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SPIN GLASS MODELS FROM THE POINT OF VIEW OF SPIN DISTRIBUTIONS

BY DMITRY PANCHENKO[1]

Texas A&M University

In many spin glass models, due to the symmetry among sites, any limiting joint distribution of spins under the annealed Gibbs measure admits the Aldous–Hoover representation encoded by a function $\sigma : [0, 1]^4 \to \{-1, +1\}$, and one can think of this function as a generic functional order parameter of the model. In a class of diluted models, and in the Sherrington–Kirkpatrick model, we introduce novel perturbations of the Hamiltonian that yield certain invariance and self-consistency equations for this generic functional order parameter and we use these invariance properties to obtain representations for the free energy in terms of σ. In the setting of the Sherrington–Kirkpatrick model, the self-consistency equations imply that the joint distribution of spins is determined by the joint distributions of the overlaps, and we give an explicit formula for σ under the Parisi ultrametricity hypothesis. In addition, we discuss some connections with the Ghirlanda–Guerra identities and stochastic stability and describe the expected Parisi ansatz in the diluted models in terms of σ.

1. Introduction and main results. In various mean-field spin glass models, such as the Sherrington–Kirkpatrick model and diluted p-spin and p-sat models that we will focus on in this paper, one considers a random Hamiltonian $H_N(\boldsymbol{\sigma})$ indexed by spin configurations $\boldsymbol{\sigma} \in \Sigma_N = \{-1, +1\}^N$ and defines the corresponding Gibbs measure G_N as a random probability measure on Σ_N given by

$$G_N(\boldsymbol{\sigma}) = \frac{1}{Z_N} \exp(-H_N(\boldsymbol{\sigma})), \tag{1.1}$$

where the normalizing factor Z_N is called the partition function. Let $(\boldsymbol{\sigma}^l)_{l\geq 1}$ be an i.i.d. sequence of replicas from measure G_N. Let μ_N denote the joint distribution of the array of all spins on all replicas, $(\sigma_i^l)_{1\leq i\leq N, 1\leq l}$, under the annealed product Gibbs measure $\mathbb{E}G_N^{\otimes\infty}$ which means that for any choice of signs $a_i^l \in \{-1, +1\}$, and for any $n \geq 1$,

$$\begin{aligned}\mu_N(\{\sigma_i^l = a_i^l : 1 \leq i \leq N, 1 \leq l \leq n\})\\ = \mathbb{E}G_N^{\otimes n}(\{\sigma_i^l = a_i^l : 1 \leq i \leq N, 1 \leq l \leq n\}).\end{aligned} \tag{1.2}$$

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In most mean-field spin glass models this distribution has the following two symmetries. Clearly, it is always invariant under the permutation of finitely many replica indices $l \geq 1$, but in most models μ_N is also invariant under the permutation of coordinates $i \in \{1, \ldots, N\}$ since the distribution of $H_N(\boldsymbol{\sigma})$ is symmetric under the permutation of coordinates of $\boldsymbol{\sigma}$, and this invariance of μ_N is called symmetry among sites. Let us think of μ_N as a distribution on (σ_i^l) for all $i, l \geq 1$ simply by setting $\sigma_i^l = 0$ for $i > N$. It is usually not known how to prove that the sequence (μ_N) converges (in the sense of convergence of finite-dimensional distributions) and, in fact, even the answer to a much less general question whether the distribution of one overlap $N^{-1}\sum_{i\leq N}\sigma_i^1\sigma_i^2$ under $\mathbb{E}G_N^{\otimes 2}$ converges is known only in the Sherrington–Kirkpatrick model with all p-spin interaction terms present, the proof of which relies on the Parisi formula for the free energy; see [27, 28]. As a result, we will consider a family $\mathcal{M}$ of all possible limits over the subsequences of (μ_N). Whenever we have symmetry among sites, any limiting distribution $\mu \in \mathcal{M}$ will be invariant under the permutations of both row and column coordinates l and i. Such two-dimensional arrays are called exchangeable arrays and the representation result of Aldous [2] and Hoover [15] (see also [5]) states that there exists a measurable function $\sigma_\mu : [0,1]^4 \to \mathbb{R}$ such that the distribution μ coincides with the distribution of the array (s_i^l) given by

$$s_i^l = \sigma_\mu(w, u_l, v_i, x_{i,l}), \tag{1.3}$$

where random variables w, (u_l), (v_i), $(x_{i,l})$ are i.i.d. uniform on $[0, 1]$. This function σ_μ is defined uniquely up to some measure-preserving transformations (Theorem 2.1 in [16]) so we can identify the distribution μ of array (s_i^l) with the function σ_μ. Since we only consider the case when spins and thus σ_μ take values in $\{-1, +1\}$, the distribution μ is completely encoded by the function

$$\bar{\sigma}_\mu(w, u, v) = \mathbb{E}_x \sigma_\mu(w, u, v, x), \tag{1.4}$$

where $\mathbb{E}_x$ is the expectation in x only and we can think of this last coordinate as a dummy variable that generates a Bernoulli r.v. with expectation $\bar{\sigma}_\mu(w, u, v)$. However, keeping in mind that a function of three variables $\bar{\sigma}_\mu$ encodes the distribution of the array (1.3), for convenience of notation we will sometimes not identify a Bernoulli distribution with its expectation (especially, in the diluted models) and work with the function $\sigma_\mu(w, u, v, x)$.

One can think of a function σ_μ (or $\bar{\sigma}_\mu$) as what physicists might call a generic "functional order parameter" of the model, and it is easy to see that information encoded by σ_μ is equivalent to the limiting joint distribution of all multi-overlaps

$$R^N_{l_1,\ldots,l_n} = N^{-1} \sum_{1\leq i\leq N} \sigma_i^{l_1} \cdots \sigma_i^{l_n} \tag{1.5}$$

for all $n \geq 1$ and all $l_1, \ldots, l_n \geq 1$ under μ_N, which may be a more familiar object than the joint distribution of spins. Indeed, by expanding the powers of (1.5) in terms of products of spins and using symmetry among sites, in the limit one can express the joint moments of multi-overlaps in terms of the joint moments of spins and vice versa. By comparing these moments, the asymptotic joint distribution of (1.5) over a subsequence of μ_N converging to μ coincides with the joint distribution of

$$R^{\infty}_{l_1,\ldots,l_n} = \mathbb{E}_v \bar{\sigma}(w, u_{l_1}, v) \cdots \bar{\sigma}(w, u_{l_n}, v) \tag{1.6}$$

for $\bar{\sigma} = \bar{\sigma}_\mu$, for all $n \geq 1$ and all $l_1, \ldots, l_n \geq 1$, where $\mathbb{E}_v$ is the expectation in the last coordinate v only. For $n = 2$, the corresponding quantity

$$R^{\infty}_{l,l'} = \mathbb{E}_v \bar{\sigma}(w, u_l, v)\bar{\sigma}(w, u_{l'}, v) \tag{1.7}$$

is the asymptotic version of the overlap $N^{-1}\sum_{i\leq N}\sigma_i^l\sigma_i^{l'}$. With these notations it is clear that the famous Parisi ultrametricity conjecture, which says that $R^{\infty}_{2,3} \geq \min(R^{\infty}_{1,2}, R^{\infty}_{1,3})$ with probability one, can be expressed in terms of $\bar{\sigma}_\mu$ by saying that for all $w \in [0, 1]$ the family of functions $v \to \bar{\sigma}_\mu(w, u, v)$ parametrized by $u \in [0, 1]$ is ultrametric in $\mathcal{L}^2([0, 1], dv)$.

