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POLY-ADIC FILTRATIONS, STANDARDNESS, COMPLEMENTABILITY AND MAXIMALITY

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Given some essentially separable filtration $(\mathcal{Z}_n)_{n\le 0}$ indexed by the nonpositive integers, we define the notion of complementability for the filtrations contained in $(\mathcal{Z}_n)_{n\le 0}$. We also define and characterize the notion of maximality for the poly-adic sub-filtrations of $(\mathcal{Z}_n)_{n\le 0}$. We show that any poly-adic sub-filtration of $(\mathcal{Z}_n)_{n\le 0}$ which can be complemented by a Kolmogorovian filtration is maximal in $(\mathcal{Z}_n)_{n\le 0}$. We show that the converse is false, and we prove a partial converse, which generalizes Vershik's lacunary isomorphism theorem for poly-adic filtrations.

Introduction. We fix a standard Borel probability space $(\Omega, \mathcal{A}, \mathbf{P})$, namely $(\Omega, \mathcal{A})$ is the Borel space associated to some Polish space. Our study focuses on filtrations indexed by the nonpositive integers, for which interesting phenomena occur near the time $-\infty$. We work modulo the negligible events: we say that two events B and C are almost surely equal if $\mathbf{P}[B\triangle C] = 0$. Two sub-$\sigma$-fields $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ are equal modulo $\mathbf{P}$ (denoted by $\mathcal{B} = \mathcal{C} \bmod \mathbf{P}$) if every event of $\mathcal{B}$ is almost surely equal to some event of $\mathcal{C}$ and conversely. Two filtrations $(\mathcal{F}_n)_{n\le 0}$ and $(\mathcal{G}_n)_{n\le 0}$ of $(\Omega, \mathcal{A}, \mathbf{P})$ are equal modulo $\mathbf{P}$ if for every $n \le 0$, $\mathcal{F}_n = \mathcal{G}_n \bmod \mathbf{P}$.

To each sequence $(X_n)_{n\le 0}$ of random variables defined on $(\Omega, \mathcal{A}, \mathbf{P})$, we associate its natural filtration $\mathcal{F}^X = (\mathcal{F}^X_n)_{n\le 0}$, defined by $\mathcal{F}^X_n = \sigma((X_k)_{k\le n})$. We call *product-type* filtration any filtration that can be generated (modulo the negligible events) by some sequence of independent random variables.

Product-type filtrations are the simplest ones to understand. It is also worthwhile to consider filtrations sharing some weaker properties. A filtration $(\mathcal{F}_n)_{n\le 0}$ of $(\Omega, \mathcal{A}, \mathbf{P})$ is called *Kolmogorovian*[1] when its tail σ-field

$$\mathcal{F}_{-\infty} = \bigcap_{n\le 0} \mathcal{F}_n$$

contains only events of probability 0 or 1; one also says that $\mathcal{F}_{-\infty}$ is *trivial*. A filtration $(\mathcal{F}_n)_{n\le 0}$ of $(\Omega, \mathcal{A}, \mathbf{P})$ has *independent increments* when it possesses some *sequence of innovations*, that is, a sequence $(I_n)_{n\le 0}$ of random variables such that

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[1]This terminology has been introduced by Laurent in reference to Kolmogorov's 0–1 law.

for every $n \le 0$, $\mathcal{F}_n = \mathcal{F}_{n-1} \vee \sigma(I_n)$ mod $\mathbf{P}$, with I_n independent of $\mathcal{F}_{n-1}$. Furthermore, if each random variable I_n is uniform on some finite set of size r_n, the filtration $(\mathcal{F}_n)_{n\le 0}$ is called $(r_n)_{n\le 0}$-*adic*. To say that the innovations $(I_n)_{n\le 0}$ are uniform on some finite sets without specifying the sizes, one says that $(\mathcal{F}_n)_{n\le 0}$ is *poly-adic*.

Every product-type filtration is Kolmogorovian (by Kolmogorov's 0–1 law) and has independent increments, but the converse is false. In particular, equalities like $\mathcal{F}_n = \mathcal{F}_{n-1} \vee \sigma(I_n)$ mod $\mathbf{P}$ for every $n \le 0$ (with I_n independent of $\mathcal{F}_{n-1}$) and $\mathcal{F}_{-\infty} = \{\varnothing, \Omega\}$ mod $\mathbf{P}$ do not ensure that $\mathcal{F}_n = \sigma((I_k)_{k\le n})$ mod $\mathbf{P}$ for every $n \le 0$. Worse, it may happen that $(\mathcal{F}_n)_{n\le 0}$ is not a product-type filtration, meaning that no sequence of independent random variables can generate $(\mathcal{F}_n)_{n\le 0}$.

The first examples of Kolmogorovian poly-adic filtrations which are not product-type were provided by Vershik in [15]. Vershik introduced the notion of standard filtration (which coincides with the notion of product-type filtrations in the case of poly-adic filtrations) and gave a criterion to check the standardness or nonstandardness of filtrations.

Actually, Vershik worked with decreasing sequences of measurable partitions indexed by the nonnegative integers, but this makes no difference. The notion of standardness was translated into a probabilistic framework by Émery and Schachermayer in [6].

Most of the problems involving filtrations indexed by the nonpositive integers arise from the fact that the supremum (of σ-fields) is not distributive with regard to the decreasing countable intersections: given a sub-σ-field $\mathcal{F}$ of $\mathcal{A}$ and a filtration $(\mathcal{G}_n)_{n\le 0}$ on $(\Omega, \mathcal{A}, \mathbf{P})$, the inclusion

$$\mathcal{F} \vee \bigcap_{n\le 0} \mathcal{G}_n \subset \bigcap_{n\le 0} (\mathcal{F} \vee \mathcal{G}_n)$$

may be strict modulo $\mathbf{P}$. When equality holds modulo $\mathbf{P}$, one says that the *exchange property* (of the supremum and the intersection) holds. This is the case when $\mathcal{F}$ and $\mathcal{G}_0$ are independent. The exchange property has been studied in [16]: von Weizsäcker gives necessary and sufficient conditions for the exchange property to hold. The exchange property is involved in many situations (e.g., hidden Markov models, see [14]) and the reader will not be surprised to meet it here.

The notion of immersion of filtrations plays a crucial role in this study. Let us briefly recall some useful facts (a more complete exposition can be found in [6]). By definition, one says that a filtration $(\mathcal{F}_n)_{n\le 0}$ is immersed in a filtration $(\mathcal{G}_n)_{n\le 0}$ if one of the following equivalent conditions holds:

- Every martingale in $(\mathcal{F}_n)_{n\le 0}$ is still a martingale in $(\mathcal{G}_n)_{n\le 0}$.
- For every $n \le 0$, $\mathcal{F}_n \subset \mathcal{G}_n$ and $\mathcal{F}_0$ is independent of $\mathcal{G}_n$ conditionally on $\mathcal{F}_n$.
- For every $n \le 0$, and for every $\mathcal{F}_0$-measurable random variable X taking values in some Polish space, the conditional laws $\mathcal{L}(X|\mathcal{F}_n)$ and $\mathcal{L}(X|\mathcal{G}_n)$ are almost surely equal.

Lemma 5 in [6] shows that if $(\mathcal{F}_n)_{n\le 0}$ is immersed in $(\mathcal{G}_n)_{n\le 0}$, then for every $n \le 0$, $\mathcal{F}_n = \mathcal{F}_0 \cap \mathcal{G}_n$; therefore, $(\mathcal{F}_n)_{n\le 0}$ and $(\mathcal{G}_n)_{n\le 0}$ are equal modulo $\mathbf{P}$ as soon as $\mathcal{F}_0 = \mathcal{G}_0$ mod $\mathbf{P}$.

The simplest nontrivial example of immersion is produced by an independent enlargement: if $(\mathcal{F}_n)_{n\le 0}$ and $(\mathcal{H}_n)_{n\le 0}$ are independent filtrations, then $(\mathcal{F}_n)_{n\le 0}$ is immersed in $(\mathcal{F}_n \vee \mathcal{H}_n)_{n\le 0}$.

We will meet another example: if $(\mathcal{F}_n)_{n\le 0}$ is included in $(\mathcal{G}_n)_{n\le 0}$, and if $(\mathcal{F}_n)_{n\le 0}$ and $(\mathcal{G}_n)_{n\le 0}$ possess a common sequence of innovations, then $(\mathcal{F}_n)_{n\le 0}$ is immersed in $(\mathcal{G}_n)_{n\le 0}$.

Content of the paper. In the present paper, we consider a filtration $(\mathcal{Z}_n)_{n\le 0}$ on the standard Borel probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Working on Polish spaces allows us to define conditional probabilities on $\mathcal{A}$ (or on $\mathcal{Z}_0$). It also ensures that $\mathcal{A}$ is essentially separable, namely $\mathcal{A}$ can be generated modulo $\mathbf{P}$ by countably many events in $\mathcal{A}$ (or by some real $\mathcal{A}$-measurable random variable). Equivalently, the Hilbert space $L^2(\mathcal{A}, \mathbf{P})$ is separable. Therefore, every sub-σ-field of $\mathcal{A}$ is also essentially separable.

Most of the time, we work with poly-adic filtrations. This restriction simplifies our study, notably because of the result below provided by Vershik's theory.

THEOREM 0.1 (Vershik [15]). *If $(\mathcal{Z}_n)_{n\le 0}$ is a product-type filtration such that the final σ-field $\mathcal{Z}_0$ is essentially separable, then every poly-adic filtration immersed in $(\mathcal{Z}_n)_{n\le 0}$ is product-type.*

We introduce the notions of complementability (for filtrations indexed by the nonpositive integers) and maximality (only for poly-adic filtrations). These notions are very close to the eponymous notions concerning 1-dimensional Brownian filtrations immersed in a 2-dimensional Brownian filtration (see [2, 3]).

Besides, J. P. Thouvenot drew my attention to the striking analogy with the notions of complementability and maximality that existed for a long time in ergodic theory for the factors of an automorphism (see [10, 11]).

1. *Complementability*. In Section 1, we study the notion of complementability. Given a filtration included in $(\mathcal{Z}_n)_{n\le 0}$, we focus on the existence and the properties of an independent complement.

DEFINITION 0.1. Let $(\mathcal{U}_n)_{n\le 0}$ be a filtration included in $(\mathcal{Z}_n)_{n\le 0}$. An independent complement of $(\mathcal{U}_n)_{n\le 0}$ in $(\mathcal{Z}_n)_{n\le 0}$ is a filtration $(\mathcal{V}_n)_{n\le 0}$, independent of $(\mathcal{U}_n)_{n\le 0}$, such that $\mathcal{U}_n \vee \mathcal{V}_n = \mathcal{Z}_n$ mod $\mathbf{P}$ for every $n \le 0$. One says that $(\mathcal{U}_n)_{n\le 0}$ is complementable in $(\mathcal{Z}_n)_{n\le 0}$ if it possesses some independent complement.

Since independent enlargements of a filtration always produce filtrations in which the initial filtration is immersed, $(\mathcal{U}_n)_{n\le 0}$ needs to be immersed in $(\mathcal{Z}_n)_{n\le 0}$ to possess an independent complement.

Our next result shows that if $(\mathcal{U}_n)_{n\le 0}$ is complementable in $(\mathcal{Z}_n)_{n\le 0}$, then the independent complements are isomorphic.[2]

PROPOSITION 0.1. *Let $(\mathcal{U}_n)_{n\le 0}$ and $(\mathcal{V}_n)_{n\le 0}$ be two independent filtrations of $(\Omega, \mathcal{A}, \mathbf{P})$, such that $\mathcal{U}_0 \vee \mathcal{V}_n = \mathcal{U}_0 \vee \mathcal{Z}_n \mod \mathbf{P}$ for every $n \le 0$. Let U be any random variable valued in some measurable space $(E, \mathcal{E})$, generating $\mathcal{U}_0$, and $(\mathbf{P}_u)_{u\in E}$ a regular version of the conditional probability $\mathbf{P}$ given U. Then, for $U(\mathbf{P})$-almost every $u \in E$, the filtered probability space $(\Omega, \mathcal{A}, (\mathcal{Z}_n)_{n\le 0}, \mathbf{P}_u)$ is isomorphic to the filtered probability space $(\Omega, \mathcal{A}, (\mathcal{V}_n)_{n\le 0}, \mathbf{P})$. In particular, this result applies when $(\mathcal{V}_n)_{n\le 0}$ is an independent complement of $(\mathcal{U}_n)_{n\le 0}$ in $(\mathcal{Z}_n)_{n\le 0}$.*

This result shows that if $(\mathcal{U}_n)_{n\le 0}$ is complementable by some product-type filtration, then $(\mathcal{Z}_n)_{n\le 0}$ is product-type under $\mathbf{P}_u$ for $U(\mathbf{P})$-almost every $u \in E$. Furthermore, we show that if $(\mathcal{Z}_n)_{n\le 0}$ is poly-adic or product-type, then every filtration which is complementable in $(\mathcal{Z}_n)_{n\le 0}$ and its complement are also poly-adic or product-type.

PROPOSITION 0.2. *Let $(c_n)_{n\le 0}$ be a sequence of positive integers and $(\mathcal{Z}_n)_{n\le 0}$ a $(c_n)_{n\le 0}$-adic filtration of $(\Omega, \mathcal{A}, \mathbf{P})$. Let $(\mathcal{U}_n)_{n\le 0}$ and $(\mathcal{V}_n)_{n\le 0}$ be independent filtrations such that $\mathcal{U}_n \vee \mathcal{V}_n = \mathcal{Z}_n \mod \mathbf{P}$ for every $n \le 0$. Then*:

1. *there exist two sequences $(a_n)_{n\le 0}$ and $(b_n)_{n\le 0}$ of positive integers such that $a_n b_n = c_n$ for every $n \le 0$, $(\mathcal{U}_n)_{n\le 0}$ is $(a_n)_{n\le 0}$-adic and $(\mathcal{V}_n)_{n\le 0}$ is $(b_n)_{n\le 0}$-adic*;
2. *if $(\mathcal{Z}_n)_{n\le 0}$ is product-type, then $(\mathcal{U}_n)_{n\le 0}$ and $(\mathcal{V}_n)_{n\le 0}$ are also product-type.*

The second part of Proposition 0.2 relies on Vershik's theorem which ensures that every poly-adic filtration immersed in some essentially separable product-type filtration is also product-type.

