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CENTRAL LIMIT THEOREM FOR AN ADDITIVE FUNCTIONAL OF THE FRACTIONAL BROWNIAN MOTION

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We prove a central limit theorem for an additive functional of the d-dimensional fractional Brownian motion with Hurst index $H \in (\frac{1}{1+d}, \frac{1}{d})$, using the method of moments, extending the result by Papanicolaou, Stroock and Varadhan in the case of the standard Brownian motion.

1. Introduction. Let $\{B(t) = (B^1(t), \dots, B^d(t)), t \geq 0\}$ be a d-dimensional fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$. If $Hd < 1$, then the local time of B exists (see, e.g., [4–6]) and can be defined as

$$L_t(x) = \int_0^t \delta\bigl(B(s) - x\bigr)\, ds, \qquad t \geq 0, x \in \mathbb{R}^d,$$

where δ is the Dirac delta function. The above local time is jointly continuous with respect to t and x; see [4]. For any integrable function $f : \mathbb{R}^d \to \mathbb{R}$, one can easily show the following convergence in law in the space $C([0, \infty))$, as n tends to infinity:

$$\left(n^{Hd-1} \int_0^{nt} f\bigl(B(s)\bigr)\, ds, t \geq 0\right) \xrightarrow{\mathcal{L}} \left(L_t(0) \int_{\mathbb{R}^d} f(x)\, dx, t \geq 0\right). \tag{1.1}$$

In fact, making the change of variable $s = nu$, and using the scaling property of the fBm, we see that the process $(n^{Hd-1} \int_0^{nt} f(B(s))\, ds, t \geq 0)$ has the same law as

$$\begin{aligned} n^{Hd} \int_0^t f\bigl(n^H B(u)\bigr)\, du &= n^{Hd} \int_{\mathbb{R}^d} f\bigl(n^H x\bigr) L_t(x)\, dx \\ &= \int_{\mathbb{R}^d} f(x) L_t\bigl(n^{-H} x\bigr)\, dx, \qquad t \geq 0. \end{aligned}$$

From here it is straightforward to verify (1.1).

If we assume that $\int_{\mathbb{R}^d} f(x)\, dx = 0$, then we see $n^{Hd-1} \int_0^{nt} f(B(s))\, ds$ converges to 0. It is interesting to know if there is a $\beta > Hd - 1$ such that

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$n^\beta \int_0^{nt} f(B(s))\,ds$ converges to a nonzero process. This will be proved to be true. In order to formulate this result we introduce the following space of functions. Fix a number $\beta > 0$, and denote

$$H_0^\beta = \left\{ f \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} |f(x)||x|^\beta \,dx < \infty \text{ and } \int_{\mathbb{R}^d} f(x)\,dx = 0 \right\}.$$

For any $f \in H_0^\beta$, and assuming $\beta \in (0, 2)$, by Lemma A.1, the quantity

$$\|f\|_\beta^2 := -\int_{\mathbb{R}^{2d}} f(x) f(y) |x-y|^\beta \,dx\,dy \tag{1.2}$$

is finite and nonnegative. The next theorem is the main result of this paper.

THEOREM 1.1. *Suppose $\frac{1}{d+1} < H < \frac{1}{d}$ and $f \in H_0^{1/H-d}$. Then*

$$\left(n^{(Hd-1)/2} \int_0^{nt} f(B(s))\,ds, t \ge 0 \right) \xrightarrow{\mathcal{L}} \left(\sqrt{C_{H,d}} \|f\|_{1/H-d} W(L_t(0)), t \ge 0 \right)$$

in the space $C([0,\infty))$, as n tends to infinity, where "$\xrightarrow{\mathcal{L}}$" denotes the convergence in law, W is a real-valued standard Brownian motion independent of B and

$$\begin{aligned} C_{H,d} &= \frac{2}{(2\pi)^{d/2}} \int_0^\infty w^{-Hd} \left(1 - \exp\left(-\frac{1}{2w^{2H}} \right) \right) dw \\ &= \frac{2^{1-1/(2H)}}{(1-Hd)\pi^{d/2}} \Gamma\left(\frac{Hd + 2H - 1}{2H} \right). \end{aligned}$$

Notice that $\frac{Hd-1}{2} > Hd - 1$ since $H < \frac{1}{d}$. When $d = 1$ and $H = \frac{1}{2}$, the above theorem is obtained by Papanicolaou, Stroock and Varadhan in [14] with $C_{1/2,1} = 2$. On the other hand, the constant $C_{H,d}$ is finite for any $H > \frac{1}{d+2}$. We conjecture that our result also holds for $\frac{1}{d+2} < H < \frac{1}{d}$. But we have not been able to show our result in the case $H \le \frac{1}{d+1}$. The main reason is that in the proof of Proposition 3.4 we need $H > \frac{1}{d+1}$; see the remark at the end of Section 3.

In the critical case $Hd = 1$, the local time does not exist. For the Brownian motion case ($H = \frac{1}{2}$ and $d = 2$), Kallianpur and Robbins [7] proved that for any bounded and integrable function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\frac{1}{\log n} \int_0^n f(B_s)\,ds \xrightarrow{\mathcal{L}} \frac{Z}{2\pi} \int_{\mathbb{R}^2} f(x)\,dx$$

as n tends to infinity, where Z is a random variable with exponential distribution of parameter 1. A functional version of this result was given by Kasahara and Kotani in [9], where they also proved the second-order results when $\int_{\mathbb{R}^2} f(x)\,dx = 0$. The Kallianpur–Robbins law was extended to the fBm by Kôno in [10], and the corresponding functional version was obtained by Kasahara and Kosugi in [8].

However, second-order results for the fBm in the critical case $Hd = 1$ have not been yet proved. On the other hand, we refer to Biane [3] for some extensions of these results to the case of functionals of k independent Brownian motions.

The limit theorem proved in this paper, where an independent source of noise appears in the limit, might be connected to the central limit theorems for weighted power variations of the fractional Brownian motion in the critical cases $H = \frac{1}{4}$ and $H = \frac{1}{6}$ (see the works by Nourdin and Réveillac [12] and Nourdin, Réveillac and Swanson [13]), where a similar phenomenon happens. However, the method of proof in these papers relies on the techniques of Malliavin calculus, and it is related to a general central limit theorem for multiple Skorohod integrals obtained by Nourdin and Nualart in [11]. These techniques do not seem to work for the class of additive functionals considered in this paper.

We would like to give a heuristic explanation of the fact that an independent noise appears in the limit, and also to indicate the main ideas of the proof. First, by the scaling property of the fractional Brownian motion, we can consider the continuous process

$$F_n(t) := n^{(1+Hd)/2} \int_0^t f\big(n^H B(s)\big)\, ds. \tag{1.3}$$

Making the change of variables $u_1 = n(s_2 - s_1)$ and $u_2 = s_2$, we can write formally

$$\begin{aligned} F_n(t)^2 &= 2n^{1+Hd} \int_0^t \int_0^{s_2} f\big(n^H B(s_1)\big) f\big(n^H B(s_2)\big)\, ds_1\, ds_2 \\ &= 2n^{Hd} \int_0^t \int_0^{nu_2} f\left(n^H B\left(u_2 - \frac{u_1}{n}\right)\right) f\big(n^H B(u_2)\big)\, du_1\, du_2 \\ &= 2n^{Hd} \int_0^t \int_0^{nu_2} \int_{\mathbb{R}^d} f\left(n^H \left(B\left(u_2 - \frac{u_1}{n}\right) - B(u_2)\right) + n^H z\right) \\ &\qquad \times f\big(n^H z\big) \delta\big(B(u_2) - z\big)\, dz\, du_1\, du_2. \end{aligned}$$

The change of variable $n^H z = x$ yields

$$\begin{aligned} F_n(t)^2 = 2 \int_0^t \int_0^{nu_2} \int_{\mathbb{R}^d} & f\big(B_n(u_1, u_2) + x\big) \\ & \times f(x) \delta\left(B(u_2) - \frac{x}{n^H}\right) dx\, du_1\, du_2, \end{aligned} \tag{1.4}$$

where $B_n(u_1, u_2) = n^H (B(u_2 - \frac{u_1}{n}) - B(u_2))$. Notice that $B_n(u_1, u_2)$ is a d-dimensional centered Gaussian vector whose components are independent and with variance u_1^{2H}. Using the covariance function of the fractional Brownian motion, it is easy to show that for any $u_1, u_2, u_3 \geq 0$, $(B_n(u_1, u_2), B(u_3))$ (assuming $u_1 \leq nu_2$) converges in law to

$$\big(B_\infty(u_1, u_2), B(u_3)\big),$$

where $B_\infty(u_1, u_2)$ is independent of $B(u_3)$. As a consequence,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\biggl(f\bigl(B_n(u_1,u_2)+x\bigr)\delta\biggl(B(u_2)-\frac{x}{n^H}\biggr)\biggr) \\ = \mathbb{E}\bigl(f\bigl(B_\infty(u_1,u_2)+x\bigr)\bigr)\mathbb{E}\bigl(\delta\bigl(B(u_2)\bigr)\bigr).\end{aligned}$$

Assuming that we can commute the expectation with the integrals, this formally yields

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\bigl(F_n(t)^2\bigr) = \frac{2}{(2\pi)^{d/2}} \mathbb{E}\biggl(\int_0^t \delta\bigl(B(u_2)\bigr)\,du_2\biggr) \\ \times \int_0^\infty \int_{\mathbb{R}^{2d}} f(x+y)f(x) \\ \times u_1^{-Hd}\bigl(e^{-|y|^2/(2u_1^{2H})}-1\bigr)\,dy\,dx\,du_1,\end{aligned}$$

where we can add the term -1 because the integral of f is zero. The right-hand side of the above expression is equal to

$$\begin{aligned}&\frac{2}{(2\pi)^{d/2}}\mathbb{E}\bigl(W\bigl(L_t(0)\bigr)^2\bigr) \\ &\quad\times \int_0^\infty \int_{\mathbb{R}^{2d}} f(x+y)f(x)u_1^{-Hd}\bigl(e^{-|y|^2/(2u_1^{2H})}-1\bigr)\,dy\,dx\,du_1 \\ &\quad= \frac{2}{(2\pi)^{d/2}}\mathbb{E}\bigl(W\bigl(L_t(0)\bigr)^2\bigr) \\ &\qquad\times \int_0^\infty w^{-Hd}\bigl(1-e^{-1/(2w^{2H})}\bigr)\,dw \int_{\mathbb{R}^{2d}} -f(x+y)f(x)|y|^{1/H-d}\,dy\,dx \\ &\quad= C_{H,d}\|f\|_{1/H-d}^2 \mathbb{E}\bigl(W\bigl(L_t(0)\bigr)^2\bigr),\end{aligned}$$

and this shows the convergence of the moments of order two. Roughly speaking, the term $B(u_2)$ appearing in (1.4) contributes to the local time at zero whereas $B_n(u_1, u_2)$ becomes independent of B in the limit and contributes to the constant $C_{H,d}$. The main technical difficulty is the commutation of the expectation with the limit, and for this we will implement a convenient truncation argument. The idea is to replace the interval $[0, nu_2]$ by a compact set $[0, K]$, and then show that the integral over $[K, nu_2]$ converges to zero as K tends to infinity, uniformly in n. However, this convergence holds only if we integrate over $[K, \frac{nu_2}{2}]$, and for this reason, we need to show (see Proposition 3.3) that the integral over $[\frac{nu_2}{2}, nu_2]$ tends to zero as n tends to infinity. We will make this heuristic computation rigorous when we compute the limit of the moments of even order of a vector of increments of the process $F_n(t)$. In this case, we will have an even product of factors, and for each couple of consecutive factors, we will make the above change of variables.

The basic idea of the approach used in this paper is to apply the method of moments to an additive functional, and when dealing with an integral on $[0,t]^{2m}$, with respect to the measure $ds_1 \cdots ds_{2m}$, we make the change of variables $u_{2k-1} = n(s_{2k} - s_{2k-1})$ and $u_{2k} = s_{2k}$, $1 \le k \le m$. Then the increments of B in small intervals will be responsible for the independent noise appearing in the limit. This methodology could be applied to other examples of additive functionals and processes.

After some preliminaries in Section 2, Section 3 is devoted to the proof of Theorem 1.1, based on the method of moments. Throughout this paper, if not mentioned otherwise, the letter c, with or without a subscript, denotes a generic positive finite constant whose exact value is independent of n and may change from line to line. We use ι to denote $\sqrt{-1}$.

2. Preliminaries. Let $\{B(t) = (B^1(t), \dots, B^d(t)), t \ge 0\}$ be a d-dimensional fractional Brownian motion with Hurst index $H \in (0,1)$, defined on some probability space $(\Omega, \mathcal{F}, P)$. That is, the components of B are independent centered Gaussian processes with covariance

$$\mathbb{E}\big(B^i(t)B^i(s)\big) = \tfrac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big).$$

We refer to [2] for a detailed analysis of this process.

The next lemma gives a formula for the moments of the increments of the process $\{W(L_t(0)) : t \ge 0\}$ on disjoint intervals, where W is a real-valued standard Brownian motion independent of B.

