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INTEGRAL IDENTITY AND MEASURE ESTIMATES FOR STATIONARY FOKKER–PLANCK EQUATIONS

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We consider a Fokker–Planck equation in a general domain in $\mathbb{R}^n$ with L^p_{loc} drift term and $W^{1,p}_{\mathrm{loc}}$ diffusion term for any $p > n$. By deriving an integral identity, we give several measure estimates of regular stationary measures in an exterior domain with respect to diffusion and Lyapunov-like or anti-Lyapunov-like functions. These estimates will be useful to problems such as the existence and nonexistence of stationary measures in a general domain as well as the concentration and limit behaviors of stationary measures as diffusion vanishes.

1. Introduction. Consider the stationary Fokker–Planck equation

$$
\begin{cases} Lu(x) =: \partial^2_{ij}\big(a^{ij}(x)u(x)\big) - \partial_i\big(V^i(x)u(x)\big) = 0, & x \in \mathcal{U}, \\ u(x) \geq 0, & \displaystyle\int_{\mathcal{U}} u(x)\,\mathrm{d}x = 1, \end{cases} \tag{1.1}
$$

where $\mathcal{U}$ is a connected open set in $\mathbb{R}^n$ which can be bounded, unbounded, or the entire space $\mathbb{R}^n$, L is the Fokker–Planck operator, $A = (a^{ij})$ is an everywhere positive semidefinite, $n \times n$-matrix valued function on $\mathcal{U}$, called the diffusion matrix, and $V = (V^i)$ is a vector field on $\mathcal{U}$ valued in $\mathbb{R}^n$, called the drift field. This equation is in fact the one satisfied by stationary solutions of the Fokker–Planck equation

$$
\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = Lu(x,t), & x \in \mathcal{U}, t > 0, \\ u(x,t) \geq 0, & \displaystyle\int_{\mathcal{U}} u(x,t)\,\mathrm{d}x = 1. \end{cases} \tag{1.2}
$$

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In the above and also throughout the rest of the paper, we use short notation $\partial_i = \partial/\partial x_i$, $\partial^2_{ij} = \partial^2/\partial x_i\,\partial x_j$, and we also adopt the usual summation convention on $i, j = 1, 2, \dots, n$ whenever applicable.

Following [10, 12, 13], etc., we make the following standard hypothesis:

(A) $a^{ij} \in W^{1,p}_{\mathrm{loc}}(\mathcal{U})$, $V^i \in L^p_{\mathrm{loc}}(\mathcal{U})$ for all $i, j = 1, \dots, n$, where $p > n$ is fixed.

Under the regularity condition (A), in the weakest situation one considers measure solutions of (1.1), called *stationary measures* of the Fokker–Planck equation (1.2), which are Borel probability measures μ satisfying

$$V^i \in L^1_{\mathrm{loc}}(\mathcal{U}, \mu), \qquad i = 1, 2, \dots, n \quad \text{and} \tag{1.3}$$

$$\int_{\mathcal{U}} \mathcal{L} f(x)\,\mathrm{d}\mu(x) = 0 \qquad \text{for all } f \in C^\infty_0(\mathcal{U}), \tag{1.4}$$

where

$$\mathcal{L} = a^{ij}\,\partial^2_{ij} + V^i\,\partial_i$$

is the adjoint Fokker–Planck operator and $C^\infty_0(\mathcal{U})$ denotes the space of C^∞ functions on $\mathcal{U}$ with compact supports. If a stationary measure μ of (1.2) is *regular* with density u, that is, $\mathrm{d}\mu(x) = u(x)\,\mathrm{d}x$ for some $u \in C(\mathcal{U})$, then it is clear that u must be a *weak stationary solution* of (1.2), that is,

$$\begin{cases} \displaystyle\int_{\mathcal{U}} \mathcal{L} f(x) u(x)\,\mathrm{d}x = 0, & \text{for all } f \in C^\infty_0(\mathcal{U}), \\ u(x) \geq 0, & \displaystyle\int_{\mathcal{U}} u(x)\,\mathrm{d}x = 1. \end{cases} \tag{1.5}$$

In fact, under condition (A) and that $A = (a^{ij})$ is everywhere positive definite in $\mathcal{U}$, it follows from a regularity theorem due to Bogachev–Krylov–Röckner [9] that any stationary measure μ of (1.2) must admit a positive density $u \in W^{1,p}_{\mathrm{loc}}(\mathcal{U})$.

The purpose of the present paper is to provide several useful measure estimates, in an exterior domain $\mathcal{U} \setminus K$ for a compact subset K of $\mathcal{U}$, of regular stationary measures of (1.2) with densities lying in $W^{1,p}_{\mathrm{loc}}(\mathcal{U})$. Such exterior estimates are evidentally important especially when $\mathcal{U}$ is unbounded (e.g., $\mathcal{U} = \mathbb{R}^n, \mathbb{R}^n_+$) or (a^{ij}) is degenerate on the boundary of $\mathcal{U}$. The measure estimates contained in this paper are nontrivial because they do not follow from the existing theory of elliptic equations even if (a^{ij}) is everywhere positive definite in $\mathcal{U}$. Indeed, as to be seen in the paper, measure estimates we give in this paper crucially rely on an integral identity (Theorem 2.1) which reveals fundamental natures of stationary Fokker–Planck equations and enables one to estimate the measure in a subdomain by making use of information of noise distributions on the boundary of the domain. In fact, the integral identity plays a similar role as the Pohozaev identity does to semilinear elliptic equations. It is because of this identity that our essential measure estimates can be made regardless of the positive definiteness of (a^{ij}) in $\mathcal{U}$.

Our measure estimates in an exterior domain will be made with respect to diffusions and derivatives of a Lyapunov-like or an anti-Lyapunov-like function which is primarily a compact function in the domain.

DEFINITION 1.1. A nonnegative function $U \in C(\mathcal{U})$ is said to be a *compact function in* $\mathcal{U}$ if:

(i) $U(x) < \rho_M$, $x \in \mathcal{U}$; and
(ii) $\lim_{x \to \partial\mathcal{U}} U(x) = \rho_M$,

where $\rho_M = \sup_{x \in \mathcal{U}} U(x)$ is called the *essential upper bound of* U.

When $\mathcal{U}$ is unbounded, $\partial\mathcal{U}$ and the limit $x \to \partial\mathcal{U}$ in (ii) above should be understood under the topology which is defined through a fixed homeomorphism between the extended Euclidean space $\mathbb{E}^n = \mathbb{R}^n \cup \partial\mathbb{R}^n$ and the closed unit ball $\bar{\mathbb{B}}^n = \mathbb{B}^n \cup \partial\mathbb{B}^n$ in $\mathbb{R}^n$ which identifies $\mathbb{R}^n$ with B^n and $\partial\mathbb{R}^n$ with $\mathbb{S}^{n-1}$, and in particular, identifies each $x_* \in \mathbb{S}^{n-1}$ with the infinity element $x_*^\infty \in \partial\mathbb{R}^n$ of the ray through x_*. Consequently, if $\mathcal{U} = \mathbb{R}^n$, then $x \to \partial\mathbb{R}^n$ under this topology simply means $x \to \infty$ in the usual sense, and it is easy to see that an unbounded, nonnegative function $U \in C(\mathbb{R}^n)$ is a compact function in $\mathbb{R}^n$ iff

$$\lim_{x \to \infty} U(x) = +\infty. \tag{1.6}$$

For simplicity, we will use the same symbol Ω_ρ to denote the ρ-sublevel set $\{x \in \mathcal{U} : U(x) < \rho\}$ of any compact function U on $\mathcal{U}$.

DEFINITION 1.2. Let U be a C^2 compact function in $\mathcal{U}$.

(i) U is called a *Lyapunov function* (resp., *anti-Lyapunov function*) in $\mathcal{U}$ with respect to $\mathcal{L}$, if there is a $\rho_m \in (0, \rho_M)$, called *essential lower bound of* U, and a constant $\gamma > 0$, called *Lyapunov constant* (resp., *anti-Lyapunov constant*) of U, such that

$$\mathcal{L}U(x) \le -\gamma \qquad (\text{resp.}, \ge \gamma), x \in \tilde{\mathcal{U}} = \mathcal{U} \setminus \bar{\Omega}_{\rho_m}, \tag{1.7}$$

where $\tilde{\mathcal{U}}$ is called *essential domain of* U.

(ii) If $\gamma = 0$ in (1.7), then U is referred to as a *weak Lyapunov function* (resp., *weak anti-Lyapunov function*) in $\mathcal{U}$ with respect to $\mathcal{L}$.