An ultimate goal would be to show that the set of possible limits $\mu \in \mathcal{M}$ and their representations σ_μ are described by the Parisi ultrametric ansatz. Even though this goal is out of reach at the moment, in the setting of the Sherrington–Kirkpatrick and diluted models we will obtain several results which demonstrate that the point of view based on the Aldous–Hoover representation (1.3) provides a useful framework for studying the asymptotic behavior of these models. First, we will narrow down possible limits $\mathcal{M}$ to some well-defined class of distributions $\mathcal{M}_{\mathrm{inv}}$ that will be described via invariance and self-consistency equations on σ_μ. The proof of these invariance properties will be based on some standard cavity computations; however, justification of these computations will rely on certain properties of convergence of measures μ_N that are not intuitive or, at least, do not easily follow from known results. In both types of models we will introduce a novel perturbation of the Hamiltonian that will force the sequence (μ_N) to satisfy these properties, and the ideas behind these perturbations will constitute the main technical contribution of the paper.

Besides giving some constructive description of possible limits $\mathcal{M}$, the invariance equations will play a significant role in other ways. First, using these equations we will be able to prove representations for the limit of the free energy $F_N = N^{-1}\mathbb{E}\log Z_N$ in terms of σ_μ for $\mu \in \mathcal{M}_{\mathrm{inv}}$ which will automatically coincide with the corresponding Parisi formulas for the free energy if one can show that all measures in $\mathcal{M}_{\mathrm{inv}}$ satisfy the predictions of the Parisi ansatz. These representations, proved in Sections 2.2, 2.3 for diluted models and in Sections 3.2, 3.3 for the Sherrington–Kirkpatrick model, will arise from an application of the Aizenman–Sims–Starr scheme introduced in [1] and, what is crucial, thanks to the invariance equations we will only use this scheme with one cavity coordinate whereas all previous applications of this scheme (e.g., in [1, 10] or [17]) only worked when the number of cavity coordinates goes to infinity.

In the setting of the Sherrington–Kirkpatrick model we will utilize a Gaussian nature of the Hamiltonian to give other important applications of the invariance properties of $\mu \in \mathcal{M}$. First, we will prove in Theorem 5 below that the joint distributions of all spins, and thus measure μ, are completely determined by the joint distribution of the overlaps (1.7). Then in Section 1.3 we will show that all limits $\mu \in \mathcal{M}$ that satisfy the Parisi ultrametricity hypothesis correspond to σ_μ given by certain specific realizations of the Ruelle probability cascades. This means that, under ultrametricity, we obtain a more detailed asymptotic description of the model which includes the joint distribution of all spins or multi-overlaps and not only overlaps, as in the usual description of the Parisi ansatz. Motivated by this special form of σ_μ in the Sherrington–Kirkpatrick model, in the second part of Section 1.3 we will try to formulate a more general Parisi ansatz expected to hold in the diluted models in terms of the Aldous–Hoover representation (1.3).

Finally, we would like to mention recent work [4] where the authors study asymptotic behavior of spin glass models in the framework of random overlap structures, or ROSts, which in our notation correspond to the $\mathcal{L}^2([0,1], dv)$ structure of the family of functions $v \to \bar{\sigma}_\mu(w, u, v)$. They obtain a number of interesting properties of ROSts and prove several results which are similar in spirit to ours, for example, the Parisi formula in the Sherrington–Kirkpatrick model under the assumption of ultrametricity.

1.1. *Diluted models.* To illustrate the main new ideas we will start with the case of the diluted models where many technical details will be simpler. We will consider the following class of diluted models as in [22]. Let $p \geq 2$ be an *even* integer, and let $\alpha > 0$. Consider a random function $\theta : \{-1, +1\}^p \to \mathbb{R}$ and a sequence $(\theta_k)_{k\geq 1}$ of independent copies of θ. Consider an i.i.d. sequence of indices $(i_{l,k})_{l,k\geq 1}$ with uniform distribution on $\{1, \ldots, N\}$, and let $\pi(\alpha N)$ be a Poisson r.v. with mean αN. Let us define the Hamiltonian $H_N(\boldsymbol{\sigma})$ on Σ_N by

$$-H_N(\boldsymbol{\sigma}) = \sum_{k \leq \pi(\alpha N)} \theta_k(\sigma_{i_{1,k}}, \ldots, \sigma_{i_{p,k}}). \tag{1.8}$$

Clearly, any such model has symmetry between sites. We will make the following assumptions on the random function θ. We assume that there exists a random function $f : \{-1, +1\} \to \mathbb{R}$ [i.e., $f(\sigma) = f' + f''\sigma$ for some random (f', f'')] such that

$$\exp \theta(\sigma_1, \ldots, \sigma_p) = a\bigl(1 + b f_1(\sigma_1) \cdots f_p(\sigma_p)\bigr), \tag{1.9}$$

where $f_1, \ldots, f_p$ are independent copies of f, b is a r.v. independent of $f_1, \ldots, f_p$ that satisfies the condition

$$\forall n \geq 1 \qquad \mathbb{E}(-b)^n \geq 0, \tag{1.10}$$

and a is an arbitrary r.v. such that $\mathbb{E}|\log a| < \infty$. Finally, we assume that

$$|b f_1(\sigma_1) \cdots f_p(\sigma_p)| < 1 \qquad \text{a.s.}, \tag{1.11}$$

and θ satisfies some mild integrability conditions

$$-\infty < \mathbb{E}\min_{\sigma}\theta(\sigma_1,\dots,\sigma_p), \qquad \mathbb{E}\max_{\sigma}\theta(\sigma_1,\dots,\sigma_p) < +\infty. \tag{1.12}$$

Two well-known models in this class of models are the p-spin and K-sat models.

EXAMPLE 1 (p-spin model). Consider $\beta > 0$ and a symmetric r.v. J. The p-spin model corresponds to the choice of

$$\theta(\sigma_1,\dots,\sigma_p) = \beta J \sigma_1 \cdots \sigma_p.$$

Equation (1.9) holds with $a = \mathrm{ch}(\beta J)$, $b = \mathrm{th}(\beta J)$ and $f(\sigma) = \sigma$ and condition (1.10) holds since we assume that the distribution of J is symmetric. Equation (1.12) holds if $\mathbb{E}|J| < \infty$.

EXAMPLE 2 (K-sat model). Consider $\beta > 0$ and a sequence of i.i.d. Bernoulli r.v. $(J_l)_{l\ge 1}$ with $\mathbb{P}(J_l = \pm 1) = 1/2$. The K-sat model (with $K = p$) corresponds to

$$\theta(\sigma_1,\dots,\sigma_p) = -\beta \prod_{l\le p} \frac{1 + J_l\sigma_l}{2}.$$

Equation (1.9) holds with $a = 1$, $b = e^{-\beta} - 1$ and $f_l(\sigma_l) = (1 + J_l\sigma_l)/2$, and (1.10) holds since $b < 0$.