LEMMA 2.1. *Fix a finite number of disjoint intervals $(a_i, b_i]$ in $[0,\infty)$, where $i = 1, \dots, N$ and $b_i \le a_{i+1}$. Consider a multi-index $\mathbf{m} = (m_1, \dots, m_N)$, where $m_i \ge 1$ and $1 \le i \le N$. Then*

$$\mathbb{E}\left(\prod_{i=1}^{N} \big[W\big(L_{b_i}(0)\big) - W\big(L_{a_i}(0)\big)\big]^{m_i}\right) \tag{2.1}$$

is equal to

$$\left(\prod_{i=1}^{N} \frac{m_i!}{2^{m_i/2}(2\pi)^{m_i d/4}(m_i/2)!}\right) \int_{\prod_{i=1}^{N}[a_i,b_i]^{m_i/2}} \det\big(A(w)\big)^{-1/2}\, dw,$$

if all m_i are even and 0 otherwise, where $A(w)$ is the covariance matrix of the Gaussian random vector

$$\left(B\big(w_k^i\big) : 1 \le i \le N \text{ and } 1 \le k \le \frac{m_i}{2}\right).$$

PROOF. It is easy to see that when one of m_i is odd, the expectation is 0. Suppose now that all m_i are even. Denote by $\mathcal{F}^B$ the σ-algebra generated by the fractional Brownian motion B. Since W is a standard Brownian motion independent of B, we have

$$
\begin{aligned}
&\mathbb{E}\prod_{i=1}^{N}\big[W\big(L_{b_i}(0)\big)-W\big(L_{a_i}(0)\big)\big]^{m_i}\\
&\quad=\mathbb{E}\left\{\mathbb{E}\left(\prod_{i=1}^{N}\big[W\big(L_{b_i}(0)\big)-W\big(L_{a_i}(0)\big)\big]^{m_i}\Big|\mathcal{F}^B\right)\right\}\\
&\quad=\left[\prod_{i=1}^{N}\frac{m_i!}{2^{m_i/2}(m_i/2)!}\right]\mathbb{E}\prod_{i=1}^{N}\big[L_{b_i}(0)-L_{a_i}(0)\big]^{m_i/2}\\
&\quad=\left(\prod_{i=1}^{N}\frac{m_i!}{2^{m_i/2}(2\pi)^{m_i d/4}(m_i/2)!}\right)\int_{\prod_{i=1}^{N}[a_i,b_i]^{m_i/2}}\det\big(A(w)\big)^{-1/2}\,dw.
\end{aligned}
$$

This completes the proof. □

We shall use the following local nondeterminism property of the fractional Brownian motion (see [1]): for any $n\ge 2$ there exists a positive constant k_H depending on n, such that for any $0=s_0<s_1\le\cdots\le s_n<\infty$ and $u_1,\dots,u_n\in\mathbb{R}^d$,

$$
\operatorname{Var}\left(\sum_{i=1}^{n}u_i\cdot\big(B(s_i)-B(s_{i-1})\big)\right)\ge k_H\sum_{i=1}^{n}|u_i|^2(s_i-s_{i-1})^{2H}. \tag{2.2}
$$

This can also be written as

$$
\operatorname{Var}\left(\sum_{i=1}^{n}u_i\cdot B(s_i)\right)\ge k_H\sum_{i=1}^{n}\left|\sum_{j=i}^{n}u_j\right|^2(s_i-s_{i-1})^{2H}. \tag{2.3}
$$

We claim that the law of the random vector $(W(L_{b_i}(0))-W(L_{a_i}(0)):1\le i\le N)$ is determined by the moments computed in Lemma 2.1. This is a consequence of the following estimates. Fix an even integer $n=2k$, and set $D_k=\{s\in[0,t]^k:0<s_1<s_2<\cdots<s_k<t\}$. Let $A_k(s)$ be the covariance matrix of Gaussian random vector $(B(s_1),B(s_2),\dots,B(s_k))$. Then the local nondeterminism property (2.2) implies that

$$
\big(\det A_k(s)\big)^{-1/2}\le c^k\prod_{i=1}^{k}(s_i-s_{i-1})^{-Hd}\qquad\text{for some constant } c. \tag{2.4}
$$

As a consequence of (2.1) and (2.4),

$$
\begin{aligned}
\mathbb{E}\big[W\big(L_t(0)\big)\big]^n&\le c^k n!\int_{D_k}s_1^{-Hd}(s_2-s_1)^{-Hd}\cdots(s_k-s_{k-1})^{-Hd}\,ds\\
&=c^k n!\int_{\{0<u_1+\cdots+u_k<t\}}\prod_{i=1}^{k}u_i^{-Hd}\,du\\
&=c^k n! t^{k(1-Hd)}\frac{\Gamma^k(1-Hd)}{\Gamma(k(1-Hd)+1)}.
\end{aligned}
$$

Therefore, $\mathbb{E}[W(L_t(0))]^n$ is bounded by $c^k n!/\Gamma(k(1-Hd)+1)$, and this easily implies the desired characterization of the law of the increments of the process $\{W(L_t(0)) : t \geq 0\}$ on disjoint intervals by its moments.

3. Proof of Theorem 1.1. By the scaling property of the fractional Brownian motion we see that, as processes indexed by $t \geq 0$,

$$n^{(Hd-1)/2} \int_0^{nt} f\big(B(s)\big)\, ds \overset{\mathcal{L}}{=} n^{(1+Hd)/2} \int_0^t f\big(n^H B(s)\big)\, ds.$$

Therefore, it suffices to show the theorem for the continuous process defined in (1.3). The proof of Theorem 1.1 will be done in two steps. We first show tightness, and then establish the convergence of moments. Tightness will be deduced from the following result.

PROPOSITION 3.1. *For any $0 \leq a < b \leq t$ and any integer $m \geq 1$,*

$$\mathbb{E}\big[\big(F_n(b) - F_n(a)\big)^{2m}\big] \leq C\bigg((b-a)^{1-Hd} \int_{\mathbb{R}^{2d}} \big|f(x)f(y)\big| |y|^{1/H-d}\, dx\, dy\bigg)^m,$$

where C is a constant depending only on H and m.

PROOF. Define

$$D = \big\{s \in \mathbb{R}^{2m} : a < s_1 < s_2 < \cdots < s_{2m} < b\big\}. \tag{3.1}$$

Using the following identity for $f \in H_0^{1/H-d}$

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{\iota x\cdot\xi} \int_{\mathbb{R}^d} e^{-\iota\xi\cdot y} f(y)\, dy\, d\xi,$$

we then have

$$\begin{aligned}
&\mathbb{E}\big[\big(F_n(b) - F_n(a)\big)^{2m}\big] \\
&\quad = (2m)!\, n^{m(1+Hd)} \mathbb{E}\bigg(\int_D \prod_{i=1}^{2m} f\big(n^H B(s_i)\big)\, ds\bigg) \\
&\quad = \frac{(2m)!}{(2\pi)^{2md}} n^{m(1-Hd)} \\
&\qquad \times \int_{\mathbb{R}^{2md}} \int_D \int_{\mathbb{R}^{2md}} \prod_{i=1}^{2m} f(y_i) \exp\bigg(-\frac{1}{2} \operatorname{Var}\bigg(\sum_{i=1}^{2m} \xi_i \cdot B(s_i)\bigg) \\
&\qquad\qquad - \iota \sum_{i=1}^{2m} \frac{y_i \cdot \xi_i}{n^H}\bigg)\, d\xi\, ds\, dy
\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&= \frac{(2m)!}{(2\pi)^{2md}} n^{m(1-Hd)} \\
&\quad \times \int_{\mathbb{R}^{2md}} \int_D \int_{\mathbb{R}^{2md}} \prod_{i=1}^{2m} f(y_i) \exp\biggl(-\frac{1}{2} \operatorname{Var}\biggl(\sum_{i=1}^{2m} \xi_i \cdot B(s_i)\biggr)\biggr) \\
&\qquad \times \prod_{i=1}^{2m} \bigl(e^{-\iota y_i \cdot \xi_i / n^H} - 1\bigr) \, d\xi \, ds \, dy,
\end{aligned}
$$

where in the last equality we used the fact that

$$
\int_{\mathbb{R}^d} f(x) \, dx = 0.
$$

By the local nondeterminism property (2.3), with the convention $s_0 = 0$ and $\eta_{2m+1} = 0$, we can write

$$
\begin{aligned}
&\mathbb{E}\bigl[\bigl(F_n(b) - F_n(a)\bigr)^{2m}\bigr] \\
&\quad \le c_1 n^{m(1-Hd)} \\
&\qquad \times \int_{\mathbb{R}^{2md}} \int_D \biggl|\prod_{i=1}^{2m} f(y_i)\biggr| \int_{\mathbb{R}^{2md}} \exp\biggl(-\frac{\kappa_H}{2} \sum_{i=1}^{2m} \biggl|\sum_{j=i}^{2m} \xi_j\biggr|^2 (s_i - s_{i-1})^{2H}\biggr) \\
&\qquad\qquad \times \prod_{i=1}^{2m} \bigl|e^{-\iota y_i \cdot \xi_i / n^H} - 1\bigr| \, d\xi \, ds \, dy \\
&\quad = c_1 n^{m(1-Hd)} \\
&\qquad \times \int_{\mathbb{R}^{2md}} \int_D \biggl|\prod_{i=1}^{2m} f(y_i)\biggr| \int_{\mathbb{R}^{2md}} \exp\biggl(-\frac{\kappa_H}{2} \sum_{i=1}^{2m} |\eta_i|^2 (s_i - s_{i-1})^{2H}\biggr) \\
&\qquad\qquad \times \prod_{i=1}^{2m} \biggl|\exp\biggl(\iota \frac{y_i}{n^H} \cdot (\eta_{i+1} - \eta_i)\biggr) - 1\biggr| \, d\eta \, ds \, dy,
\end{aligned}
$$

where we made the change of variables

$$
\eta_i = \sum_{j=i}^{2m} \xi_j
$$

for $i = 1, \dots, 2m$ in the last equality.

Let

$$
x_i = \eta_i (s_i - s_{i-1})^H \quad \text{and} \quad u_i = s_i - s_{i-1}
$$

for $i = 1, \dots, 2m$. Then

$$
\begin{aligned}
&\mathbb{E}\big[\big(F_n(b) - F_n(a)\big)^{2m}\big] \\
&\quad \le c_1 n^{m(1-Hd)} \\
&\qquad \times \int_{\mathbb{R}^{2md}} \int_{[a,b]\times[0,b-a]^{2m-1}} \int_{\mathbb{R}^{2md}} \prod_{i=1}^{2m} |f(y_i)| \left(\prod_{i=1}^{2m} u_i^{-Hd}\right) \\
&\qquad\qquad \times \exp\left(-\frac{\kappa_H}{2}\sum_{i=1}^{2m}|x_i|^2\right) \\
&\qquad\qquad \times \prod_{i=1}^{2m}\left|\exp\left(\iota \frac{y_i}{n^H}\cdot\left(\frac{x_{i+1}}{u_{i+1}^H} - \frac{x_i}{u_i^H}\right)\right) - 1\right| dx\, du\, dy,
\end{aligned}
$$

where $x_{2m+1} = 0$ and the integral on $[a,b]\times[0,b-a]^{2m-1}$ means that $u_1 \in [a,b]$ and $u_i \in [0, b-a]$ for $i = 2, \dots, 2m$.

Let $\sqrt{\kappa_H}X_1, \dots, \sqrt{\kappa_H}X_{2m}$ be independent copies of a d-dimensional standard normal random vector and $X_{2m+1} = 0$. Then the above inequality can be rewritten as

$$
\begin{aligned}
&\mathbb{E}\big[\big(F_n(b) - F_n(a)\big)^{2m}\big] \\
&\quad \le c_2 n^{m(1-Hd)} \\
&\qquad \times \int_{\mathbb{R}^{2md}} \int_{[a,b]\times[0,b-a]^{2m-1}} \left(\prod_{i=1}^{2m}|f(y_i)| u_i^{-Hd}\right) \\
&\qquad\qquad \times \mathbb{E}\left(\prod_{i=1}^{2m}\left|\exp\left(\iota\frac{y_i\cdot X_{i+1}}{n^H u_{i+1}^H} - \iota\frac{y_i\cdot X_i}{n^H u_i^H}\right) - 1\right|\right) du\, dy.
\end{aligned}
\tag{3.3}
$$

Notice that

$$
\begin{aligned}
&\big|e^{(\iota y_i/n^H)\cdot(X_{i+1}/u_{i+1}^H - X_i/u_i^H)} - 1\big| \\
&\qquad \le 2 \wedge \big(\big|e^{\iota y_i\cdot X_{i+1}/(n^H u_{i+1}^H)} - 1\big| + \big|e^{\iota y_i\cdot X_i/(n^H u_i^H)} - 1\big|\big).
\end{aligned}
$$

For each factor in the product inside the expectation in (3.3), we choose the upper bound 2 when i is even and the upper bound

$$
\big|e^{\iota y_i\cdot X_{i+1}/(n^H u_{i+1}^H)} - 1\big| + \big|e^{\iota y_i\cdot X_i/(n^H u_i^H)} - 1\big|,
$$

when i is odd. Thus we have

$$\begin{aligned}
&\mathbb{E}\big[\big(F_n(b)-F_n(a)\big)^{2m}\big]\\
&\quad\le c_3\int_{\mathbb{R}^{2md}}\int_{[a,b]\times[0,b-a]^{2m-1}}\mathbb{E}\Bigg[\prod_{i=1}^m\big(n^{1-Hd}\big|f(y_{2i-1})f(y_{2i})\big|\big(u_{2i-1}^{-Hd}u_{2i}^{-Hd}\big)\\
&\qquad\qquad\times\big(\big|e^{\iota y_{2i-1}\cdot X_{2i}/(n^H u_{2i}^H)}-1\big|\\
&\qquad\qquad+\big|e^{\iota y_{2i-1}\cdot X_{2i-1}/(n^H u_{2i-1}^H)}-1\big|\big)\big)\Bigg]\,du\,dy.
\end{aligned}$$

Since the above random factors are independent, we have

$$\begin{aligned}
&\mathbb{E}\big[\big(F_n(b)-F_n(a)\big)^{2m}\big]\\
&\quad\le c_3\int_{\mathbb{R}^{2md}}\int_{[a,b]\times[0,b-a]^{2m-1}}\prod_{i=1}^m\big(n^{1-Hd}\big|f(y_{2i-1})f(y_{2i})\big|\big(u_{2i-1}^{-Hd}u_{2i}^{-Hd}\big)\\
&\qquad\qquad\times\mathbb{E}\big(\big|e^{\iota y_{2i-1}\cdot X_{2i}/(n^H u_{2i}^H)}-1\big|\\
&\qquad\qquad+\big|e^{\iota y_{2i-1}\cdot X_{2i-1}/(n^H u_{2i-1}^H)}-1\big|\big)\big)\,du\,dy.
\end{aligned}$$

With the change of variables $x=y_{2i-1}$, $y=y_{2i}$, $u=u_{2i}$ and $v=u_{2i-1}$, the above inequality can be rewritten as

$$\begin{aligned}
&\mathbb{E}\big[\big(F_n(b)-F_n(a)\big)^{2m}\big]\\
&\quad\le c_3\int_{\mathbb{R}^{2d}}\int_a^b\int_0^{b-a}n^{1-Hd}\big|f(x)f(y)\big|(uv)^{-Hd}\\
&\qquad\qquad\times\mathbb{E}\big(\big|e^{\iota y\cdot X_2/(n^H u^H)}-1\big|\\
&\qquad\qquad+\big|e^{\iota y\cdot X_1/(n^H v^H)}-1\big|\big)\,du\,dv\,dx\,dy\\
&\qquad\qquad\times\bigg(2\int_{\mathbb{R}^{2d}}\int_0^{b-a}\int_0^{b-a}n^{1-Hd}\big|f(x)f(y)\big|(uv)^{-Hd}\\
&\qquad\qquad\times\mathbb{E}\big(\big|e^{\iota y\cdot X_1/(n^H u^H)}-1\big|\big)\,du\,dv\,dx\,dy\bigg)^{m-1}.
\end{aligned}\tag{3.4}$$

Notice that $\int_a^b u^{-Hd}\,du\le c_4(b-a)^{1-Hd}$. Then, by Lemma A.2, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}}\int_a^b\int_0^{b-a}n^{1-Hd}\big|f(x)f(y)\big|(uv)^{-Hd}\\
&\qquad\times\mathbb{E}\big(\big|e^{\iota y\cdot X_2/(n^H u^H)}-1\big|+\big|e^{\iota y\cdot X_1/(n^H v^H)}-1\big|\big)\,du\,dv\,dx\,dy\\
&\quad\le c_5(b-a)^{1-Hd}\int_{\mathbb{R}^{2d}}\big|f(x)f(y)\big||y|^{1/H-d}\,dx\,dy
\end{aligned}\tag{3.5}$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^{2d}} \int_0^{b-a} \int_0^{b-a} n^{1-Hd} \big| f(x) f(y) \big| (uv)^{-Hd} \\
&\qquad\qquad \times \mathbb{E}\big(\big| e^{\iota y \cdot X_1/(n^H u^H)} - 1 \big| \big)\, du\, dv\, dx\, dy \qquad (3.6)\\
&\quad \le c_6 (b-a)^{1-Hd} \int_{\mathbb{R}^{2d}} \big| f(x) f(y) \big| |y|^{1/H-d}\, dx\, dy.
\end{aligned}
$$

Now Proposition 3.1 follows from (3.4), (3.5) and (3.6). □

Now we prove that the moments of $F_n(t)$ converge to the corresponding moments of $W(L_t(0))$.