Below, for any C^1 compact function U on $\mathcal{U}$ with essential upper bound ρ_M, we let h, H be two nonnegative, locally bounded functions on $[0, \rho_M)$ such that

$$h(\rho) \le a^{ij}(x)\,\partial_i U(x)\,\partial_j U(x) \le H(\rho), \qquad x \in U^{-1}(\rho), \rho \in [0, \rho_M), \tag{1.8}$$

where $U^{-1}(\rho)$ denotes the ρ-level set of U. For instance, $h(\rho)$, respectively, $H(\rho)$, can be taken as the infimum, respectively, the supremum, of $a^{ij}(x)\,\partial_i U(x)\,\partial_j U(x)$ on $U^{-1}(\rho)$. For simplicity, the dependency of h, H on U will be made implicit.

For a regular stationary measure of (1.2) with density lying in $W^{1,p}_{\mathrm{loc}}(\mathcal{U})$, the result below gives some upper bound estimates of the measure in the essential domain of a Lyapunov-like function in $\mathcal{U}$ with respect to $\mathcal{L}$.

THEOREM A. *Assume* (A) *and that there is either a Lyapunov or a weak Lyapunov function U in $\mathcal{U}$ with respect to $\mathcal{L}$ with essential lower, upper bound ρ_m, ρ_M, respectively. Then the following hold for any regular stationary measure μ of* (1.2) *with density lying in $W^{1,p}_{\mathrm{loc}}(\mathcal{U})$*:

(a) *If U is a Lyapunov function with Lyapunov constant γ, then for any $\rho_0 \in (\rho_m, \rho_M)$, there exists a constant $C_{\rho_m,\rho_0} > 0$ depending only on ρ_m, ρ_0 such that*

$$\mu(\mathcal{U} \setminus \Omega_{\rho_0}) \le \gamma^{-1} C_{\rho_m,\rho_0}\Big(\sup_{(\rho_m,\rho_0)} H\Big)\mu(\Omega_{\rho_0} \setminus \Omega_{\rho_m}),$$

where H is as in (1.8).

(b) *If, in addition, the Lyapunov function U in* (a) *satisfies*

$$\nabla U(x) \neq 0 \qquad \forall x \in U^{-1}(\rho) \text{ for a.e. } \rho \in [\rho_m, \rho_M), \tag{1.9}$$

then

$$\mu(\mathcal{U} \setminus \Omega_\rho) \le \mathrm{e}^{-\gamma \int_{\rho_m}^{\rho} 1/H(t)\,\mathrm{d}t}, \qquad \rho \in [\rho_m, \rho_M),$$

where H is as in (1.8).

(c) *If U is a weak Lyapunov function such that in* (1.8) *h is positive and H is continuous on $[\rho_m, \rho_M)$, then for any $\rho_0 \in (\rho_m, \rho_M)$,*

$$\mu(\mathcal{U} \setminus \Omega_{\rho_m}) \le \mu(\Omega_{\rho_0} \setminus \Omega_{\rho_m})\mathrm{e}^{\int_{\rho_0}^{\rho_M} 1/\tilde{H}(\rho)\,\mathrm{d}\rho},$$

where $\tilde{H}(\rho) = h(\rho)\int_{\rho_m}^{\rho} 1/(H(s))\,\mathrm{d}s$, $\rho \in [\rho_m, \rho_M)$.

We note by Sard's theorem that if $U \in C^n(\mathcal{U})$, then the set of regular values of U is of full Lebesgue measure in $[\rho_m, \rho_M)$, that is, (1.9) is automatically satisfied when $U \in C^n(\mathcal{U})$.

For a regular stationary measure of (1.2) with density lying in $W^{1,p}_{\mathrm{loc}}(\mathcal{U})$, the result below gives some lower bound estimates of the measure in the essential domain of an anti-Lyapunov-like function in $\mathcal{U}$ with respect to $\mathcal{L}$.

THEOREM B. *Assume* (A) *and that there is either an anti-Lyapunov or a weak anti-Lyapunov function U in $\mathcal{U}$ with respect to $\mathcal{L}$ with essential lower, upper bound ρ_m, ρ_M, respectively. Then the following hold for any regular stationary measure μ of* (1.2) *with density lying in $W^{1,p}_{\mathrm{loc}}(\mathcal{U})$ and any $\rho_0 \in (\rho_m, \rho_M)$*:

(a) *If U is an anti-Lyapunov function with anti-Lyapunov constant γ such that* (1.9) *holds, then*

$$\mu\big(\Omega_\rho \setminus \Omega^*_{\rho_m}\big) \ge \mu\big(\Omega_{\rho_0} \setminus \Omega^*_{\rho_m}\big)\mathrm{e}^{\gamma \int_{\rho_0}^{\rho} 1/H(t)\,\mathrm{d}t}, \qquad \rho \in (\rho_0, \rho_M),$$

where $\Omega^*_{\rho_m} = \Omega_{\rho_m} \cup U^{-1}(\rho_m)$ and H is as in (1.8).

(b) *If U is a weak anti-Lyapunov function such that h in* (1.8) *is positive and continuous on $[\rho_m, \rho_M)$, then*

$$\mu(\Omega_\rho \setminus \Omega_{\rho_m}) \geq \mu(\Omega_{\rho_0} \setminus \Omega_{\rho_m}) \mathrm{e}^{\int_{\rho_0}^{\rho} 1/\tilde{H}(t)\,\mathrm{d}t}, \qquad \rho \in [\rho_0, \rho_M),$$

where $\tilde{H}(\rho) = H(\rho) \int_{\rho_m}^{\rho} \frac{1}{h(s)}\,\mathrm{d}s$, $\rho \in [\rho_m, \rho_M)$.

Following the pioneering work of Has'minskiĭ [17, 18] for locally Lipschitz coefficients, the existence and uniqueness of regular stationary measures of (1.2) in $\mathbb{R}^n$ have been extensively studied when (a^{ij}) is everywhere positive definite (see, e.g., [1–4, 6, 8–15] and [22–25]). In particular, Veretennikov [24] showed the existence when (a^{ij}) is continuous and bounded under sup-norm, and V is measurable, locally bounded in $\mathbb{R}^n$ and satisfies

$$V(x) \cdot x \leq -\gamma, \qquad |x| \gg 1$$

for some positive constant γ depending on (a^{ij}). Later, Bogachev–Röckner [10] showed the existence and uniqueness under condition (A) when there exists an unbounded Lyapunov function in $\mathbb{R}^n$ with respect to $\mathcal{L}$ such that

$$\lim_{x \to \infty} \mathcal{L}U(x) = -\infty.$$

In this work, (a^{ij}) is even allowed to be degenerate in $\mathbb{R}^n$ for the existence of a stationary measure that is not necessarily regular. Recently, Arapostathis–Borkar–Ghosh [2], Theorem 2.6.10, showed the existence when $(a^{ij}), (V^i)$ are locally Lipschitz and do not grow faster than linearly at ∞, and there exists a so-called inf-compact function satisfying (1.7) in $\mathbb{R}^n$ with "$\leq$" sign for some $\gamma > 0$. Bogachev–Röckner–Shaposhnikov [11] proved the existence under condition (A) when there exists an unbounded Lyapunov function U in $\mathbb{R}^n$ with respect to $\mathcal{L}$.

As shown in our work [19], the measure estimates contained in Theorems A, B above are useful in dealing with problems of the existence and nonexistence of stationary measures of (1.2) in a general domain $\mathcal{U}$ involving Lyapunov and weak Lyapunov functions for the existence and anti-Lyapunov and weak anti-Lyapunov functions for the nonexistence. Also, as explored in our works [20, 21], these estimates play important roles in characterizing the concentration of stationary measures at both global and local levels as well as in studying limit behaviors of a family of stationary measures as diffusion matrices vanish. In particular, even when we consider local concentration of stationary measures defined in the entire space $\mathbb{R}^n$, the stationary measures can be restricted to a subdomain in order to apply these estimates. This is another motivation for us to consider these estimates in a general domain.

This paper is organized as follows. In Section 2, we derive two identities—an integral identity and a derivative formula, which are of fundamental importance to the level set method to be adopted in this paper. We prove Theorem A(a) in

Section 3, Theorem A(b) and Theorem B(a) in Section 4, and Theorem A(c) and Theorem B(b) in Section 5.

Throughout the rest of the paper, for simplicity, we will use the same symbol $|\cdot|$ to denote absolute value of a number, cardinality of a set and norm of a vector or a matrix.

2. Ingredients of level set method. Our measure estimates will be carried out using the level set method. In this section, we will prove two fundamental identities involved in the level set method for conducting measure estimates of stationary measures of (1.2). One is an integral identity which will play a crucial role in capturing information of a weak stationary solution of (1.2) in each sublevel set of a Lyapunov-like or an anti-Lyapunov-like function from its boundary. The other one is a derivative formula which will be particularly useful in the measure estimates of a stationary measure of (1.2) with respect to functions h, H in (1.8).