In practice, the data are two filtrations $(\mathcal{U}_n)_{n\le 0} \subset (\mathcal{Z}_n)_{n\le 0}$ and we wonder whether $(\mathcal{U}_n)_{n\le 0}$ is complementable in $(\mathcal{Z}_n)_{n\le 0}$. The immersion of $(\mathcal{U}_n)_{n\le 0}$ is the main necessary condition. The paper focuses on the cases of poly-adic filtrations $(\mathcal{Z}_n)_{n\le 0}$. In this case, Proposition 0.2 shows that the poly-adicity of $(\mathcal{U}_n)_{n\le 0}$ is a necessary condition, and Proposition 0.2 along with Proposition 0.1 shows that is also necessary that $(\mathcal{Z}_n)_{n\le 0}$ is poly-adic conditionally to $\mathcal{U}_0$. In fact, we will see in Section 5 (Corollary 5.1) that the conditional poly-adicity of $(\mathcal{Z}_n)_{n\le 0}$ always holds whenever $(\mathcal{U}_n)_{n\le 0}$ is a poly-adic filtration immersed in $(\mathcal{Z}_n)_{n\le 0}$. We will also see in Section 5 that, similarly to standardness, the complementability for poly-adic filtrations is an asymptotic property at time $-\infty$ (Corollary 5.2).

2. *Maximality*. In Section 2, we study the notion of maximality for poly-adic filtrations. We fix a sequence $(a_n)_{n\le 0}$ of positive integers.

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[2] The definition of isomorphism of filtered probability spaces is given in [1] and in [6].

DEFINITION 0.2. Let $(\mathcal{U}_n)_{n\le 0}$ be an $(a_n)_{n\le 0}$-adic filtration which is immersed in $(\mathcal{Z}_n)_{n\le 0}$. One says that $(\mathcal{U}_n)_{n\le 0}$ is maximal in $(\mathcal{Z}_n)_{n\le 0}$ if every $(a_n)_{n\le 0}$-adic filtration containing $(\mathcal{U}_n)_{n\le 0}$ and immersed in $(\mathcal{Z}_n)_{n\le 0}$ is almost surely equal to $(\mathcal{U}_n)_{n\le 0}$.

We show that any $(a_n)_{n\le 0}$-adic filtration immersed in $(\mathcal{Z}_n)_{n\le 0}$ is included in some unique maximal filtration, which can be constructed explicitly.

THEOREM 0.2. *Assume that $(\mathcal{U}_n)_{n\le 0}$ is an $(a_n)_{n\le 0}$-adic filtration immersed in $(\mathcal{Z}_n)_{n\le 0}$. For every $n \le 0$, set*

$$\mathcal{U}'_n = \bigcap_{s\le 0} (\mathcal{Z}_s \vee \mathcal{U}_n).$$

Then

1. *any sequence of innovations of the filtration $(\mathcal{U}_n)_{n\le 0}$ is still a sequence of innovations ot the filtration $(\mathcal{U}'_n)_{n\le 0}$.*
2. *$(\mathcal{U}'_n)_{n\le 0}$ is the largest $(a_n)_{n\le 0}$-adic filtration containing $(\mathcal{U}_n)_{n\le 0}$ and immersed in $(\mathcal{Z}_n)_{n\le 0}$. Thus, $(\mathcal{U}'_n)_{n\le 0}$ is maximal in $(\mathcal{Z}_n)_{n\le 0}$.*

REMARK 0.1. Applying the same procedure to $(\mathcal{U}'_n)_{n\le 0}$ instead of $(\mathcal{U}_n)_{n\le 0}$ does not yield anything new: for every $n \le 0$,

$$\bigcap_{s\le 0} (\mathcal{Z}_s \vee \mathcal{U}'_n) = \mathcal{U}'_n.$$

Note that this is a true equality, and not only an equality modulo $\mathbf{P}$.

Theorem 0.2 provides many characterizations of maximality.

COROLLARY 0.1. *Assume that $(\mathcal{U}_n)_{n\le 0}$ is an $(a_n)_{n\le 0}$-adic filtration immersed in $(\mathcal{Z}_n)_{n\le 0}$. Let U be a random variable taking values in some measurable space $(E, \mathcal{E})$ such that $\sigma(U) = \mathcal{U}_0 \bmod \mathbf{P}$, and $(\mathbf{P}_u)_{u\in E}$ a regular version of the conditional probability $\mathbf{P}$ given U. Consider the statements*:

1. *$(\mathcal{U}_n)_{n\le 0}$ is maximal in $(\mathcal{Z}_n)_{n\le 0}$.*
2. *For every $n \le 0$, $\mathcal{U}_n = \bigcap_{s\le 0}(\mathcal{Z}_s \vee \mathcal{U}_n) \bmod \mathbf{P}$.*
3. *$\mathcal{U}_0 = \bigcap_{s\le 0}(\mathcal{Z}_s \vee \mathcal{U}_0) \bmod \mathbf{P}$.*
4. *For $U(\mathbf{P})$-almost every $u \in E$, $\mathbf{P}_u$ is trivial on $\mathcal{Z}_{-\infty}$.*

Then (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftarrow$ (4).

Moreover, if for every $s \le 0$, $\mathcal{Z}_s$ is $(\mathbf{P}_u)_{u\in E}$-separable [*there exists some sub-σ-field $\mathcal{H}_s$ of $\mathcal{Z}_0$, generated by some countable family of events, such that $U(\mathbf{P})$-almost every $u \in E$, $\mathcal{Z}_s = \mathcal{H}_s \bmod \mathbf{P}_u$*], *then the implication* (3) $\Longrightarrow$ (4) *also holds.*

Note that the inclusions below always hold:

$$\mathcal{U}_0 \subset \mathcal{Z}_{-\infty} \vee \mathcal{U}_0 \subset \bigcap_{s \le 0} (\mathcal{Z}_s \vee \mathcal{U}_0).$$

Thus, condition (3) can be decomposed into the following two sub-conditions:

- (3a) the inclusion $\mathcal{Z}_{-\infty} \subset \mathcal{U}_0$ mod $\mathbf{P}$,
- (3b) the exchange property for the σ-field $\mathcal{U}_0$ and the filtration $(\mathcal{Z}_s)_{s \le 0}$.

Conditions (3b) and (4) above are nothing but a reformulation in our set-up of conditions (a) and (e) in Theorem 1 of [16]. The implication (4) $\Longrightarrow$ (3) does not follow from Theorem 1 of [16], but it is easily proved, so we prove it directly. The partial converse (3) $\Longrightarrow$ (4) under the additional assumption of $(\mathbf{P}_u)_{u \in E}$-separability of the σ-fields $\mathcal{Z}_s$ is more involved and follows from the implication (a) $\Longrightarrow$ (e) in Theorem 1 of [16]. We will not use it in the rest of the paper.

Note that the $(\mathbf{P}_u)_{u \in E}$-separability of a sub-σ-field $\mathcal{G}$ of $\mathcal{A}$ is much stronger than the essential separability of $\mathcal{G}$ under $\mathbf{P}_u$ for $U(\mathbf{P})$-almost every $u \in E$. Moreover, the $(\mathbf{P}_u)_{u \in E}$-separability is not implied by the $\mathbf{P}$-separability (a counterexample is given in the Annex).

The characterization given in Corollary 0.1 provides a necessary condition for a poly-adic filtration to be complemented by some Kolmogorovian filtration.

COROLLARY 0.2. *Assume that $(\mathcal{U}_n)_{n \le 0}$ is an $(a_n)_{n \le 0}$-adic filtration immersed in $(\mathcal{Z}_n)_{n \le 0}$. If $(\mathcal{U}_n)_{n \le 0}$ can be complemented by some Kolmogorovian filtration, then $(\mathcal{U}_n)_{n \le 0}$ is maximal in $(\mathcal{Z}_n)_{n \le 0}$.*

Corollary 0.1 shows also that the notion of maximality is invariant by extraction of subsequences.

COROLLARY 0.3. *Assume that $(\mathcal{U}_n)_{n \le 0}$ is a poly-adic filtration immersed in $(\mathcal{Z}_n)_{n \le 0}$. Let $(t_n)_{n \le 0}$ be any subdivision of $\mathbf{Z}_-$. Then $(\mathcal{U}_n)_{n \le 0}$ is maximal in $(\mathcal{Z}_n)_{n \le 0}$ if and only if $(\mathcal{U}_{t_n})_{n \le 0}$ is maximal in $(\mathcal{Z}_{t_n})_{n \le 0}$.*

Indeed, this directly follows from the equalities $\mathcal{U}_{t_0} = \mathcal{U}_0$ and

$$\bigcap_{m \le 0} (\mathcal{Z}_{t_m} \vee \mathcal{U}_{t_0}) = \bigcap_{s \le 0} (\mathcal{Z}_s \vee \mathcal{U}_0).$$

3. *Counter-examples to the converse of Corollary* 0.2. The converse of Corollary 0.2 is false, and we give two counter-examples in Section 3.

The first counter-example is the randomised dyadic split-words process, a variant of the dyadic split-words process introduced by Smorodinsky in [12], which was inspired by Vershik's example 3 in [15]. Poly-adic split-words processes have been studied by Laurent [8] and Ceillier [4]. The second counter-example, which uses finite fields, is a variant of an example given in [5], which was inspired by unpublished notes of Tsirelson [13].

To prove that these filtrations actually provide counter-examples, we use Propositions 0.1 and 0.2, together with the implication (4) $\Longrightarrow$ (1) of Corollary 0.1.

4. *Kantorovitch–Rubinstein pseudo-metrics*. Although the converse of Corollary 0.2 is false, a partial converse holds. The proof of this result involves some properties of Kantorovitch–Rubinstein metrics, that we establish in Section 4.

5. *Poly-adic filtrations and piecewise complementability*. In Section 5, we work with two poly-adic filtrations $(\mathcal{U}_n)_{n\le 0}$ and $(\mathcal{Z}_n)_{n\le 0}$; the former being immersed in the latter.

First, we show in Lemma 5.1 that for every sequence $(U_n)_{n\le 0}$ of innovations of $(\mathcal{U}_n)_{n\le 0}$, one can construct a sequence $(V_n)_{n\le 0}$, independent of $\mathcal{U}_0$ such that $((U_n, V_n))_{n\le 0}$ is a sequence of innovations of $(\mathcal{Z}_n)_{n\le 0}$.

Lemma 5.1 has interesting consequences (see Corollaries 5.1 and 5.2 to have precise statements):

- conditionally on $\mathcal{U}_0$, the filtration $(\mathcal{Z}_n)_{n\le 0}$ is poly-adic;
- the complementability of $(\mathcal{U}_n)_{n\le 0}$ in $(\mathcal{Z}_n)_{n\le 0}$ is an asymptotic property at time $-\infty$;
- if we worked with filtrations indexed by the positive integers instead of the nonpositive integers, then $(\mathcal{U}_n)_{n\ge 1}$ would necessarily be complementable in $(\mathcal{Z}_n)_{n\ge 1}$.

Lemma 5.1 is also used together with the results of Section 4 to prove the main result of Section 5, which is the following partial converse of Corollary 0.2.

THEOREM 0.3. *Let $(\mathcal{U}_n)_{n\le 0}$ be a maximal filtration in some poly-adic filtration $(\mathcal{Z}_n)_{n\le 0}$. There exists some subdivision $(t_n)_{n\le 0}$ of $\mathbf{Z}_-$ such that the extracted filtration $(\mathcal{U}_{t_n})_{n\le 0}$ can be complemented by some product-type filtration in the filtration $(\mathcal{Z}_{t_n})_{n\le 0}$.*

Theorem 0.3 generalizes Vershik's lacunary isomorphism theorem (see [15]) in the case of poly-adic filtrations: indeed, one can take $\mathcal{U}_n = \{\varnothing, \Omega\}$ and $a_n = 1$ for all $n \le 0$. Yet, our theorem cannot be easily deduced from Vershik's lacunary isomorphism theorem.

In the light of the counter-examples given in Section 3, Theorem 0.3 shows that the notion of complementability is not invariant by extraction of subsequences, unlike the notion of maximality (Corollary 0.3).

1. Complementability.

1.1. *Proof of Proposition* 0.1. We refer the reader to [1] or [6] for the definition of isomorphisms.

For every $n \le 0$, choose two random variables V_n and Z_n which generate respectively $\mathcal{V}_n$ and $\mathcal{Z}_n$ (modulo $\mathbf{P}$). As $\mathcal{U}_0 \vee \mathcal{V}_n = \mathcal{U}_0 \vee \mathcal{Z}_n \bmod \mathbf{P}$, one gets $\mathbf{P}$-almost surely $Z_n = f_n(U, V_n)$ and $V_n = g_n(U, Z_n)$, where f_n and g_n are some measurable functions. These equalities hold $\mathbf{P}_u$-almost surely for $U(\mathbf{P})$-almost every $u \in E$.

Besides, the independence of U and V_0 entails that, for almost every $u \in E$, the law of V_0 under $\mathbf{P}_u$ coincides with the law of V_0 under $\mathbf{P}$.

Call E' the set of all $u \in E$ such that:

1. V_0 has the same law under $\mathbf{P}_u$ as under $\mathbf{P}$.
2. for all $n \le 0$, $Z_n = f_n(U, V_n)$ $\mathbf{P}_u$-almost surely.
3. for all $n \le 0$, $V_n = g_n(U, Z_n)$ $\mathbf{P}_u$-almost surely.

Then $U(\mathbf{P})(E') = 1$.

Now, fix $u \in E'$. Denote by $\mathcal{L}^0(\mathcal{V}_0, \mathbf{P})$ the space of all real $\mathcal{V}_0$-measurable random variables and $L^0(\mathcal{V}_0, \mathbf{P})$ its quotient by $\mathbf{P}$-almost sure equality. Define $L^0(\mathcal{Z}_0, \mathbf{P}_u)$ by the same way.