Fix a finite number of disjoint intervals $(a_i, b_i]$ with $i = 1, \dots, N$ and $b_i \le a_{i+1}$. Let $\mathbf{m} = (m_1, \dots, m_N)$ be a fixed multi-index with $m_i \in \mathbb{N}$ for $i = 1, \dots, N$. Set $\sum_{i=1}^N m_i = |\mathbf{m}|$ and $\prod_{i=1}^N m_i! = \mathbf{m}!$. We need to consider the following sequence of random variables:

$$
G_n = \prod_{i=1}^N \big(F_n(b_i) - F_n(a_i) \big)^{m_i}
$$

and compute $\lim_{n\to\infty} \mathbb{E}(G_n)$. Notice that the expectation of G_n can be written as

$$
\mathbb{E}(G_n) = \mathbf{m}! n^{|\mathbf{m}|(1+Hd)/2} \mathbb{E}\left(\int_{D_{\mathbf{m}}} \prod_{i=1}^N \prod_{j=1}^{m_i} f\big(n^H B\big(s_j^i\big)\big)\, ds \right),
$$

where

$$
D_{\mathbf{m}} = \big\{ s \in \mathbb{R}^{|\mathbf{m}|} : a_i < s_1^i < \cdots < s_{m_i}^i < b_i, 1 \le i \le N \big\}. \qquad (3.7)
$$

Here and in the sequel we denote the coordinates of a point $s \in \mathbb{R}^{|\mathbf{m}|}$ as $s = (s_j^i)$, where $1 \le i \le N$ and $1 \le j \le m_i$.

For simplicity of notation, we define

$$
J_0 = \big\{ (i, j) : 1 \le i \le N, 1 \le j \le m_i \big\}.
$$

For any (i_1, j_1) and $(i_2, j_2) \in J_0$, we define the following dictionary ordering:

$$
(i_1, j_1) \le (i_2, j_2),
$$

if $i_1 < i_2$ or $i_1 = i_2$ and $j_1 \le j_2$. For any (i, j) in J_0, under the above ordering, (i, j) is the $(\sum_{k=1}^{i-1} m_k + j)$th element in J_0, and we define $\#(i, j) = \sum_{k=1}^{i-1} m_k + j$.

PROPOSITION 3.2. *Suppose that at least one of the exponents m_i is odd. Then*

$$\lim_{n\to\infty} \mathbb{E}(G_n) = 0.$$

PROOF. The proof will be done in several steps.

Step 1. Using a similar argument to that in (3.2), we obtain

$$\begin{aligned}
\mathbb{E}(G_n) = {} & \frac{\mathbf{m}!}{(2\pi)^{|\mathbf{m}|d}} n^{|\mathbf{m}|(1-Hd)/2} \\
& \times \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \left(\prod_{i=1}^{N} \prod_{j=1}^{m_i} f\big(y_j^i\big) \right) \\
& \qquad \times \exp\left(-\frac{1}{2} \operatorname{Var}\left(\sum_{i=1}^{N} \sum_{j=1}^{m_i} \xi_j^i \cdot B\big(s_j^i\big) \right) \right. \\
& \qquad\qquad \left. - \iota \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{y_j^i \cdot \xi_j^i}{n^H} \right) d\xi\, ds\, dy \\
= {} & \frac{\mathbf{m}!}{(2\pi)^{|\mathbf{m}|d}} n^{|\mathbf{m}|(1-Hd)/2} \\
& \times \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \left(\prod_{i=1}^{N} \prod_{j=1}^{m_i} f\big(y_j^i\big) \right) \\
& \qquad \times \exp\left(-\frac{1}{2} \operatorname{Var}\left(\sum_{i=1}^{N} \sum_{j=1}^{m_i} \xi_j^i \cdot B\big(s_j^i\big) \right) \right) \\
& \qquad \times \prod_{i=1}^{N} \prod_{j=1}^{m_i} \big(e^{-\iota y_j^i \cdot \xi_j^i / n^H} - 1 \big)\, d\xi\, ds\, dy,
\end{aligned}$$

where we used the fact $\int_{\mathbb{R}^d} f(x)\, dx = 0$ in the last equality.

By the local nondeterminism property (2.3), with the convention $s_0^i = s_{m_{i-1}}^{i-1}$ for $2 \le i \le N$ and $s_0^1 = 0$,

$$\operatorname{Var}\left(\sum_{i=1}^{N} \sum_{j=1}^{m_i} \xi_j^i \cdot B\big(s_j^i\big) \right) \ge \kappa_H \sum_{i=1}^{N} \sum_{j=1}^{m_i} \left| \sum_{(l,k)\ge(i,j)} \xi_k^l \right|^2 \big(s_j^i - s_{j-1}^i\big)^{2H}.$$

Let $F(y) = \prod_{i=1}^{N} \prod_{j=1}^{m_i} f(y_j^i)$, and make the change of variables

$$\eta_j^i = \sum_{(\ell,k)\ge(i,j)} \xi_k^\ell$$

for $1 \le i \le N$ and $1 \le j \le m_i$. Then we can estimate $E(G_n)$ as follows:

$$\begin{aligned}\big|\mathbb{E}(G_n)\big| &\le c_1 n^{|\mathbf{m}|(1-Hd)/2}\\ &\quad\times \int_{\mathbb{R}^{|\mathbf{m}|d}}\int_{D_{\mathbf{m}}}\int_{\mathbb{R}^{|\mathbf{m}|d}} |F(y)| \exp\Bigg(-\frac{\kappa_H}{2}\sum_{i=1}^{N}\sum_{j=1}^{m_i}|\eta_j^i|^2(s_j^i - s_{j-1}^i)^{2H}\Bigg)\\ &\quad\times \Bigg(\prod_{i=1}^{N}\prod_{j=1}^{m_i}\bigg|\exp\bigg(-\iota\frac{y_j^i}{n^H}\cdot(\eta_j^i - \eta_{j+1}^i)\bigg) - 1\bigg|\Bigg)\,d\eta\,ds\,dy.\end{aligned}$$

Making the change of variable $\eta_j^i = (s_j^i - s_{j-1}^i)^H \xi_j^i$ yields

$$\begin{aligned}&\big|\mathbb{E}(G_n)\big|\\ &\quad\le c_1 n^{|\mathbf{m}|(1-Hd)/2}\\ &\qquad\times \int_{\mathbb{R}^{|\mathbf{m}|d}}\int_{D_{\mathbf{m}}}\int_{\mathbb{R}^{|\mathbf{m}|d}} |F(y)|\\ &\qquad\quad\times \Bigg(\prod_{i=1}^{N}\prod_{j=1}^{m_i}(s_j^i - s_{j-1}^i)^{-Hd}\Bigg)\exp\Bigg(-\frac{\kappa_H}{2}\sum_{i=1}^{N}\sum_{j=1}^{m_i}|\xi_j^i|^2\Bigg)\\ &\qquad\quad\times \Bigg(\prod_{i=1}^{N}\prod_{j=1}^{m_i}\bigg|\exp\bigg(\iota\frac{y_j^i\cdot\xi_{j+1}^i}{n^H(s_{j+1}^i - s_j^i)^H}\\ &\qquad\qquad - \iota\frac{y_j^i\cdot\xi_j^i}{n^H(s_j^i - s_{j-1}^i)^H}\bigg) - 1\bigg|\Bigg)\,d\xi\,ds\,dy\end{aligned}$$

with the convention $\xi_{m_N+1}^N = 0$, $\xi_{m_i+1}^i = \xi_1^{i+1}$ for $1 \le i \le N-1$. For the probabilistic argument to be used below, it is convenient to express the integral with respect to $d\xi$ as an expectation. In this way we can write

$$\begin{aligned}&\big|\mathbb{E}(G_n)\big|\\ &\quad\le c_2 n^{|\mathbf{m}|(1-Hd)/2}\\ &\qquad\times \int_{\mathbb{R}^{|\mathbf{m}|d}}\int_{D_{\mathbf{m}}} |F(y)|\Bigg(\prod_{i=1}^{N}\prod_{j=1}^{m_i}(s_j^i - s_{j-1}^i)^{-Hd}\Bigg)\\ &\qquad\quad\times \mathbb{E}\Bigg(\prod_{i=1}^{N}\prod_{j=1}^{m_i}\bigg|\exp\bigg(\iota\frac{y_j^i\cdot X_{j+1}^i}{n^H(s_{j+1}^i - s_j^i)^H}\\ &\qquad\qquad - \iota\frac{y_j^i\cdot X_j^i}{n^H(s_j^i - s_{j-1}^i)^H}\bigg) - 1\bigg|\Bigg)\,ds\,dy,\end{aligned}\tag{3.8}$$

where $\sqrt{\kappa_H} X_j^i$ ($1 \le i \le N$, $1 \le j \le m_i$) are independent copies of a d-dimensional standard normal random vector, and as before, we use the convention $X_{m_N+1}^N = 0$, $X_{m_i+1}^i = X_1^{i+1}$ for $1 \le i \le N-1$.

Denote the expectation in (3.8) by I. That is,

$$\begin{aligned} I &= \mathbb{E}\biggl(\prod_{i=1}^{N}\prod_{j=1}^{m_i}\biggl|\exp\biggl(\iota \frac{y_j^i \cdot X_{j+1}^i}{n^H(s_{j+1}^i - s_j^i)^H} - \iota \frac{y_j^i \cdot X_j^i}{n^H(s_j^i - s_{j-1}^i)^H}\biggr) - 1\biggr|\biggr) \\ &= \mathbb{E}\biggl(\prod_{(i,j)\in J_0} I_{i,j}\biggr), \end{aligned}$$

where

$$I_{i,j} = \biggl|\exp\biggl(\iota \frac{y_j^i \cdot X_{j+1}^i}{n^H(s_{j+1}^i - s_j^i)^H} - \iota \frac{y_j^i \cdot X_j^i}{n^H(s_j^i - s_{j-1}^i)^H}\biggr) - 1\biggr|$$

for $1 \le i \le N$ and $1 \le j \le m_i$.

Notice that the random variables $I_{i,j}$ for $(i,j) \in J_0$ are dependent. We are going to choose a proper subset of J_0 in the following way. Assume that m_ℓ is the first odd exponent. Then we choose all the factors $I_{i,j}$ such that $\#(i,j) < \#(\ell, m_\ell)$ and $\#(i,j)$ is odd. Then we choose all the factors $I_{i,j}$ such that $\#(i,j) > \#(\ell, m_\ell) + 1$, and $\#(i,j)$ is even. Notice that all these factors are mutually independent, and they are also independent of the product $I_{\ell,m_\ell} I_{\ell+1,1}$. The lack of independence of the two factors I_{ℓ,m_ℓ} and $I_{\ell+1,1}$ will be compensated by the fact that the integral of $(s_1^{\ell+1} - s_{m_\ell}^\ell)^{-\beta}$ is finite for any $\beta < 2$ because we have the constraint $s_{m_\ell}^\ell < b_\ell < s_1^{\ell+1}$. To make this argument more precise, let us define

$$J_\ell = J_{\ell,1} \cup J_{\ell,2},$$

where

$$J_{\ell,1} = \{(i,j) \in J_0 : \#(i,j) < \#(\ell, m_\ell) \text{ and } \#(i,j) \text{ odd}\}$$

and

$$J_{\ell,2} = \{(i,j) \in J_0 : \#(i,j) > \#(\ell, m_\ell) + 1 \text{ and } \#(i,j) \text{ even}\}.$$

Notice that $I_{i,j} \le 2$ for all $(i,j) \in J_0$. Then

$$I \le \begin{cases} c_2 \mathbb{E}\biggl(I_{\ell,m_\ell} I_{\ell+1,1} \prod_{(i,j)\in J_\ell} I_{i,j}\biggr), & \text{if } \ell \ne N, \\ c_2 \mathbb{E}\biggl(I_{\ell,m_\ell} \prod_{(i,j)\in J_\ell} I_{i,j}\biggr), & \text{if } \ell = N. \end{cases}$$

Step 2. We first consider the case $\ell \neq N$. In this case, the number of elements in J_ℓ is $[\frac{|\mathbf{m}|}{2}] - 1$ and

$$\big|\mathbb{E}(G_n)\big| \le c_3 n^{|\mathbf{m}|(1-Hd)/2} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}} |F(y)| \Big(\prod_{(i,j)\in J_0} (s_j^i - s_{j-1}^i)^{-Hd} \Big) \times \mathbb{E}(I_{\ell,m_\ell} I_{\ell+1,1}) \prod_{(i,j)\in J_\ell} \mathbb{E}(I_{i,j})\, ds\, dy.$$

In the last inequality, we used the fact that all random variables $I_{\ell,m_\ell} I_{\ell+1,1}$ and $I_{i,j}$ for $(i,j) \in J_\ell$ are independent.