We call a bounded open set Ω in $\mathbb{R}^n$ a *generalized Lipschitz domain* if (i) it is a disjoint union of finitely many Lipschitz subdomains; and (ii) intersections of boundaries among these Lipschitz subdomains only occur at finitely many points.

THEOREM 2.1 (Integral identity). *Assume that* (A) *holds in a domain* $\Omega\subset\mathbb{R}^n$ *and let* $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ *be a weak stationary solution of* (1.2) *in* Ω. *Then for any generalized Lipschitz domain* $\Omega'\subset\subset\Omega$ *and any function* $F\in C^2(\bar{\Omega}')$ *with* $F|_{\partial\Omega'}=$ *constant*,

$$\int_{\Omega'}(\mathcal{L}F)u\,\mathrm{d}x=\int_{\partial\Omega'}\bigl(a^{ij}\,\partial_i F\nu_j\bigr)u\,\mathrm{d}s, \tag{2.1}$$

where for a.e. $x\in\partial\Omega'$, $(\nu_j(x))$ *denotes the unit outward normal vector of* $\partial\Omega'$ *at* x.

PROOF. Let $F|_{\partial\Omega'}=c$ and Ω_* be a smooth domain such that $\Omega'\subset\subset\Omega_*\subset\subset\Omega$. Consider the function

$$\tilde{F}(x)=\begin{cases}F(x)-c, & x\in\Omega',\\ 0, & x\in\Omega\setminus\Omega'.\end{cases}$$

Clearly, $\tilde{F}\in W^{1,\infty}(\Omega)$ and $\operatorname{supp}(\tilde{F})\subset\bar{\Omega}'$. For any $0<h<1$, we let $\tilde{F}_h$ be the regularization of $\tilde{F}$ in Ω, that is,

$$\tilde{F}_h(x)=h^{-n}\int_{\Omega}\xi\left(\frac{x-y}{h}\right)\tilde{F}(y)\,\mathrm{d}y,$$

where the function ξ is a mollifier—a nonnegative C^∞ function in $\mathbb{R}^n$ vanishing outside of the unit ball of $\mathbb{R}^n$ centered at the origin and satisfying $\int_{\mathbb{R}^n}\xi(x)\,\mathrm{d}x=1$. Then $\tilde{F}_h\in C_0^\infty(\Omega)$, $\operatorname{supp}(\tilde{F}_h)\subset\bar{\Omega}_*$ as $0<h\ll 1$, and $\tilde{F}_h\to\tilde{F}$ in $W^{1,q}(\Omega_*)$, as $h\to 0$, for any $0<q<\infty$. Since u is a weak stationary solution of (1.2) in Ω,

$$\int_{\Omega}\bigl(a^{ij}\,\partial^2_{ij}\tilde{F}_h+V^i\,\partial_i\tilde{F}_h\bigr)u\,\mathrm{d}x=0\qquad\text{as }0<h\ll 1. \tag{2.2}$$

We note that $a^{ij}, u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $i, j = 1, 2, \ldots, n$. We have by passing to the limit $h \to 0$ that

$$\begin{aligned}\int_\Omega ua^{ij}\,\partial^2_{ij}\tilde{F}_h\,\mathrm{d}x &= \int_{\Omega_*} ua^{ij}\,\partial^2_{ij}\tilde{F}_h\,\mathrm{d}x = -\int_{\Omega_*}\partial_j\big(ua^{ij}\big)(\partial_i\tilde{F}_h)\,\mathrm{d}x\\ &\to -\int_{\Omega_*}\partial_j\big(ua^{ij}\big)(\partial_i\tilde{F})\,\mathrm{d}x = -\int_{\Omega'}\partial_j\big(ua^{ij}\big)(\partial_i F)\,\mathrm{d}x\\ &= \int_{\Omega'} ua^{ij}\,\partial^2_{ij}F\,\mathrm{d}x - \int_{\partial\Omega'} ua^{ij}\,\partial_i F\nu_j\,\mathrm{d}s.\end{aligned}\tag{2.3}$$

On the other hand, we note by the Sobolev embedding theorem that $u \in C(\bar{\Omega}_*)$, and hence $V^i u \in L^p(\Omega_*)$, $i = 1, 2, \ldots, n$. Thus, we can also pass to the limit $h \to 0$ to obtain

$$\begin{aligned}\int_\Omega uV^i\,\partial_i\tilde{F}_h\,\mathrm{d}x &= \int_{\Omega_*} uV^i\,\partial_i\tilde{F}_h\,\mathrm{d}x\\ &\to \int_{\Omega_*} uV^i\,\partial_i\tilde{F}\,\mathrm{d}x = \int_{\Omega'} uV^i\,\partial_i\tilde{F}\,\mathrm{d}x = \int_{\Omega'} uV^i\,\partial_i F\,\mathrm{d}x.\end{aligned}\tag{2.4}$$

The theorem now follows from (2.2)–(2.4). $\square$

REMARK 2.1. 1. We note that the theorem does not require (a^{ij}) to be even positive semidefinite. It also holds for less regular (a^{ij}), (V^i), and u, as long as $a^{ij}u \in W^{1,\alpha}_{\mathrm{loc}}(\Omega)$ and $V^i u \in L^\alpha_{\mathrm{loc}}(\Omega)$, $\forall\, i, j, = 1, 2, \ldots, n$, for some $\alpha > 1$.

2. In applying the integral identity (2.1), one typically chooses Ω' as a sublevel set of a Lyapunov-like or an anti-Lyapunov-like function U. Of course, Ω', being such a sublevel set, need not be a generalized Lipschitz domain. As will be seen in the next section, a technique to get around that is to use the approximation of U by Morse functions.

THEOREM 2.2 (Derivative formula). *Let μ be a Borel probability measure with density $u \in C(\mathcal{U})$. For a compact function $U \in C^1(\mathcal{U})$, consider the measure function*

$$y(\rho) := \mu(\Omega_\rho) = \int_{\Omega_\rho} u\,\mathrm{d}x, \qquad \rho \in (0, \rho_M),$$

and the open set

$$\mathcal{I} =: \{\rho \in (0, \rho_M) : \nabla U(x) \neq 0, x \in U^{-1}(\rho)\},\tag{2.5}$$

where ρ_M is the essential upper bound of U and Ω_ρ is the ρ-sublevel set of U for each $\rho \in (0, \rho_M)$. Then y is of the class C^1 on $\mathcal{I}$ with derivatives

$$y'(\rho) = \int_{\partial\Omega_\rho} \frac{u}{|\nabla U|}\,\mathrm{d}s, \qquad \rho \in \mathcal{I}.\tag{2.6}$$

PROOF. Since U is a compact function on $\mathcal{U}$, it is easy to see that $\partial\Omega_\rho \subset U^{-1}(\rho)$ for all $\rho \in (0, \rho_M)$. Let $\rho \in \mathcal{I}$. Then $\nabla U(x) \neq 0$, $x \in \partial\Omega_\rho$. Hence, $\partial\Omega_\rho$ is a C^1 hypersurface which coincides with $U^{-1}(\rho)$.

Let $\mathcal{T}| = \{(x, e_j) : j = 1, 2, \dots, n\}$ be an orientation preserving, orthonormal, moving frame defined over $\partial\Omega_\rho$ such that for each $x \in \partial\Omega_\rho$, $e_j = e_j(x)$, $j = 1, 2, \dots, n-1$, are tangent vectors, and $e_n = e_n(x)$ is the outward unit normal vector, of $\partial\Omega_\rho$ at x. We denote $\{(x, \omega^j) : j = 1, 2, \dots, n\}$ as the dual frame of $\mathcal{T}$ defined over $\partial\Omega_\rho$, that is, $\omega^i(e_j) = \delta^i_j$, $i, j = 1, 2, \dots, n$. Since, for each $x \in \partial\Omega_\rho$, $e_n = \nabla U/|\nabla U|$, we have $\omega^n = \mathrm{d}U/|\nabla U|$. Therefore,

$$\mathrm{d}x = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = \mathrm{d}s \wedge \omega^n = \frac{1}{|\nabla U|}\,\mathrm{d}s\,\mathrm{d}U,$$

where $\mathrm{d}s = \omega^1 \wedge \cdots \wedge \omega^{n-1}$ is a volume form defined on $\partial\Omega_\rho$, from which (2.6) easily follows.

Continuity of $y'(\rho)$ on $\mathcal{I}$ follows from (2.6). □

REMARK 2.2. In fact, the derivative formula (2.6) is known when $|\nabla U(x)| \geq c > 0$ a.e. in $\mathbb{R}^n$ (see [5], Proposition 5.8.34), and is already used in [7], Proposition 2, for level set estimates concerning functions that satisfy (1.6).