It is well known that under the above conditions the sequence NF_N is superadditive, and, therefore, the limit of F_N exists; see, for example, [10]. If we knew that (μ_N) has a unique limit, that is, $\mathcal{M} = \{\mu\}$, then computing the limit of the free energy in terms of σ_μ in (1.3) would be rather straightforward as will become clear in Section 2.2. However, since we do not know how to prove that (μ_N) converges, this will create some obstacles. Moreover, if (μ_{N_k}) converges to μ over some subsequence (N_k) we do not know how to show that (μ_{N_k+n}) converges to the same limit for a fixed shift $n \ge 1$, even though we can show that it does converge simply by treating n of the coordinates as cavity coordinates. Even if we knew that μ_N converges, we would still like to have some description of what the limit looks like. To overcome some of these obstacles, we will utilize the idea of adding a "small" perturbation to the Hamiltonian (1.8) that will not affect the limit of the free energy but at the same time ensure that (μ_{N_k+n}) and (μ_{N_k}) converge to the same limit. In some sense, this is similar to the idea of adding p-spin perturbation terms in the Sherrington–Kirkpatrick model to force the overlap distribution to satisfy the Ghirlanda–Guerra identities [13]; see also [11]. The perturbation for diluted models will be defined as follows.

Consider a sequence (c_N) such that $c_N \to \infty$, $c_N/N \to 0$ and $|c_{N+1} - c_N| \to 0$. Consider an i.i.d. sequence of indices $(i_{j,k,l})_{j,k,l\ge 1}$ with uniform distribution on $\{1,\dots,N\}$, let $\pi(c_N)$ be a Poisson r.v. with mean c_N, $(\pi_l(\alpha p))$ be i.i.d. Poisson with mean αp and $(\theta_{k,l})$ be a sequence of i.i.d. copies of θ. All these random variables are assumed to be independent of each other and of everything else. Whenever we introduce a new random variable, by default it is assumed to be independent of all other random variables. Let us define the perturbation Hamiltonian $H_N^p(\boldsymbol{\sigma})$ on Σ_N by

$$-H_N^p(\boldsymbol{\sigma}) = \sum_{l \le \pi(c_N)} \log \mathrm{Av}_\varepsilon \exp \sum_{k \le \pi_l(\alpha p)} \theta_{k,l}(\varepsilon, \sigma_{i_{1,k,l}}, \dots, \sigma_{i_{p-1,k,l}}), \tag{1.13}$$

where Av_ε will denote uniform average over $\varepsilon \in \{-1, +1\}$ as well as replicas (ε_l) below. Let us redefine the Hamiltonian in (1.8) by

$$-H_N(\boldsymbol{\sigma}) = \sum_{k \le \pi(\alpha N)} \theta_k(\sigma_{i_{1,k}}, \dots, \sigma_{i_{p,k}}) - H_N^p(\boldsymbol{\sigma}), \tag{1.14}$$

and from now on we assume that (μ_N) and $\mathcal{M}$ are defined for this perturbed Hamiltonian. Obviously, condition (1.12) implies that the perturbation term does not affect the limit of free energy since $c_N = o(N)$. The benefits of adding this perturbation term will first appear in Lemma 3 below where it will be shown that thanks to this term (μ_{N_k}) and (μ_{N_k+n}) converge to the same limit for any fixed shift $n \ge 1$. Another important consequence will appear in Theorem 1 below where the perturbation will force the limiting distributions $\mu \in \mathcal{M}$ to satisfy some important invariance properties that will play crucial role in the proof of the representation for the free energy in Theorem 2.

Let us introduce some notations. We will usually work with σ_μ for a fixed distribution $\mu \in \mathcal{M}$ so for simplicity of notation we will omit subscript μ and simply write σ. Let $(v_{i_1,\dots,i_n})$, $(x_{i_1,\dots,i_n})$ be i.i.d. sequences uniform on $[0,1]$ for $n \ge 1$ and $i_1, \dots, i_n \ge 1$, and let

$$s_{i_1,\dots,i_n} = \sigma(w, u, v_{i_1,\dots,i_n}, x_{i_1,\dots,i_n}). \tag{1.15}$$

The role of multi-indices $(i_1, \dots, i_n)$ will be simply to select various subsets of array (1.3) with disjoint coordinate indices i without worrying about how to enumerate them. Let $(\theta_{i_1,\dots,i_n})$ be the copies of random function θ independent over different sets of indices. In addition, let $\hat{v}, \hat{x}, \hat{\theta}$ be independent copies of the above sequences, and let

$$\hat{s}_{i_1,\dots,i_n} = \sigma(w, u, \hat{v}_{i_1,\dots,i_n}, \hat{x}_{i_1,\dots,i_n}). \tag{1.16}$$

Notice that we keep the same w and u in both s and $\hat{s}$. Throughout the paper let us denote by $\pi(\lambda)$ Poisson random variables with mean λ which will always be independent from all other random variables and from each other. For example, if we write $\pi(\alpha)$ and $\pi(\beta)$, we assume them to be independent even if $\alpha = \beta$. Let $(\pi_j(\lambda))$ be independent copies of these r.v. for $j \ge 1$. Let

$$A_i(\varepsilon) = \sum_{k \le \pi_i(p\alpha)} \theta_{k,i}(\varepsilon, s_{1,i,k}, \dots, s_{p-1,i,k}) \tag{1.17}$$

for $i \geq 1$ and $\varepsilon \in \{-1, +1\}$, and let

$$B_i = \sum_{k \leq \pi_i((p-1)\alpha)} \hat{\theta}_{k,i}(\hat{s}_{1,i,k}, \ldots, \hat{s}_{p,i,k}). \tag{1.18}$$

We will express invariance and self-consistency properties of distributions $\mu \in \mathcal{M}$ in terms of equations for the joint moments of arbitrary subset of spins in the array (1.3). Take arbitrary $n, m, q, r \geq 1$ such that $n \leq m$. In the equations below, the index q will correspond to the number of replicas selected, m will be the total number of coordinates and n the number of cavity coordinates considered and r will be the number of perturbation terms of certain type. For each replica index $l \leq q$ we consider an arbitrary subset of coordinates $C_l \subseteq \{1, \ldots, m\}$ and split them into the cavity and noncavity coordinates

$$C_l^1 = C_l \cap \{1, \ldots, n\}, \qquad C_l^2 = C_l \cap \{n+1, \ldots, m\}. \tag{1.19}$$

Let $\mathbb{E}'$ denote the expectation in u and in sequences x and $\hat{x}$, and let

$$U_l = \mathbb{E}' \operatorname{Av}_\varepsilon \prod_{i \in C_l^1} \varepsilon_i \exp \sum_{i \leq n} A_i(\varepsilon_i) \prod_{i \in C_l^2} s_i \exp \sum_{k \leq r} \hat{\theta}_k(\hat{s}_{1,k}, \ldots, \hat{s}_{p,k}) \tag{1.20}$$

and

$$V = \mathbb{E}' \operatorname{Av}_\varepsilon \exp \sum_{i \leq n} A_i(\varepsilon_i) \exp \sum_{k \leq r} \hat{\theta}_k(\hat{s}_{1,k}, \ldots, \hat{s}_{p,k}). \tag{1.21}$$

Then the following holds.