Since V_0 generates $\mathcal{V}_0$, every real $\mathcal{V}_0$-measurable random variable has the same law under $\mathbf{P}_u$ as under $\mathbf{P}$. Thus, the inclusion map from $\mathcal{L}^0(\mathcal{V}_0, \mathbf{P})$ to $\mathcal{L}^0(\mathcal{Z}_0, \mathbf{P}_u)$ induces an injective morphism (for the composition with Borelian functions from $\mathbf{R}^{\mathbf{N}}$ to $\mathbf{R}$) Ψ from $L^0(\mathcal{V}_0, \mathbf{P})$ to $L^0(\mathcal{Z}_0, \mathbf{P}_u)$.

The last point to be proved is that for every $n \le 0$, Ψ maps $L^0(\mathcal{V}_n, \mathbf{P})$ onto $L^0(\mathcal{Z}_n, \mathbf{P}_u)$. The inclusion $\Psi(L^0(\mathcal{V}_n, \mathbf{P})) \subset L^0(\mathcal{Z}_n, \mathbf{P}_u)$ follows from the $\mathbf{P}_u$-almost sure equalities $V_n = g_n(U, Z_n) = g_n(u, Z_n)$. The reverse inclusion follows from the $\mathbf{P}_u$-almost sure equalities $Z_n = f_n(U, V_n) = f_n(u, V_n)$.

1.2. *Proof of Proposition* 0.2. First, we establish the following characterization of $(c_n)_{n \le 0}$-adic filtrations.

LEMMA 1.1. *Let $(\mathcal{Z}_n)_{n \le 0}$ be a filtration of $(\Omega, \mathcal{A}, \mathbf{P})$. For every $n \le 0$, fix a real random variable R_n which generates $\mathcal{Z}_n$. The following statements are equivalent*:

1. *The filtration $(\mathcal{Z}_n)_{n \le 0}$ is $(c_n)_{n \le 0}$-adic.*
2. *For every $n \le 0$, almost surely, the conditional law $\mathcal{L}(R_n | \mathcal{Z}_{n-1})$ is uniform on some finite set with size c_n.*

PROOF. Fix $n \le 0$.

If $\mathcal{Z}_n = \mathcal{Z}_{n-1} \vee \sigma(I_n) \bmod \mathbf{P}$, where I_n is independent of $\mathcal{Z}_{n-1}$ and uniform on $[\![1, c_n]\!] = [1, c_n] \cap \mathbf{N}$, then there exists some measurable maps f_n and g_n such that $R_n = f_n(R_{n-1}, I_n)$ and $I_n = g_n(R_n)$ almost surely. Almost surely, $\mathcal{L}(R_n | \mathcal{Z}_{n-1})$ is the push-forward by $f_n(R_{n-1}, \cdot)$ of the uniform law on $[\![1, c_n]\!]$; conversely, the uniform law on $[\![1, c_n]\!]$ is the push-forward by g_n of $\mathcal{L}(R_n | \mathcal{Z}_{n-1})$. Hence, $\mathcal{L}(R_n | \mathcal{Z}_{n-1})$ is uniform on some finite set with size c_n.

Conversely, assume that $\mathcal{L}(R_n | \mathcal{Z}_{n-1})$ is almost surely the uniform law on some random finite set with size c_n. Note $X_1 < \cdots < X_{c_n}$ the atoms of this conditional law in increasing order. The random variables $X_1, \dots, X_{c_n}$ are $\mathcal{Z}_{n-1}$-measurable, and $R_n \in \{X_1, \dots, X_{c_n}\}$ almost surely. Note I_n the unique index $i \in [\![1, c_n]\!]$ such that $R_n = X_i$. Then for every $i \in [\![1, c_n]\!]$,

$$\mathbf{P}[I_n = i | \mathcal{Z}_{n-1}] = \mathbf{P}[R_n = X_i | \mathcal{Z}_{n-1}] = 1/c_n,$$

so I_n is independent of $\mathcal{Z}_{n-1}$ and uniform on $[\![1, c_n]\!]$. Furthermore, $\mathcal{Z}_n = \mathcal{Z}_{n-1} \vee \sigma(I_n)$ mod $\mathbf{P}$, since I_n is $\mathcal{Z}_n$-measurable (modulo the negligible events) and $R_n = X_{I_n}$ almost surely. □

We now prove Proposition 0.2.

PROOF OF PROPOSITION 0.2. Fix $n \leq 0$, and let U_n and V_n be real random variables generating $\mathcal{U}_n$ and $\mathcal{V}_n$ (modulo $\mathbf{P}$). By independence of $\mathcal{U}_n$ and $\mathcal{V}_n$, one has almost surely

$$\mathcal{L}\big((U_n, V_n) | \mathcal{Z}_{n-1}\big) = \mathcal{L}(U_n | \mathcal{U}_{n-1}) \otimes \mathcal{L}(V_n | \mathcal{V}_{n-1}).$$

As (U_n, V_n) generates $\mathcal{Z}_n$ (modulo $\mathbf{P}$), the conditional law $\mathcal{L}((U_n, V_n)|\mathcal{Z}_{n-1})$ is almost surely uniform on some finite set C_n with size c_n.

Hence, the laws $\mathcal{L}(U_n|\mathcal{U}_{n-1})$ and $\mathcal{L}(V_n|\mathcal{V}_{n-1})$ are almost surely uniform on some finite sets A_n and B_n such that $A_n \times B_n = C_n$. The sizes of the sets A_n and B_n are independent random variables whose product equals c_n almost surely, so they are almost surely constant. Call a_n and b_n these constants.

Lemma 1.1 shows that filtrations $(\mathcal{U}_n)_{n\leq 0}$ and $(\mathcal{V}_n)_{n\leq 0}$ are respectively $(a_n)_{n\leq 0}$-adic and $(b_n)_{n\leq 0}$-adic. As these filtrations are immersed in $(\mathcal{Z}_n)_{n\leq 0}$, Vershik's theorem (recalled in the Introduction) ensures that if $(\mathcal{Z}_n)_{n\leq 0}$ is product-type, then $(\mathcal{U}_n)_{n\leq 0}$ and $(\mathcal{V}_n)_{n\leq 0}$ are also product-type. □

2. Maximality.

2.1. *Proof of Theorem* 0.2. We begin with a general result involving two $(a_n)_{n\leq 0}$-adic filtrations, one immersed in the other.

LEMMA 2.1. *Let $(\mathcal{U}_n)_{n\leq 0}$ and $(\mathcal{X}_n)_{n\leq 0}$ be $(a_n)_{n\leq 0}$-adic filtrations, such that $(\mathcal{U}_n)_{n\leq 0}$ is immersed in $(\mathcal{X}_n)_{n\leq 0}$. Then every sequence of innovations for the filtration $(\mathcal{U}_n)_{n\leq 0}$ is still a sequence of innovations for the filtration $(\mathcal{X}_n)_{n\leq 0}$.*

PROOF. Let $(U_n)_{n\leq 0}$ be any sequence of innovations for the filtration $(\mathcal{U}_n)_{n\leq 0}$. By composing each U_n with some deterministic bijection, one may assume that U_n is uniform on $[\![1, a_n]\!] = [1, a_n] \cap \mathbf{N}$. By immersion of $(\mathcal{U}_n)_{n\geq 0}$ in $(\mathcal{X}_n)_{n\geq 0}$, one almost surely has

$$\mathcal{L}(U_n | \mathcal{X}_{n-1}) = \mathcal{L}(U_n | \mathcal{U}_{n-1}) = \mathrm{Unif}\big([\![1, a_n]\!]\big),$$

so U_n is independent of $\mathcal{Z}_{n-1}$.

Now, let (X_n) be some sequence of innovations for the filtration $(\mathcal{X}_n)_{n\le 0}$, taking values in the sets $([\![1,a_n]\!])_{n\le 0}$. For each $n \le 0$, choose a random variable ξ_n generating $\mathcal{X}_n$ (modulo $\mathbf{P}$). As U_n is $\mathcal{X}_n$-measurable and $\mathcal{X}_n = \sigma(\xi_{n-1}) \vee \sigma(X_n) \bmod \mathbf{P}$, there exists some measurable function h_n from $\mathbf{R} \times [\![1,a_n]\!]$ to $[\![1,a_n]\!]$ such that $U_n = h_n(\xi_{n-1}, X_n)$ almost surely. Thus,

$$\mathcal{L}(U_n|\mathcal{X}_{n-1}) = h_n(\xi_{n-1},\cdot)\bigl(\mathrm{Unif}\bigl([\![1,a_n]\!]\bigr)\bigr) \qquad \text{a.s.}$$

Hence, almost surely, the map $h_n(\xi_{n-1},\cdot)$ is some permutation of $[\![1,a_n]\!]$ and $X_n = h_n(\xi_{n-1},\cdot)^{-1}(U_n)$. The equalities

$$\mathcal{X}_n = \sigma(\xi_{n-1}) \vee \sigma(X_n) = \sigma(\xi_{n-1}) \vee \sigma(U_n) \bmod \mathbf{P}$$

show that $(U_n)_{n\le 0}$ is a sequence of innovations for the filtration $(\mathcal{X}_n)_{n\le 0}$. $\square$

We now prove Theorem 0.2.

PROOF OF THEOREM 0.2. One directly checks the inclusions $\mathcal{U}'_{n-1} \subset \mathcal{U}'_n$ and $\mathcal{U}_n \subset \mathcal{U}'_n \subset \mathcal{Z}_n \vee \mathcal{U}_n = \mathcal{Z}_n$. This shows that $(\mathcal{U}'_n)_{n\le 0}$ is a filtration containing $(\mathcal{U}_n)_{n\le 0}$ and contained in $(\mathcal{Z}_n)_{n\le 0}$.

Let $(U_n)_{n\le 0}$ be a sequence of innovations for the $(a_n)_{n\le 0}$-adic filtration $(\mathcal{U}_n)_{n\le 0}$. By assumption and by immersion of $(\mathcal{U}_n)_{n\le 0}$ in $(\mathcal{Z}_n)_{n\le 0}$, U_n is uniform on some finite set A_n of size a_n, and

$$\mathcal{L}(U_n|\mathcal{Z}_{n-1}) = \mathcal{L}(U_n|\mathcal{U}_{n-1}) = \mathrm{Unif}(A_n),$$

so U_n is independent of $\mathcal{Z}_{n-1}$ and a fortiori of $\mathcal{U}'_{n-1}$. Thus, the exchange property applies to the σ-field $\sigma(U_n)$ and the filtration $(\mathcal{Z}_s \vee \mathcal{U}_{n-1})_{s\le n-1}$, so

$$\begin{aligned}
\mathcal{U}'_n &= \bigcap_{s\le n-1} (\mathcal{Z}_s \vee \mathcal{U}_n) = \bigcap_{s\le n-1} \bigl(\mathcal{Z}_s \vee \mathcal{U}_{n-1} \vee \sigma(U_n)\bigr) \bmod \mathbf{P} \\
&= \bigcap_{s\le n-1} (\mathcal{Z}_s \vee \mathcal{U}_{n-1}) \vee \sigma(U_n) \bmod \mathbf{P} \\
&= \mathcal{U}'_{n-1} \vee \sigma(U_n) \bmod \mathbf{P}.
\end{aligned}$$

Hence, $(U_n)_{n\le 0}$ is a sequence of innovations for the filtration $(\mathcal{U}_n)_{n\le 0}$, which is therefore $(a_n)_{n\le 0}$-adic.

To prove that $(\mathcal{U}'_n)_{n\le 0}$ is immersed in $(\mathcal{Z}_n)_{n\le 0}$, one has to check that for every $n \le 0$ and every random variable $R \in L^1(\mathcal{U}'_0)$, $\mathbf{E}[R|\mathcal{Z}_n] = \mathbf{E}[R|\mathcal{U}'_n]$. Since $\mathcal{U}'_0 = \mathcal{U}'_n \vee \sigma(U_{n+1},\dots,U_0)$, one only needs to consider random variables that can be written $R = R_1R_2$ where R_1 is $\mathcal{U}'_n$-measurable and $R_2 \in L^1(\sigma(U_{n+1},\dots,U_0))$. In this case,

$$\mathbf{E}[R|\mathcal{Z}_n] = R_1\mathbf{E}\bigl[R_2|\sigma(\mathcal{Z}_n)\bigr] = R_1\mathbf{E}[R_2],$$

which is $\mathcal{U}'_n$-measurable, yielding the equality to be proved.

Now, let $(\mathcal{X}_n)_{n\le 0}$ be any $(a_n)_{n\le 0}$-adic filtration containing $(\mathcal{U}_n)_{n\le 0}$ and immersed in $(\mathcal{Z}_n)_{n\le 0}$. One checks that $(\mathcal{U}_n)_{n\le 0}$ is immersed in $(\mathcal{X}_n)_{n\le 0}$. Lemma 2.1 shows that $(U_n)_{n\le 0}$ is a sequence of innovations for the filtration $(\mathcal{X}_n)_{n\le 0}$. Thus, for every $s \le n \le 0$,

$$\mathcal{X}_n = \mathcal{X}_s \vee \sigma(U_{s+1}, \dots, U_n) \subset \mathcal{Z}_s \vee \mathcal{U}_n \bmod \mathbf{P}.$$

Taking intersection over s, one gets $\mathcal{X}_n \subset \mathcal{U}'_n$, which completes the proof. $\square$

2.2. *Proof of Corollaries* 0.1 *and* 0.2. First, we prove Corollary 0.1, keeping the notation of Theorem 0.2.

PROOF OF COROLLARY 0.1. The equivalence between statement 1 [maximality of $(\mathcal{U}_n)_{n\le 0}$], statement 2 [equality of the filtrations $(\mathcal{U}_n)_{n\le 0}$ and $(\mathcal{U}'_n)_{n\le 0}$] and statement 3 (equality of $\mathcal{U}_0$ and $\mathcal{U}'_0$) follows from Theorem 0.2 and the classical fact that two filtrations, one immersed in the other, are equal as soon as their final σ-fields coincide (see [6], Lemma 5).