3. Proof of Theorem A(a). Let U be a Lyapunov function in $\mathcal{U}$ with respect to $\mathcal{L}$ with Lyapunov constant γ and essential lower bound ρ_m and upper bound ρ_M and let Ω_ρ denote the ρ-sublevel set of U for each $\rho \in [\rho_m, \rho_M)$.

Given $\rho \in [\rho_m, \rho_M)$, we fix a $\rho^* \in (\rho, \rho_M)$. Since Morse functions are dense in $C^2(\mathcal{U})$, there is a sequence $U_k \in C^2(\mathcal{U})$, $k = 1, 2, \dots,$ of Morse functions such that $U_k \to U$ in $C^2(\mathcal{U})$, in particular, $U_k \to U$ in $C^2(\bar{\Omega}_{\rho^*})$, as $k \to \infty$. For each k, denote

$$\Omega^k_\rho = \bigl\{x \in \Omega_{\rho^*} : \text{ either } U_k(x) < \rho \text{ or } x \text{ is a local maximal point of } U_k \text{ lying in } U_k^{-1}(\rho)\bigr\}.$$

It is obvious that Ω^k_ρ's are nonempty open sets for all $k \geq 1$.

LEMMA 3.1. *There is a positive integer $k(\rho)$ such that $\Omega^k_\rho \subset\subset \Omega_{\rho^*}$ for all $k \geq k(\rho)$.*

PROOF. If this is not true, then there are sequences $k_i \to \infty$, $x_i \in \partial\Omega^{k_i}_\rho$, $i = 1, 2, \dots,$ such that $x_i \in \partial\Omega_{\rho^*}$. Then $U(x_i) = \rho^*$ for all i. Since $\bar{\Omega}_{\rho^*}$ is compact, we may assume without loss of generality that $\{x_i\}$ converges, say, to some $\bar{x} \in \bar{\Omega}_{\rho^*}$. On one hand, we have $U(\bar{x}) = \rho^*$. But on the other hand, since $\rho \geq U_{k_i}(x_i)$ and $U_{k_i} \to U$ uniformly on $\bar{\Omega}_{\rho^*}$, taking limit $i \to \infty$ yields that $\rho \geq U(\bar{x})$. It follows that $\rho \geq \rho^*$, a contradiction. □

LEMMA 3.2. *Ω_ρ^k is a generalized Lipschitz domain for each $k \geq k(\rho)$.*

PROOF. We only consider the case $n > 1$ because the case with $n = 1$ is trivial. Let $k \geq k(\rho)$ be fixed. We note by claim 1 that $\partial\Omega_\rho^k$ is compact and contained in $U_k^{-1}(\rho)$. Consider a point $x_0 \in \partial\Omega_\rho^k$. If $\nabla U_k(x_0) \neq 0$, then the implicit function theorem implies that, in a neighborhood of x_0, $\partial\Omega_\rho^k$ is actually a C^2 hypersurface which coincides with $U_k^{-1}(\rho)$. Let $\nabla U_k(x_0) = 0$. Then the Hessian $D^2U_k(x_0)$ is nondegenerate because U_k is a Morse function. If $D^2U_k(x_0)$ is positive definite, then x_0 is a local minimal point of U_k, and thus it cannot lie in $\bar{\Omega}_\rho^k$. If $D^2U_k(x_0)$ is negative definite, then x_0 is a local maximal point of U_k and thus it must lie in the interior Ω_ρ^k. Hence, $D^2U_k(x_0)$ must be a hyperbolic matrix. Let $1 \leq M < n$ be the number of positive eigenvalues of $D^2U_k(x_0)$. Then by the Morse lemma [16], there is a C^2 local change of coordinates $v = (v_1, \dots, v_n) = \Phi(x)$ in a neighborhood of x_0 under which $\Phi(x_0) = 0$ and

$$U_k\big(\Phi^{-1}(v)\big) = \rho + v_1^2 + \cdots + v_M^2 - v_{M+1}^2 - \cdots - v_n^2.$$

It follows that, near x_0, $\partial\Omega_\rho^k = U_k^{-1}(\rho)$ is a union of two Lipschitz hypersurfaces intersecting at x_0; each belongs to the boundary of a component of Ω_ρ^k. Since all nondegenerate critical points of U_k are isolated and $\partial\Omega_\rho^k$ is a compact set, the number of critical points of U_k on $\partial\Omega_\rho^k$ must be finite. Consequently, the number of connected components of Ω_ρ^k which contain nondegenerate critical points on their boundaries are finite. The number of connected components of Ω_ρ^k which contain no critical points on their boundaries is also finite, because each such a component is separated from the rest of Ω_ρ^k. Thus, Ω_ρ^k is a generalized Lipschitz domain. $\square$

PROOF OF THEOREM A(A). Let μ be a regular stationary measure of (1.2) with density $u \in W^{1,p}_{\mathrm{loc}}(\mathcal{U})$.

For given $\rho_0 \in (\rho_m, \rho_M)$, we consider a fixed monotonically increasing function $\phi \in C^2(\mathbb{R}_+)$ satisfying

$$\phi(t) = \begin{cases} 0, & \text{if } t \in [0, \rho_m]; \\ t, & \text{if } t \in [\rho_0, +\infty). \end{cases}$$

We note that $\phi''(t) = 0$ for all $t \in [0, \rho_m] \cup [\rho_0, +\infty)$.

Let $\rho \in (\rho_0, \rho_M)$ and $\rho^* \in (\rho, \rho_M)$. Since $\phi \circ U_k \equiv \rho$ on $\partial\Omega_\rho^k$, using Lemmas 3.1, 3.2, we can apply Theorem 2.1 with $F = \phi \circ U_k$, $\Omega = \Omega_{\rho^*}$, and $\Omega' = \Omega_\rho^k$ for each $k \geq k(\rho)$ to obtain the identity

$$\int_{\Omega_\rho^k} \big(a^{ij}\,\partial_{ij}^2\phi(U_k) + V^i\,\partial_i\phi(U_k)\big)u\,\mathrm{d}x = \int_{\partial\Omega_\rho^k} ua^{ij}\,\partial_i\phi(U_k)\nu_j\,\mathrm{d}s,$$

that is,

$$\begin{aligned}\int_{\Omega_\rho^k} \phi'(U_k)(\mathcal{L}U_k)u\,\mathrm{d}x + \int_{\Omega_\rho^k} \phi''(U_k)\big(a^{ij}\,\partial_i U_k\,\partial_j U_k\big)u\,\mathrm{d}x \\ = \int_{\partial\Omega_\rho^k} \phi'(U_k)ua^{ij}\,\partial_i U_k \nu_j\,\mathrm{d}s = \int_{\partial\Omega_\rho^k} ua^{ij}\,\partial_i U_k \nu_j\,\mathrm{d}s,\end{aligned} \tag{3.1}$$

where (ν_j) denote the unit outward normal vectors of $\partial\Omega_\rho^k$. For each $k \geq k(\rho)$, if $\nabla U_k(x_0) \neq 0$ at some $x_0 \in \partial\Omega_\rho^k$, then the implicit function theorem implies that there is a neighborhood of x_0 on $\partial\Omega_\rho^k$ such that $\nu(x) = (\nabla U_k(x))/(|\nabla U_k(x)|)$ within the neighborhood. Thus,

$$a^{ij}(x)\,\partial_i U_k(x)\nu_j(x) \geq 0, \qquad x \in \partial\Omega_\rho^k.$$