THEOREM 1. *For any limiting distribution $\mu \in \mathcal{M}$ and $\sigma = \sigma_\mu$, we have*

$$\mathbb{E} \prod_{l \leq q} \prod_{i \in C_l} s_i^l = \mathbb{E} \prod_{l \leq q} \mathbb{E}' \prod_{i \in C_l} s_i = \mathbb{E} \frac{\prod_{l \leq q} U_l}{V^q}. \tag{1.22}$$

We will say a few words about various interpretations of (1.22) below, but first let us describe the promised representation for the free energy. Let $\mathcal{M}_{\mathrm{inv}}$ denote the set of distributions of exchangeable arrays generated by functions $\sigma : [0, 1]^4 \to \{-1, +1\}$ as in (1.3) that satisfy invariance equations (1.22) for all possible choices of parameters. Theorem 1 proves that $\mathcal{M} \subseteq \mathcal{M}_{\mathrm{inv}}$. Let

$$A(\varepsilon) = \sum_{k \leq \pi(p\alpha)} \theta_k(\varepsilon, s_{1,k}, \ldots, s_{p-1,k})$$

for $\varepsilon \in \{-1, +1\}$,

$$B = \sum_{k \leq \pi((p-1)\alpha)} \theta_k(s_{1,k}, \ldots, s_{p,k})$$

and let

$$\mathcal{P}(\mu) = \log 2 + \mathbb{E} \log \mathbb{E}' \operatorname{Av}_\varepsilon \exp A(\varepsilon) - \mathbb{E} \log \mathbb{E}' \exp B. \tag{1.23}$$

The following representation holds.

THEOREM 2. *We have*

$$\lim_{N\to\infty} F_N = \inf_{\mu\in\mathcal{M}} \mathcal{P}(\mu) = \inf_{\mu\in\mathcal{M}_{\mathrm{inv}}} \mathcal{P}(\mu). \tag{1.24}$$

One can simplify the last term in (1.23) since we will show at the end of Section 2.3 that

$$\mathbb{E}\log\mathbb{E}'\exp B = (p-1)\alpha\mathbb{E}\log\mathbb{E}'\exp\theta(s_1,\dots,s_p) \tag{1.25}$$

for $\mu\in\mathcal{M}_{\mathrm{inv}}$. To better understand (1.22) let us describe several special cases. Let us define

$$A_i = \log \mathrm{Av}_\varepsilon \exp A_i(\varepsilon). \tag{1.26}$$

First, if we set $r=0$ and let sets C_l be such that $C_l\subseteq\{n+1,\dots,m\}$ for all $l\le q$, then (1.22) becomes

$$\mathbb{E}\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l} s_i = \mathbb{E}\frac{\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l} s_i \exp\sum_{i\le n} A_i}{(\mathbb{E}'\exp\sum_{i\le n} A_i)^q}. \tag{1.27}$$

On the other hand, if we set $n=0$, then (1.22) becomes

$$\mathbb{E}\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l} s_i = \mathbb{E}\frac{\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l} s_i \exp\sum_{i\le r}\hat\theta_i(\hat s_{1,i},\dots,\hat s_{p,i})}{(\mathbb{E}'\exp\sum_{i\le r}\hat\theta_i(\hat s_{1,i},\dots,\hat s_{p,i}))^q}. \tag{1.28}$$

These equations can be interpreted as the invariance of the distribution of (s_i^l) under various changes of density, and they will both play an important role in the proof of Theorem 2. Another consequence of (1.22) are the following self-consistency equations for the distribution of spins. Let us set $r=0$ and $n=m$. Let

$$s_i^A = \frac{\mathrm{Av}_\varepsilon\,\varepsilon\exp A_i(\varepsilon)}{\mathrm{Av}_\varepsilon\exp A_i(\varepsilon)}.$$

Then (1.22) becomes

$$\mathbb{E}\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l} s_i = \mathbb{E}\frac{\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l} s_i^A \exp\sum_{i\le n} A_i}{(\mathbb{E}'\exp\sum_{i\le n} A_i)^q}. \tag{1.29}$$

This means that the distribution of spins (s_i^l) coincides with the distribution of "new" spins $(s_i^{A,l})$ under a certain change of density. Even though we cannot say more about the role (1.29) might play in the diluted models, its analog in the Sherrington–Kirkpatrick model will play a very important role in proving that the joint overlap distribution under μ determines μ and in constructing the explicit formula for $\bar\sigma$ under the Parisi ultrametricity hypothesis.

It will become clear from the arguments below that, in essence, the representation (1.24) is the analog of the Aizenman–Sims–Starr scheme in the Sherrington–Kirkpatrick model [1] with one cavity coordinate. Previous applications of this scheme (e.g., in [1, 10] or [17]) only worked when the number of cavity coordinates goes to infinity, since considering one cavity coordinate in general yields only a lower bound on the free energy. This lower bound expressed in terms of the generic functional order parameter σ_μ will be proved in Section 2.2. Then the main new ideas of the paper—the roles played by the perturbation Hamiltonian (1.13) and the consequent invariance in (1.22)—will help us justify that this lower bound is exact and, moreover, represent it via a well-defined family $\mathcal{M}_{\mathrm{inv}}$. First, following the arguments in [12, 22], in Section 2.3 we will prove a corresponding Franz–Leone type upper bound which will depend on an arbitrary function σ that defines an exchangeable array as in (1.3). For a general σ, this upper bound will depend on N. However, we will show that for σ_μ for $\mu \in \mathcal{M}_{\mathrm{inv}}$ the invariance of Theorem 1 implies that the upper bound is independent of N and matches the lower bound. This is the main point where the invariance properties will come into play. The same ideas will work in the Sherrington–Kirkpatrick model with the appropriate choice of the perturbation Hamiltonian.

1.2. *The Sherrington–Kirkpatrick model.* Let us consider mixed p-spin Sherrington–Kirkpatrick Hamiltonian

$$-H_N(\boldsymbol{\sigma}) = -\sum_{p\geq 1} \beta_p H_{N,p}(\boldsymbol{\sigma}), \tag{1.30}$$

where

$$-H_{N,p}(\boldsymbol{\sigma}) = \frac{1}{N^{(p-1)/2}} \sum_{1\leq i_1,\ldots,i_p\leq N} g_{i_1,\ldots,i_p}\sigma_{i_1}\cdots\sigma_{i_p}, \tag{1.31}$$

the sum is over $p = 1$ and even $p \geq 2$ and $(g_{i_1,\ldots,i_p})$ are standard Gaussian independent for all $p \geq 1$ and all $(i_1,\ldots,i_p)$. The covariance of (1.30) is given by

$$\mathbb{E}H_N(\boldsymbol{\sigma}^1)H_N(\boldsymbol{\sigma}^2) = N\xi(R_{1,2}), \tag{1.32}$$

where $\xi(x) = \sum_{p\geq 1}\beta_p^2 x^p$, and we assume that the sequence (β_p) satisfies $\sum_{p\geq 1} 2^p\beta_p^2 < \infty$. Let us start by introducing the analog of the perturbation Hamiltonian (1.13) for the Sherrington–Kirkpatrick model. Consider independent Gaussian processes $G_{\xi'}(\boldsymbol{\sigma})$ and $G_\theta(\boldsymbol{\sigma})$ on $\Sigma_N = \{-1,+1\}^N$ with covariances

$$\mathbb{E}G_{\xi'}(\boldsymbol{\sigma}^1)G_{\xi'}(\boldsymbol{\sigma}^2) = \xi'(R_{1,2}), \qquad \mathbb{E}G_\theta(\boldsymbol{\sigma}^1)G_\theta(\boldsymbol{\sigma}^2) = \theta(R_{1,2}), \tag{1.33}$$

where $\theta(x) = x\xi'(x) - \xi(x)$, and let $G_{\xi',k}(\boldsymbol{\sigma})$ and $G_{\theta,k}(\boldsymbol{\sigma})$ be their independent copies for $k \geq 1$. For (c_N) as above, let us add the following perturbation to the Hamiltonian (1.30):