Proof of implication (4) $\Longrightarrow$ (3). Assume that for $U(\mathbf{P})$-almost every $u \in E$, $\mathbf{P}_u$ is trivial on $\mathcal{Z}_{-\infty}$. We have to show that $\mathcal{U}'_0 \subset \mathcal{U}_0 \bmod \mathbf{P}$.

Let $X \in \mathcal{L}^1(\mathcal{U}'_0)$. For each $s \le 0$, fix some real random variable R_s which generates $\mathcal{Z}_s$ (modulo $\mathbf{P}$). Since $\mathcal{U}'_0 \subset \mathcal{Z}_s \vee \mathcal{U}_0 = \sigma(R_s, U) \bmod \mathbf{P}$, one has $X = f_s(R_s, U)$ $\mathbf{P}$-almost surely for some measurable function f_s.

Call E' the set of all $u \in E$ such that $\mathbf{P}_u$ is trivial on $\mathcal{Z}_{-\infty}$ and for every $n \le 0$, $\mathbf{P}_u[X = f_s(R_s, u)] = 1$. Then $U(\mathbf{P})(E') = 1$.

Fix $u \in E'$. The random variable

$$X_u = \limsup_{s \to -\infty} f_s(R_s, u)$$

is $\mathcal{Z}_{-\infty}$-measurable, hence it is $\mathbf{P}_u$-almost surely constant. Since $X = X_u$ $\mathbf{P}_u$-almost surely, $X = \mathbf{E}_u[X]$ $\mathbf{P}_u$-almost surely.

Hence, $X = \mathbf{E}[X|\mathcal{U}_0]$ $\mathbf{P}$-almost surely, which yields the inclusion $\mathcal{U}'_0 \subset \mathcal{U}_0 \bmod \mathbf{P}$.

Proof of implication (3) $\Longrightarrow$ (4). under the additional hypothesis. Assume that the σ-fields $(\mathcal{Z}_s)_{s\le 0}$ are $(\mathbf{P}_u)_{u\in E}$-separable and that condition (3) holds. Condition (3) provides (3a) the inclusion $\mathcal{Z}_{-\infty} \subset \mathcal{U}_0 \bmod \mathbf{P}$ and (3b) the exchange property

$$\mathcal{Z}_{-\infty} \vee \mathcal{U}_0 = \bigcap_{s\le 0} (\mathcal{Z}_s \vee \mathcal{U}_0).$$

The implication (a) $\Longrightarrow$ (d) in Theorem 1 of [16] ensures that the σ-field $\mathcal{Z}_{-\infty}$ is also $(\mathbf{P}_u)_{u\in E}$-separable: there exists some sub-σ-field $\mathcal{H}$ of $\mathcal{Z}_{-\infty}$, generated by countably many events, such that for $U(\mathbf{P})$-almost every $u \in E$, $\mathcal{Z}_{-\infty} = \mathcal{H} \bmod \mathbf{P}_u$. Thus, one has to show that for $U(\mathbf{P})$-almost every $u \in E$, for every $A \in \mathcal{H}$, $\mathbf{P}_u(A) \in \{0, 1\}$. Since $\mathcal{H}$ is generated by countably many events, one needs only to check the property on these generating events, so one can exchange the order of for "$U(\mathbf{P})$-almost every $u \in E$" and "for every $A \in \mathcal{H}$." So let $A \in \mathcal{H}$. As $\mathcal{H} \subset \mathcal{Z}_{-\infty}$ and $\mathcal{Z}_{-\infty} \subset \mathcal{U}_0$ mod $\mathbf{P}$, one gets $\mathbf{P}_U(A) = \mathbf{E}[\mathbf{1}_A|\mathcal{U}_0] = \mathbf{1}_A$ $\mathbf{P}$-almost surely, which completes the proof. □

Now, we prove Corollary 0.2.

PROOF OF COROLLARY 0.2. Assume that $(\mathcal{U}_n)_{n\le 0}$ possesses some independent complement $(\mathcal{V}_n)_{n\le 0}$ in $(\mathcal{Z}_n)_{n\le 0}$. For all $s \le 0$, $\mathcal{U}_0 \vee \mathcal{Z}_s = \mathcal{U}_0 \vee \mathcal{U}_s \vee \mathcal{V}_s = \mathcal{U}_0 \vee \mathcal{V}_s$ mod $\mathbf{P}$. Therefore, independence of $\mathcal{U}_0$ and $\mathcal{V}_0$ yields the exchange property

$$\mathcal{U}_0 \vee \mathcal{V}_{-\infty} = \bigcap_{s\le n}(\mathcal{U}_0 \vee \mathcal{V}_s) = \bigcap_{s\le 0}(\mathcal{U}_0 \vee \mathcal{Z}_s) \text{ mod } \mathbf{P}.$$

If $(\mathcal{V}_n)_{n\le 0}$ is Kolmogorovian, statement (3) of Corollary 0.1 holds, proving the maximality of $(\mathcal{U}_n)_{n\le 0}$.

Alternative proof: let U be a random variable valued in some measurable space $(E, \mathcal{E})$, generating $\mathcal{U}_0$, and $(\mathbf{P}_u)_{u\in E}$ a regular version of the conditional probability $\mathbf{P}$ given U. If $(\mathcal{V}_n)_{n\le 0}$ is Kolmogorovian, then Proposition 0.1 shows that for $U(\mathbf{P})$-almost every $u \in E$, $(\mathcal{Z}_n)_{n\le 0}$ is Kolmogorovian under $\mathbf{P}_u$. The maximality of $(\mathcal{U}_n)_{n\le 0}$ follows, by the implication (4) $\Longrightarrow$ (1) of Corollary 0.1. □

3. Counter-examples to the converse of Corollary 0.2. In this section, we give two examples where some poly-adic filtration is maximal in some other poly-adic filtration, but does not possess any independent complement. Both constructions rely on the following statement.

PROPOSITION 3.1. *Let $(\mathcal{Z}_n)_{n\le 0}$ be a $(c_n)_{n\le 0}$-adic product-type filtration, and $(\mathcal{U}_n)_{n\le 0}$ be an $(a_n)_{n\le 0}$-adic filtration immersed in $(\mathcal{Z}_n)_{n\le 0}$. Let U be a random variable taking values in some measurable space $(E, \mathcal{E})$ such that $\sigma(U) = \mathcal{U}_0$ mod $\mathbf{P}$, and $(\mathbf{P}_u)_{u\in E}$ a regular version of the conditional probability $\mathbf{P}$ given U. If, for every u in some subset $B \subset E$ such that $\mathbf{P}[U \in B] > 0$, the filtration $(\mathcal{Z}_n)_{n\le 0}$ is not product-type under $\mathbf{P}_u$, then $(\mathcal{U}_n)_{n\le 0}$ is not complementable in $(\mathcal{Z}_n)_{n\le 0}$.*

PROOF. The noncomplementability is proved by reduction to the absurd: if the filtration $(\mathcal{U}_n)_{n\le 0}$ had an independent complement $(\mathcal{V}_n)_{n\le 0}$ in $(\mathcal{Z}_n)_{n\le 0}$, then $(\mathcal{V}_n)_{n\le 0}$ would be product-type by Proposition 0.2. But at the same time, Proposition 0.1 would tell that for $U(\mathbf{P})$-almost every u, the filtration $(\mathcal{V}_n)_{n\le 0}$ seen under $\mathbf{P}$ is isomorphic to $(\mathcal{Z}_n)_{n\le 0}$ seen under $\mathbf{P}_u$, leading to a contradiction. □

Keeping the notation of Proposition 3.1, we also know by Corollary 0.1 that if, for $U(\mathbf{P})$-almost every u, the filtration $(\mathcal{Z}_n)_{n\le 0}$ is Kolmogorovian under $\mathbf{P}_u$, then $(\mathcal{U}_n)_{n\le 0}$ is maximal in $(\mathcal{Z}_n)_{n\le 0}$.

So we will get counterexamples by randomising suitably processes generating Kolmogorovian but not product-type filtrations.

3.1. *Randomised dyadic split-words process.* The main result of this subsection, namely Proposition 3.3, relies on the nonstandardness of the dyadic split-word filtration (see [12] or [4]).

For each $n \le 0$, call E_n the set of all subsets of $[\![1, 2^{|n|+1}]\!]$ with $2^{|n|}$ elements. The size of E_n is $c_n = \binom{2^{|n|+1}}{2^{|n|}}$.

DEFINITION 3.1. A randomised dyadic split-words process on the alphabet $\{a, b\}$ is an inhomogeneous Markov chain $(W_n, I_n)_{n\le 0}$ such that for every $n \le 0$:

1. (W_n, I_n) is uniform on $\{a, b\}^{2^{|n|}} \times E_n$,
2. I_n is independent of (W_{n-1}, I_{n-1}),
3. W_n is obtained from W_{n-1} by keeping the letters of index in I_n.

More precisely, if $W_{n-1} = (x_1, \dots, x_{2^{|n|+1}})$ and if $i_1 < \cdots < i_{2^{|n|}}$ are the elements of I_n, then $W_n = (x_{i_1}, \dots, x_{i_{2^{|n|}}})$.

Such a process exists and is unique in law, since the family of uniform laws on the sets $(\{a, b\}^{2^{|n|}} \times E_n)_{n\le 0}$ form an entrance law for the transitions given by conditions 2 and 3. Conditions 2 and 3 show that $(I_n)_{n\le 0}$ is a sequence of innovations for the filtration $(\mathcal{F}^{W,I})_{n\le 0}$, which is therefore $(c_n)_{n\le 0}$-adic.

Note that W_n contains one half of the letters of W_{n-1}, like in the dyadic split-words process. But here, this half is given by an arbitrary subsets of $[\![1, 2^{|n|+1}]\!]$ with $2^{|n|}$ elements, whereas in the dyadic split-words process W_n is the left half (i.e., the letters of index in $[\![1, 2^{|n|}]\!]$) the right half (i.e., letters of index in $[\![2^{|n|} + 1, 2^{|n|+1}]\!]$) of W_{n-1}.

Actually, the randomised dyadic split-words process is close to the erased-words process studied by Laurent, in which each word is obtained from the previous one by deleting a letter whose position is chosen uniformly. Laurent [7, 9] proved that the erased-words filtration is product-type. From this, one can deduce the following result.

PROPOSITION 3.2. *The filtration of the randomised split-words process on the alphabet $\{a, b\}$ is product-type.*

PROOF. Since the randomised dyadic split-words filtration is poly-adic, it suffices to show that it can be immersed in product-type filtration, thanks to Vershik's theorem recalled in the Introduction. We actually show that it can be immersed in some filtration extracted from the erased-words filtration.

By definition, the erased-words process on the alphabet $\{a, b\}$ is an inhomogeneous Markov chain $(M_n, \eta_n)_{n\le 0}$ such that for every $n \le 0$:

1. (M_n, η_n) is uniform on $\{a, b\}^{|n|} \times [\![1, |n| + 1]\!]$,
2. η_n is independent of (M_{n-1}, η_{n-1}),
3. M_n is obtained from M_{n-1} by deleting the η_nth letter.

From a erased-words process $(M_n, \eta_n)_{n\le 0}$, one gets a randomised dyadic split-words by setting $W_n = M_{-2^{|n|}}$ and by calling I_n the subset of $[\![1, 2^{|n|+1}]\!]$ indicating which letters of $M_{-2^{|n-1|}}$ are kept to form the word $M_{-2^{|n|}}$. This subset is some deterministic function of the $2^{|n|}$-uple $(\eta_{-2^{|n|+1}+1}, \dots, \eta_{-2^{|n|}})$. One checks that the filtration $(\mathcal{F}^{W,I}_n)_{n\le 0}$ is immersed in $(\mathcal{F}^{M,\eta}_{-2^{|n|}})_{n\le 0}$: the only difference between these two filtrations is that the former "forgets" the order in which letters are deleted during each time interval $[\![-2^{|n|+1} + 1, -2^{|n|}]\!]$, and this order is independent of the process $(W_n, I_n)_{n\le 0}$. $\square$

PROPOSITION 3.3. *For every $n \le 0$, call U_n the partition of $[\![1, 2^{|n|+1}]\!]$ given by $U_n = \{I_n, [\![1, 2^{|n|+1}]\!] \setminus I_n\}$. Then $U = (U_n)_{n\le 0}$ is a sequence of independent random variables. For $U(\mathbf{P})$-almost every $u \in E$, the filtration $\mathcal{F}^{W,I}$ is Kolmogorovian but not product-type under $\mathbf{P}_u = \mathbf{P}[\cdot|U = u]$. Therefore, the $(c_n/2)_{n\le 0}$-adic filtration $(\mathcal{F}^U)_{n\le 0}$ is maximal but not complementable in $(\mathcal{F}^{W,I})_{n\le 0}$.*

PROOF. Let F_n be the set of all partitions of $[\![1, 2^{|n|+1}]\!]$ with two blocks of size $2^{|n|}$. The map $h_n : A \mapsto \{A, [\![1, 2^{|n|+1}]\!] \setminus A\}$ from E_n to F_n is two to one: each element of F_n possesses exactly two pre-images in E_n, namely the block containing 1 and its complement in $[\![1, 2^{|n|+1}]\!]$. Since $U_n = h_n(I_n)$, the random variables $(U_n)_{n\le 0}$ are independent and generate a $(c_n/2)_{n\le 0}$-adic filtration.

Set $V_n = \mathbf{1}_{[1\notin I_n]}$. Then V_n is uniform on $\{0, 1\}$, U_n and V_n are independent and $\sigma(U_n, V_n) = \sigma(I_n)$. The independence of the sequences $(U_n)_{n\le 0}$ and $(V_n)_{n\le 0}$ shows that the filtration $(\mathcal{F}^U_n)_{n\le 0}$ is immersed in $(\mathcal{F}^I)_{n\le 0}$ and in $(\mathcal{F}^{W,I})_{n\le 0}$.