It then follows from (3.1) that

$$\int_{\Omega_\rho^k} \phi'(U_k)(\mathcal{L}U_k)u\,\mathrm{d}x + \int_{\Omega_\rho^k} \phi''(U_k)\big(a^{ij}\,\partial_i U_k\,\partial_j U_k\big)u\,\mathrm{d}x \geq 0,$$

that is,

$$\begin{aligned}&\int_{\Omega_{\rho^*}\setminus U^{-1}(\rho)} \chi_{\Omega_\rho^k}\phi'(U_k)(\mathcal{L}U_k)u\,\mathrm{d}x + \int_{U^{-1}(\rho)\cap\Omega_\rho^k} \phi'(U_k)(\mathcal{L}U_k)u\,\mathrm{d}x \\ &\quad \geq -\int_{\Omega_{\rho^*}\setminus U^{-1}(\rho)} \chi_{\Omega_\rho^k}\phi''(U_k)\big(a^{ij}\,\partial_i U_k\,\partial_j U_k\big)u\,\mathrm{d}x \\ &\qquad - \int_{U^{-1}(\rho)\cap\Omega_\rho^k} \phi''(U_k)\big(a^{ij}\,\partial_i U_k\,\partial_j U_k\big)u\,\mathrm{d}x,\end{aligned} \tag{3.2}$$

where for any Borel set $E \subset \Omega_{\rho^*}$, χ_E denotes the indicator function of E in Ω_{ρ^*}. Since U is a Lyapunov function and $\rho \in (\rho_0, \rho_M)$, we have

$$\begin{aligned}&\int_{U^{-1}(\rho)\cap\Omega_\rho^k} \phi'(U_k)(\mathcal{L}U_k)u\,\mathrm{d}x \\ &\quad \leq \int_{U^{-1}(\rho)\cap\Omega_\rho^k} \big|\phi'(U_k)\mathcal{L}U_k - \phi'(U)\mathcal{L}U\big|u\,\mathrm{d}x + \int_{U^{-1}(\rho)\cap\Omega_\rho^k} (\mathcal{L}U)u\,\mathrm{d}x \\ &\quad \leq \big(\big|\phi'(U_k) - \phi'(U)\big|_{C(\Omega_{\rho^*})}|U|_{C^2(\Omega_{\rho^*})} + \big|\phi'(U_k)\big|_{C(\Omega_{\rho^*})}|U_k - U|_{C^2(\Omega_{\rho^*})}\big) \\ &\qquad \times \int_{\Omega_{\rho^*}} \big(|A| + |V|\big)u\,\mathrm{d}x - \gamma\mu\big(U^{-1}(\rho)\cap\Omega_\rho^k\big).\end{aligned}$$

It follows from the facts $u \in C(\bar{\Omega}_{\rho^*})$ and $U_k \to U$ in $C^2(\bar{\Omega}_{\rho^*})$ that

$$\limsup_{k\to\infty} \int_{U^{-1}(\rho)\cap\Omega_\rho^k} \phi'(U_k)(\mathcal{L}U_k)u\,\mathrm{d}x \leq 0. \tag{3.3}$$

Since $\phi''(\rho) = 0$, we also have

$$\begin{aligned}
&\lim_{k\to\infty} \int_{U^{-1}(\rho)\cap\Omega_\rho^k} |\phi''(U_k)||a^{ij}\,\partial_i U_k\,\partial_j U_k|u\,\mathrm{d}x \\
&\quad\le \lim_{k\to\infty} \int_{U^{-1}(\rho)} |\phi''(U_k)||a^{ij}\,\partial_i U_k\,\partial_j U_k|u\,\mathrm{d}x \\
&\quad\le |\phi''(\rho)||A|_{C(U^{-1}(\rho))}|\nabla U|^2_{C(U^{-1}(\rho))} = 0.
\end{aligned} \tag{3.4}$$

Using the uniform convergence of $U_k \to U$ in Ω_{ρ^*}, it is easy to see that as $k \to \infty$,

$$\chi_{\Omega_\rho^k}(x) \to \chi_{\Omega_\rho}(x), \qquad x \in \Omega_{\rho^*} \setminus U^{-1}(\rho). \tag{3.5}$$

By taking limit $k \to \infty$ in (3.2) and using (3.3)–(3.5) and the dominated convergence theorem, we now have

$$\begin{aligned}
\int_{\Omega_\rho} \phi'(U)(\mathcal{L}U)u\,\mathrm{d}x &= \int_{\Omega_{\rho^*}\setminus U^{-1}(\rho)} \chi_{\Omega_\rho}\phi'(U)(\mathcal{L}U)u\,\mathrm{d}x \\
&\ge -\int_{\Omega_{\rho^*}\setminus U^{-1}(\rho)} \chi_{\Omega_\rho}\phi''(U)\big(a^{ij}\,\partial_i U\,\partial_j U\big)u\,\mathrm{d}x \\
&= -\int_{\Omega_\rho} \phi''(U)\big(a^{ij}\,\partial_i U\,\partial_j U\big)u\,\mathrm{d}x,
\end{aligned}$$

which, by definition of ϕ, is equivalent to

$$\int_{\Omega_\rho\setminus\Omega_{\rho m}} \phi'(U)(\mathcal{L}U)u\,\mathrm{d}x \ge -\int_{\Omega_\rho\setminus\Omega_{\rho m}} \phi''(U)\big(a^{ij}\,\partial_i U\,\partial_j U\big)u\,\mathrm{d}x.$$

Letting $\rho \to \rho_M$ in the above, we obtain

$$\int_{\mathcal{U}\setminus\Omega_{\rho m}} \phi'(U)(\mathcal{L}U)u\,\mathrm{d}x \ge -\int_{\mathcal{U}\setminus\Omega_{\rho m}} \phi''(U)\big(a^{ij}\,\partial_i U\,\partial_j U\big)u\,\mathrm{d}x. \tag{3.6}$$

We note that $\phi'(t) \ge 0$ and $\phi'(t) = 1$ as $t \ge \rho_0$. Using the fact that U is a Lyapunov function, we clearly have

$$\begin{aligned}
\int_{\mathcal{U}\setminus\Omega_{\rho m}} \phi'(U)(\mathcal{L}U)u\,\mathrm{d}x &\le -\gamma\int_{\mathcal{U}\setminus\Omega_{\rho m}} \phi'(U)u\,\mathrm{d}x \\
&= -\gamma\int_{\mathcal{U}\setminus\Omega_{\rho_0}} u\,\mathrm{d}x - \gamma\int_{\Omega_{\rho_0}\setminus\Omega_{\rho m}} \phi'(U)u\,\mathrm{d}x \\
&\le -\gamma\int_{\mathcal{U}\setminus\Omega_{\rho_0}} u\,\mathrm{d}x \\
&= -\gamma\mu(\mathcal{U}\setminus\Omega_{\rho_0}).
\end{aligned} \tag{3.7}$$

Denote $C_{\rho_m,\rho_0} = \max_{\rho_m \le \rho \le \rho_0} |\phi''(\rho)|$. Then it is also clear that

$$\begin{aligned}\int_{\mathcal{U}\setminus\Omega_{\rho_m}} |\phi''(U)|\big(a^{ij}\,\partial_i U\,\partial_j U\big)u\,\mathrm{d}x &= \int_{\Omega_{\rho_0}\setminus\Omega_{\rho_m}} |\phi''(U)|\big(a^{ij}\,\partial_i U\,\partial_j U\big)u\,\mathrm{d}x \\ &\le C_{\rho_m,\rho_0}\Big(\sup_{\rho\in(\rho_m,\rho_0)} H(\rho)\Big)\int_{\Omega_{\rho_0}\setminus\Omega_{\rho_m}} u\,\mathrm{d}x \\ &= C_{\rho_m,\rho_0}\Big(\sup_{\rho\in(\rho_m,\rho_o)} H(\rho)\Big)\mu(\Omega_{\rho_0}\setminus\Omega_{\rho_m}).\end{aligned} \tag{3.8}$$

The theorem now follows from (3.6)–(3.8). □

4. Proof of Theorem A(b) and Theorem B(a). Let U be either a Lyapunov function or an anti-Lyapunov function in $\mathcal{U}$ with respect to $\mathcal{L}$ with either Lyapunov constant or anti-Lyapunov constant γ and essential lower, upper bound ρ_m, ρ_M, respectively, which satisfies (1.9). Also let H be as in (1.8) and denote Ω_ρ as the ρ-sublevel set of U for each $\rho \in [\rho_m, \rho_M)$. Let μ be a regular stationary measure of (1.2) with density $u \in W^{1,p}_{\mathrm{loc}}(\mathcal{U})$.

Consider the set $\mathcal{I} = \{\rho \in (\rho_m, \rho_M) : \nabla U(x) \neq 0, x \in U^{-1}(\rho)\}$. Then for each $\eta \in \mathcal{I}$, Ω_η is a C^2 domain, whose boundary $\partial\Omega_\eta$ coincides with $U^{-1}(\eta)$, and the outward unit normal vector $\nu(x)$ of $\partial\Omega_\eta$ at each x is well defined and equals $(\nabla U(x))/(|\nabla U(x)|)$. Since $\mathcal{I}$ is open,

$$\mathcal{I} = \bigcup_{1\le k<I} (a_k, b_k),$$

where I can be a positive integer or $+\infty$, and the intervals (a_k, b_k), $1 \le k < I$, are pairwise disjoint.