Since $|e^{\iota(z_1-z_2)} - 1| \le |e^{\iota z_1} - 1| + |e^{\iota z_2} - 1|$ for all $z_1, z_2 \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}(I_{\ell,m_\ell} I_{\ell+1,1}) \le \mathbb{E}\big\{ &\big(\big|e^{\iota y_{m_\ell}^\ell \cdot X_1^{\ell+1}/(n^H (s_1^{\ell+1} - s_{m_\ell}^\ell)^H)} - 1\big| \\ &+ \big|e^{\iota y_{m_\ell}^\ell \cdot X_{m_\ell}^\ell/(n^H (s_{m_\ell}^\ell - s_{m_\ell-1}^\ell)^H)} - 1\big|\big) \\ &\times \big(\big|e^{\iota y_1^{\ell+1} \cdot X_2^{\ell+1}/(n^H (s_2^{\ell+1} - s_1^{\ell+1})^H)} - 1\big| \\ &+ \big|e^{\iota y_1^{\ell+1} \cdot X_1^{\ell+1}/(n^H (s_1^{\ell+1} - s_{m_\ell}^\ell)^H)} - 1\big|\big)\big\}. \end{aligned}$$

Notice that $X_2^{\ell+1}$, $X_1^{\ell+1}$ and $X_{m_\ell}^\ell$ are independent. As a consequence, we can write

$$\mathbb{E}(I_{\ell,m_\ell} I_{\ell+1,1}) \le I_1^\ell + I_2^\ell,$$

where

$$\begin{aligned} I_1^\ell = {}& \mathbb{E}\big|e^{\iota y_{m_\ell}^\ell \cdot X_1^{\ell+1}/(n^H (s_1^{\ell+1} - s_{m_\ell}^\ell)^H)} - 1\big| \mathbb{E}\big|e^{\iota y_1^{\ell+1} \cdot X_2^{\ell+1}/(n^H (s_2^{\ell+1} - s_1^{\ell+1})^H)} - 1\big| \\ &+ \mathbb{E}\big|e^{\iota y_{m_\ell}^\ell \cdot X_{m_\ell}^\ell/(n^H (s_{m_\ell}^\ell - s_{m_\ell-1}^\ell)^H)} - 1\big| \mathbb{E}\big|e^{\iota y_1^{\ell+1} \cdot X_2^{\ell+1}/(n^H (s_2^{\ell+1} - s_1^{\ell+1})^H)} - 1\big| \\ &+ \mathbb{E}\big|e^{\iota y_{m_\ell}^\ell \cdot X_{m_\ell}^\ell/(n^H (s_{m_\ell}^\ell - s_{m_\ell-1}^\ell)^H)} - 1\big| \mathbb{E}\big|e^{\iota y_1^{\ell+1} \cdot X_1^{\ell+1}/(n^H (s_1^{\ell+1} - s_{m_\ell}^\ell)^H)} - 1\big| \end{aligned}$$

and

$$I_2^\ell = \mathbb{E}\big(\big|e^{\iota y_{m_\ell}^\ell \cdot X_1^{\ell+1}/(n^H (s_1^{\ell+1} - s_{m_\ell}^\ell)^H)} - 1\big|\big|e^{\iota y_1^{\ell+1} \cdot X_1^{\ell+1}/(n^H (s_1^{\ell+1} - s_{m_\ell}^\ell)^H)} - 1\big|\big). \tag{3.9}$$

Therefore,

$$\big|\mathbb{E}(G_n)\big| \le c_3 n^{|\mathbf{m}|(1-Hd)/2} \int_{\mathbb{R}^{|\mathbf{m}|d}} \big|F(y)\big| (G_{1,n} + G_{2,n})\, dy, \tag{3.10}$$

where

$$G_{k,n} = \int_{D_{\mathbf{m}}} \Big(\prod_{(i,j)\in J_0} (s_j^i - s_{j-1}^i)^{-Hd} \Big) I_k^\ell \prod_{(i,j)\in J_\ell} \mathbb{E}(I_{i,j})\, ds$$

for $k = 1, 2$.

We claim that

$$G_{1,n} \le c_4 n^{([|\mathbf{m}|/2]+1)(Hd-1)} |y_1^{\ell+1}|^{1/H-d} |y_{m_\ell}^\ell|^{1/H-d} \prod_{(i,j)\in J_\ell} |y_j^i|^{1/H-d}.$$

In fact, making the change of variables $v_j^i = s_j^i - s_{j-1}^i$ for all $(i,j) \in J_0$, and defining

$$a_{i,j} = \begin{cases} (v_j^i v_{j+1}^i)^{-Hd} \mathbb{E}(I_{i,j}), & \text{if } (i,j) \in J_\ell \text{ and } (i,j) \neq (N, m_N); \\ (v_{m_N}^N)^{-Hd} \mathbb{E}(I_{N,m_N}), & \text{if } (i,j) \in J_\ell \text{ and } (i,j) = (N, m_N), \end{cases}$$

and $a_1^\ell = (v_2^{\ell+1} v_1^{\ell+1} v_{m_\ell}^\ell)^{-Hd} I_1^\ell$, we obtain

$$\begin{aligned} G_{1,n} &\le \int_{[0,b_N]^{|\mathbf{m}|}} a_1^\ell \prod_{(i,j)\in J_\ell} a_{i,j}\, dv \\ &= \int_{[0,b_N]^3} a_1^\ell \, dv_2^{\ell+1}\, dv_1^{\ell+1}\, dv_{m_\ell}^\ell \prod_{(i,j)\in J_\ell} \int_{[0,b_N]^2} a_{i,j}\, dv_j^i\, dv_{j+1}^i \qquad (3.11) \\ &\le c_5 n^{([|\mathbf{m}|/2]+1)(Hd-1)} |y_1^{\ell+1}|^{1/H-d} |y_{m_\ell}^\ell|^{1/H-d} \prod_{(i,j)\in J_\ell} |y_j^i|^{1/H-d}. \end{aligned}$$

Here we used Lemma A.2 in the last inequality $[\frac{|\mathbf{m}|}{2}]+1$ times.

For any $\beta \in [0,1]$, we have $|e^{\iota z} - 1| \le c_\beta |z|^\beta$ for all $z \in \mathbb{R}$. Recall the definition of I_2^ℓ in (3.9). We then have

$$\begin{aligned} I_2^\ell &\le c_6 n^{-2H\beta} |y_{m_\ell}^\ell|^\beta |y_1^{\ell+1}|^\beta (s_1^{\ell+1} - s_{m_\ell}^\ell)^{-2H\beta} \mathbb{E}|X_1^{\ell+1}|^{2\beta} \\ &\le c_7 n^{-2H\beta} |y_{m_\ell}^\ell|^\beta |y_1^{\ell+1}|^\beta (s_1^{\ell+1} - s_{m_\ell}^\ell)^{-2H\beta}. \end{aligned}$$

So

$$\begin{aligned} G_{2,n} &\le c_7 n^{-2H\beta} |y_{m_\ell}^\ell|^\beta |y_1^{\ell+1}|^\beta \\ &\quad \times \int_{D_{\mathbf{m}}} (s_1^{\ell+1} - s_{m_\ell}^\ell)^{-2H\beta} \Bigg(\prod_{(i,j)\in J_0} (s_j^i - s_{j-1}^i)^{-Hd} \Bigg) I_2^\ell \prod_{(i,j)\in J_\ell} \mathbb{E}(I_{i,j})\, ds. \end{aligned}$$

Define $J_{\ell,3} = \{(i,j) \in J_0 : \#(i,j) < \#(\ell, m_\ell)\}$ and

$$D_{\mathbf{m}}^\ell = \{a_i < s_1^i < \cdots < s_{m_i}^i < b_i, 1 \le i \le \ell - 1; a_\ell < s_1^\ell < \cdots < s_{m_\ell - 1}^\ell < b_\ell\}.$$

Integrating the above integral with respect to s_j^i for $(i,j) \in J_{\ell,2}$ and using Lemma A.2,

$$\begin{aligned} G_{2,n} &\le c_8 n^{-2H\beta} n^{\# J_{\ell,2}(Hd-1)} |y_{m_\ell}^\ell|^\beta |y_1^{\ell+1}|^\beta \prod_{(i,j)\in J_{\ell,2}} |y_j^i|^{1/H-d} \\ &\quad \times \int_{D_{\mathbf{m}}^\ell} I_3^\ell \prod_{(i,j)\in J_{\ell,3}} (s_j^i - s_{j-1}^i)^{-Hd} \prod_{(i,j)\in J_{\ell,1}} \mathbb{E}(I_{i,j})\, ds, \end{aligned}$$

where $\#J_{\ell,2}$ is the cardinality of $J_{\ell,2}$ and

$$I_3^\ell = \int_{s^\ell_{m_\ell-1}}^{b_l} \int_{a_{\ell+1}}^{b_{l+1}} \int_{s_1^{\ell+1}}^{b_{l+1}} \left(s_2^{\ell+1} - s_1^{\ell+1}\right)^{-Hd} \left(s_1^{\ell+1} - s^\ell_{m_\ell}\right)^{-Hd-2H\beta} \\ \times \left(s^\ell_{m_\ell} - s^\ell_{m_\ell-1}\right)^{-Hd} \, ds_2^{\ell+1} \, ds_1^{\ell+1} \, ds^\ell_{m_\ell}.$$

We observe that if $1 - Hd < 2H\beta \le 2 - 2Hd$,

$$\begin{aligned} I_3^\ell &\le c_9 \int_{s^\ell_{m_\ell-1}}^{b_l} \int_{a_{\ell+1}}^{b_{l+1}} \left(s_1^{\ell+1} - s^\ell_{m_\ell}\right)^{-Hd-2H\beta} \left(s^\ell_{m_\ell} - s^\ell_{m_\ell-1}\right)^{-Hd} \, ds_1^{\ell+1} \, ds^\ell_{m_\ell} \\ &\le c_{10} \int_{s^\ell_{m_\ell-1}}^{b_l} \left(a_{\ell+1} - s^\ell_{m_\ell}\right)^{1-Hd-2H\beta} \left(s^\ell_{m_\ell} - s^\ell_{m_\ell-1}\right)^{-Hd} \, ds^\ell_{m_\ell} \\ &\le c_{11} (a_{\ell+1} - a_\ell)^{2-2Hd-2H\beta}. \end{aligned}$$

As a consequence,

$$\begin{aligned} G_{2,n} &\le c_{12} n^{-2H\beta} n^{\#J_{\ell,2}(Hd-1)} \left|y^\ell_{m_\ell}\right|^\beta \left|y_1^{\ell+1}\right|^\beta \prod_{(i,j)\in J_{\ell,2}} \left|y_j^i\right|^{1/H-d} \\ &\qquad \times \int_{D^\ell_{\mathbf{m}}} \prod_{(i,j)\in J_{\ell,3}} \left(s_j^i - s_{j-1}^i\right)^{-Hd} \prod_{(i,j)\in J_{\ell,1}} \mathbb{E}(I_{i,j}) \, ds \\ &\le c_{13} n^{-2H\beta} n^{\#J_\ell(Hd-1)} \left|y^\ell_{m_\ell}\right|^\beta \left|y_1^{\ell+1}\right|^\beta \prod_{(i,j)\in J_\ell} \left|y_j^i\right|^{1/H-d} \\ &= c_{13} n^{-2H\beta} n^{([|\mathbf{m}|/2]-1)(Hd-1)} \left|y^\ell_{m_\ell}\right|^\beta \left|y_1^{\ell+1}\right|^\beta \prod_{(i,j)\in J_\ell} \left|y_j^i\right|^{1/H-d}. \end{aligned} \tag{3.12}$$

Choosing $\beta = \frac{1}{H} - d$ in (3.12),

$$G_{2,n} \le c_{13} n^{([|\mathbf{m}|/2]+1)(Hd-1)} \left|y^\ell_{m_\ell}\right|^{1/H-d} \left|y_1^{\ell+1}\right|^{1/H-d} \\ \times \prod_{(i,j)\in J_\ell} \left|y_j^i\right|^{1/H-d}. \tag{3.13}$$

Substituting (3.11) and (3.13) into (3.10) yields

$$\left|\mathbb{E}(G_n)\right| \le c_{14} n^{-(1-Hd)/2} \\ \times \int_{\mathbb{R}^{|\mathbf{m}|d}} \left|F(y)\right| \left|y^\ell_{m_\ell}\right|^{1/H-d} \left|y_1^{\ell+1}\right|^{1/H-d} \\ \times \prod_{(i,j)\in J_\ell} \left|y_j^i\right|^{1/H-d} \, dy. \tag{3.14}$$

Step 3. Now we consider the case $\ell = N$. In this case, $J_\ell = J_{\ell,1}$ and

$$\begin{aligned}\left|\mathbb{E}(G_n)\right| \le c_{15} n^{|\mathbf{m}|(1-Hd)/2} \\ \times \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}} |F(y)| \prod_{(i,j)\in J_0} \left(s_j^i - s_{j-1}^i\right)^{-Hd} \\ \times \mathbb{E}(I_{N,m_N}) \prod_{(i,j)\in J_{N,1}} \mathbb{E}(I_{i,j})\, ds\, dy.\end{aligned}$$

Define $J_{N,3} = \{(i,j) \in J_0 : \#(i,j) < \#(N, m_N)\}$ and

$$D_{\mathbf{m}}^N = \left\{a_i < s_1^i < \cdots < s_{m_i}^i < b_i, 1 \le i \le N-1; a_\ell < s_1^N < \cdots < s_{m_N-1}^N < b_N\right\}.$$

Integrating the above integral with respect to $s_{m_N}^N$ and using Lemma A.2 yield

$$\begin{aligned}\left|\mathbb{E}(G_n)\right| \le c_{16} n^{(|\mathbf{m}|-2)(1-Hd)/2} \\ \times \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}^N} |F(y)| \left|y_{m_N}^N\right|^{1/H-d} \\ \times \prod_{(i,j)\in J_{N,3}} \left(s_j^i - s_{j-1}^i\right)^{-Hd} \prod_{(i,j)\in J_{N,1}} \mathbb{E}(I_{i,j})\, ds\, dy.\end{aligned}$$

Using arguments similar to those in step 2,

$$\begin{aligned}\left|\mathbb{E}(G_n)\right| \le c_{17} n^{-(1-Hd)/2} \\ \times \int_{\mathbb{R}^{|\mathbf{m}|d}} |F(y)| \left|y_{m_N}^N\right|^{1/H-d} \prod_{(i,j)\in J_{N,1}} \left|y_j^i\right|^{1/H-d} dy.\end{aligned} \tag{3.15}$$

Step 4. Recall that $f \in H_0^{1/H-d}$. Then from (3.14) and (3.15), we see that $|\mathbb{E}(G_n)|$ is bounded by a multiple of $n^{-(1-Hd)/2}$. Our result now follows from taking the limit. $\square$

In the sequel, we consider the convergence of moments when all exponents m_i are even. Recall the definition of $D_{\mathbf{m}}$ in (3.7). For $1 \le \ell \le N$ and $1 \le k \le \frac{m_\ell}{2}$, we define

$$O_k^\ell = D_{\mathbf{m}} \cap \left\{ \frac{s_{2k}^\ell - s_{2k-2}^\ell}{2} < s_{2k}^\ell - s_{2k-1}^\ell \right\}.$$

The following result tells us that the integrals over the domain O_k^ℓ do not contribute to the limit of the moments. This result will play a fundamental role in computing the limits of even moments.