$$-H_N^p(\boldsymbol{\sigma}) = \sum_{k\leq \pi(c_N)} \log \operatorname{ch} G_{\xi',k}(\boldsymbol{\sigma}) + \sum_{k\leq \pi'(c_N)} G_{\theta,k}(\boldsymbol{\sigma}), \tag{1.34}$$

where $\pi(c_N)$ and $\pi'(c_N)$ are independent Poisson random variables with means c_N. Clearly, this Hamiltonian does not affect the limit of the free energy since $c_N = o(N)$. We will see that this choice of perturbation ensures the same nice properties of convergence as the perturbation (1.13) in the setting of the diluted models. As a consequence, we will get the following analog of the invariance of Theorem 1. Given a measurable function $\bar{\sigma} : [0,1]^3 \to [-1,1]$, for any $w \in [0,1]$, let $g_{\xi'}(\bar{\sigma}(w,u,\cdot))$ be a Gaussian process indexed by functions $v \to \bar{\sigma}(w,u,\cdot)$ for $u \in [0,1]$ with covariance

$$\mathrm{Cov}(g_{\xi'}(\bar{\sigma}(w,u,\cdot)), g_{\xi'}(\bar{\sigma}(w,u',\cdot))) = \xi'(\mathbb{E}_v \bar{\sigma}(w,u,v)\bar{\sigma}(w,u',v)) \tag{1.35}$$

and $g_\theta(\bar{\sigma}(w,u,\cdot))$ be a Gaussian process independent of $g_{\xi'}(\bar{\sigma}(w,u,\cdot))$ with covariance

$$\mathrm{Cov}(g_\theta(\bar{\sigma}(w,u,\cdot)), g_\theta(\bar{\sigma}(w,u',\cdot))) = \theta(\mathbb{E}_v \bar{\sigma}(w,u,v)\bar{\sigma}(w,u',v)). \tag{1.36}$$

Let us consider independent standard Gaussian random variables z and z' and define

$$G_{\xi'}(\bar{\sigma}(w,u,\cdot)) = g_{\xi'}(\bar{\sigma}(w,u,\cdot)) + z\big(\xi'(1) - \xi'(\mathbb{E}_v \bar{\sigma}(w,u,v)^2)\big)^{1/2} \tag{1.37}$$

and

$$G_\theta(\bar{\sigma}(w,u,\cdot)) = g_\theta(\bar{\sigma}(w,u,\cdot)) + z'\big(\theta(1) - \theta(\mathbb{E}_v \bar{\sigma}(w,u,v)^2)\big)^{1/2}. \tag{1.38}$$

For simplicity of notation we will keep the dependence of $G_{\xi'}$ and G_θ on z or z' implicit. Let $G_{\xi',i}$ and $G_{\theta,i}$ be independent copies of these processes. Random variables z, and z' will play the role of replica variables similarly to u and for this reason in the Sherrington–Kirkpatrick model we will denote by $\mathbb{E}'$ the expectation in u, z and z'. The main purpose of introducing the second term in (1.37) and (1.38) is to match the variances of these Gaussian processes, $\xi'(1)$ and $\theta(1)$, to variances in (1.33) for $\boldsymbol{\sigma}^1 = \boldsymbol{\sigma}^2$.

As in the setting of diluted models, consider arbitrary $n, m, q, r \geq 1$ such that $n \leq m$. For each $l \leq q$ consider an arbitrary subset $C_l \subseteq \{1, \ldots, m\}$, and let C_l^1 and C_l^2 be defined as in (1.19). Let $\bar{\sigma}_i = \bar{\sigma}(w,u,v_i)$. For $l \leq q$ define

$$U_l = \mathbb{E}' \prod_{i \in C_l^1} \mathrm{th}\, G_{\xi',i}(\bar{\sigma}(w,u,\cdot)) \prod_{i \in C_l^2} \bar{\sigma}_i \mathcal{E}_{n,r}, \tag{1.39}$$

where

$$\mathcal{E}_{n,r} = \exp\Big(\sum_{i \leq n} \log \mathrm{ch}\, G_{\xi',i}(\bar{\sigma}(w,u,\cdot)) + \sum_{k \leq r} G_{\theta,k}(\bar{\sigma}(w,u,\cdot))\Big), \tag{1.40}$$

and let $V = \mathbb{E}' \mathcal{E}_{n,r}$. If $\mathcal{M}$ denotes the set of possible limits of μ_N corresponding to the Hamiltonian (1.30) perturbed by (1.34), then the following holds.

THEOREM 3. *For any $\mu \in \mathcal{M}$ and $\bar{\sigma} = \bar{\sigma}_\mu$ we have*

$$\mathbb{E} \prod_{l \leq q} \mathbb{E}' \prod_{i \in C_l} \bar{\sigma}_i = \mathbb{E} \frac{\prod_{l \leq q} U_l}{V^q}. \tag{1.41}$$

Let $\mathcal{M}_{\text{inv}}$ be the family of distributions defined by the invariance properties (1.41), so that Theorem 3 proves that $\mathcal{M} \subseteq \mathcal{M}_{\text{inv}}$. If we define

$$\begin{aligned}\mathcal{P}(\mu) = \log 2 + \mathbb{E}\log \mathbb{E}' \operatorname{ch} G_{\xi'}(\bar{\sigma}_\mu(w, u, \cdot)) \\ - \mathbb{E}\log \mathbb{E}' \exp G_\theta(\bar{\sigma}_\mu(w, u, \cdot)),\end{aligned} \tag{1.42}$$

then we have the following representation for the free energy in the Sherrington–Kirkpatrick model.

THEOREM 4. *We have*

$$\lim_{N\to\infty} F_N = \inf_{\mu\in\mathcal{M}} \mathcal{P}(\mu) = \inf_{\mu\in\mathcal{M}_{\text{inv}}} \mathcal{P}(\mu). \tag{1.43}$$

As in the case of diluted models above, let us describe several special cases of (1.41). If $r = 0$ and sets C_l are such that $C_l \subseteq \{n+1, \dots, m\}$ for all $l \le q$, then (1.41) becomes

$$\mathbb{E}\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l}\bar{\sigma}_i = \mathbb{E}\frac{\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l}\bar{\sigma}_i \prod_{i\le n}\operatorname{ch} G_{\xi',i}(\bar{\sigma}(w,u,\cdot))}{(\mathbb{E}'\prod_{i\le n}\operatorname{ch} G_{\xi',i}(\bar{\sigma}(w,u,\cdot)))^q}. \tag{1.44}$$

If we set $n = 0$, then (1.41) becomes

$$\mathbb{E}\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l}\bar{\sigma}_i = \mathbb{E}\frac{\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l}\bar{\sigma}_i \exp\sum_{k\le r} G_{\theta,k}(\bar{\sigma}(w,u,\cdot))}{(\mathbb{E}'\exp\sum_{k\le r} G_{\theta,k}(\bar{\sigma}(w,u,\cdot)))^q}. \tag{1.45}$$

Again, these equations can be interpreted as the invariance of the spin distributions under various random changes of density. Finally, if we set $r = 0$ and $n = m$, then (1.41) becomes

$$\begin{aligned}&\mathbb{E}\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l}\bar{\sigma}_i \\ &\quad = \mathbb{E}\frac{\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l}\operatorname{th} G_{\xi',i}(\bar{\sigma}(w,u,\cdot))\prod_{i\le n}\operatorname{ch} G_{\xi',i}(\bar{\sigma}(w,u,\cdot))}{(\mathbb{E}'\prod_{i\le n}\operatorname{ch} G_{\xi',i}(\bar{\sigma}(w,u,\cdot)))^q}.\end{aligned} \tag{1.46}$$

The meaning of this self-consistency equation is that the joint distribution of spins generated by a function $\bar{\sigma}(w, u, v)$ coincides with the distribution of spins generated by $\operatorname{th} G_{\xi'}(\bar{\sigma}(w, u, \cdot))$ under a properly interpreted random change of density, and we will discuss this interpretation in more detail below under the Parisi ultrametricity hypothesis. The choice of parameters in (1.46), most importantly $n = m$, will be the key to the following special property of the Sherrington–Kirkpatrick model.