PROOF OF THEOREM A(B). Let $\eta^* \in (\rho_m, \rho_M) \cap \mathcal{I}$. For any $\eta \in (\rho_m, \eta^*) \cap \mathcal{I}$, applications of Theorem 2.1 with $F = U$ on $\Omega' = \Omega_{\eta^*}, \Omega_\eta$, respectively, yield that

$$\begin{aligned}\int_{\partial\Omega_\eta} ua^{ij}\frac{\partial_i U\,\partial_j U}{|\nabla U|}\,\mathrm{d}s &+ \int_{\Omega_{\eta^*}\setminus\Omega_\eta} \big(a^{ij}\,\partial^2_{ij} U + V^i\,\partial_i U\big)u\,\mathrm{d}x \\ &= \int_{\partial\Omega_{\eta^*}} ua^{ij}\frac{\partial_i U\,\partial_j U}{|\nabla U|}\,\mathrm{d}s.\end{aligned}$$

Since the right-hand side of the above is nonnegative, applications of (1.8) to the first term of the left-hand side of above and the definition of Lyapunov function to the second term of the left-hand side of above yield that

$$\gamma\int_{\Omega_{\eta^*}\setminus\Omega_\eta} u\,\mathrm{d}x \le H(\eta)\int_{\partial\Omega_\eta} \frac{u}{|\nabla U|}\,\mathrm{d}s, \qquad \eta \in \big[\rho_m, \eta^*\big) \cap \mathcal{I}. \tag{4.1}$$

Consider the function

$$y(\eta) = \mu(\Omega_{\eta^*}\setminus\Omega_\eta) = \int_{\Omega_{\eta^*}\setminus\Omega_\eta} u\,\mathrm{d}x, \qquad \eta \in \big(\rho_m, \eta^*\big) \cap \mathcal{I}.$$

By Theorem 2.2, $y(\eta)$ is of the class C^1 on $(\rho_m, \eta^*) \cap \mathcal{I}$ and

$$y'(\eta) = -\int_{\partial\Omega_\eta} \frac{u}{|\nabla U|}\,\mathrm{d}s, \qquad \eta \in (\rho_m, \eta^*) \cap \mathcal{I}.$$

Hence, by (4.1),

$$y'(\eta) + \frac{\gamma}{H(\eta)} y(\eta) \le 0, \qquad \eta \in (\rho_m, \eta^*) \cap \mathcal{I}. \tag{4.2}$$

Let $1 \le k < I$ be fixed. For any $\tilde{\eta}, \eta \in (a_k, b_k)$ with $\tilde{\eta} < \eta < \eta^*$, integrating (4.2) in the interval $[\tilde{\eta}, \eta]$ yields that

$$\mu(\Omega_{\eta^*} \setminus \Omega_\eta) \le \mu(\Omega_{\eta^*} \setminus \Omega_{\tilde{\eta}}) \mathrm{e}^{-\gamma \int_{\tilde{\eta}}^{\eta} 1/H(t)\,\mathrm{d}t}.$$

In (4.2), we have assumed without loss of generality that H is a positive function. If not, we can replace H in (4.1) [hence in (4.2)] by $H + \varepsilon$, $0 < \varepsilon \ll 1$, so that the above estimate holds with $H + \varepsilon$ in place of H. Since $y(\rho)$ is independent of ε, the estimate in fact holds for H after taking $\varepsilon \to 0$.

Since $\mathcal{I}$ is dense in $[\rho_m, \rho_M)$ by (1.9), letting $\eta^* \to \rho_M$ in the above yields that

$$\mu(\mathcal{U} \setminus \Omega_\eta) \le \mu(\mathcal{U} \setminus \Omega_{\tilde{\eta}}) \mathrm{e}^{-\gamma \int_{\tilde{\eta}}^{\eta} 1/H(t)\,\mathrm{d}t}. \tag{4.3}$$

By taking $\tilde{\eta} \to a_k$, or $\eta \to b_k$, and noting that function $\mu(\mathcal{U} \setminus \Omega_t)$ is monotone in $t \in [\rho_m, \rho_M]$, we see that (4.3) in fact holds for all $\tilde{\eta}, \eta \in [a_k, b_k]$ with $\tilde{\eta} \le \eta$.

Next, let $\rho_*, \rho^* \in \mathcal{I}$ with $\rho_* < \rho^*$. We can find $1 \le \ell < I$ such that $\rho_*, \rho^* \in \bigcup_{k=1}^{\ell} (a_k, b_k)$. Denote $I_\ell = \{i \in \{1, 2, \dots, \ell\} : (a_i, b_i) \cap [\rho_*, \rho^*] \ne \varnothing\}$ and $\tau = |I_\ell|$. Then $I_\ell = \{i_1, i_2, \dots, i_\tau : b_{i_s} \le a_{i_{s+1}}, s = 1, 2, \dots, \tau - 1\}$, $\rho_* \in (a_{i_1}, b_{i_1})$, and $\rho^* \in (a_{i_\tau}, b_{i_\tau})$. By a recursive application of (4.3) for $k = i_1, i_2, \dots, i_\tau$ respectively, we have

$$\begin{aligned}
\mu(\mathcal{U} \setminus \Omega_{\rho^*}) &\le \mu(\mathcal{U} \setminus \Omega_{a_{i_\tau}}) \mathrm{e}^{-\gamma \int_{a_{i_\tau}}^{\rho^*} 1/H(t)\,\mathrm{d}t} \\
&\le \mu(\mathcal{U} \setminus \Omega_{b_{i_{\tau-1}}}) \mathrm{e}^{-\gamma \int_{a_{i_\tau}}^{\rho^*} 1/H(t)\,\mathrm{d}t} \\
&\le \mu(\mathcal{U} \setminus \Omega_{a_{i_{\tau-1}}}) \mathrm{e}^{-\gamma \int_{a_{i_{\tau-1}}}^{b_{i_{\tau-1}}} 1/H(t)\,\mathrm{d}t} \cdot \mathrm{e}^{-\gamma \int_{a_{i_\tau}}^{\rho^*} 1/H(t)\,\mathrm{d}t} \\
&= \mu(\mathcal{U} \setminus \Omega_{a_{i_{\tau-1}}}) \mathrm{e}^{-\gamma \int_{[\rho_*, \rho^*] \cap \bigcup_{k=\tau-1}^{\tau} (a_{i_k}, b_{i_k})} 1/H(t)\,\mathrm{d}t} \\
&\le \cdots \le \mu(\mathcal{U} \setminus \Omega_{a_{i_2}}) \mathrm{e}^{-\gamma \int_{[\rho_*, \rho^*] \cap \bigcup_{k=2}^{\tau} (a_{i_k}, b_{i_k})} 1/H(t)\,\mathrm{d}t} \\
&\le \mu(\mathcal{U} \setminus \Omega_{b_{i_1}}) \mathrm{e}^{-\gamma \int_{[\rho_*, \rho^*] \cap \bigcup_{k=2}^{\tau} (a_{i_k}, b_{i_k})} 1/H(t)\,\mathrm{d}t} \\
&\le \mu(\mathcal{U} \setminus \Omega_{\rho_*}) \mathrm{e}^{-\gamma \int_{\rho_*}^{b_{i_1}} 1/H(t)\,\mathrm{d}t} \cdot \mathrm{e}^{-\gamma \int_{[\rho_*, \rho^*] \cap \bigcup_{k=2}^{\tau} (a_{i_k}, b_{i_k})} 1/H(t)\,\mathrm{d}t} \\
&= \mu(\mathcal{U} \setminus \Omega_{\rho_*}) \mathrm{e}^{-\gamma \int_{[\rho_*, \rho^*] \cap \bigcup_{k=1}^{\tau} (a_{i_k}, b_{i_k})} 1/H(t)\,\mathrm{d}t}
\end{aligned}$$

$$= \mu(\mathcal{U} \setminus \Omega_{\rho_*}) e^{-\gamma \int_{[\rho_*, \rho^*] \cap \bigcup_{k=1}^{\ell} (a_k, b_k)} 1/H(t)\,dt}$$
$$\leq e^{-\gamma \int_{[\rho_*, \rho^*] \cap \bigcup_{k=1}^{\ell} (a_k, b_k)} 1/H(t)\,dt}.$$

Let $\ell \to I$ in the above. Since $[\rho_*, \rho^*] \cap \bigcup_{k=1}^{\ell}(a_k, b_k) \to [\rho_*, \rho^*] \cap \mathcal{I}$ which is a full Lebesgue measure subset of $[\rho_*, \rho^*]$, we obtain

$$\mu(\mathcal{U} \setminus \Omega_{\rho^*}) \leq e^{-\gamma \int_{\rho_*}^{\rho^*} 1/H(t)\,dt}. \tag{4.4}$$

Now for any $\rho \in [\rho_m, \rho_M)$, we let ρ_i^*, ρ_*^i be sequences in $\mathcal{I}$ such that $\rho_i^* \nearrow \rho$ and $\rho_*^i \searrow \rho_m$ as $i \to \infty$. Since (4.4) holds with ρ_*^i, ρ_i^* in place of ρ_*, ρ^* respectively for all i, the proof is complete by taking $i \to \infty$. $\square$