To prove Proposition 3.3, it is sufficient, by Corollary 0.1 and Proposition 3.1, to check that, conditionally on U, the process $(W_n, I_n)_{n\le 0}$ generates a dyadic split-words filtration, which is known to be Kolmogorovian but not standard (see [12]). For this purpose, we use at each time $n \le 0$ the random variables $U_{n+1}, \dots, U_0$ to modify the order of the letters of W_n. The idea is to put on the left the letters $W_n(i)$ for i belonging to the same block as 1, and to put on the right the other ones.

For every $n \le 0$ and $j \in [\![1, 2^{|n|}]\!]$, denote by $U_n(j)$ the block of the partition U_n which contains j and $R_n(j) = \mathrm{Card}(U_n(j) \cap [1, j])$ the rank of j in the block $U_n(j)$. One defines by recursion a random map Σ_n from $[\![1, 2^{|n|}]\!]$ to $[\![1, 2^{|n|}]\!]$ by $\Sigma_0 = \mathrm{id}_{\{1\}}$ and, for every $n \le 0$ and $j \in [\![1, 2^{|n|+1}]\!]$,

$$\Sigma_{n-1}(j) = \Sigma_n\bigl(R_n(j)\bigr) + 2^{|n|}\mathbf{1}_{1\notin U_n(j)}.$$

A recursion shows that the maps Σ_n are permutations. Indeed, if Σ_n is a random permutation of $[\![1, 2^{|n|}]\!]$, the recursion formula shows that Σ_{n-1} induces an increasing bijection from $U_n(1)$ to $[\![1, 2^{|n|}]\!]$ and an increasing bijection from $[\![1, 2^{|n|+1}]\!] \setminus U_n(1)$ to $[\![2^{|n|} + 1, 2^{|n|+1}]\!]$. Thus, Σ_{n-1} is a random permutation of $[\![1, 2^{|n|+1}]\!]$. Moreover, since $U_n(1) = \Sigma^{-1}_{n-1}([\![1, 2^{|n|}]\!])$, the knowledge of Σ_{n-1} enables us to recover U_n and Σ_n. A recursion shows that for every $n \le 0$, $\sigma(\Sigma_n) = \sigma(U_{n+1}, \dots, U_0)$.

Since W_n, $(U_{n+1},\dots,U_0)$ and $(V_{n+1},\dots,V_0)$ are independent, and W_n is uniform on $\{a,b\}^{2^{|n|}}$, one gets that the random variable $W'_n = W_n \circ \Sigma_n^{-1}$ is uniform on $\{a,b\}^{2^{|n|}}$ and that W'_n, $(U_{n+1},\dots,U_0)$ and $(V_{n+1},\dots,V_0)$ are independent. Thus, the process $(W'_n, V_n)_{n\le 0}$ is independent of the process $(U_n)_{n\le 0}$. Furthermore, for every $n \le 0$,

$$\mathcal{F}_0^U \vee \mathcal{F}_n^{W',V} = \mathcal{F}_0^U \vee \mathcal{F}_n^{W,I} \bmod \mathbf{P}.$$

By Proposition 0.1, for $U(\mathbf{P})$-almost every u, the filtration $(\mathcal{F}^{W,I})_{n\le 0}$ seen under $\mathbf{P}_u$ is isomorphic to the natural filtration of $(W'_n, V_n)_{n\le 0}$ seen under $\mathbf{P}$.

Let us check that $(W'_n, V_n)_{n\le 0}$ is a dyadic split-words process. For each $n \le 0$, denote by $\Phi_n(1) < \cdots < \Phi_n(2^{|n|})$ the elements of I_n. Then for every $r \in [\![1, 2^{|n|}]\!]$,

$$\Sigma_{n-1}\big(\Phi_n(r)\big) = \Sigma_n(r) + 2^{|n|} V_n.$$

For every $i \in [\![1, 2^{|n|}]\!]$,

$$\Sigma_{n-1}\big(\Phi_n\big(\Sigma_n^{-1}(i)\big)\big) = i + 2^{|n|} V_n,$$

hence, as $W_n = W_{n-1} \circ \Phi_n$,

$$\begin{aligned} W'_n(i) &= W_n\big(\Sigma_n^{-1}(i)\big) \\ &= W_{n-1}\big(\Phi_n\big(\Sigma_n^{-1}(i)\big)\big) \\ &= W_{n-1}\big(\Sigma_{n-1}^{-1}\big(i + 2^{|n|} V_n\big)\big) \\ &= W'_{n-1}\big(i + 2^{|n|} V_n\big). \end{aligned}$$

In other words, W'_n is the left half or the right half of W'_{n-1} according that V_n equals 0 or 1, which shows that $(W'_n, V_n)_{n\le 0}$ is a dyadic split-words process. The proof is complete. $\square$

Let us mention a question raised by the referee. The problem of identifying sequences $(t_n)_{n\le 0}$ satisfying Theorem 0.3, namely such that the extracted filtration $(\mathcal{U}_{t_n})_{n\le 0}$ is complementable in $(\mathcal{F}_{t_n}^{W,I})_{n\le 0}$, arises naturally. Since conditionally on U, the process $(W_n, I_n)_{n\le 0}$ generates an $(r_n)_{n\le 0}$-adic split-words filtration, with $r_n = 2^{t_n - t_{n-1}}$, Proposition 3.1 shows that an $(r_n)_{n\le 0}$-adic split-words filtration needs to be product-type for $(\mathcal{U}_{t_n})_{n\le 0}$ to be complementable in $(\mathcal{F}_{t_n}^{W,I})_{n\le 0}$. This necessary condition is equivalent to

$$\sum_{n\le 0} \frac{t_n - t_{n-1}}{2^{|t_n|}} = +\infty,$$

thanks to the known characterization of standard filtrations among split-words filtrations (see [4]). Is this condition also sufficient?

3.2. *An example using finite fields.* In this subsection, we randomise a process studied in [5], which was inspired by an example constructed by Tsirelson [13].

Let $q \geq 5$ be a prime number or some power of a prime number. For every $n \leq 0$, denote by K_n the field with $q^{2^{|n|}}$ elements. Let $(Z_n)_{n\leq 0}$ be a sequence of independent random variables such that for every $n \leq 0$,

$$Z_{2n-1} = E_n \text{ is uniform on } K_n \quad \text{and} \quad Z_{2n} = (X_n, Y_n) \text{ is uniform on } K_n^2.$$

Set $c_{2n-1} = q^{2^{|n|}}$ and $c_{2n} = q^{2^{|n|+1}}$. The filtration $(\mathcal{F}_n^Z)_{n\leq 0}$ is $(c_n)_{n\leq 0}$-adic.

For every $n \leq 0$, K_{n-1} can be seen as a 2-dimensional vector space on K_n. Since the random variable $Z_{2n-2} = (X_{n-1}, Y_{n-1})$ is uniform on K_{n-1}^2, it can be identified with some random variable (A_n, B_n, C_n, D_n) uniformly distributed on K_n^4. Set

$$U_{2n-1} = 0_{K_n} \quad \text{and} \quad U_{2n} = Y_n - A_n X_n^4 - B_n X_n^3 - C_n X_n^2 - D_n X_n - E_n \in K_n.$$

The random variables X_n and Y_n are uniform on K_n, and $\mathcal{F}_{2n-1}^Z, X_n, Y_n$ are independent, whereas $A_n X_n^4 + B_n X_n^3 + C_n X_n^2 + D_n X_n + E_n$ is $\mathcal{F}_{2n-1}^Z \vee \sigma(X_n)$-measurable. Hence, U_{2n} is independent of $\mathcal{F}_{2n-1}^Z \vee \sigma(X_n)$ and uniform on K_n, so (X_n, U_{2n}) is independent of $\mathcal{F}_{2n-1}^Z$ and uniform on K_n^2.

Therefore, the random variables $(U_n)_{n\leq 0}$ are independent, the filtration $(\mathcal{F}_n^U)_{n\leq 0}$ is immersed in $(\mathcal{F}_n^Z)_{n\leq 0}$ and $(a_n)_{n\leq 0}$-adic with $a_{2n-1} = 1$ and $a_{2n} = q^{2^{|n|}}$ for every $n \leq 0$.

PROPOSITION 3.4. *Set $U = (U_n)_{n\leq 0}$ and see U as a random variable taking values in some product space E. For $U(\mathbf{P})$-almost every $u \in E$, the filtration $(\mathcal{F}_n^Z)_{n\leq 0}$ is Kolmogorovian but not product-type under $\mathbf{P}_u = \mathbf{P}[\cdot|U = u]$. Therefore, the $(a_n)_{n\leq 0}$-adic filtration $(\mathcal{F}_n^U)_{n\leq 0}$ is maximal but not complementable in $(\mathcal{F}_n^Z)_{n\leq 0}$.*

PROOF. Call υ the Haar measure of the additive group $G = \prod_{n\leq 0} K_n$. By construction, the random sequence $(Y_n - A_n X_n^4 - B_n X_n^3 - C_n X_n^2 - D_n X_n)_{n\leq 0}$ is $\mathcal{F}_0^{X,Y}$-measurable, whereas the random variable $(E_n)_{n\leq 0}$ is independent of $\mathcal{F}_0^{X,Y}$ with law υ. By difference, $(U_{2n})_{n\leq 0}$ is also independent of $\mathcal{F}_0^{X,Y}$ with law υ.

Since the sequence $(U_{2n-1})_{n\leq 0}$ is deterministic, the process $(Z_{2n})_{n\leq 0} = ((X_n, Y_n))_{n\leq 0}$ is independent of the random variable $U = (U_n)_{n\leq 0}$. Besides, the process $(Z_{2n-1})_{n\leq 0}$ is a deterministic function of the processes $(Z_{2n})_{n\leq 0}$ and $(U_n)_{n\leq 0}$, since for every $n \leq 0$,

$$Z_{2n-1} = E_n = \left(Y_n - A_n X_n^4 - B_n X_n^3 - C_n X_n^2 - D_n X_n\right) - U_{2n}.$$

Let $(\mathbf{P}_u)_{u\in E}$ be a regular version of the conditional probability $\mathbf{P}$ given U. The law of $(Z_n)_{n\leq 0}$ under $\mathbf{P}_u$ is described as follows: the random variables $(Z_{2n})_{n\leq 0}$

are independent and uniform on the sets $(K_n^2)_{n\le 0}$, and for every $n \le 0$, Z_{2n-1} is a deterministic function of (Z_{2n-2}, Z_{2n}) given by

$$Z_{2n-1} = \big(Y_n - A_n X_n^4 - B_n X_n^3 - C_n X_n^2 - D_n X_n\big) - u_{2n}.$$

Under $\mathbf{P}_u$, the filtration $(\mathcal{F}^Z_{2n})_{n\le 0}$ is product-type so the asymptotic σ-field $\mathcal{F}^Z_{-\infty}$ is trivial. Yet, we prove below that $(\mathcal{F}^Z_n)_{n\le 0}$ is not product-type by negating the I-cosiness criterion, like in [5].

Let $(Z'_n)_{n\le 0}$ and $(Z''_n)_{n\le 0}$ be copies defined on some probability space $(\overline{\Omega}, \overline{\mathcal{A}}, \overline{\mathbf{P}}_u)$ of the process $(Z_n)_{n\le 0}$ seen under $\mathbf{P}_u$, such that both natural filtrations $(\mathcal{F}^{Z'}_n)_{n\le 0}$ and $(\mathcal{F}^{Z''}_n)_{n\le 0}$ are immersed in $(\mathcal{F}^{Z'}_n \vee \mathcal{F}^{Z''}_n)_{n\le 0}$. The key step of the proof is the inequality, for every $n \le 0$,

$$\overline{\mathbf{P}}_u\big[Z'_{2n} \ne Z''_{2n} | \mathcal{F}^{Z'}_{2n-2} \vee \mathcal{F}^{Z''}_{2n-2}\big] \ge \big(1 - 4/q^{2^{|n|}}\big)\mathbf{1}_{[Z'_{2n-2}\ne Z''_{2n-2}]}.$$

Indeed, with obvious notation, the point $Z'_{2n} = (X'_n, Y'_n)$ belongs to the graph of the polynomial function $x \mapsto A'_n x^4 + B'_n x^3 + C'_n x^2 + D'_n x + E'_n + u_{2n}$ and the similar statement holds for $Z''_{2n} = (X''_n, Y''_n)$. If $(A'_n, B'_n, C'_n, D'_n) \ne (A''_n, B''_n, C''_n, D''_n)$, the two graphs intersect in at most four points, and X'_n must be the first component of one of these points to make the equality $Z'_{2n} = Z''_{2n}$ possible. But since $(\mathcal{F}^{Z'}_n)_{n\le 0}$ is immersed in $(\mathcal{F}^{Z'}_n \vee \mathcal{F}^{Z''}_n)_{n\le 0}$, the random variable X'_n is uniform on K_n under $\overline{\mathbf{P}}_u[\cdot | \mathcal{F}^{Z'}_{2n-2} \vee \mathcal{F}^{Z''}_{2n-2}]$. The inequality follows.

A recursion yields, for every $n \le 0$,

$$\overline{\mathbf{P}}_u\big[Z'_0 \ne Z''_0 | \mathcal{F}^{Z'}_{2n} \vee \mathcal{F}^{Z''}_{2n}\big] \ge \prod_{n-1\le k\le 0} \big(1 - 4/q^{2^{|k|}}\big)\mathbf{1}_{[Z'_{2n}\ne Z''_{2n}]}.$$

If $(Z'_n)_{n\le 0}$ and $(Z''_n)_{n\le 0}$ are independent until some deterministic time $N > -\infty$, one gets (by taking the expectations in both sides and passing to the limit as $n \to -\infty$)

$$\overline{\mathbf{P}}_u\big[Z'_0 \ne Z''_0\big] \ge \prod_{k\le 0} \big(1 - 4/q^{2^{|k|}}\big) > 0,$$

which shows that the random variable Z_0 does not satisfy the I-cosiness criterion.