THEOREM 5. *For any $\mu \in \mathcal{M}_{\text{inv}}$, the joint distribution of $(R^\infty_{l,l'})_{l,l'\ge 1}$ defined in* (1.7) *for $\bar{\sigma} = \bar{\sigma}_\mu$ uniquely determines μ and thus the joint distribution of all multi-overlaps.*

The fact that the joint distribution of overlaps determines μ leads to a natural addition to the statement of Theorem 4. It will be clear early in the proof of Theorem 4 that $\mathcal{P}(\mu)$ for $\mu \in \mathcal{M}$ depends only on the distribution of the array (1.7) for $\bar{\sigma} = \bar{\sigma}_\mu$, and, as a result, one can express the free energy in (1.43) as the infimum over a family of measures $\mathcal{M}'_{\mathrm{inv}}$ defined completely in terms of the invariance of the joint overlap distribution and such that $\mathcal{M}_{\mathrm{inv}} \subseteq \mathcal{M}'_{\mathrm{inv}}$. For this purpose one does not need the self-consistency part of the equations (1.41), so we will only use the case when $C_l^2 = C_l$ in (1.19) for all l. Let us consider processes $G_{\xi'}$ and G_θ in (1.37), (1.38) defined in terms of replicas (u_l), (z_l) and (z'_l) of u, z and z', namely,

$$G_{\xi'}(\bar{\sigma}(w,u_l,\cdot)) = g_{\xi'}(\bar{\sigma}(w,u_l,\cdot)) + z_l\bigl(\xi'(1) - \xi'(\mathbb{E}_v \bar{\sigma}(w,u_l,v)^2)\bigr)^{1/2} \tag{1.47}$$

and

$$G_\theta(\bar{\sigma}(w,u_l,\cdot)) = g_\theta(\bar{\sigma}(w,u_l,\cdot)) + z'_l\bigl(\theta(1) - \theta(\mathbb{E}_v \bar{\sigma}(w,u_l,v)^2)\bigr)^{1/2}. \tag{1.48}$$

Let $F = F((R^\infty_{l,l'})_{l,l' \le q})$ be an arbitrary continuous function of the overlaps on q replicas. Let

$$U = \mathbb{E}' F \prod_{l \le q} \exp\biggl(\sum_{i \le n} \log \operatorname{ch} G_{\xi',i}(\bar{\sigma}(w,u_l,\cdot)) + \sum_{k \le r} G_{\theta,k}(\bar{\sigma}(w,u_l,\cdot))\biggr). \tag{1.49}$$

Then the condition

$$\mathbb{E}F = \mathbb{E}(U/V^q) \tag{1.50}$$

for all q, n, r and all continuous bounded functions F defines the family $\mathcal{M}'_{\mathrm{inv}}$. Equation (1.50) is obviously implied by (1.41) which contains the case of polynomial F simply by making sure that $C_l^2 = C_l$, so $\mathcal{M}_{\mathrm{inv}} \subseteq \mathcal{M}'_{\mathrm{inv}}$. Then one can add

$$\lim_{N\to\infty} F_N = \inf_{\mu \in \mathcal{M}'_{\mathrm{inv}}} \mathcal{P}(\mu) \tag{1.51}$$

to the statement of Theorem 4. This together with Theorem 5 shows that in the Sherrington–Kirkpatrick model the role of the order parameter is played by the joint distribution of overlaps rather than the joint distribution of all multi-overlaps or the generic functional order parameter $\bar{\sigma}_\mu$. This gives an idea about how close this point of view takes us to the Parisi ansatz [24] where the order parameter is the distribution of one overlap. Since we can always ensure that the Ghirlanda–Guerra identities [13] hold by adding a mixed p-spin perturbation term [see (1.52) below], the remaining gap is the ultrametricity of the overlaps, since it is well known that the Ghirlanda–Guerra identities and ultrametricity determine the joint distribution of overlaps from the distribution of one overlap; see, for example, [6] or [8]. If one can generalize the results in [19] and [29] to show that the Ghirlanda–Guerra identities always imply ultrametricity, (1.43) would coincide with the Parisi formula proved in [27].

The Ghirlanda–Guerra identities and stochastic stability. Let us mention that the Ghirlanda–Guerra identities and stochastic stability can also be expressed in terms of the generic functional order parameter $\bar{\sigma}$. We will use a version of both properties in the formulation proved in [29]. Let us now consider a different perturbation term

$$H_N^{\delta}(\boldsymbol{\sigma}) = \delta_N \sum_{p\geq 1} \beta_{N,p} H'_{N,p}(\boldsymbol{\sigma}), \tag{1.52}$$

where

$$-H'_{N,p}(\boldsymbol{\sigma}) = \frac{1}{N^{(p-1)/2}} \sum_{1\leq i_1,\dots,i_p\leq N} g'_{i_1,\dots,i_p}\sigma_{i_1}\cdots\sigma_{i_p} \tag{1.53}$$

are independent copies of (1.31). When $\delta_N \to 0$ this perturbation term is of smaller order than (1.30) and does not affect the limit of the free energy. However, the arguments in the proof of the Ghirlanda–Guerra identities and stochastic stability in [29] require that δ_N does not go to zero too fast; for example, the choice of $\delta_N = N^{-1/16}$ works. Then, Theorem 2.5 in [29] states that one can choose a sequence $\boldsymbol{\beta}_N = (\beta_{N,p})$ such that $|\beta_{N,p}| \leq 2^{-p}$ for all N and such that the following properties hold. First of all, if $\langle\cdot\rangle$ is the Gibbs average corresponding to the sum

$$-H'_N(\boldsymbol{\sigma}) = -H_N(\boldsymbol{\sigma}) - H_N^{\delta}(\boldsymbol{\sigma}) \tag{1.54}$$

of the Hamiltonians (1.30) and (1.52), and F is a continuous function of finitely many multi–overlaps (1.5) on replicas $\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n$, then the Ghirlanda–Guerra identities

$$\lim_{N\to\infty}\left|\mathbb{E}\langle F R_{1,n+1}^p\rangle - \frac{1}{n}\mathbb{E}\langle F\rangle\mathbb{E}\langle R_{1,2}^p\rangle - \frac{1}{n}\sum_{l=2}^{n}\mathbb{E}\langle F R_{1,l}^p\rangle\right| = 0 \tag{1.55}$$

hold for all $p \geq 1$. Now, for $p \geq 1$, let $G_p(\boldsymbol{\sigma})$ be a Gaussian process on Σ_N with covariance

$$\mathbb{E}G_p(\boldsymbol{\sigma}^1)G_p(\boldsymbol{\sigma}^2) = R_{1,2}^p, \tag{1.56}$$

and for $t > 0$ let $\langle\cdot\rangle_t$ denote the Gibbs average corresponding to the Hamiltonian

$$-H'_{N,t}(\boldsymbol{\sigma}) = -H'_N(\boldsymbol{\sigma}) - tG_p(\boldsymbol{\sigma}).$$