PROOF OF THEOREM B(A). Let $\eta_* \in (\rho_m, \rho_M) \cap \mathcal{I}$ and $\eta \in (\eta_*, \rho_M) \cap \mathcal{I}$ be arbitrarily chosen. Applying Theorem 2.1 with $F = U$ on $\Omega' = \Omega_\eta, \Omega_{\eta_*}$, respectively, we have

$$\int_{\partial\Omega_{\eta_*}} u a^{ij} \frac{\partial_i U\, \partial_j U}{|\nabla U|}\,ds + \int_{\Omega_\eta \setminus \Omega_{\eta_*}} (a^{ij}\,\partial_{ij}^2 U + V^i\,\partial_i U) u\,dx$$
$$= \int_{\partial\Omega_\eta} u a^{ij} \frac{\partial_i U\, \partial_j U}{|\nabla U|}\,ds.$$

Since the first term in the left-hand side of above is nonnegative, applications of the definition of anti-Lyapunov function to the second term of the left-hand side of above and (1.8) to the right-hand side of above yield that

$$\gamma \int_{\Omega_\eta \setminus \Omega_{\eta_*}} u\,dx \leq H(\eta) \int_{\partial\Omega_\eta} \frac{u}{|\nabla U|}\,ds. \tag{4.5}$$

Consider the function

$$y(\eta) = \mu(\Omega_\eta \setminus \Omega_{\eta_*}) = \int_{\Omega_\eta \setminus \Omega_{\eta_*}} u\,dx, \qquad \eta \in (\eta_*, \rho_M).$$

Then by Theorem 2.2, $y(\eta)$ is of class C^1 at each $\eta \in \mathcal{I} \cap (\eta_*, \rho_M)$ with derivative

$$y'(\eta) = \int_{\partial\Omega_\eta} \frac{u}{|\nabla U|}\,ds.$$

Hence, (4.5) yields that

$$y'(\eta) - \frac{\gamma}{H(\eta)} y(\eta) \geq 0, \qquad \eta \in (\eta_*, \rho_M) \cap \mathcal{I}. \tag{4.6}$$

Here, we have again assumed without loss of generality that H is a positive function, via the same reasoning as in the proof of Theorem A(b) above.

Fix $1 \leq k < I$. For any $\eta, \tilde{\eta} \in (a_k, b_k)$ with $\tilde{\eta} < \eta$, we may assume that $\eta_* < \tilde{\eta}$. Integrating (4.6) in the interval $[\tilde{\eta}, \eta]$ yields that

$$\mu(\Omega_\eta \setminus \Omega_{\eta_*}) \geq \mu(\Omega_{\tilde{\eta}} \setminus \Omega_{\eta_*}) e^{\gamma \int_{\tilde{\eta}}^{\eta} 1/H(t)\,dt}.$$

By (1.9), $\mathcal{I}$ is dense in $[\rho_m, \rho_M)$. Then by letting $\eta_* \searrow \rho_m$ in the above and noting that $\lim_{\eta_*\searrow\rho_m} \mu(\Omega_{\eta_*}) = \mu(\Omega^*_{\rho_m})$, we have

$$\mu(\Omega_\eta \setminus \Omega^*_{\rho_m}) \geq \mu(\Omega_{\tilde\eta} \setminus \Omega^*_{\rho_m}) e^{\gamma \int_{\tilde\eta}^{\eta} 1/H(t)\,dt}, \qquad a_k < \tilde\eta < \eta < b_k. \tag{4.7}$$

We note that (4.7) also holds when $\tilde\eta = a_k$ or $\eta = b_k$ by the monotonicity of the function $t \in [\rho_m, \rho_M] \mapsto \mu(\Omega_t \setminus \Omega^*_{\rho_m})$.

Next, let $\rho_*, \rho^* \in \mathcal{I}$ with $\rho_* < \rho^*$. We fix $1 \leq \ell < I$ such that $\rho_*, \rho^* \in \bigcup_{k=1}^{\ell}(a_k, b_k)$. Denote $I_\ell = \{i \in \{1, 2, \ldots, \ell\} : (a_i, b_i) \cap [\rho_*, \rho^*] \neq \varnothing\}$ and $\tau = |I_\ell|$. Then $I_\ell = \{i_1, i_2, \ldots, i_\tau : b_{i_s} \leq a_{i_{s+1}}, s = 1, 2, \ldots, \tau - 1\}$, $\rho_* \in (a_{i_1}, b_{i_1})$, and $\rho^* \in (a_{i_\tau}, b_{i_\tau})$. In the case $\tau \geq 2$, by a recursive application of (4.7) for $k = i_1, i_2, \ldots, i_\tau$ respectively, we have

$$\begin{aligned}
\mu(\Omega_{\rho^*} \setminus \Omega^*_{\rho_m}) &\geq \mu(\Omega_{a_{i_\tau}} \setminus \Omega^*_{\rho_m}) e^{\gamma \int_{a_{i_\tau}}^{\rho^*} 1/H(t)\,dt} \\
&\geq \mu(\Omega_{b_{i_{\tau-1}}} \setminus \Omega^*_{\rho_m}) e^{\gamma \int_{a_{i_\tau}}^{\rho^*} 1/H(t)\,dt} \\
&\geq \mu(\Omega_{a_{i_{\tau-1}}} \setminus \Omega^*_{\rho_m}) e^{\gamma \int_{a_{i_{\tau-1}}}^{b_{i_{\tau-1}}} 1/H(t)\,dt} \cdot e^{\gamma \int_{a_{i_\tau}}^{\rho^*} 1/H(t)\,dt} \\
&= \mu(\Omega_{a_{i_{\tau-1}}} \setminus \Omega^*_{\rho_m}) e^{\gamma \int_{[\rho_*,\rho^*]\cap\bigcup_{k=\tau-1}^{\tau}(a_{i_k}, b_{i_k})} 1/H(t)\,dt} \\
&\geq \cdots \geq \mu(\Omega_{a_{i_2}} \setminus \Omega^*_{\rho_m}) e^{\gamma \int_{[\rho_*,\rho^*]\cap\bigcup_{k=2}^{\tau}(a_{i_k}, b_{i_k})} 1/H(t)\,dt} \\
&\geq \mu(\Omega_{b_{i_1}} \setminus \Omega^*_{\rho_m}) e^{\gamma \int_{[\rho_*,\rho^*]\cap\bigcup_{k=2}^{\tau}(a_{i_k}, b_{i_k})} 1/H(t)\,dt} \\
&\geq \mu(\Omega_{\rho_*} \setminus \Omega^*_{\rho_m}) e^{\gamma \int_{\rho_*}^{b_{i_1}} 1/H(t)\,dt} \cdot e^{\gamma \int_{[\rho_*,\rho^*]\cap\bigcup_{k=2}^{\tau}(a_{i_k}, b_{i_k})} 1/H(t)\,dt} \\
&= \mu(\Omega_{\rho_*} \setminus \Omega^*_{\rho_m}) e^{\gamma \int_{[\rho_*,\rho^*]\cap\bigcup_{k=1}^{\tau}(a_{i_k}, b_{i_k})} 1/H(t)\,dt} \\
&= \mu(\Omega_{\rho_*} \setminus \Omega^*_{\rho_m}) e^{\gamma \int_{[\rho_*,\rho^*]\cap\bigcup_{k=1}^{\ell}(a_k, b_k)} 1/H(t)\,dt}.
\end{aligned}$$

Let $\ell \to I$ in above. Since $[\rho_*, \rho^*] \cap \mathcal{I}$ is of full Lebesgue measure in $[\rho_*, \rho^*]$, we have

$$\mu(\Omega_{\rho^*} \setminus \Omega^*_{\rho_m}) \geq \mu(\Omega_{\rho_*} \setminus \Omega^*_{\rho_m}) e^{\gamma \int_{\rho_*}^{\rho^*} 1/H(t)\,dt}. \tag{4.8}$$

In the case $\tau = 1$, (4.8) follows directly from (4.7).

Now for any $\rho_m < \rho_0 < \rho < \rho_M$, we let ρ_*^i, ρ_i^* be sequences in $\mathcal{I}$ such that $\rho_i^* \nearrow \rho$ and $\rho_*^i \searrow \rho_0$ as $i \to \infty$. Since (4.8) holds with ρ_*^i, ρ_i^* in place of ρ_*, ρ^* respectively for all i, the proof is complete by taking $i \to \infty$. $\square$

5. Proof of Theorem A(c) and Theorem B(b). Let U be either a weak Lyapunov function or a weak anti-Lyapunov function in $\mathcal{U}$ with respect to $\mathcal{L}$ with essential lower, upper bound ρ_m, ρ_M, respectively. Also let h, H be as in (1.8) and denote Ω_ρ as the ρ-sublevel set of U for each $\rho \in [\rho_m, \rho_M)$.

