \| L_t^x \|_{2(q-1)}^{(q-1)/2} \, \mathrm{d}x < \infty,$$

the Vitali convergence theorem implies that the integral on the right-hand side of (4.12) converges to zero as well, completing the proof. □

REMARK 4.2. In the following, we continue using the shorthand $\Delta_x^h L_t^x = L_t^{x+h} - L_t^x$:

1. For $q = 2$, Theorem 1.2 reads

$$\frac{1}{h^{3/2}} \left(\int_{-\infty}^{\infty} (\Delta_x^h L_t^x)^2 \, \mathrm{d}x - 4 \int_{-\infty}^{\infty} \int_x^{x+h} L_t^u \, \mathrm{d}u \, \mathrm{d}x \right)$$
$$\xrightarrow{d} \sqrt{\frac{64}{3} \int_{-\infty}^{\infty} (L_t^x)^2 \, \mathrm{d}x} \, Z,$$

and as by Fubini and the occupation times formula

$$4 \int_{-\infty}^{\infty} \int_x^{x+h} L_t^u \, \mathrm{d}u \, \mathrm{d}x = 4 \int_{-\infty}^{\infty} \int_{u-h}^{u} \mathrm{d}x \, L_t^u \, \mathrm{d}u = 4ht,$$

we recover the second-order result (1.1) from [6].

2. For $q=3$, Theorem 1.2 reads

$$\frac{1}{h^2}\left(\int_{-\infty}^{\infty}(\Delta_x^h L_t^x)^3\,\mathrm{d}x - 12\int_{-\infty}^{\infty}\Delta_x^h L_t^x\int_x^{x+h}L_t^u\,\mathrm{d}u\,\mathrm{d}x\right)$$
$$\xrightarrow{d}\sqrt{192\int_{-\infty}^{\infty}(L_t^x)^3\,\mathrm{d}x}\,Z$$

and as

$$\int_{-\infty}^{\infty}\Delta_x^h L_t^x\int_x^{x+h}L_t^u\,\mathrm{d}u\,\mathrm{d}x = \lim_{n\to\infty}\int_{-n}^{n}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{x+h}L_t^u\,\mathrm{d}u\right)^2\mathrm{d}x = 0, \tag{4.13}$$

we recover the third-order result (1.2) from [22].

3. For $q=4$, Theorem 1.2 becomes

$$\begin{aligned}&\frac{1}{h^{5/2}}\left(\int_{-\infty}^{\infty}(\Delta_x^h L_t^x)^4\,\mathrm{d}x - 24\int_{-\infty}^{\infty}(\Delta_x^h L_t^x)^2\int_x^{x+h}L_t^u\,\mathrm{d}u\,\mathrm{d}x\right.\\&\qquad\left.+48\int_{-\infty}^{\infty}\left(\int_x^{x+h}L_t^u\,\mathrm{d}u\right)^2\mathrm{d}x\right)\xrightarrow{d}c_4\sqrt{\int_{-\infty}^{\infty}(L_t^x)^4\,\mathrm{d}x}\,Z.\end{aligned}\tag{4.14}$$

Compared to Rosen's Conjecture 1.1, we see that our compensator differs from the conjectured

$$-24h\int_{-\infty}^{\infty}(\Delta_x^h L_t^x)^2 L_t^x\,\mathrm{d}x + 48h^2\int_{-\infty}^{\infty}(L_t^x)^2\,\mathrm{d}x - \int_{-\infty}^{\infty}(\Delta_x^h L_t^x)L_t^x\,\mathrm{d}x.$$

In view of (4.13), we would recover this conjectured compensator from (4.14) if we could replace the term $\frac{1}{h}\int_x^{x+h}L_t^u\,\mathrm{d}u$ by its limit L_t^x. However, as by the mean value theorem,

$$\left|\frac{1}{h}\int_x^{x+h}L_t^u\,\mathrm{d}u - L_t^x\right|\le h^{\varepsilon}$$

for any $\varepsilon\in(0,1/2)$, but be would need an order greater than $h^{1/2}$ to do the replacement, proving that our compensator is equal to the conjectured one (up to negligible terms) does not seem to be straightforward.

4. It is natural to ask whether, as in the cases $q=2$ and $q=3$, a central limit theorem continues to hold for $q\ge 4$. It turns out that this is equivalent to asking whether $R_{q,h}$ can be replaced by its expectation in the statement of Theorem 1.2. Indeed, by inspecting the proof of Theorem 1.2, we see that the expectations of $R_{q,h}$ and $\int_{-\infty}^{\infty}(L_t^{x+h}-L_t^x)^q\,\mathrm{d}x$ have the same order of convergence. Furthermore, if q is odd, both of these expectations are zero by symmetry. Therefore, to obtain a central limit theorem for $q\ge 4$, one has to show for odd $q\ge 5$ that $R_{q,h}/h^{(q+1)/2}\xrightarrow{d}0$ and for even $q\ge 4$ that

$$\frac{1}{h^{(q+1)/2}}(R_{q,h}-\mathrm{E}[R_{q,h}])\xrightarrow{d}0.$$

Unfortunately, we have to leave this question open for further research.

Our space approach allows to generalize Theorem 1.2 in several directions. For example, in view of Lemmas 3.6 and 3.7, a careful examination of the proof of Theorem 1.2 immediately yields the following result.