PROPOSITION 3.3. *For any* $1 \le \ell \le N$ *and* $1 \le k \le \frac{m_\ell}{2}$,

$$\lim_{n\to\infty} n^{|\mathbf{m}|(1+Hd)/2} \mathbb{E}\left(\int_{O_k^\ell} \prod_{i=1}^N \prod_{j=1}^{m_i} f\left(n^H B(s_j^i)\right) ds \right) = 0.$$

PROOF. Using the arguments and notation in the proof of Proposition 3.2, we obtain

$$
\begin{aligned}
n^{|\mathbf{m}|(1+Hd)/2} & \left|\mathbb{E}\left(\int_{O_k^\ell} \prod_{i=1}^N \prod_{j=1}^{m_i} f\big(n^H B(s_j^i)\big)\,ds\right)\right| \\
&\le c_1 n^{|\mathbf{m}|(1-Hd)/2} \\
&\quad \times \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{O_k^\ell} |F(y)| \left(\prod_{i=1}^N \prod_{j=1}^{m_i} (s_j^i - s_{j-1}^i)^{-H}\right) \\
&\qquad \times \mathbb{E}\left(\prod_{i=1}^N \prod_{j=1}^{m_i} \left|\exp\left(\iota \frac{y_j^i \cdot X_{j+1}^i}{n^H (s_{j+1}^i - s_j^i)^H} \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. - \iota \frac{y_j^i \cdot X_j^i}{n^H (s_j^i - s_{j-1}^i)^H}\right) - 1\right|\right) ds\,dy,
\end{aligned}
$$

where $X_{m_N+1}^N = 0$, $X_{m_i+1}^i = X_1^{i+1}$ for $1 \le i \le N-1$, and $\sqrt{\kappa_H} X_j^i$ ($1 \le i \le N$, $1 \le j \le m_i$) are independent copies of a d-dimensional standard normal random vector.

We make the change of variables $v_j^i = s_j^i - s_{j-1}^i$ for all $(i,j) \in J_0$. The integral domain O_k^ℓ becomes

$$
D_k^\ell = \left\{ v \in \mathbb{R}_+^{|\mathbf{m}|} : a_1 < \sum_{(i,j)\in J_0} v_j^i < b_N,\, v_{2k-1}^\ell < v_{2k}^\ell \right\}.
$$

For $(i,j) \in J_0$, define

$$
I_{i,j} = \left|\exp\left(\iota \frac{y_j^i \cdot X_{j+1}^i}{n^H (v_{j+1}^i)^H} - \iota \frac{y_j^i \cdot X_j^i}{n^H (v_j^i)^H}\right) - 1\right|.
$$

Then

$$
\begin{aligned}
n^{|\mathbf{m}|(1+Hd)/2} & \left|\mathbb{E}\left(\int_{O_k^\ell} \prod_{i=1}^N \prod_{j=1}^{m_i} f\big(n^H B(s_j^i)\big)\,ds\right)\right| \\
&\le c_1 n^{|\mathbf{m}|(1-Hd)/2} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_k^\ell} F(y) \mathbb{E}\left(\prod_{(i,j)\in J_0} (v_j^i)^{-Hd} I_{i,j}\right) dv\,dy.
\end{aligned}
\tag{3.16}
$$

Next we estimate the expectation in (3.16). We are going to use an argument similar to the one used in the proof of Proposition 3.2, based on the selection of some factors in the above product. Here, the dependent product that will play a basic role will be $I_{\ell,2k} I_{\ell,2k-1}$, due to the definition of the set O_k^ℓ. Define

$$
J_k^\ell = J_{k,1}^\ell \cup J_{k,2}^\ell,
$$

where

$$J_{k,1}^{\ell} = \{(i,j) \in J_0 : \#(i,j) < \#(\ell, 2k-2), \#(i,j) \text{ odd}\},$$
$$J_{k,2}^{\ell} = \{(i,j) \in J_0 : \#(i,j) > \#(\ell, 2k), \#(i,j) \text{ even}\}.$$

Since all exponents m_i are even, the number of elements in J_k^{ℓ} is $\frac{|\mathbf{m}|-2}{2} = \frac{|\mathbf{m}|}{2} - 1$. From the definition of $I_{i,j}$, we know that random variables $I_{\ell,2k} I_{\ell,2k-1}$ and $I_{i,j}$ for $(i,j) \in J_k^{\ell}$ are independent. Then

$$\begin{aligned} n^{|\mathbf{m}|(1+Hd)/2} & \left| \mathbb{E}\left(\int_{O_k^{\ell}} \prod_{i=1}^{N} \prod_{j=1}^{m_i} f\big(n^H B(s_j^i)\big)\, ds \right) \right| \\ & \le c_2 n^{|\mathbf{m}|(1-Hd)/2} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_k^{\ell}} F(y) \mathbb{E}(I_{\ell,2k} I_{\ell,2k-1}) \\ & \qquad \times \prod_{(i,j)\in J_0} (v_j^i)^{-Hd} \prod_{(i,j)\in J_k^{\ell}} \mathbb{E}(I_{i,j})\, dv\, dy. \end{aligned}$$

For $(i,j) \in J_k^{\ell}$ and $(i,j) \neq (N, m_N)$, define

$$a_{i,j} = \big(v_j^i v_{j+1}^i\big)^{-Hd} \mathbb{E}(I_{i,j})$$

and $a_{N,m_N} = (v_{m_N}^N)^{-Hd} \mathbb{E}(I_{N,m_N})$. From Lemma A.2, we obtain

$$\int_{[0,b_N]^2} a_{i,j}\, dv_j^i\, dv_{j+1}^i \le c_3 n^{Hd-1} |y_j^i|^{1/H-d}$$

for all $(i,j) \in J_k^{\ell}$. Therefore,

$$\begin{aligned} n^{|\mathbf{m}|(1+Hd)/2} & \left| \mathbb{E}\left(\int_{O_k^{\ell}} \prod_{i=1}^{N} \prod_{j=1}^{m_i} f\big(n^H B(s_j^i)\big)\, ds \right) \right| \\ & \le c_4 n^{1-Hd} \int_{\mathbb{R}^{|\mathbf{m}|d}} F(y) \prod_{(i,j)\in J_k^{\ell}} |y_j^i|^{1/H-d} I_k^{\ell}\, dy, \end{aligned} \tag{3.17}$$

where

$$I_k^{\ell} = \int_0^{b_N} \int_{v_{2k-1}^{\ell}}^{b_N} \int_0^{b_N} \big(v_{2k+1}^{\ell} v_{2k}^{\ell} v_{2k-1}^{\ell}\big)^{-Hd} \mathbb{E}(I_{\ell,2k} I_{\ell,2k-1})\, dv_{2k+1}^{\ell}\, dv_{2k}^{\ell}\, dv_{2k-1}^{\ell}.$$

Notice that $|e^{\iota(z_1-z_2)} - 1| \le |e^{\iota z_1} - 1| + |e^{\iota z_2} - 1|$ for all $z_1, z_2 \in \mathbb{R}$. Using the independence of X_{2k-1}^{ℓ}, X_{2k}^{ℓ} and X_{2k+1}^{ℓ}, we obtain

$$\mathbb{E}(I_{\ell,2k-1} I_{\ell,2k}) \le A_{k,1}^{\ell} + A_{k,2}^{\ell},$$

where

$$A_{k,1}^{\ell} = \mathbb{E}\big|e^{\iota y_{2k-1}^{\ell}\cdot X_{2k}^{\ell}/(n^H (v_{2k}^{\ell})^H)} - 1\big|\mathbb{E}\big|e^{\iota y_{2k}^{\ell}\cdot X_{2k+1}^{\ell}/(n^H (v_{2k+1}^{\ell})^H)} - 1\big|$$
$$+ \mathbb{E}\big|e^{\iota y_{2k-1}^{\ell}\cdot X_{2k-1}^{\ell}/(n^H (v_{2k-1}^{\ell})^H)} - 1\big|\mathbb{E}\big|e^{\iota y_{2k}^{\ell}\cdot X_{2k+1}^{\ell}/(n^H (v_{2k+1}^{\ell})^H)} - 1\big|$$
$$+ \mathbb{E}\big|e^{\iota y_{2k-1}^{\ell}\cdot X_{2k-1}^{\ell}/(n^H (v_{2k-1}^{\ell})^H)} - 1\big|\mathbb{E}\big|e^{\iota y_{2k}^{\ell}\cdot X_{2k}^{\ell}/(n^H (v_{2k}^{\ell})^H)} - 1\big|$$

and

$$A_{k,2}^{\ell} = \mathbb{E}\big(\big|e^{\iota y_{2k-1}^{\ell}\cdot X_{2k}^{\ell}/(n^H (v_{2k}^{\ell})^H)} - 1\big|\big|e^{\iota y_{2k}^{\ell}\cdot X_{2k}^{\ell}/(n^H (v_{2k}^{\ell})^H)} - 1\big|\big).$$

Now we have

$$I_k^{\ell} = I_{k,1}^{\ell} + I_{k,2}^{\ell}, \tag{3.18}$$

where

$$I_{k,i}^{\ell} = \int_0^{b_N}\int_{v_{2k-1}^{\ell}}^{b_N}\int_0^{b_N} \big(v_{2k+1}^{\ell} v_{2k}^{\ell} v_{2k-1}^{\ell}\big)^{-Hd} A_{k,i}^{\ell}\, dv_{2k+1}^{\ell}\, dv_{2k}^{\ell}\, dv_{2k-1}^{\ell}$$

for $i = 1, 2$. By Lemma A.2,

$$I_{k,1}^{\ell} \le c_5 n^{-2(1-Hd)} \big|y_{2k-1}^{\ell}\big|^{1/H-d} \big|y_{2k}^{\ell}\big|^{1/H-d}. \tag{3.19}$$

For any $\beta \in [0, 1]$, we have $|e^{\iota z} - 1| \le c_\beta |z|^\beta$ for all $z \in \mathbb{R}$. Then

$$A_{k,2}^{\ell} \le c_6 n^{-2H\beta} \big(v_{2k}^{\ell}\big)^{-2H\beta} \big|y_{2k-1}^{\ell}\big|^{\beta} \big|y_{2k}^{\ell}\big|^{\beta}.$$

Therefore, if $1 - Hd < 2H\beta < 2 - 2Hd$,

$$\begin{aligned} I_{k,2}^{\ell} &\le c_6 n^{-2H\beta} \big|y_{2k-1}^{\ell}\big|^{\beta} \big|y_{2k}^{\ell}\big|^{\beta} \\ &\quad \times \int_0^{b_N}\int_{v_{2k-1}^{\ell}}^{b_N}\int_0^{b_N} \big(v_{2k+1}^{\ell}\big)^{-Hd} \big(v_{2k}^{\ell}\big)^{-Hd-2H\beta} \\ &\qquad \times \big(v_{2k-1}^{\ell}\big)^{-Hd}\, dv_{2k+1}^{\ell}\, dv_{2k}^{\ell}\, dv_{2k-1}^{\ell} \\ &\le c_7 n^{-2H\beta} \big|y_{2k-1}^{\ell}\big|^{\beta} \big|y_{2k}^{\ell}\big|^{\beta} \int_0^{b_N} \big(v_{2k-1}^{\ell}\big)^{1-2Hd-2H\beta}\, dv_{2k-1}^{\ell} \\ &\le c_8 n^{-2H\beta} \big|y_{2k-1}^{\ell}\big|^{\beta} \big|y_{2k}^{\ell}\big|^{\beta}. \end{aligned}$$

Choose $\beta = \frac{3(1-Hd)}{4H}$,

$$I_{k,2}^{\ell} \le c_8 n^{-3(1-Hd)/2} \big|y_{2k-1}^{\ell}\big|^{3(1-Hd)/(4H)} \big|y_{2k}^{\ell}\big|^{3(1-Hd)/(4H)}. \tag{3.20}$$

Substituting (3.19) and (3.20) into (3.18), we obtain

$$\begin{aligned} I_k^{\ell} \le c_9 n^{-3(1-Hd)/2} \big(&\big|y_{2k-1}^{\ell}\big|^{3(1-Hd)/(4H)} \big|y_{2k}^{\ell}\big|^{3(1-Hd)/(4H)} \\ &+ \big|y_{2k-1}^{\ell}\big|^{1/H-d} \big|y_{2k}^{\ell}\big|^{1/H-d}\big). \end{aligned} \tag{3.21}$$

Our result now follows easily from (3.17), (3.21) and the assumption $f \in H_0^{1/H-d}$. $\square$

Consider now the convergence of moments when all exponents m_i are even. On each portion of the coordinates $a_i < s_1^i < \cdots < s_{m_i}^i < b_i$ we make the following change of variables:

$$u_{2k}^i = s_{2k}^i \quad \text{and} \quad u_{2k-1}^i = n(s_{2k}^i - s_{2k-1}^i) \qquad \text{where } 1 \le k \le m_i/2$$

with the convention $u_0^i = s_0^i = a_i$. The idea is to couple each variable with an odd subindex with the next one. In this way we obtain

$$\begin{aligned}\mathbb{E}(G_n) = \mathbf{m}! n^K \mathbb{E}\biggl(\int_{D_{\mathbf{m}}^n} \prod_{i=1}^N \prod_{k=1}^{m_i/2} f\bigl(n^H B(u_{2k}^i)\bigr) \\ \times f\biggl(n^H B\biggl(u_{2k}^i - \frac{u_{2k-1}^i}{n}\biggr)\biggr) du\biggr),\end{aligned} \tag{3.22}$$

where K and $D_{\mathbf{m}}^n$ are as follows:

$$K = \frac{|\mathbf{m}| H d}{2}$$

and

$$\begin{aligned} D_{\mathbf{m}}^n = \biggl\{ u \in \mathbb{R}^{|\mathbf{m}|} : a_i < u_2^i < u_4^i < \cdots < u_{m_i}^i < b_i; \\ 0 < u_{2k-1}^i < n(u_{2k}^i - u_{2k-2}^i), 1 \le k \le \frac{m_i}{2} \biggr\}. \end{aligned}$$