Hence, the filtration $(\mathcal{F}^Z_n)_{n\le 0}$ is Kolmogorovian but not product-type under the probabilities $\mathbf{P}_u$. By Corollary 0.1 and Proposition 3.1, $(\mathcal{F}^U_n)_{n\le 0}$ is maximal but not complementable in $(\mathcal{F}^Z_n)_{n\le 0}$. □

4. Kantorovitch–Rubinstein pseudo-metrics. Kantorovitch–Rubinstein (or Wasserstein L^1) metrics play an important role in Vershik's theory, and one will not be surprised to meet them here. We recall the definition, and we establish some lemmas that will be used in Section 5.

Throughout the present section, $(E, \mathcal{E})$ denotes a measurable space and d a measurable bounded pseudo-metric on $(E, \mathcal{E})$. We call Δ the diameter of (E, d).

If μ and ν are probability measures on $(E, \mathcal{E})$, one defines the Kantorovitch–Rubinstein pseudo-distance between μ and ν by

$$d(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{E^2} d(x, y) \, \mathrm{d}\pi(x, y),$$

where $\Pi(\mu, \nu)$ is the set of all probability measures on E^2 with marginals μ and ν. One gets a pseudo-metric on the set of all probability measures on $(E, \mathcal{E})$.

We begin with an elementary lemma.

LEMMA 4.1. *Let μ, ν be probability measures on E carried by some finite set $\{c_1, \dots, c_\ell\}$. Then*

$$d(\mu, \nu) \le \Delta \sum_{k=1}^{\ell} \bigl[\mu(c_k) - \nu(c_k)\bigr]_+ = \frac{\Delta}{2} \sum_{k=1}^{\ell} \bigl|\mu(c_k) - \nu(c_k)\bigr|.$$

PROOF. One can build on a same probability space two random variables X and Y taking values in $\{c_1, \dots, c_\ell\}$, with respective laws μ and ν, such that

$$\mathbf{P}[X = Y] = \sum_{k=1}^{\ell} \mu(c_k) \wedge \nu(c_k).$$

Since $d(X, Y) \le \Delta \mathbf{1}_{[X \neq Y]}$, one has

$$d(\mu, \nu) = \mathbf{E}\bigl[d(X, Y)\bigr] \le \Delta \mathbf{P}[X \neq Y] = \Delta \sum_{k=1}^{\ell} \bigl(\mu(c_k) - \mu(c_k) \wedge \nu(c_k)\bigr).$$

The statement follows. $\square$

Most of the time, it is hard to calculate $d(\mu, \nu)$. But when μ and ν are iso-barycenter of n Dirac masses, the following classical result holds.

LEMMA 4.2. *Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be points in E, nonnecessarily distinct*:

$$\textit{If } \mu = \frac{1}{n} \sum_{k=1}^{n} \delta_{a_k} \textit{ and } \nu = \frac{1}{n} \sum_{k=1}^{n} \delta_{b_k} \textit{ then } d(\mu, \nu) = \min_{\sigma \in \mathfrak{S}_n} \frac{1}{n} \sum_{k=1}^{n} d(a_k, b_{\sigma(k)}).$$

PROOF. Let υ be the uniform law on $[\![1, n]\!]$. One checks that the image by the map $(i, j) \mapsto (a_i, b_j)$ of any probability measure $\gamma \in \Pi(\upsilon, \upsilon)$ is a probability measure $\pi \in \Pi(\mu, \nu)$, and conversely: it suffices to set

$$\gamma(i, j) = \pi(a_i, b_j) / \bigl|\bigl\{(k, l) \in [\![1, n]\!]^2 : (a_k, b_l) = (a_i, b_j)\bigr\}\bigr|.$$

But the probability measures of $\Pi(\upsilon, \upsilon)$ are exactly the measures given by $\gamma(i, j) = m_{i,j}/n$, where $M = (m_{i,j})_{1\le i,j\le n}$ is some bi-stochastic matrix. Hence,

$$d(\mu, \nu) = \inf_M \frac{1}{n} \sum_{1\le i,j\le n} d(a_i, b_j) m_{i,j},$$

where M ranges in the set of all bi-stochastic matrices. The quantity to minimize depends linearly of M. As the set of all bi-stochastic matrices is the convex hull of the permutation matrices, the greatest lower bound above is achieved at some permutation matrix. □

In the proof of Theorem 0.3, we will use approximation of probability measures by isobarycenters of Dirac measures.

LEMMA 4.3. *Assume that* (E, d) *is pre-compact* (*for every* $\varepsilon > 0$, *one can cover* E *by finitely many balls of radius* ε). *To any probability measure* μ *on a* $(E, \mathcal{E})$, *one can associate a family of points* $(r_{n,k}(\mu))_{n\ge k\ge 1}$ *of* E, *which depends measurably on* μ, *such that the probability measures*

$$\nu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{r_{n,k}(\mu)}$$

satisfy $d(\mu, \nu_n) \to 0$ *as* $n \to +\infty$.

PROOF. By pre-compacity of E, one can build a sequence of finite sets $F_n = \{c_{n,1}, \dots, c_{n,\ell(n)}\}$ such that

$$1 \ll \ell_n \ll n \quad \text{and} \quad D_n := \sup_{x\in E} d(x, F_n) \to 0 \qquad \text{as } n \to +\infty.$$

For each $n \ge 1$, we define a map f_n from E to E by

$$f_n(x) = c_{n,K_n(x)} \qquad \text{where } K_n(x) = \min\bigl\{k \in [\![1, \ell_n]\!] : d(x, c_{n,k}) = d(x, F_n)\bigr\}.$$

Since f_n is measurable and $d(x, f_n(x)) \le D_n$ for every $x \in E$, one has $d(\mu, f_n(\mu)) \le D_n$. Set (denoting by $\lfloor\cdot\rfloor$ the integer part)

$$f_n(\mu) = \sum_{k=1}^{\ell_n} \alpha_{n,k} \delta_{c_{n,k}} \quad \text{and} \quad \nu_n = \sum_{k=1}^{\ell_n} \frac{\lfloor n\alpha_{n,k}\rfloor}{n} \delta_{c_{n,k}} + \left(1 - \sum_{k=1}^{\ell_n} \frac{\lfloor n\alpha_{n,k}\rfloor}{n}\right) \delta_{c_{n,1}}.$$

But by Lemma 4.1, $d(f_n(\mu), \nu_n) \le \Delta \ell_n/n$. Hence, $d(\mu, \nu_n) \to 0$ as $n \to +\infty$.

Besides, ν_n is the isobarycenter of n Dirac masses, in points $r_{n,1}, \dots, r_{n,n}$ given by repeating as many time as necessary $c_{n,1}, \dots, c_{n,\ell(n)}$. These points depend of μ only through the values $\mu[K_n = k]$ for $k \in [\![1, \ell_n]\!]$, and the formulas above show that this dependence is measurable. □

We will apply Lemma 4.3 to the space $[0,1]$ endowed with various pseudo-metrics. If we worked only with the usual metric, it would have been sufficient to define $r_{n,k}(\mu)$ as the image of $(2k-1)/(2n)$ by the quantile function of μ.

The pseudo-metrics that we will introduce on $[0,1]$ are provided by the compositions of some distance with some bounded Borel map. The next lemma will be useful in this context.

LEMMA 4.4. *Let H be some bounded Borel map from* $[0,1]$ *to some metric space* (E,d). *One defines a pseudo-distance d_H on* $[0,1]$ *by* $d_H(r,r') = d(H(r), H(r'))$. *Take two probability measures μ and ν on* $[0,1]$, *and call $H(\mu)$ and $H(\nu)$ the images by H of μ and ν. Then $d_H(\mu,\nu) = d(H(\mu), H(\nu))$.*

PROOF. If $\pi \in \Pi(\mu,\nu)$, the image of π by the map $(x,y) \mapsto (H(x), H(y))$ belongs to $\Pi(H(\mu), H(\nu))$. To prove the equality above, one only needs to show that any coupling of $\Pi(H(\mu), H(\nu))$ can be obtained by this way.

Let (U,V) be a random variable taking values in $E \times E$ with marginals $H(\mu)$ and $H(\nu)$. We want to build some random variable (X,Y) such that $(H(X), H(Y)) = (U,V)$ almost surely. A possible way to do this is to use an auxiliary random variable T, independent of (U,V) and uniform on $]0,1[$, and to set

$$(X,Y) = \bigl(F^{\leftarrow}(U,T), G^{\leftarrow}(V,T)\bigr),$$

where $F(u,\cdot)$ and $G(v,\cdot)$ are the conditional distribution functions of X given $H(X) = u$ and of Y given $H(Y) = v$, and $F^{\leftarrow}(u,\cdot)$ and $G^{\leftarrow}(v,\cdot)$ their left-continuous pseudo-inverses: for every $t \in]0,1[$,

$$F^{\leftarrow}(u,t) = \inf\{x \in \mathbf{R} : F(u,x) \geq t\}, \qquad G^{\leftarrow}(v,t) = \inf\{x \in \mathbf{R} : G(v,x) \geq t\}.$$

Indeed, by independence of U and T, one gets for P_U-almost every $u \in E$,

$$\mathcal{L}\bigl(F^{\leftarrow}(U,T)|U = u\bigr) = \mathcal{L}\bigl(F^{\leftarrow}(u,T)\bigr) = \mathcal{L}\bigl(X|H(X) = u\bigr).$$

Since U and $H(X)$ are equi-distributed, $(F^{\leftarrow}(U,T), U)$ and $(X, H(X))$ are also equi-distributed. Thus, the law of $F^{\leftarrow}(U,T)$ is μ and $H(F^{\leftarrow}(U,T)) = U$ almost surely. The same arguments work with V. $\square$

We will also use the following fact.

LEMMA 4.5. *Assume that* (E,d) *is a pre-compact metric space. Let* $(\mu_n)_{n\geq 1}$ *and μ be probability measures on E* (*endowed with the Borel σ-field*). *If $\mu_n \to \mu$ narrowly, then $d(\mu,\mu_n) \to 0$.*

PROOF. Assume that $\mu_n \to \mu$ weakly, and fix $\varepsilon > 0$. Then (E,d) can be covered by finitely many open balls with radius ε. Call $c_1, \dots, c_\ell$ their centers. For each $k \in [\![1, \ell_n]\!]$, denote by D_k the set of all atoms of the image of μ by the map $x \mapsto d(c_k, x)$ from E to $\mathbf{R}$. Call D the union of the D_k. Then D is countable, and for every $r \in \mathbf{R}_+ \setminus D$, the union of the spheres $S(c_k, r)$ is μ-negligible.

Fix $r \in [\varepsilon, 2\varepsilon] \setminus D$. The Borel sets

$$A_1 = B(c_1, r), A_2 = B(c_2, r) \setminus B(c_1, r), \dots, A_\ell = B(c_\ell, r) \Big\backslash \bigcap_{k=1}^{\ell-1} B(c_k, r)$$

form a partition of E and for every $k \in [\![1, \ell_n]\!]$, $\partial A_k \subset S(c_1, r) \cup \cdots \cup S(c_k, r)$ so $\mu(\partial A_k) = 0$ and $\mu_n(A_k) \to \mu(A_k)$ as $n \to +\infty$.

Define a map f from E in E by $f(x) = c_k$ if $x \in A_k$. Then $d(x, f(x)) \le r$ for every $x \in E$, so for every probability measure ν on E, $d(\nu, f(\nu)) \le r \le 2\varepsilon$. Call Δ the diameter of (E, d). By Lemma 4.1, one gets for every $n \ge 1$,

$$d(\mu, \mu_n) \le 4\varepsilon + d\big(f(\mu), f(\mu_n)\big) \le 4\varepsilon + \frac{\Delta}{2} \sum_{k=1}^{\ell} \big|\mu(A_k) - \mu_n(A_k)\big|.$$

Hence, $d(\mu, \mu_n) \le 5\varepsilon$ eventually, which completes the proof. $\square$

5. Poly-adic filtrations and piecewise complementability. In the whole section, we fix a $(c_n)_{n\le 0}$-adic filtration $(\mathcal{Z}_n)_{n\le 0}$, and an $(a_n)_{n\le 0}$-adic filtration $(\mathcal{U}_n)_{n\le 0}$ immersed in $(\mathcal{Z}_n)_{n\le 0}$.

5.1. *Completing the innovations.* First, we show that any sequence of innovations of $(\mathcal{U}_n)_{n\le 0}$ can be completed to provide a sequence of innovations of $(\mathcal{Z}_n)_{n\le 0}$.

LEMMA 5.1. *Let $(U_n)_{n\le 0}$ be a sequence of innovations of $(\mathcal{U}_n)_{n\le 0}$, taking values in the sets $([\![1, a_n]\!])_{n\le 0}$. Then for every $n \le 0$, one has $c_n = a_n b_n$ with $b_n \in \mathbf{N}$. Furthermore, there exists some sequence $(V_n)_{n\le 0}$ of uniform random variables on the sets $([\![1, b_n]\!])_{n\le 0}$ such that for every $n \le 0$, $\mathcal{Z}_n = \mathcal{Z}_{n-1} \vee \sigma(U_n, V_n)$ mod $\mathbf{P}$, with $\mathcal{Z}_{n-1}, U_n, V_n$ independent. In particular, $((U_n, V_n))_{n\le 0}$ is a sequence of innovations of the filtration $(\mathcal{Z}_n)_{n\le 0}$.*

REMARK 5.1. Actually, the proof below shows that the statement given in Lemma 5.1 reduce to results involving two-times filtrations.

PROOF OF LEMMA 5.1. We fix $n \le 0$ and we look at the filtrations at times n and $n-1$. By immersion of $(\mathcal{U}_n)_{n\ge 0}$ in $(\mathcal{Z}_n)_{n\ge 0}$,

$$\mathcal{L}(U_n|\mathcal{Z}_{n-1}) = \mathcal{L}(U_n|\mathcal{U}_{n-1}) = \mathrm{Unif}\big([\![1, a_n]\!]\big) \quad \text{a.s.} \tag{1}$$

In particular, U_n is independent of $\mathcal{Z}_{n-1}$.

Let $(Z_n)_{n\le 0}$ be some sequence of innovations of the filtration $(\mathcal{Z}_n)_{n\le 0}$, taking values in the sets $([\![1,c_n]\!])_{n\le 0}$. For each $n \le 0$, fix some real random variable R_n which generates $\mathcal{Z}_n$ modulo $\mathbf{P}$.