For each $\rho \in [\rho_m, \rho_M)$, since $h(\rho) > 0$ in (1.8), $\nabla U(x) \neq 0$ for all $x \in U^{-1}(\rho)$ and Ω_ρ is a C^2 domain with

$$\partial\Omega_\rho = U^{-1}(\rho). \tag{5.1}$$

Consider a regular stationary measure μ of (1.2) with density $u(x) \in W^{1,p}_{\mathrm{loc}}(\mathcal{U})$. Then by Theorem 2.2, the function

$$y(\rho) = \int_{\Omega_\rho \setminus \Omega_{\rho_m}} u\,\mathrm{d}x, \qquad \rho \in (\rho_m, \rho_M)$$

is of the class C^1 and

$$y'(\rho) = \int_{\partial\Omega_\rho} \frac{u}{|\nabla U|}\,\mathrm{d}s, \qquad \rho \in (\rho_m, \rho_M).$$

For $t \in [\rho_m, \rho_M)$, consider

$$H_*(t) = \begin{cases} \displaystyle\int_{\rho_m}^{t} \int_{\rho_m}^{\tau} 1 H(s)\,\mathrm{d}s\,\mathrm{d}\tau, & \text{in the case of Theorem A(c)}, \\ \displaystyle\int_{\rho_m}^{t} \int_{\rho_m}^{\tau} 1 h(s)\,\mathrm{d}s\,\mathrm{d}\tau, & \text{in the case of Theorem B(b)}. \end{cases}$$

Since H is positive and continuous on $[\rho_m, \rho_M)$ in the case of Theorem A(c) so is h in the case of Theorem B(b), H_* is a C^2 function on $[\rho_m, \rho_M)$. We extend H_* to a C^2 function on $[0, \rho_M)$ and still denote it by H_*.

LEMMA 5.1. *For each* $\rho \in [\rho_m, \rho_M)$,

$$\int_{\Omega_\rho \setminus \Omega_{\rho_m}} (a^{ij}\,\partial^2_{ij} F + V^i\,\partial_i F) u\,\mathrm{d}x = H_*'(\rho) \int_{\partial\Omega_\rho} u a^{ij} \frac{\partial_i U\,\partial_j U}{|\nabla U|}\,\mathrm{d}s. \tag{5.2}$$

PROOF. Let $F = H_* \circ U$. For any $\rho \in (\rho_m, \rho_M)$, we note that $F \in C^2(\bar{\Omega}_\rho)$ and $F|_{\partial\Omega_\rho} \equiv H_*(\rho)$. We apply Theorem 2.1 to F with Ω' being Ω_ρ, Ω_{ρ_m}, respectively. By using the fact that the unit outward normal vector $\nu(x)$ is well defined and equals $\frac{\nabla U(x)}{|\nabla U(x)|}$ for any $x \in \partial\Omega_\rho \cup \partial\Omega_{\rho_m}$, we have

$$\begin{aligned} &H_*'(\rho_m) \int_{\partial\Omega_{\rho_m}} u a^{ij} \frac{\partial_i U\,\partial_j U}{|\nabla U|}\,\mathrm{d}s + \int_{\Omega_\rho \setminus \Omega_{\rho_m}} (a^{ij}\,\partial^2_{ij} F + V^i\,\partial_i F) u\,\mathrm{d}x \\ &\quad = H_*'(\rho) \int_{\partial\Omega_\rho} u a^{ij} \frac{\partial_i U\,\partial_j U}{|\nabla U|}\,\mathrm{d}s. \end{aligned}$$

Since $H_*'(\rho_m) = 0$, the lemma holds. $\square$

PROOF OF THEOREM A(C). To estimate the left-hand side of (5.2), we let $x \in \Omega_\rho \setminus \bar{\Omega}_{\rho_m}$ and denote $\rho' =: U(x)$. Clearly, $\rho' \in (\rho_m, \rho)$. Then $H_*''(\rho') =$

$H^{-1}(\rho')$, $H_*'(\rho') \geq 0$ and $\mathcal{L}U(x) = a^{ij}(x)\,\partial_{ij}^2 U(x) + V^i(x)\,\partial_i U(x) \leq 0$. It then follows from (1.8) that

$$\begin{aligned} a^{ij}(x)\,\partial_{ij}^2 F(x) + V^i(x)\,\partial_i F(x) &= H_*'(\rho')\big(a^{ij}(x)\,\partial_{ij}^2 U(x) + V^i(x)\,\partial_i U(x)\big) \\ &\quad + H_*''(\rho')a^{ij}(x)\partial_i U(x)\partial_j U(x) \\ &\leq 0 + \frac{1}{H(\rho')}H(\rho') = 1. \end{aligned}$$

Also by (1.8), the right-hand side of (5.2) simply satisfies

$$H_*'(\rho)\int_{\partial\Omega_\rho} u a^{ij}\frac{\partial_i U\,\partial_j U}{|\nabla U|}\,\mathrm{d}s \geq \int_{\rho_m}^{\rho}\frac{1}{H(s)}\,\mathrm{d}s\,h(\rho)\int_{\partial\Omega_\rho}\frac{u}{|\nabla U|}\,\mathrm{d}s.$$

Hence, by (5.2),

$$\int_{\Omega_\rho\setminus\Omega_{\rho_m}} u(x)\,\mathrm{d}x \geq \tilde{H}(\rho)\int_{\partial\Omega_\rho}\frac{u}{|\nabla U|}\,\mathrm{d}s, \qquad \rho\in(\rho_m,\rho_M),$$

that is,

$$y'(\rho) \leq \frac{1}{\tilde{H}(\rho)}y(\rho), \qquad \rho\in(\rho_m,\rho_M).$$

For any $\rho_0 \in (\rho_m, \rho_M)$ and $\rho \in [\rho_0, \rho_M)$, a direct integration of the above inequality yields that

$$y(\rho) \leq y(\rho_0)\mathrm{e}^{\int_{\rho_0}^{\rho} 1/\tilde{H}(r)\,\mathrm{d}r}, \qquad \rho\in(\rho_0,\rho_M).$$

The proof is complete simply by taking limit $\rho \to \rho_M$ in the above. □

PROOF OF THEOREM B(B). To estimate the left-hand side of (5.2), we note that $H_*'(t) \geq 0$, $H_*''(t) = h^{-1}(t)$ when $t > \rho_m$, and $\mathcal{L}U = a^{ij}\partial_{ij}^2 U + V^i\partial_i U \geq 0$ in $\Omega_\rho \setminus \bar{\Omega}_{\rho_m}$. It then follows from (1.8) that

$$\begin{aligned} a^{ij}\partial_{ij}^2 F + V^i\partial_i F &= H_*'(U)\big(a^{ij}\partial_{ij}^2 U + V^i\partial_i U\big) + H_*''(U)a^{ij}\partial_i U\partial_j U \\ &\geq 0 + \frac{1}{h(U)}h(U) = 1. \end{aligned}$$

Also by (1.8), the right-hand side of (5.2) simply satisfies

$$H_*'(\rho)\int_{\partial\Omega_\rho} u a^{ij}\frac{\partial_i U\,\partial_j U}{|\nabla U|}\,\mathrm{d}s \leq \int_{\rho_m}^{\rho}\frac{1}{h(s)}\,\mathrm{d}s\,H(\rho)\int_{\partial\Omega_\rho}\frac{u}{|\nabla U|}\,\mathrm{d}s.$$

Hence, (5.2) becomes

$$\begin{aligned} \int_{\Omega_\rho\setminus\Omega_{\rho_m}} u(x)\,\mathrm{d}x &\leq \int_{\rho_m}^{\rho}\frac{1}{h(s)}\,\mathrm{d}s\,H(\rho)\int_{\partial\Omega_\rho}\frac{u}{|\nabla U|}\,\mathrm{d}s \\ &= \tilde{H}(\rho)\int_{\partial\Omega_\rho}\frac{u}{|\nabla U|}\,\mathrm{d}s, \qquad \rho\in(\rho_m,\rho_M), \end{aligned}$$

that is,

$$\frac{1}{\tilde{H}(\rho)} y(\rho) \le y'(\rho), \qquad \rho \in (\rho_m, \rho_M).$$

For any $\rho_0 \in (\rho_m, \rho_M)$, let $\rho \in [\rho_0, \rho_M)$ be fixed. The proof is complete by a direct integration of the above in the interval $[\rho_0, \rho]$. $\square$

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