We compute the expectation (3.22) in the following way. Define the **m**-dimensional Gaussian random vector $X(u)$ by

$$X_{2k}^i(u) = B(u_{2k}^i) \quad \text{and} \quad X_{2k-1}^i(u) = n^H\biggl(B\biggl(u_{2k}^i - \frac{u_{2k-1}^i}{n}\biggr) - B(u_{2k}^i)\biggr),$$

where $1 \le k \le \frac{m_i}{2}$. The covariance matrix and the probability density function of the Gaussian random vector $X(u)$ are denoted by $Q_n(u)$ and

$$p_n(x) = (2\pi)^{-|\mathbf{m}|d/2} (\det Q_n(u))^{-1/2} \exp\bigl(-\tfrac{1}{2} x Q_n(u)^{-1} x^T\bigr),$$

respectively. With the above notation we can write

$$\mathbb{E}(G_n) = \mathbf{m}! n^K \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}} \prod_{i=1}^N \prod_{k=1}^{m_i/2} f\bigl(n^H x_{2k}^i\bigr) f\bigl(n^H x_{2k}^i + x_{2k-1}^i\bigr) p_n(x)\, du\, dx.$$

Making the change of variables $y_j^i = n^H x_j^i$ if j is even, and $y_j^i = x_j^i$ if j is odd, we then obtain

$$\mathbb{E}(G_n) = \mathbf{m}! \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}^n} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} f(y_{2k}^i) f(y_{2k}^i + y_{2k-1}^i) p_n(y(n))\, du\, dy, \tag{3.23}$$

where $y_j^i(n) = n^{-H} y_j^i$ if j is even and $y_j^i(n) = y_j^i$ if j is odd.

PROPOSITION 3.4. *Suppose that all exponents m_i are even. Then*

$$\lim_{n\to\infty} \mathbb{E}(G_n) = C_{H,d}^{|\mathbf{m}|/2} \|f\|_{1/H-d}^{|\mathbf{m}|} \mathbb{E}\left(\prod_{i=1}^{N} (W(L_{b_i}(0)) - W(L_{a_i}(0)))^{m_i}\right). \tag{3.24}$$

PROOF. Notice that we can find a sequence of functions f_N, which are infinitely differentiable with compact support, such that $\int_{\mathbb{R}^d} f_N(x)\, dx = 0$ and

$$\lim_{N\to\infty} \int_{\mathbb{R}^d} |f(x) - f_N(x)| \big(|x|^{1/H-d} \vee 1\big)\, dx = 0.$$

So, by Proposition 3.1, we can assume that f is infinitely differentiable with compact support and $\int_{\mathbb{R}^d} f(x)\, dx = 0$.

The equation (3.23) can be written as

$$\mathbb{E}(G_n) = \mathbf{m}! \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}^n} F(y) p_n(y(n))\, du\, dy, \tag{3.25}$$

where

$$F(y) = \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} f(y_{2k}^i) f(y_{2k}^i + y_{2k-1}^i).$$

The proof will be done in several steps.

Step 1. Let us compute the limit of the density $p_n(y(n))$ as n tends to infinity. We split the random vector $X(u)$ into two random vectors $X(u) = (Y(u), Z_n(u))$, where $Y(u)$ contains the components of $X(u)$ with even subindices, and $Z_n(u)$ contains the components with odd subindices. That is, $Y(u)$ is an $\frac{|\mathbf{m}|d}{2}$-dimensional random vector, such that $Y_k^i(u) = B(u_{2k}^i)$ for $1 \le i \le N$ and $1 \le k \le \frac{m_i}{2}$. We denote by $A(u)$ the covariance matrix of $Y(u)$, which does not depend on n. On the other hand, the covariance matrix between the components of $Z_n(u)$ and $Y(u)$ converges to the zero matrix, and the covariance matrix of the random vector $Z_n(u)$ converges to a diagonal matrix with entries equal to $(u_{2k-1}^i)^{2H}$, $1 \le i \le N$, and $1 \le k \le \frac{m_i}{2}$. Therefore,

$$\lim_{n\to\infty} p_n(y(n)) = (2\pi)^{-|\mathbf{m}|d/2} (\det A(u))^{-1/2} \times \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} (u_{2k-1}^i)^{-Hd} \exp\left(-\frac{|y_{2k-1}^i|^2}{2(u_{2k-1}^i)^{2H}}\right).$$

On the other hand, the region $D^n_{\mathbf{m}}$ converges, as n tends to infinity, to

$$\left\{u \in \mathbb{R}^{|\mathbf{m}|} : a_i < u_2^i < \cdots < u_{m_i}^i < b_i; 0 < u_{2k-1}^i < \infty; 1 \le k \le \frac{m_i}{2}, 1 \le i \le N\right\}.$$

Notice that we can add a term -1 because $\int_{\mathbb{R}^d} F(y)\, dy_{2k-1}^i = 0$ for any i, k, and

$$\begin{aligned}\int_0^\infty & u^{-Hd}\big[e^{-|y_{2k-1}^i|^2/(2u^{2H})} - 1\big]\, du \\ &= -\big|y_{2k-1}^i\big|^{1/H-d} \int_0^\infty u^{-Hd}\big[1 - e^{-1/(2u^{2H})}\big]\, du.\end{aligned}$$

Therefore, provided that we can interchange the limit with the integrals in the expression (3.25), we obtain

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}(G_n) = {} & \mathbf{m}! 2^{-|\mathbf{m}|/2} (2\pi)^{-|\mathbf{m}|d/4} C_{H,d}^{|\mathbf{m}|/2} \|f\|_{1/H-d}^{|\mathbf{m}|} \\ & \times \int_{O_{\mathbf{m}/2}} \big(\det A(w)\big)^{-1/2}\, dw,\end{aligned} \tag{3.26}$$

where

$$O_{\mathbf{m}/2} = \{w \in \mathbb{R}^{|\mathbf{m}|/2} : a_i < w_1^i < \cdots < w_{m_i/2}^i < b_i, 1 \le i \le N\},$$

and $A(w) = A(u)$ with the change of variable $w_k^i = u_{2k}^i$. Finally, the right-hand side of (3.26) can also be written as

$$\begin{aligned}& \left(\prod_{i=1}^N \frac{m_i!}{2^{m_i/2}(m_i/2)!} C_{H,d}^{m_i/2} \|f\|_{1/H-d}^{m_i}\right) \\ & \quad \times \int_{\prod_{i=1}^N [a_i,b_i]^{m_i/2}} (2\pi)^{-|\mathbf{m}|d/4} \big(\det A(w)\big)^{-1/2}\, dw,\end{aligned} \tag{3.27}$$

and, taking into account Lemma 2.1, this would finish the proof.

Step 2. In order to justify the passage of the limit inside the integrals, we decompose the region $D^n_{\mathbf{m}}$ into two components as follows. For $K > 0$, we define

$$D^n_{\mathbf{m},K,1} = \left\{u \in D^n_{\mathbf{m}} : 0 < u_{2k-1}^i < K \wedge n\big(u_{2k}^i - u_{2k-2}^i\big); 1 \le k \le \frac{m_i}{2}\right\}$$

and $D^n_{\mathbf{m},K,2} = D^n_{\mathbf{m}} - D^n_{\mathbf{m},K,1}$. Then, $\mathbb{E}(G_n) = I^1_{n,K} + I^2_{n,K}$, where

$$I^1_{n,K} = \mathbf{m}! \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D^n_{\mathbf{m},K,1}} F(y) p_n\big(y(n)\big)\, du\, dy,$$

$$I^2_{n,K} = \mathbf{m}! \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D^n_{\mathbf{m},K,2}} F(y) p_n\big(y(n)\big)\, du\, dy.$$

The region $D^n_{\mathbf{m},K,1}$ is uniformly bounded in n, and we can then interchange the limit and the integral with respect to u, provided that we have a uniform integrability condition. To do this we need the following estimate of the density $p_n(y(n))$.

For any $\xi \in \mathbb{R}^{|\mathbf{m}|}$ with components (ξ^i_j), $1 \le i \le N$, $1 \le j \le m_i$, we can write

$$\begin{aligned}
\langle \xi, X\rangle &= \sum_{i=1}^{N}\Biggl(\sum_{j=1}^{m_i/2} \xi^i_{2j}\cdot B(u^i_{2j})\\
&\qquad + \sum_{k=1}^{m_i/2} \xi^i_{2k-1}\cdot n^H\biggl(B\biggl(u^i_{2k}-\frac{u^i_{2k-1}}{n}\biggr)-B(u^i_{2k-2})\biggr)\Biggr)\\
&= \sum_{i=1}^{N}\sum_{k=1}^{m_i/2}\biggl(\sum_{(\ell,2j)\ge(i,2k)} \xi^\ell_{2j}\biggr)\cdot\biggl(B\biggl(u^i_{2k}-\frac{u^i_{2k-1}}{n}\biggr)-B(u^i_{2k-2})\biggr)\\
&\quad + \sum_{i=1}^{N}\sum_{k=1}^{m_i/2}\biggl(\sum_{(\ell,2j)\ge(i,2k)} \xi^\ell_{2j} - n^H \xi^i_{2k-1}\biggr)\cdot\biggl(B(u^i_{2k})-B\biggl(u^i_{2k}-\frac{u^i_{2k-1}}{n}\biggr)\biggr).
\end{aligned}$$

Here we have used the ordering $(\ell, 2j) \ge (i, 2k)$ if $\ell > i$ or $\ell = i$ and $j \ge k$.

By the local nondeterminism property (2.3),

$$\begin{aligned}
\operatorname{Var}\langle \xi, X\rangle &\ge k_H\Biggl[\sum_{i=1}^{N}\sum_{k=1}^{m_i/2}\biggl|\sum_{(\ell,2j)\ge(i,2k)} \xi^\ell_{2j}\biggr|^2\cdot\biggl(u^i_{2k}-\frac{u^i_{2k-1}}{n}-u^i_{2k-2}\biggr)^{2H}\\
&\qquad + \sum_{i=1}^{N}\sum_{k=1}^{m_i/2}\biggl|\sum_{(\ell,2j)\ge(i,2k)} \xi^\ell_{2j} - n^H\xi^i_{2k-1}\biggr|^2\biggl(\frac{u^i_{2k-1}}{n}\biggr)^{2H}\Biggr]\\
&= k_H\Biggl[\sum_{i=1}^{N}\sum_{k=1}^{m_i/2}|\eta^i_{2k}|^2\biggl(u^i_{2k}-\frac{u^i_{2k-1}}{n}-u^i_{2k-2}\biggr)^{2H} \qquad (3.28)\\
&\qquad + \sum_{i=1}^{N}\sum_{k=1}^{m_i/2}|\eta^i_{2k}-n^H\eta^i_{2k-1}|^2\biggl(\frac{u^i_{2k-1}}{n}\biggr)^{2H}\Biggr]\\
&=: k_H R(\eta),
\end{aligned}$$

where we have made the change of variables

$$\eta^i_{2k} = \sum_{(\ell,2j)\ge(i,2k)} \xi^\ell_{2j} \quad\text{and}\quad \eta^i_{2k-1} = \xi^i_{2k-1}.$$

This implies that

$$\begin{aligned}
(\det Q_n)^{-1/2} &= (2\pi)^{-|\mathbf{m}|d/2}\int_{\mathbb{R}^{|\mathbf{m}|d}} \exp\biggl(-\frac{1}{2}\operatorname{Var}\langle\xi, X\rangle\biggr)\,d\xi\\
&\le (2\pi)^{-|\mathbf{m}|d/2}\int_{\mathbb{R}^{|\mathbf{m}|d}} \exp\biggl(-\frac{k_H}{2}R(\eta)\biggr)\,d\eta\\
&= c_1\prod_{i=1}^{N}\prod_{k=1}^{m_i/2}(u^i_{2k-1})^{-Hd}\biggl(u^i_{2k}-\frac{u^i_{2k-1}}{n}-u^i_{2k-2}\biggr)^{-Hd}.
\end{aligned}$$

Therefore,

$$p_n(y(n)) \le c_2 \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} (u_{2k-1}^i)^{-Hd} \left(u_{2k}^i - \frac{u_{2k-1}^i}{n} - u_{2k-2}^i \right)^{-Hd}. \tag{3.29}$$

As a consequence of (3.29) and the inequality (A.5) in Lemma A.5,

$$\begin{aligned}
&\int_{D^n_{\mathbf{m},K,1}} p_n(y(n))\,du \\
&\quad \le c_3 \int_{D^n_{\mathbf{m},K,1}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} (u_{2k-1}^i)^{-Hd} \left(u_{2k}^i - \frac{u_{2k-1}^i}{n} - u_{2k-2}^i \right)^{-Hd} du \\
&\quad \le c_4,
\end{aligned}$$

where c_4 is a constant independent of n and y. Thus, taking into account that the function $F(y)$ is integrable, by the dominated convergence theorem we obtain

$$\lim_{n\to\infty} I^1_{n,K} = \mathbf{m}! \int_{\mathbb{R}^{|\mathbf{m}|d}} F(y) \left(\lim_{n\to\infty} \int_{D^n_{\mathbf{m},K,1}} p_n(y(n))\,du \right) dy.$$

On the other hand, again by (3.29) and Lemma A.5, there exists $p > 1$ such that

$$\sup_n \int_{D^n_{\mathbf{m},K,1}} |p_n(y(n))|^p\,du < \infty, \tag{3.30}$$

which implies

$$\lim_{n\to\infty} I^1_{n,K} = \mathbf{m}! \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{\mathbb{R}^{|\mathbf{m}|}} F(y) \lim_{n\to\infty} \mathbf{1}_{D^n_{\mathbf{m},K,1}}(u) p_n(y(n))\,du\,dy.$$

With the same notation as above we get

$$\begin{aligned}
\lim_{n\to\infty} I^1_{n,K} &= \mathbf{m}!(2\pi)^{-|\mathbf{m}|d/2} \left(\int_{O_{\mathbf{m}/2}} (\det A(w))^{-1/2}\,dw \right) \\
&\quad \times \int_{\mathbb{R}^{|\mathbf{m}|d}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} \Big(f(y_{2k}^i) f(y_{2k}^i + y_{2k-1}^i) \\
&\qquad \times \int_0^K u^{-Hd} \left(e^{-|y_{2k-1}^i|^2/(2u^{2H})} - 1 \right) du \Big)\,dy.
\end{aligned} \tag{3.31}$$

The right-hand side of the above equality converges to the term in (3.27) as K tends to infinity.