Since U_n is $\mathcal{Z}_n$-measurable and $\mathcal{Z}_n = \sigma(R_{n-1}) \vee \sigma(Z_n)$ mod $\mathbf{P}$, there exists some measurable function f_n from $\mathbf{R} \times [\![1,c_n]\!]$ to $[\![1,a_n]\!]$ such that $U_n = f_n(R_{n-1}, Z_n)$ almost surely. But R_{n-1} is $\mathcal{Z}_{n-1}$-measurable whereas Z_n is independent of $\mathcal{Z}_{n-1}$ and uniform in $[\![1,c_n]\!]$, hence

$$\mathcal{L}(U_n|\mathcal{Z}_{n-1}) = f_n(R_{n-1},\cdot)\big(\mathrm{Unif}([\![1,c_n]\!])\big) \qquad \text{a.s.} \tag{2}$$

Equalities (1) and (2) show that for every $u \in [\![1,a_n]\!]$,

$$\frac{1}{a_n} = \frac{1}{c_n}\big|f_n(R_{n-1},\cdot)^{-1}(u)\big| \qquad \text{a.s.}$$

Thus, $c_n = a_n b_n$ with $b_n \in \mathbf{N}$, and for $R_{n-1}(\mathbf{P})$-almost every $r \in \mathbf{R}$, the map $f_n(r,\cdot)$ is b_n to one: each element $v \in [\![1,a_n]\!]$ has exactly b_n antecedents.

For each $z \in [\![1,c_n]\!]$, denote by $g_n(r,z)$ the rank of z among the b_n antecedents of $f(r,z)$. Then the map $z \mapsto (f_n(r,z), g_n(r,z))$ is a bijection from $[\![1,c_n]\!]$ to $[\![1,a_n]\!] \times [\![1,b_n]\!]$, which depends measurably on r. Set $V_n = g_n(R_{n-1}, Z_n)$. The random variable (U_n, V_n) is obtained from Z_n by applying the random bijection $(f_n(R_{n-1},\cdot), g_n(R_{n-1},\cdot))$, which depends only on R_{n-1}. Hence, (U_n, V_n) is independent of R_{n-1} and uniform on $[\![1,a_n]\!] \times [\![1,b_n]\!]$, and $\mathcal{Z}_n = \mathcal{Z}_{n-1} \vee \sigma(U_n, V_n)$ mod $\mathbf{P}$. We are done. $\square$

REMARK 5.2. *The sequence $(V_n)_{n\le 0}$ provided by Lemma 5.1 is independent of $\mathcal{U}'_0$ (with the notation of Theorem 0.2).*

PROOF. Fix $n \le 0$. By Theorem 0.2, $\mathcal{U}'_0 = \mathcal{U}'_n \vee \sigma(U_{n+1},\dots,U_0)$ mod $\mathbf{P}$. Since $\mathcal{Z}_n$, $(U_{n+1},\dots,U_0)$ and $(V_{n+1},\dots,V_0)$ are independent and since $\mathcal{U}'_n \subset \mathcal{Z}_n$, $(V_{n+1},\dots,V_0)$ is independent of $\mathcal{U}'_0$, which completes the proof. $\square$

5.2. *Consequences of Lemma 5.1.* We now mention two consequences of Lemma 5.1 which are interesting by themselves although they will not be used in the proof of Theorem 0.3.

COROLLARY 5.1. *Let X be a random variable taking values in some measurable space $(E,\mathcal{E})$ such that $\sigma(X) = \mathcal{U}_0$ mod $\mathbf{P}$, and $(\mathbf{P}_x)_{x\in E}$ a regular version of the conditional probability $\mathbf{P}$ given X. Then for $X(\mathbf{P})$-almost every x, the filtration $(\mathcal{Z}_n)_{n\le 0}$ is $(b_n)_{n\le 0}$-adic under $\mathbf{P}_x$, where $c_n = a_n b_n$ for every $n \le 0$.*

PROOF. Let $(U_n)_{n\le 0}$ be a sequence of innovations of $(\mathcal{U}_n)_{n\le 0}$, taking values in the sets $([\![1,a_n]\!])_{n\le 0}$, and $(V_n)_{n\le 0}$ the sequence of the complements of innovations provided by Lemma 5.1.

By assumption, the σ-field generated by the random variables $(U_n)_{n\le 0}$ is included (and possibly strictly included) in $\mathcal{U}_0$, which equals $\sigma(X)$ modulo $\mathbf{P}$. Therefore, $U_n = u_n(X)$ almost surely for some measurable function u_n from E to $[\![1, a_n]\!]$.

For each $n \le 0$, let R_n be a real random variable such that $\sigma(R_n) = \mathcal{Z}_n$ mod $\mathbf{P}$. Since $\sigma(V_n) \subset \sigma(R_n) \subset \sigma(R_{n-1}, U_n, V_n)$ mod $\mathbf{P}$, there exist some measurable functions φ_n from $\mathbf{R}$ to $[\![1, b_n]\!]$ and ψ_n from $\mathbf{R} \times [\![1, a_n]\!] \times [\![1, b_n]\!]$ to $\mathbf{R}$ such that $V_n = \varphi_n(R_n)$ and $R_n = \psi_n(R_{n-1}, U_n, V_n)$ almost surely.

Since $V_n = \varphi_n(\psi_n(R_{n-1}, U_n, V_n))$ almost surely, the independence of R_{n-1}, U_n and V_n shows that for $R_{n-1}(\mathbf{P})$-almost every $r \in \mathbf{R}$ and for every $u \in [\![1, a_n]\!]$, the map $\varphi_n \circ \psi_n(r, u, \cdot)$ is the identity map on $[\![1, b_n]\!]$, so $\psi_n(r, u, \cdot)$ is injective.

Furthermore, for $X(\mathbf{P})$-almost every x,

$$R_n = \psi_n\big(R_{n-1}, u_n(X), V_n\big) = \psi_n\big(R_{n-1}, u_n(x), V_n\big), \qquad \mathbf{P}_x\text{-almost surely.}$$

The equality $\sigma(R_{n-1}) \vee \sigma(X) = \sigma(R_{n-1}, U_n, U_{n+1}, \dots, U_0)$ mod $\mathbf{P}$ and the independence of the random variables $R_{n-1}, U_n, V_n, U_{n+1}, \dots, U_0$ show that V_n is independent of $\sigma(R_{n-1}) \vee \sigma(X)$, so R_{n-1} and V_n are independent conditionally on X. Denoting by $\mathcal{L}_x(\cdot|\mathcal{Z}_{n-1})$ the conditional law given $\mathcal{Z}_{n-1}$ computed under $\mathbf{P}_x$, this yields

$$\mathcal{L}_x(R_n|\mathcal{Z}_{n-1}) = \psi_n\big(R_{n-1}, u_n(x), \cdot\big)\big(\mathrm{Unif}\big([\![1, b_n]\!]\big)\big), \qquad \mathbf{P}_x\text{-almost surely.}$$

Thus, for $X(\mathbf{P})$-almost every x, the law $\mathcal{L}_x(R_n|\mathcal{Z}_{n-1})$ is $\mathbf{P}_x$-almost surely uniform on some finite set with size b_n. The result follows by Lemma 1.1. $\square$

Lemma 5.1 and Remark 5.1 have also two striking consequences. The first one shows that in the world of poly-adic filtrations indexed by the nonpositive integers, complementability is an asymptotic property at time $-\infty$. The second one shows that it is not worth studying complementability in the world of poly-adic filtrations indexed by the positive integers.

COROLLARY 5.2. 1. *$(\mathcal{U}_n)_{n\le 0}$ is complementable in $(\mathcal{Z}_n)_{n\le 0}$ if and only if $(\mathcal{U}_n)_{n\le -1}$ is complementable in $(\mathcal{Z}_n)_{n\le -1}$.*

2. *If the filtrations were indexed by the positive integers, then $(\mathcal{U}_n)_{n\ge 1}$ would automatically be complementable in $(\mathcal{Z}_n)_{n\ge 1}$.*

PROOF. 1. Let U_0 be an innovation of the filtration $(\mathcal{U}_n)_{n\le 0}$ at time 0. Remark 5.1 provides a random variable V_0, independent of U_0, such that (U_0, V_0) an innovation of the filtration $(\mathcal{Z}_n)_{n\le 0}$ at time 0. If $(\mathcal{V}_n)_{n\le -1}$ is an independent complement of $(\mathcal{U}_n)_{n\le -1}$ in $(\mathcal{Z}_n)_{n\le -1}$, then the filtration $(\mathcal{V}_n)_{n\le 0}$ given by $\mathcal{V}_0 = \mathcal{V}_{-1} \vee \sigma(V_0)$ is an independent complement of $(\mathcal{U}_n)_{n\le 0}$ in $(\mathcal{Z}_n)_{n\le 0}$. The converse does not require any proof.

2. If the filtrations were indexed by the positive integers, we could set $\mathcal{U}_0 = \mathcal{Z}_0 = \{\varnothing, \Omega\}$ to get poly-adic filtrations indexed by the nonnegative integers. Fixing a sequence $(U_n)_{n\geq 1}$ of innovations of $(\mathcal{U}_n)_{n\geq 0}$ and a sequence $(V_n)_{n\geq 1}$ of complements of innovations in $(\mathcal{Z}_n)_{n\geq 0}$, we would get an independent complement of $(\mathcal{U}_n)_{n\geq 1}$ in $(\mathcal{Z}_n)_{n\geq 1}$ by setting $\mathcal{V}_n = \sigma(V_1, \ldots, V_n)$ for every $n \geq 1$. $\square$

5.3. *Proof of Theorem* 0.3. We now assume that the $(a_n)_{n\leq 0}$-adic filtration $(\mathcal{U}_n)_{n\leq 0}$ is maximal in the $(c_n)_{n\leq 0}$-adic filtration $(\mathcal{Z}_n)_{n\leq 0}$. We have to build a subdivision $(t_n)_{n\leq 0}$ of $\mathbf{Z}_-$ such that the filtration $(\mathcal{U}_{t_n})_{n\leq 0}$ is complementable by some product-type filtration in the filtration $(\mathcal{Z}_{t_n})_{n\leq 0}$.

For convenience, we fix some sequence $(R_n)_{n\leq 0}$ of random variables taking values in $[0, 1]$ such that for every $n \leq 0$, R_n generates $\mathcal{Z}_n$ (modulo $\mathbf{P}$).

By hypothesis, the filtration $(\mathcal{U}_n)_{n\leq 0}$ possesses some sequence $(U_n)_{n\leq 0}$ of innovations, uniform on the sets $([\![1, a_n]\!])_{n\leq 0}$, and Lemma 5.1 provides a sequence $(V_n)_{n\leq 0}$ of uniform random variables on the sets $([\![1, b_n]\!])_{n\leq 0}$, independent of $(U_n)_{n\leq 0}$, such that $((U_n, V_n))_{n\leq 0}$ is a sequence of innovations of the filtration $(\mathcal{Z}_n)_{n\leq 0}$.

With the help of the properties established in Section 4, we are going to modify the sequence $(V_n)_{n\leq 0}$ to get another sequence $(V'_n)_{n\leq 0}$, independent of $(U_n)_{n\leq 0}$ and equi-distributed as $(V_n)_{n\leq 0}$, such that the natural filtration of $((U_n, V'_n))_{n\leq 0}$ coincides with the one of $(\mathcal{Z}_n)_{n\leq 0}$ at infinitely many times.

We are led to group the innovations by intervals of time: For $s \leq t \leq 0$, set $U_{s,t} = (U_{s+1}, \ldots, U_t)$ and $V_{s,t} = (V_{s+1}, \ldots, V_t)$. These random variables take values in the finite sets

$$A_{s,t} = \prod_{s<k\leq t} [\![1, a_k]\!] \quad \text{and} \quad B_{s,t} = \prod_{s<k\leq t} [\![1, b_k]\!].$$

Since $\mathcal{Z}_t = \mathcal{Z}_s \vee \sigma(U_{s,t}, V_{s,t})$ mod $\mathbf{P}$, there exists some measurable map $h_{s,t}$ from $[0, 1] \times A_{s,t} \times B_{s,t}$ to $[0, 1]$ such that $R_t = h_{s,t}(R_s, U_{s,t}, V_{s,t})$. Moreover, as $V_{s,t}$ is independent of $\mathcal{Z}_s \vee \sigma(U_{s,t})$, the conditional law of R_t given $\mathcal{Z}_s \vee \sigma(U_{s,t})$ is

$$\mathcal{L}\big(R_t|\mathcal{Z}_s \vee \sigma(U_{s,t})\big) = \frac{1}{|B_{s,t}|} \sum_{v\in B_{s,t}} \delta_{h_{s,t}(R_s, U_{s,t}, v)}.$$

We start by proving the following result.