Then, in addition to (1.55), the following stochastic stability property holds for any $t > 0$:

$$\lim_{N\to\infty}|\mathbb{E}\langle F\rangle_t - \mathbb{E}\langle F\rangle| = 0. \tag{1.57}$$

This property was also proved in [4] without perturbation (1.52) under the condition of differentiability of the limiting free energy. Let μ_N be the joint distribution of spins (1.2) corresponding to the Hamiltonian $H'_N(\boldsymbol{\sigma})$ and $\mathcal{M}$ be the set of all limits of (μ_N). Then both (1.55) and (1.57) can be expressed in the limit in terms of $\bar{\sigma} = \bar{\sigma}_\mu$ for any $\mu \in \mathcal{M}$ as follows. First of all, (1.55) becomes the exact equality in the limit by comment above (1.6),

$$\mathbb{E}F(R^\infty_{1,n+1})^p = \frac{1}{n}\mathbb{E}F\mathbb{E}(R^\infty_{1,2})^p + \frac{1}{n}\sum_{l=2}^n \mathbb{E}F(R^\infty_{1,l})^p. \tag{1.58}$$

Stochastic stability (1.57) can be expressed as follows. For $w \in [0,1]$, let $g_p(\bar{\sigma}(w,u,\cdot))$ be a Gaussian process indexed by $u \in [0,1]$ with covariance

$$\mathrm{Cov}(g_p(\bar{\sigma}(w,u,\cdot)), g_p(\bar{\sigma}(w,u',\cdot))) = (\mathbb{E}_v\bar{\sigma}(w,u,v)\bar{\sigma}(w,u',v))^p \tag{1.59}$$

and, as in (1.37), let

$$G_p(\bar{\sigma}(w,u,\cdot)) = g_p(\bar{\sigma}(w,u,\cdot)) + z\big(1 - (\mathbb{E}_v\bar{\sigma}(w,u,v)^2)^p\big)^{1/2}. \tag{1.60}$$

Then (1.57) implies the following analog of Theorem 3.

THEOREM 6. *For any $\mu \in \mathcal{M}$ and $\bar{\sigma} = \bar{\sigma}_\mu$ we have for all $p \geq 1$ and $t > 0$,*

$$\mathbb{E}\prod_{l\leq q}\mathbb{E}'\prod_{i\in C_l}\bar{\sigma}_i = \mathbb{E}\frac{\prod_{l\leq q}\mathbb{E}'\prod_{i\in C_l}\bar{\sigma}_i \exp tG_p(\bar{\sigma}(w,u,\cdot))}{(\mathbb{E}'\exp tG_p(\bar{\sigma}(w,u,\cdot)))^q}. \tag{1.61}$$

The proof that (1.57) implies (1.61) will not be detailed since it follows exactly the same argument as the proof of Theorem 3 (we will point this out at the appropriate step in Section 3.4). Note that (1.61) is more general than (1.45), which shows that the invariance of Theorem 3 is related to the stochastic stability (1.57). It is interesting to note, however, that the size of the perturbation (1.34) that ensures the invariance in (1.41) was of arbitrarily smaller order than the original Hamiltonian (1.30) since c_N could grow arbitrarily slowly while perturbation (1.52) must be large enough since δ_N cannot go to zero too fast. Moreover, the form of the perturbation (1.34) plays a crucial role in the proof of the self-consistency part (1.46) of equations (1.41) which will allow us to give an explicit construction of the functional order parameter $\bar{\sigma}$ below under the Parisi ultrametricity hypothesis. The special case of the stochastic stability (1.61) for the overlaps [rather than multi-overlaps as in (1.61)] was the starting point of the main result in [3] under certain additional assumptions on $\bar{\sigma}$.

Let us make one more comment about the Ghirlanda–Guerra identities (1.58) from the point of view of the generic functional order parameter $\bar{\sigma}$. Equation (1.55) always arises as a simple consequence of the following concentration statement either for the perturbation Hamiltonian (1.53) (see [29]),

$$\lim_{N\to\infty}\mathbb{E}\left\langle\left|\frac{H'_{N,p}}{N} - \mathbb{E}\left\langle\frac{H'_{N,p}}{N}\right\rangle\right|\right\rangle = 0 \tag{1.62}$$

or for the Hamiltonian in (1.31),

$$\lim_{N\to\infty}\mathbb{E}\left\langle\left|\frac{H_{N,p}}{N} - \mathbb{E}\left\langle\frac{H_{N,p}}{N}\right\rangle\right|\right\rangle = 0, \tag{1.63}$$

which was proved in [21] for any p such that $\beta_p \neq 0$ in (1.30) (the case of $p = 1$ was first proved in [9]). One can similarly encode the limiting Ghirlanda–Guerra identities (1.58) as a concentration statement for the Gaussian process $G_p(\bar{\sigma}(w, u, \cdot))$ in (1.60) as follows.

THEOREM 7. *Assuming* (1.61), *the following are equivalent*:

(1) *the Ghirlanda–Guerra identities* (1.58) *hold*;
(2) *for all* $p \geq 1$,

$$\begin{aligned}(1.64)\qquad &\mathbb{E}\frac{G_p(\bar{\sigma}(w,u,\cdot))^2 \exp tG_p(\bar{\sigma}(w,u,\cdot))}{\mathbb{E}' \exp tG_p(\bar{\sigma}(w,u,\cdot))} \\ &\quad - \left(\mathbb{E}\frac{G_p(\bar{\sigma}(w,u,\cdot)) \exp tG_p(\bar{\sigma}(w,u,\cdot))}{\mathbb{E}' \exp tG_p(\bar{\sigma}(w,u,\cdot))}\right)^2\end{aligned}$$

is uniformly bounded for all $t > 0$, *in which case it is equal to* 1.

The result will follow from a simple application of the Gaussian integration by parts and the main reason behind this equivalence will be very similar to the proof of the Ghirlanda–Guerra identities for Poisson–Dirichlet cascades in [30].

1.3. *Connections to the Parisi ansatz.* We will now discuss how the functional order parameter $\bar{\sigma}(w, u, v)$ fits into the picture of the "generic ultrametric Parisi ansatz" expected to hold in the Sherrington–Kirkpatrick and diluted models and believed to represent some kind of general principle in other models as well. We will begin with the case of the Sherrington–Kirkpatrick model where the joint distribution of the overlap array (1.7) under the Parisi ultrametricity conjecture is well understood, and we will use it to give an explicit construction of $\bar{\sigma}(w, u, v)$. This will serve as an illustration of a more general case that will appear in the diluted models.