Step 3. Now it suffices to show that

$$\lim_{K\to\infty} \limsup_{n\to\infty} I^2_{n,K} = 0. \tag{3.32}$$

First we observe that

$$D^n_{\mathbf{m},K,2} = \bigcup_{i=1}^{N} \bigcup_{k=1}^{m_i/2} D^n_{\mathbf{m},K,i,k},$$

where

$$D^n_{\mathbf{m},K,i,k} = \{u \in D^n_{\mathbf{m}} : u^i_{2k-1} \geq K \wedge n(u^i_{2k} - u^i_{2k-2})\}.$$

So we only need to show that

$$\lim_{K\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{\bigcup_{i=1}^N \bigcup_{k=1}^{m_i/2} D^n_{\mathbf{m},K,i,k}} F(y) p_n(y(n))\, du\, dy = 0. \tag{3.33}$$

As a consequence of Proposition 3.3, we can replace $D^n_{\mathbf{m},K,i,k}$ in (3.33) with

$$D^{n,1}_{\mathbf{m},K,i,k} = \left\{u \in D^n_{\mathbf{m}};\ K \leq u^i_{2k-1} \leq \frac{n(u^i_{2k} - u^i_{2k-2})}{2}\right\}$$

and just show that

$$\lim_{K\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{\bigcup_{i=1}^N \bigcup_{k=1}^{m_i/2} D^{n,1}_{\mathbf{m},K,i,k}} F(y) p_n(y(n))\, du\, dy = 0. \tag{3.34}$$

To do this we need more refined estimates of the density $p_n(y(n))$. By Fourier analysis

$$p_n(y(n)) = (2\pi)^{-|\mathbf{m}|d} \int_{\mathbb{R}^{|\mathbf{m}|d}} \exp\Bigg(-\frac{1}{2} \operatorname{Var}\langle \xi, X\rangle - \iota \sum_{i=1}^N \sum_{j=1}^{m_i/2} \left(\frac{\xi^i_{2j} \cdot y^i_{2j}}{n^H} + \xi^i_{2j-1} \cdot y^i_{2j-1}\right)\Bigg) d\xi.$$

We choose a set J of indexes of the form $(i, 2j-1)$, where $1 \leq i \leq N$ and $1 \leq j \leq \frac{m_i}{2}$. For each index in J we introduce the operator

$$\Delta_{i,2j-1} F(y^i_{2j-1}) = F(y^i_{2j-1}) - F(0)$$

and set $\Delta_J = \prod_{(i,2j-1)\in J} \Delta_{i,2j-1}$. Taking into account that the integral on the variable y^i_{2j-1} is zero, we can replace $p_n(y(n))$ in (3.34) by $\Delta_J p_n(y(n))$. Using (3.28), we obtain the following estimate:

$$\begin{aligned} &|\Delta_J p_n(y(n))| \\ &\quad \leq c_5 \int_{\mathbb{R}^{|\mathbf{m}|d}} \exp\Bigg(-\frac{k_H}{2} \sum_{i=1}^N \sum_{j=1}^{m_i/2} \Bigg(|\eta^i_{2j}|^2 \left(u^i_{2j} - \frac{u^i_{2j-1}}{n} - u^i_{2j-2}\right)^{2H} \\ &\qquad + |\eta^i_{2j} - n^H \eta^i_{2j-1}|^2 \left(\frac{u^i_{2j-1}}{n}\right)^{2H}\Bigg)\Bigg) \\ &\qquad \times \prod_{i=1}^N \prod_{j=1}^{m_i/2} |e^{-\iota \eta^i_{2j-1}\cdot y^i_{2j-1}} - 1|\, d\eta \end{aligned} \tag{3.35}$$

$$
\begin{aligned}
&= c_5 n^{-|\mathbf{m}|Hd/2} \\
&\quad \times \int_{\mathbb{R}^{|\mathbf{m}|d}} \exp\biggl(-\frac{k_H}{2}\sum_{i=1}^{N}\sum_{j=1}^{m_i/2}\biggl(|\eta_{2j}^i|^2\biggl(u_{2j}^i - \frac{u_{2j-1}^i}{n} - u_{2j-2}^i\biggr)^{2H} \\
&\qquad\qquad + |\eta_{2j-1}^i|^2\biggl(\frac{u_{2j-1}^i}{n}\biggr)^{2H}\biggr)\biggr) \\
&\qquad \times \prod_{i=1}^{N}\prod_{j=1}^{m_i/2}\bigl|e^{-\iota(\eta_{2j}^i-\eta_{2j-1}^i)\cdot y_{2j-1}^i/n^H} - 1\bigr|\, d\eta.
\end{aligned}
$$

This shows that

$$
\biggl|\int_{\mathbb{R}^{|\mathbf{m}|d}}\int_{\bigcup_{i=1}^{N}\bigcup_{k=1}^{m_i/2} D_{\mathbf{m},K,i,k}^{n,1}} F(y)\, p_n\bigl(y(n)\bigr)\, du\, dy\biggr| \le c_6 \sum_{i=1}^{N}\sum_{k=1}^{m_i/2} I_{\mathbf{m},K,i,k}, \tag{3.36}
$$

where

$$
\begin{aligned}
&I_{\mathbf{m},K,i,k} \\
&\quad = n^{-|\mathbf{m}|Hd/2} \\
&\qquad \times \int_{D_{\mathbf{m},K,i,k}^{n,1}}\int_{\mathbb{R}^{2|\mathbf{m}|d}} \bigl|F(y)\bigr| \\
&\qquad\qquad \times \exp\biggl(-\frac{k_H}{2}\sum_{i=1}^{N}\sum_{j=1}^{m_i/2}\biggl(|\eta_{2j}^i|^2\biggl(u_{2j}^i - \frac{u_{2j-1}^i}{n} - u_{2j-2}^i\biggr)^{2H} \\
&\qquad\qquad\qquad + |\eta_{2j-1}^i|^2\biggl(\frac{u_{2j-1}^i}{n}\biggr)^{2H}\biggr)\biggr) \\
&\qquad\qquad \times \prod_{i=1}^{N}\prod_{j=1}^{m_i/2}\bigl|e^{-\iota(\eta_{2j}^i-\eta_{2j-1}^i)\cdot y_{2j-1}^i/n^H} - 1\bigr|\, d\eta\, dy\, du.
\end{aligned}
$$

To estimate $I_{\mathbf{m},K,i,k}$ on the right-hand side of (3.36), we first consider the integral in the variables $u = u_{2k}^i$, $v = u_{2k-1}^i$, $w = \eta_{2k}^i$ and $z = \eta_{2k-1}^i$. Set $u_{2k-2}^i = u_0$ and $y_{2k-1}^i = y$. That is, we have the integral

$$
\begin{aligned}
&I_1(u_0, b_i, y) \\
&\quad = n^{-Hd}\int_{u_0}^{b_i}\int_{K}^{n(u-u_0)/2}\int_{\mathbb{R}^{2d}} \exp\biggl(-\frac{\kappa_H}{2}\biggl(|z|^2\biggl(\frac{v}{n}\biggr)^{2H} \\
&\qquad\qquad + |w|^2\biggl(u - u_0 - \frac{v}{n}\biggr)^{2H}\biggr)\biggr) \\
&\qquad\qquad \times \bigl(\bigl|e^{-\iota z\cdot y/n^H} - 1\bigr| + \bigl|e^{-\iota w\cdot y/n^H} - 1\bigr|\bigr)\, dw\, dz\, dv\, du.
\end{aligned}
$$

We see that

$$I_1(u_0, b_i, y) = I_2(u_0, b_i, y) + I_3(u_0, b_i, y),$$

where

$$\begin{aligned} I_2(u_0, b_i, y) &\\ = n^{-Hd} \int_{u_0}^{b_i} \int_K^{n(u-u_0)/2} \int_{\mathbb{R}^{2d}} &\exp\left(-\frac{\kappa_H}{2}\left(|z|^2\left(\frac{v}{n}\right)^{2H} \right.\right. \\ &\left.\left. + |w|^2\left(u - u_0 - \frac{v}{n}\right)^{2H}\right)\right) \\ &\times \left|e^{-\iota z\cdot y/n^H} - 1\right| dw\, dz\, dv\, du \end{aligned}$$

and

$$\begin{aligned} I_3(u_0, b_i, y) &\\ = n^{-Hd} \int_{u_0}^{b_i} \int_K^{n(u-u_0)/2} \int_{\mathbb{R}^{2d}} &\exp\left(-\frac{\kappa_H}{2}\left(|z|^2\left(\frac{v}{n}\right)^{2H} \right.\right. \\ &\left.\left. + |w|^2\left(u - u_0 - \frac{v}{n}\right)^{2H}\right)\right) \\ &\times \left|e^{-\iota w\cdot y/n^H} - 1\right| dw\, dz\, dv\, du. \end{aligned}$$

Integrating with respect to w and using the inequality (A.4) in the Appendix lead us to

$$\begin{aligned} I_2(u_0, b_i, y) &\\ &= cn^{-Hd} \int_{u_0}^{b_i} \int_K^{n(u-u_0)/2} \int_{\mathbb{R}^d} \left(u - u_0 - \frac{v}{n}\right)^{-Hd} \\ &\qquad \times e^{-(\kappa_H |z|^2/2)(v/n)^{2H}} \left|e^{-\iota z\cdot y/n^H} - 1\right| dz\, dv\, du \qquad (3.37) \\ &\le cK^{1-Hd-H\beta}|y|^\beta \int_{u_0}^{b_i} (u-u_0)^{-Hd}\, du \\ &\le cK^{1-Hd-H\beta}(b_i - u_0)^{1-Hd}|y|^\beta. \end{aligned}$$

In a similar way, but with the application of (A.3) instead of (A.4), we obtain

$$\begin{aligned} I_3(u_0, b_i, y) &\\ &= c \int_{u_0}^{b_i} \int_K^{n(u-u_0)/2} v^{-Hd} \int_{\mathbb{R}^d} e^{-(\kappa_H |w|^2/2)(u-u_0-v/n)^{2H}} \\ &\qquad \times \left|e^{-\iota w\cdot y/n^H} - 1\right| dw\, dv\, du \qquad (3.38) \end{aligned}$$

$$\begin{aligned}
&\leq cn^{1-Hd-H\beta}|y|^{\beta}\int_{u_0}^{b_i}(u-u_0)^{1-2Hd-H\beta}\,du\\
&\leq cn^{1-Hd-H\beta}(b_i-u_0)^{2-2Hd-H\beta}|y|^{\beta}.
\end{aligned}$$

Combining (3.37) and (3.38) gives

$$\begin{aligned}
&I_1(u_0,b_i,y)\\
&\quad\leq cK^{1-Hd-H\beta}(b_i-a_i)^{1-Hd}|y|^{\beta}\\
&\qquad+cn^{1-Hd-H\beta}(b_i-a_i)^{2-2Hd-H\beta}|y|^{\beta}.
\end{aligned}\tag{3.39}$$

Once this is done, we proceed to consider the integrals in the variables $u=u_{2l}^{j}$, $v=u_{2l-1}^{j}$, $z=\eta_{2l}^{j}$ and $w=\eta_{2l-1}^{j}$ with indices $(j,l)\neq(i,k)$. Set also $u_{2l-2}^{j}=u_0$ and $y_{2l-1}^{j}=y$. That is, we have the integral

$$\begin{aligned}
&I_4(u_0,b_j,y)\\
&\quad=n^{-Hd}\int_{u_0}^{b_j}\int_0^{n(u-u_0)}\int_{\mathbb{R}^{2d}}\exp\biggl(-\frac{\kappa_H}{2}\biggl(|z|^2\Bigl(\frac{v}{n}\Bigr)^{2H}\\
&\qquad\qquad+|w|^2\Bigl(u-u_0-\frac{v}{n}\Bigr)^{2H}\biggr)\biggr)\\
&\qquad\qquad\times\bigl(\bigl|e^{-\iota z\cdot y/n^H}-1\bigr|\\
&\qquad\qquad\quad+\bigl|e^{-\iota w\cdot y/n^H}-1\bigr|\bigr)\,dw\,dz\,dv\,du.
\end{aligned}$$

We can decompose this integral into two components,

$$I_4(u_0,b_j,y)=I_5(u_0,b_j,y)+I_6(u_0,b_j,y),$$

where

$$\begin{aligned}
&I_5(u_0,b_j,y)\\
&\quad=n^{-Hd}\int_{u_0}^{b_j}\int_0^{n(u-u_0)}\int_{\mathbb{R}^{2d}}\exp\biggl(-\frac{\kappa_H}{2}\biggl(|z|^2\Bigl(\frac{v}{n}\Bigr)^{2H}\\
&\qquad\qquad+|w|^2\Bigl(u-u_0-\frac{v}{n}\Bigr)^{2H}\biggr)\biggr)\\
&\qquad\qquad\times\bigl|e^{-\iota z\cdot y/n^H}-1\bigr|\,dw\,dz\,dv\,du\\
&\quad=cn^{-Hd}\int_{u_0}^{b_j}\int_0^{n(u-u_0)}\int_{\mathbb{R}^{d}}\Bigl(u-u_0-\frac{v}{n}\Bigr)^{-Hd}e^{-(\kappa_H|z|^2/2)(v/n)^{2H}}\\
&\qquad\qquad\times\bigl|e^{-\iota z\cdot y/n^H}-1\bigr|\,dz\,dv\,du
\end{aligned}$$

and

$$
\begin{aligned}
&I_6(u_0, b_j, y) \\
&\quad = n^{-Hd} \int_{u_0}^{b_j} \int_0^{n(u-u_0)} \int_{\mathbb{R}^{2d}} \exp\biggl(-\frac{\kappa_H}{2}\biggl(|z|^2\Bigl(\frac{v}{n}\Bigr)^{2H} \\
&\qquad\qquad + |w|^2\Bigl(u - u_0 - \frac{v}{n}\Bigr)^{2H}\biggr)\biggr) \\
&\qquad\qquad \times \bigl|e^{-\iota w\cdot y/n^H} - 1\bigr| \, dw\, dz\, dv\, du \\
&\quad = c \int_{u_0}^{b_j} \int_0^{n(u-u_0)} v^{-Hd} \int_{\mathbb{R}^d} e^{-(\kappa_H |w|^2/2)(u-u_0-v/n)^{2H}} \\
&\qquad\qquad \times \bigl|e^{-\iota w\cdot y/n^H} - 1\bigr| \, dw\, dv\, du.
\end{aligned}
$$