LEMMA 5.2. *Let $t \leq 0$ and $C_t = A_{t,0} \times B_{t,0}$. Let H be some Borel map from $[0, 1]$ into $[0, 1]^{C_t}$. Endow $\mathbf{R}^{C_t}$ with the norm $|\cdot|_1$ defined as the average of the absolute values of the components and $[0, 1]^{C_t}$ with the induced metric d_1. Define the pseudo-metric d_H on $[0, 1]$ by $d_H(r, r') = |H(r) - H(r')|_1$. As $s \to -\infty$, the conditional law $\mathcal{L}(R_t|\mathcal{Z}_s \vee \sigma(U_{s,t}))$ tends almost surely to the conditional law $\mathcal{L}(R_t|\mathcal{U}_t)$ for the Kantorovitch–Rubinstein pseudo-metric associated to d_H.*

PROOF. By Corollary 0.1, we know that for every $t \le 0$,

$$\mathcal{U}_t = \bigcap_{s \le 0} (\mathcal{U}_t \vee \mathcal{Z}_s).$$

But for every $s \le t$, $\mathcal{U}_t = \mathcal{U}_s \vee \sigma(U_{s,t})$, hence $\mathcal{U}_t \vee \mathcal{Z}_s = \mathcal{Z}_s \vee \sigma(U_{s,t})$. The reverse martingale convergence theorem ensures that, for every Borel set $B \subset [0, 1]$,

$$\mathbf{P}\big[R_t \in B | \mathcal{Z}_s \vee \sigma(U_{s,t})\big] \to \mathbf{P}[R_t \in B | \mathcal{U}_t] \qquad \text{a.s. as } s \to -\infty.$$

Therefore, for every Borel set $B \subset [0, 1]^{C_t}$,

$$\mathbf{P}\big[H(R_t) \in B | \mathcal{Z}_s \vee \sigma(U_{s,t})\big] \to \mathbf{P}\big[H(R_t) \in B | \mathcal{U}_t\big] \qquad \text{a.s. as } s \to -\infty.$$

Applying this statement to all products of intervals $]-\infty, r]$ with r rational shows that almost surely, the distribution function of the law $\mathcal{L}(H(R_t)|\mathcal{Z}_s \vee \sigma(U_{s,t}))$ tends to the distribution function of $\mathcal{L}(H(R_t)|\sigma(\mathcal{U}_t))$ at any continuity point of the latter, so

$$\mathcal{L}\big(H(R_t)|\mathcal{Z}_s \vee \sigma(U_{s,t})\big) \longrightarrow \mathcal{L}\big(H(R_t)|\mathcal{U}_t\big) \qquad \text{narrowly}.$$

All these probability measures are carried by $[0, 1]^{C_t}$, so Lemma 4.5 shows the convergence for the Kantorovitch–Rubinstein metric associated to d_1. The result follows by Lemma 4.4. $\square$

We now have all the tools to prove Theorem 0.3. We abbreviate the notation by using the symbol $\mathbf{M}$ to denote the arithmetic means: for any nonempty finite family $(x_k)_{k \in F}$ of real numbers,

$$\mathop{\mathbf{M}}_{k \in F} x_k = \frac{1}{|F|} \sum_{k \in F} x_k.$$

We now build a subdivision $(t_n)_{n \le 0}$ of $\mathbf{Z}_-$ and a sequence $(V'_{t_{n-1},t_n})_{n \le 0}$ of random variables taking values in the sets $(B_{t_{n-1},t_n})_{n \le 0}$ such that for every $n \le 0$, $(U_{t_{n-1},t_n}, V'_{t_{n-1},t_n})$ is independent of $\mathcal{Z}_{t_{n-1}}$, distributed as $(U_{t_{n-1},t_n}, V_{t_{n-1},t_n})$ and

$$\mathcal{Z}_{t_n} = \mathcal{Z}_{t_{n-1}} \vee \sigma(U_{t_{n-1},t_n}, V_{t_{n-1},t_n}) \bmod \mathbf{P}.$$

We proceed recursively and start with $t_0 = 0$.

Let $n \le 0$. Assume that $t_n < \cdots < t_0$ and $V'_{t_n,t_{n+1}}, \ldots, V'_{t_{-1},t_0}$ are already constructed. Then

$$\mathcal{Z}_0 = \mathcal{Z}_{t_n} \vee \sigma\big(U_{t_n,t_0}, V'_{t_n,t_0}\big) \bmod \mathbf{P}.$$

Thus,

$$R_0 = h_n\big(R_{t_n}, U_{t_n,t_0}, V'_{t_n,t_0}\big) \qquad \text{a.s.},$$

where h_n is some measurable map from $[0, 1] \times C_{t_n}$ to $[0, 1]$. Define a measurable map H_n from $[0, 1]$ to $[0, 1]^{C_{t_n}}$ by

$$H_n(r) = \big(h_n(r, u, v)\big)_{(u,v) \in C_{t_n}}.$$

Lemma 5.2 applied to H_n shows that almost surley, as $s \to -\infty$, the conditional law $\mathcal{L}(R_{t_n}|\mathcal{Z}_s \vee \sigma(U_{s,t_n}))$ tends to the conditional law $\mathcal{L}(R_{t_n}|\mathcal{U}_{t_n})$ for the Kantorovitch–Rubinstein pseudo-metric associated to d_{H_n}. But we already know that

$$\mathcal{L}\big(R_{t_n}|\mathcal{Z}_s \vee \sigma(U_{s,t_n})\big) = \mathop{\mathbf{M}}_{v\in B_{s,t_n}} \delta_{h_{s,t}(R_s,U_{s,t_n},v)}.$$

Besides, by Lemma 4.3, the law $\mathcal{L}(R_{t_n}|\mathcal{U}_{t_n})$ can be approached in pseudo-metric d_{H_n} by some discrete law

$$\mathop{\mathbf{M}}_{v\in B_{s,t_n}} \delta_{\Upsilon_n(v)},$$

where the $\Upsilon_n(v)$ are $\sigma(\mathcal{U}_{t_n})$-measurable random variables. Hence, almost surely

$$d_{H_n}\Big(\mathop{\mathbf{M}}_{v\in B_{s,t_n}} \delta_{h_{s,t_n}(R_s,U_{s,t_n},v)}, \mathop{\mathbf{M}}_{v\in B_{s,t_n}} \delta_{\Upsilon_n(v)}\Big) \to 0.$$

Since the pseudo-metrics d_{H_n} are bounded 1, the convergence holds also in $L^1(\mathbf{P})$. We choose $t_{n-1} < t_n$ so that the random variable

$$D_n = d_n\Big(\mathop{\mathbf{M}}_{v\in B_{t_{n-1},t_n}} \delta_{h_{t_{n-1},t_n}(R_{t_{n-1}},U_{t_{n-1},t_n},v)}, \mathop{\mathbf{M}}_{v\in B_{t_{n-1},t_n}} \delta_{\Upsilon_n(v)}\Big)$$

fulfills $\mathbf{E}[D_n] \le 2^{n-1}$. But by Lemma 4.2,

$$D_n = \min_{\sigma\in\mathfrak{S}(B_{t_{n-1},t_n})} \mathop{\mathbf{M}}_{v\in B_{t_{n-1},t_n}} d_n\big(h_{t_{n-1},t_n}(R_{t_{n-1}}, U_{t_{n-1},t_n}, v), \Upsilon_n\big(\sigma(v)\big)\big).$$

Choose a random permutation Σ_n, measurable for $\mathcal{Z}_{t_{n-1}} \vee \sigma(U_{t_{n-1},t_n})$, which achieves the minimum above, and set $V'_{t_{n-1},t_n} = \Sigma_n(V_{t_{n-1},t_n})$. Then V'_{t_{n-1},t_n} is $\mathcal{Z}_{t_n}$-measurable, independent of $\mathcal{Z}_{t_{n-1}} \vee \sigma(U_{t_{n-1},t_n})$ and uniform on B_{t_{n-1},t_n}, which enables the recursive construction.

With this notation, one gets

$$\begin{aligned} D_n &= \mathop{\mathbf{M}}_{v\in B_{t_{n-1},t_n}} d_n\big(h_{t_{n-1},t_n}(R_{t_{n-1}}, U_{t_{n-1},t_n}, v), \Upsilon_n\big(\Sigma_n(v)\big)\big) \\ &= \mathbf{E}\big[d_n\big(R_{t_n}, \Upsilon_n\big(V'_{t_{n-1},t_n}\big)\big)|\mathcal{Z}_{t_{n-1}} \vee \sigma(U_{t_{n-1},t_n})\big] \qquad \text{a.s.,} \end{aligned}$$

since $R_{t_n} = h_{t_{n-1},t_n}(R_{t_{n-1}}, U_{t_{n-1},t_n}, V_{t_{n-1},t_n})$ a.s. But

$$\begin{aligned} &d_n\big(R_{t_n}, \Upsilon_n\big(V'_{t_{n-1},t_n}\big)\big) \\ &\quad = \big|H_n(R_{t_n}) - H_n\big(\Upsilon_n\big(V'_{t_{n-1},t_n}\big)\big)\big|_1 \\ &\quad = \mathbf{E}\big[\big|h_n\big(R_{t_n}, U_{t_n,0}, V'_{t_n,0}\big) - h_n\big(\Upsilon_n\big(V'_{t_{n-1},t_n}\big), U_{t_n,0}, V'_{t_n,0}\big)\big||\mathcal{Z}_{t_n}\big], \end{aligned}$$

since $(U_{t_n,0}, V'_{t_n,0})$ is independent of $\mathcal{F}^Z_{t_n}$ and uniform on K_n.

Set $S_n = h_n(\Upsilon_n(V'_{t_{n-1},t_n}), U_{t_n,0}, V'_{t_n,0})$. Then S_n is $\mathcal{U}_0 \vee \mathcal{F}_0^{V'}$-measurable and $\mathbf{E}[D_n] = \mathbf{E}[|R_0 - S_n|]$ tends to 0 as $n \to -\infty$. Therefore, R_0 is $\mathcal{U}_0 \vee \mathcal{F}_0^{V'}$-measurable modulo $\mathbf{P}$.

The filtration $(\mathcal{U}_{t_n} \vee \mathcal{F}_{t_n}^{V'})_{n\le 0}$ is not only included but also immersed in the filtration $(\mathcal{Z}_{t_n})_{n\le 0}$, since these two filtrations admit a common sequence of innovations, namely $((U_{t_{n-1},t_n}, V'_{t_{n-1},t_n}))_{n\le 0}$. Since their final σ-field coincide modulo $\mathbf{P}$, these filtrations are almost surely equal.

Annex: About the notion of $(\mathbf{P}_u)_{u\in E}$-separability. Let U be a random variable on $(\Omega, \mathcal{A}, \mathbf{P})$, taking values in some measurable space $(E, \mathcal{E})$, and $(\mathbf{P}_u)_{u\in E}$ a regular version of the conditional probability $\mathbf{P}$ given U. According to [16], one says that a sub-σ-field $\mathcal{G}$ of $\mathcal{A}$ is $(\mathbf{P}_u)_{u\in E}$-separable if there exists some sub-σ-field $\mathcal{H}$ of $\mathcal{G}$, generated (without completion) by some countable family of events, such that for $U(\mathbf{P})$-almost every $u \in E$, $\mathcal{G} = \mathcal{H}$ mod $\mathbf{P}_u$. This definition leads to two remarks.

1. The σ-field $\mathcal{H}$ introduced above does not depend of $u \in E$, so the $(\mathbf{P}_u)_{u\in E}$-separability is not equivalent to the essential separability under $\mathbf{P}_u$ for $U(\mathbf{P})$-almost every $u \in E$.
2. For every events A and B, the equality $\mathbf{P}(A\triangle B) = 0$ implies that $\mathbf{P}_u(A\triangle B) = 0$ for $U(\mathbf{P})$-almost every $u \in E$, but this $U(\mathbf{P})$-almost sure subset of E may depend on A and B.

Actually, the $(\mathbf{P}_u)_{u\in E}$-separability is a rather subtle notion. It is much stronger than the essential separability under $\mathbf{P}_u$ for $U(\mathbf{P})$-almost every $u \in E$, and it is not implied by the essential separability under $\mathbf{P}$. We give now a counterexample illustrating these two statements.

A good way to identify the problem is to consider the simplest example where the exchange property fails. Thus, we take i.i.d. uniform random variables $(Z_n)_{n\le 0}$ on $\{-1, 1\}$, we set $U_n = Z_{n-1}Z_n$ for every $n \le 0$, $U = (U_n)_{n\le 0}$ and $\mathbf{P}_u = \mathbf{P}[\cdot|U = u]$. One checks that the whole sequence $(Z_n)_{n\le 0}$ can be recovered from U and from any random variable Z_n; moreover, Z_0 is independent of U, so the inclusion

$$\sigma(U) \vee \mathcal{F}_{-\infty}^Z \subset \bigcap_{n\le 0} \bigl(\sigma(U) \vee \mathcal{F}_n^Z\bigr) = \mathcal{F}_0^Z$$

is strict modulo $\mathbf{P}$.

By Kolmogorov's 0–1 law, the tail σ-field $\mathcal{F}_{-\infty}^Z$ is trivial (thus essentially separable) under $\mathbf{P}$. But $\mathcal{F}_{-\infty}^Z$ is not trivial anymore under the probability measures $\mathbf{P}_u$: given $U = u$, there are exactly two possibilities for the sequence $(Z_n)_{n\le 0}$, each occurs with probability $1/2$, and the choice among these two possibilities is asymptotic at time $-\infty$. More precisely, $Z_0 = \lim_{n\to-\infty} Z_n u_{n+1} \cdots u_0$ $\mathbf{P}_u$-almost surely and Z_0 generates $\mathcal{F}_{-\infty}^Z$ modulo $\mathbf{P}_u$. Thus $\mathbf{P}_u$ has two atoms on $\mathcal{F}_{-\infty}^Z$, each one of mass $1/2$, so $\mathcal{F}_{-\infty}^Z$ is still essentially separable under $\mathbf{P}_u$.

Yet, $\mathcal{F}^Z_{-\infty}$ is not $(\mathbf{P}_u)_{u\in E}$-separable. By von Weizsäcker's theorem [16], this statement follows from the $(\mathbf{P}_u)_{u\in E}$-separability of each $\mathcal{F}^Z_n$ and from the fact that the exchange property fails.

Here is a direct proof (without using von Weizsäcker's theorem) of the non-$(\mathbf{P}_u)_{u\in E}$-separability of $\mathcal{F}^Z_{-\infty}$. Let $\mathcal{H}$ be a sub-σ-field of $\mathcal{F}^Z_{-\infty}$, generated (without completion) by some sequence $(A_n)_{n\geq 0}$ of events. For every $n \geq 0$, $\mathbf{P}(A_n)$ is 0 or 1. Replacing some A_n by their complement does not modify the σ-field generated by the sequence $(A_n)_{n\geq 0}$, so one may assume that $\mathbf{P}(A_n) = 1$ for every $n \geq 0$. Since n varies in a countable set, one gets that for $U(\mathbf{P})$ almost every u, $\mathbf{P}_u(A_n) = 1$ for every $n \geq 0$, therefore, $\mathcal{H}$ is trivial under $\mathbf{P}_u$, so $\mathcal{H}$ cannot be equal to $\mathcal{F}^Z_{-\infty}$ modulo $\mathbf{P}_u$.

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