THEOREM 4.3. *Let L_t^x be Brownian local time and, for some $\alpha > 0$, let $(Y_x)_{x\in\mathbb{R}}$ be a nonnegative, uniformly bounded and almost surely α-Hölder continuous process which is adapted to $(L_t^x)_{x\in\mathbb{R}}$. Then, for integers $q \geq 2$, it holds that*

$$\frac{1}{h^{\frac{q+1}{2}}}\left(\int_{-\infty}^{\infty}(L_t^{x+h} - L_t^x)^q Y_x \,\mathrm{d}x + \widetilde{R}_{t,h}\right) \xrightarrow{d} c_q \sqrt{\int_{-\infty}^{\infty}(L_t^x)^q Y_x^2 \,\mathrm{d}x}\, Z,$$

where Z is a standard Gaussian random variable, independent of the process $((L_t^x)^q Y_x^2)_{x\in\mathbb{R}}$,

$$\widetilde{R}_{t,h} = \sum_{k=1}^{\lfloor \frac{q}{2} \rfloor} a_{q,k} \int_{-\infty}^{\infty} (L_t^{x+h} - L_t^x)^{q-2k}\left(4\int_x^{x+h} L_t^u \,\mathrm{d}u\right)^k Y_x \,\mathrm{d}x$$

and the constants $a_{q,k}$ and c_q are given by (1.5).

As an explicit example, one can take $Y_x = (L_t^x)^r$ for any $r \geq 1$.

Generalizing in another direction, we can replace the time variable t, which is never touched in our proofs, with suitable stopping times τ. A necessary condition for such a stopping time τ is that $(L_\tau^x)_{x\in\mathbb{R}}$ admits a *regular semimartingale decomposition* (see [15], Section 3) on some interval I, by which we mean the existence of a probability measure Q, a filtration $\{\mathcal{G}_x(\tau)\}_{x\in I}$ and a $(\mathcal{G}_x(\tau), Q)$-Brownian motion $(\beta_x)_{x\in I}$ such that L_τ^x is a $(\mathcal{G}_x(\tau), Q)$-semimartingale with canonical decomposition

$$L_\tau^x = \begin{cases} L_\tau^0 + 2\int_0^x \sqrt{L_\tau^u}\,\mathrm{d}\beta_u + \int_0^x A_{\tau,u}\,\mathrm{d}u, & \text{if } x \in I \cap \mathbb{R}_+, \\ L_\tau^0 - 2\int_x^0 \sqrt{L_\tau^u}\,\mathrm{d}\beta_u - \int_x^0 A_{\tau,u}\,\mathrm{d}u, & \text{if } x \in I \cap \mathbb{R}_-. \end{cases}$$

Again, a careful re-evaluation of the proof of Theorem 1.2 yields the following set of sufficient conditions.

THEOREM 4.4. *Let L_t^x be the local time of Brownian motion and τ be a stopping time such that $(L_\tau^x)_{x\in\mathbb{R}}$ admits a regular semimartingale decomposition on some interval I with a finite variation kernel $A_{\tau,u}$ which satisfies*

$$\int_I |A_{t,x}|\,\mathrm{d}x < \infty \quad \textit{and} \quad \int_I \|A_{t,x}\|_p \,\mathrm{d}x < \infty$$

for $p \geq 1$. Furthermore, assume that $\|L_\tau^*\|_p < \infty$ for $p \geq 1$. Then, for any positive integer $p \geq 2$, it holds that

$$\frac{1}{h^{\frac{q+1}{2}}}\left(\int_I (L_\tau^{x+h} - L_\tau^x)^q \,\mathrm{d}x + \widetilde{R}_{q,h}\right) \xrightarrow{d} c_q \sqrt{\int_I (L_\tau^x)^q \,\mathrm{d}x}\, Z,$$

where Z is a standard Gaussian random variable, independent of $(L_\tau^x)_{x\in\mathbb{R}}$, the random variable $\widetilde{R}_{q,h}$ is given by

$$\widetilde{R}_{q,h} = \sum_{k=1}^{\lfloor \frac{q}{2} \rfloor} a_{q,k} \int_I (L_\tau^{x+h} - L_\tau^x)^{q-2k} \left(4 \int_x^{x+h} L_\tau^u \,\mathrm{d}u\right)^k \mathrm{d}x$$

and the constants $a_{q,k}$ and c_q are defined in (1.5).

An example of a stopping time verifying the conditions of Theorem 4.4 (with $I = \mathbb{R}_+$) comes from the Ray–Knight theorem (see [19], Chapter XI): If we take

$$\tau_0 = \inf\{t \geq 0 \colon L_t^0 > 0\},$$

then $L_{\tau_0}^x$ has a regular semimartingale decomposition on $\mathbb{R}_+$ with finite variation kernel $A_{\tau_0,u} = 0$. Furthermore, $(L_{\tau_0}^x)_{x\geq 0}$ is equal in law to a squared Bessel process started in zero with dimension zero, and thus, for example by [28], we have that $\|L_{\tau_0}^*\|_p < \infty$.

Acknowledgements. I heartily want to thank Giovanni Peccati, to whom the idea to look for alternative proofs of (1.1) and (1.2) through the semimartingale decomposition of Brownian local time in space is due and who provided valuable insight. I am also indebted to Guillaume Poly for numerous fruitful discussions which improved this paper in several ways. Finally, I would like to thank Maria Eulalia Vares and an anonymous referee for comments which improved the presentation of this work.

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