Inequalities (A.2) and (A.1) imply that

$$
\begin{aligned}
I_4(u_0, b_j, y) &\le c \int_{u_0}^{b_j} (u - u_0)^{-Hd} |y|^{1/H-d} \, du \\
&= c(b_j - u_0)^{1-Hd} |y|^{1/H-d} \\
&\le c(b_j - a_j)^{1-Hd} |y|^{1/H-d}.
\end{aligned}
$$

The remaining integrals can be dealt with in the same way as $I_4(u_0, b_j, y)$. Thus statement (3.34) follows. The proof is complete. $\square$

PROOF OF THEOREM 1.1. This follows from Lemma 2.1, Propositions 3.1, 3.2 and 3.4 by the method of moments. $\square$

REMARK. Although the constant $C_{H,d}$ is finite for $H > \frac{1}{d+2}$, the proof of Theorem 1.1 only works for $H > \frac{1}{d+1}$. The reason for this is that for any $y \in \mathbb{R}^d$,

$$
\int_{\mathbb{R}^d} \exp\Bigl(-\frac{\kappa}{2}|\xi|^2 u^{2H}\Bigr)(1 - e^{\iota\xi\cdot y})\, d\xi = \Bigl(\frac{2\pi}{\kappa}\Bigr)^{d/2} u^{-Hd}\Bigl(1 - \exp\Bigl(-\frac{|y|^2}{2\kappa u^{2H}}\Bigr)\Bigr),
$$

which is bounded by $c|y|^2 u^{-H(d+2)}$, while, on the other hand,

$$
\int_{\mathbb{R}^d} \exp\Bigl(-\frac{\kappa}{2}|\xi|^2 u^{2H}\Bigr)\bigl|e^{\iota\xi\cdot y} - 1\bigr|\, d\xi \le c|y|u^{-H(d+1)}.
$$

So, any type of estimation procedure, like the one based on the local nondeterminism property used in this paper, will lead to an upper bound of the form $u^{-H(d+1)}$.

APPENDIX

Here we give some lemmas which are necessary in the proof of Theorem 1.1.

LEMMA A.1. *Let $0 < \beta < 2$. If $f \in H_0^\beta$, then $\|f\|_\beta^2$ given in (1.2) is well defined and*

$$\|f\|_\beta^2 = c_{\beta,d}^{-1} \int_{\mathbb{R}^d} |\mathcal{F}f(\xi)|^2 |\xi|^{-\beta-d}\, d\xi \geq 0,$$

where $\mathcal{F}f(\xi)$ denotes the Fourier transform of f, and

$$c_{\beta,d} = \int_{\mathbb{R}^d} (1-\cos(x\cdot\xi))|\xi|^{-\beta-d}\, d\xi > 0$$

is independent of x if $|x| = 1$.

PROOF. For any $x \in S^{d-1}$ and any $d \times d$ orthogonal matrix Q, the change of variable $\xi = Q\eta$ yields

$$\int_{\mathbb{R}^d} (1-\cos(x\cdot\xi))|\xi|^{-\beta-d}\, d\xi = \int_{\mathbb{R}^d} (1-\cos((Q^T x)\cdot\eta))|\eta|^{-\beta-d}\, d\eta > 0.$$

This shows that $\int_{\mathbb{R}^d} (1-\cos(x\cdot\xi))|\xi|^{-\beta-d}\, d\xi$ depends only on $|x|$. The substitution $\xi = \frac{1}{|x|}\eta$ gives us $\int_{\mathbb{R}^d} (1-\cos(x\cdot\xi))|\xi|^{-\beta-d}\, d\xi = c_{\beta,d}|x|^\beta$. Then an elementary result from Fourier analysis [15] yields

$$\begin{aligned}\|f\|_\beta^2 &= c_{\beta,d}^{-1} \int_{\mathbb{R}^{2d}} f(x)f(y)\left(\int_{\mathbb{R}^d} (e^{\iota(x-y)\cdot\xi} - 1)|\xi|^{-\beta-d}\, d\xi\right) dx\, dy \\ &= c_{\beta,d}^{-1} \int_{\mathbb{R}^d} |\mathcal{F}f(\xi)|^2 |\xi|^{-\beta-d}\, d\xi \geq 0. \qquad \square\end{aligned}$$

LEMMA A.2. *Assume that $1 - H < Hd < 1$. Let X be a d-dimensional centered normal random vector with covariance matrix $\sigma^2 I$. Then, for any $n \in \mathbb{N}$ and $y \in \mathbb{R}^d$, there exists a constant c depending only on H and d such that*

$$\int_0^\infty u^{-Hd}\mathbb{E}\left|\exp\left(\iota\frac{y\cdot X}{n^H u^H}\right) - 1\right| du \leq cn^{Hd-1}|y|^{1/H-d}.$$

PROOF. It suffices to show the above inequality when $y \neq 0$. Making the change of variable $v = |y|^{-1/H} nu$ gives

$$\begin{aligned}&\int_0^\infty u^{-Hd}\mathbb{E}\left|\exp\left(\iota\frac{y\cdot X}{n^H u^H}\right) - 1\right| du \\ &\quad = n^{Hd-1}|y|^{1/H-d}\int_0^\infty v^{-Hd}\mathbb{E}\left|\exp\left(\iota\frac{y\cdot X}{|y|v^H}\right) - 1\right| dv \\ &\quad \leq n^{Hd-1}|y|^{1/H-d}\int_0^\infty v^{-Hd}(2\wedge v^{-H}\mathbb{E}|X|)\, dv \\ &\quad = cn^{Hd-1}|y|^{1/H-d}. \qquad \square\end{aligned}$$

LEMMA A.3. *Assume $1 - H \le Hd < 1$ and $0 \le \beta \le 1$ with $H\beta < 1 - Hd$. Then, for all $n \in \mathbb{N}$, there exists a constant c independent of n such that*

$$\begin{aligned}(A.1)\qquad &\int_0^{ns} u^{-Hd-H\beta}\,du \int_{\mathbb{R}^d} e^{-\kappa|\xi|^2(s-u/n)^{2H}} |1 - e^{\iota\xi\cdot y/n^H}|\,d\xi \\ &\quad \le cn^{-H\beta} s^{-Hd-H\beta} |y|^{1/H-d}\end{aligned}$$

and

$$\begin{aligned}(A.2)\qquad &n^{-Hd-H\beta} \int_0^{ns} \left(s - \frac{u}{n}\right)^{-Hd-H\beta} du \int_{\mathbb{R}^d} e^{-\kappa|\xi|^2(u/n)^{2H}} |1 - e^{\iota\xi\cdot y/n^H}|\,d\xi \\ &\quad \le cn^{-H\beta} s^{-Hd-H\beta} |y|^{1/H-d},\end{aligned}$$

where κ is a positive constant.

PROOF. We first note that (A.1) follows easily from (A.2) by the change of variable. So, it suffices to prove (A.2). Denote the left-hand side of (A.2) by I. We then have

$$\begin{aligned} I &= n^{-Hd-H\beta} \int_0^{ns} \left(s - \frac{u}{n}\right)^{-Hd-H\beta} du \int_{\mathbb{R}^d} e^{-\kappa|\xi|^2(u/n)^{2H}} |1 - e^{\iota\xi\cdot y/n^H}|\,d\xi \\ &= n^{-H\beta} \int_0^{ns} \left(s - \frac{u}{n}\right)^{-Hd-H\beta} u^{-Hd} \int_{\mathbb{R}^d} e^{-\kappa|x|^2} |1 - e^{\iota x\cdot y/u^H}|\,dx\,du, \end{aligned}$$

where we made the change of variable $\xi(\frac{u}{n})^H = x$.

By Lemma A.2, we obtain

$$\begin{aligned} &\int_0^{ns/2} \left(s - \frac{u}{n}\right)^{-Hd-H\beta} u^{-Hd} \int_{\mathbb{R}^d} e^{-\kappa|x|^2} |1 - e^{\iota x\cdot y/u^H}|\,dx\,du \\ &\quad \le c_1 s^{-Hd-H\beta} \int_0^{\infty} u^{-Hd} \int_{\mathbb{R}^d} e^{-\kappa|x|^2} |1 - e^{\iota x\cdot y/u^H}|\,dx\,du \\ &\quad = c_2 s^{-Hd-H\beta} |y|^{1/H-d}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\int_{ns/2}^{ns} \left(s - \frac{u}{n}\right)^{-Hd-H\beta} u^{-Hd} \int_{\mathbb{R}^d} e^{-\kappa|x|^2} |1 - e^{\iota x\cdot y/u^H}|\,dx\,du \\ &\quad \le \int_{ns/2}^{ns} \left(s - \frac{u}{n}\right)^{-Hd-H\beta} u^{-Hd-H\beta_1} \int_{\mathbb{R}^d} e^{-\kappa|x|^2} |x \cdot y|^{\beta_1}\,dx\,du \\ &\quad \le c_3 (ns)^{-Hd-H\beta_1} |y|^{\beta_1} \int_{ns/2}^{ns} \left(s - \frac{u}{n}\right)^{-Hd-H\beta} du \\ &\quad \le c_4 n^{1-Hd-H\beta_1} s^{1-H\beta-2Hd-H\beta_1} |y|^{\beta_1}, \end{aligned}$$

where β_1 can be any constant in $[0,1]$, and we used $H\beta < 1 - Hd$ in the last inequality. Our result follows by choosing $\beta_1 = \frac{1}{H} - d$. This is possible because $1 - H \le Hd < 1$. $\square$

LEMMA A.4. *For $n \in \mathbb{N}$, we assume that $1 - H < Hd < 1$ and $0 < K < \frac{ns}{2}$. Then there exists a constant c independent of n and K such that*

$$\begin{aligned}\text{(A.3)}\qquad &\int_K^{ns/2} u^{-Hd}\,du \int_{\mathbb{R}^d} e^{-\kappa|\xi|^2(s-u/n)^{2H}} \bigl|1 - e^{\iota\xi\cdot y/n^H}\bigr|\,d\xi \\ &\quad \le cn^{1-Hd-H\beta} s^{1-2Hd-H\beta}|y|^\beta\end{aligned}$$

and

$$\begin{aligned}\text{(A.4)}\qquad &n^{-Hd}\int_K^{ns/2} \left(s - \frac{u}{n}\right)^{-Hd} du \int_{\mathbb{R}^d} e^{-\kappa|\xi|^2(u/n)^{2H}} \bigl|1 - e^{\iota\xi\cdot y/n^H}\bigr|\,d\xi \\ &\quad \le cs^{-Hd}K^{1-Hd-H\beta}|y|^\beta,\end{aligned}$$

where κ and β are positive constants with $1 - Hd < H\beta < H \wedge (2 - 2Hd)$.

PROOF. Inequality (A.3) follows easily from the proof of Lemma A.3. We shall show (A.4). Denote the left-hand side of (A.4) by I. Then we have

$$\begin{aligned} I &= n^{1-Hd}\int_{K/n}^{s/2} (s-u)^{-Hd}\,du \int_{\mathbb{R}^d} e^{-\kappa|\xi|^2u^{2H}} \bigl|1 - e^{\iota\xi\cdot y/n^H}\bigr|\,d\xi \\ &\le c|y|^\beta n^{1-Hd-H\beta}\int_{K/n}^{s/2}(s-u)^{-Hd}u^{-Hd-H\beta}\,du.\end{aligned}$$

Since $\frac{K}{n} < \frac{s}{2}$ and $1 - Hd < H\beta$, we have

$$\begin{aligned}\int_{K/n}^{s/2}(s-u)^{-Hd}u^{-Hd-H\beta}\,du &\le cs^{-Hd}\int_{K/n}^{s/2} u^{-Hd-H\beta}\,du \\ &\le cs^{-Hd}\left(\frac{K}{n}\right)^{1-Hd-H\beta}.\end{aligned}$$

Therefore,

$$I \le cs^{-Hd}K^{1-Hd-H\beta}|y|^\beta.$$

This proves the lemma. $\square$

LEMMA A.5. *For any $K > 0$ and $n \in \mathbb{N}$, there exist constants c_1 and c_2 independent of n such that*

$$\text{(A.5)}\qquad \int_0^{K\wedge nu} v^{-Hd}\left(u - \frac{v}{n}\right)^{-Hd} dv \le c_1 K^{1-Hd}u^{-Hd}$$

and

$$\int_0^u v^{-Hd}(u-v)^{-Hd}\,dv = c_2 u^{1-2Hd}. \tag{A.6}$$

PROOF. Inequality (A.6) follows easily from

$$\int_0^u v^{-Hd}(u-v)^{-Hd}\,dv = u^{1-2Hd}\int_0^1 w^{-Hd}(1-w)^{-Hd}\,dw.$$

We only need to show (A.5). Notice that

$$\int_0^{K\wedge nu} v^{-Hd}\left(u-\frac{v}{n}\right)^{-Hd} dv = u^{-Hd}(nu)^{1-Hd}\int_0^{K/(nu)\wedge 1} v^{-Hd}(1-v)^{-Hd}\,dv.$$

If $nu \le 2K$, then

$$\int_0^{K/(nu)\wedge 1} v^{-Hd}(1-v)^{-Hd}\,dv \le \int_0^1 v^{-Hd}(1-v)^{-Hd}\,dv.$$

If $nu > 2K$, then

$$\int_0^{K/(nu)\wedge 1} v^{-Hd}(1-v)^{-Hd}\,dv \le 2\int_0^{K/(nu)} v^{-Hd}\,dv \le \frac{2}{1-Hd}\left(\frac{K}{nu}\right)^{1-Hd}.$$

Therefore,

$$\int_0^{K\wedge nu} v^{-Hd}\left(u-\frac{v}{n}\right)^{-Hd} dv \le c_3 K^{1-Hd}u^{-Hd}. \qquad \square$$

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