Parisi ansatz in the Sherrington–Kirkpatrick model. Let us go back to the self-consistency equations (1.46) and show that they can be used to give an explicit formula for the function $\bar{\sigma}$, or the distribution of spins, under the Parisi ultrametricity hypothesis and the Ghirlanda–Guerra identities. In this section we will assume that the reader is familiar with the Ruelle probability cascades [25] and refer to extensive literature on the subject for details. Equation (1.7) defines some realization of the directing measure of the overlap array in the following sense. If we think of $\bar{\sigma}(w, u, \cdot)$ as a function in $H = \mathcal{L}^2([0, 1], dv)$, then the image of the Lebesgue measure on $[0, 1]$ by the map $u \to \bar{\sigma}(w, u, \cdot)$ defines a random probability measure η_w on H. Equation (1.7) states that the overlaps can be generated by scalar products in H of an i.i.d. sequence from this random measure. Any such measure η_w defined on an arbitrary Hilbert space is called the directing measure of the overlap array $(R^{\infty}_{l,l'})$. It is defined uniquely up to a random isometry; see, for example, Lemma 4 in [20], or in the case of discrete overlap the end of the proof of Theorem 4 in [19]. By Theorem 2 in [19], the Ghirlanda–Guerra identities imply that

$$\mathbb{E}_v \bar{\sigma}(w,u,v)^2 = q^* \qquad \text{a.s.,} \tag{1.65}$$

where q^* is the largest point in the support of the distribution of $R^{\infty}_{1,2}$, and, therefore, equation (1.46) can be slightly simplified by getting rid of the last term in (1.37),

$$\begin{aligned}\mathbb{E}\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l}\bar{\sigma}_i \\ = \mathbb{E}\frac{\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l}\operatorname{th} g_{\xi',i}(\bar{\sigma}(w,u,\cdot))\prod_{i\le n}\operatorname{ch} g_{\xi',i}(\bar{\sigma}(w,u,\cdot))}{(\mathbb{E}'\prod_{i\le n}\operatorname{ch} g_{\xi',i}(\bar{\sigma}(w,u,\cdot)))^q}.\end{aligned} \tag{1.66}$$

The key observation now is that the right-hand side of (1.66) does not depend on the particular realization of the directing measure since the Gaussian process $g_{\xi'}$ is defined by its covariance function (1.35) which depends only on the $\mathcal{L}^2([0,1],dv)$ structure of the family $\bar{\sigma}(w,u,\cdot)$. Let us first interpret the right-hand side of (1.66) when the overlap distribution is discrete,

$$\mathbb{P}(R^{\infty}_{1,2} = q_l) = m_{l+1} - m_l \tag{1.67}$$

for some $0 \le q_1 < q_2 < \cdots < q_k = q^* \le 1$ and $0 = m_1 < \cdots < m_k < m_{k+1} = 1$. In this case it is well known that one directing measure of the overlaps is given by the Ruelle probability cascades, of course, assuming the Ghirlanda–Guerra identities and ultrametricity (see, e.g., [3, 19, 29] or [30]) and, therefore, $(g_{\xi',i})$ are the usual Gaussian fields associated with the cascades. The Ruelle probability cascades is a discrete random measure with Poisson–Dirichlet weights (w_α) customarily indexed by $\alpha \in \mathbb{N}^k$, where k is the number of atoms in (1.67), so that the Gaussian fields are also indexed by α, $(g_{\xi',i}(\alpha))$. By definition of the directing measure η_w, the expectation $\mathbb{E}'$ in u plays the role of averaging with respect to these weights, so that the right-hand side of (1.66) can be rewritten as

$$\mathbb{E}\frac{\prod_{l\le q}\sum_\alpha w_\alpha \prod_{i\in C_l}\operatorname{th} g_{\xi',i}(\alpha)\prod_{i\le n}\operatorname{ch} g_{\xi',i}(\alpha)}{(\sum_\alpha w_\alpha\prod_{i\le n}\operatorname{ch} g_{\xi',i}(\alpha))^q}. \tag{1.68}$$

This in its turn can be rewritten using well-known properties of the Ruelle probability cascades, in particular, Lemma 1.2 in [23] which is a recursive application of Proposition A.2 in [7]. If we denote

$$w'_\alpha = \frac{w_\alpha\prod_{i\le n}\operatorname{ch} g_{\xi',i}(\alpha)}{\sum_\alpha w_\alpha\prod_{i\le n}\operatorname{ch} g_{\xi',i}(\alpha)},$$

then the point processes

$$(w'_\alpha, (g_{\xi',i}(\alpha))_{i\le n})_{\alpha\in\mathbb{N}^k} \stackrel{d}{=} (w_\alpha, (g'_{\xi',i}(\alpha))_{i\le n})_{\alpha\in\mathbb{N}^k} \tag{1.69}$$

have the same distribution, where $(g'_{\xi',i}(\alpha))$ is a random field (no longer Gaussian) associated with the Ruelle probability cascades defined from the Gaussian field $(g_{\xi',i}(\alpha))$ by an explicit change of density; see equation (7) in [23]. Therefore, (1.66) can be rewritten as

$$\mathbb{E}\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l}\bar{\sigma}_i=\mathbb{E}\prod_{l\le q}\sum_{\alpha}w_\alpha\prod_{i\in C_l}\operatorname{th}g'_{\xi',i}(\alpha),\tag{1.70}$$

which can now be interpreted as the explicit construction of $\bar{\sigma}(w,u,v)$. The first coordinate w corresponds to generating the weights $(w_\alpha)_{\alpha\in\mathbb{N}^k}$ of the Ruelle probability cascade with the parameters $0=m_1<\cdots<m_k<1$, the second coordinate u plays the role of sampling an index α according to the weights (w_α) and the last coordinate v corresponds to generating the random field $(g'_{\xi'}(\alpha))$, so that the directing measure η_w carries weight w_α at the point $\operatorname{th}g'_{\xi'}(\alpha)$ in $\mathcal{L}^2([0,1],dv)$. Another way to write this is to consider a partition $(C_\alpha)_{\alpha\in\mathbb{N}^k}$ of $[0,1]$ into intervals of length $|C_\alpha|=w_\alpha$ and let

$$\bar{\sigma}(w,u,v)=\sum_{\alpha\in\mathbb{N}^k}I(u\in C_\alpha)\operatorname{th}g'_{\xi'}(\alpha),\tag{1.71}$$

where we keep the dependence of (C_α) on w and $(g'_{\xi'}(\alpha))$ on v implicit. In particular, (1.70) implies that the limiting distribution of the Gibbs averages $\langle\sigma_i\rangle$ of finitely many spins $1\le i\le n$ coincides with the distribution of

$$\sum_{\alpha}w_\alpha\operatorname{th}g'_{\xi',i}(\alpha)\qquad\text{for } 1\le i\le n.\tag{1.72}$$

This can be thought of as the generalization of the high temperature result (Theorem 2.4.12 in [26]) under the assumption of the Parisi ultrametricity. It will be clear from the proof of Theorem 3 that the right-hand side of (1.66) is continuous with respect to the distribution of the overlap array (1.7) and, on the other hand, it is well known that ultrametricity allows one to approximate any overlap array by a discretized overlap array satisfying (1.67) uniformly while preserving ultrametricity and the Ghirlanda–Guerra identities. Therefore, one can think of the case of an arbitrary distribution of the overlap simply as the limiting case of the above construction for discrete overlaps.

Parisi ansatz in the diluted models. To make a transition to the case of diluted models let us look more closely at equation (1.71). Original Gaussian field $(g_{\xi'}(\alpha))$ indexed by $\alpha=(\alpha_1,\dots,\alpha_k)\in\mathbb{N}^k$ associated to the Ruelle probability cascades is of the form [3]

$$g_{\xi'}(\alpha)=g_{\xi'}(\alpha_1)+g_{\xi'}(\alpha_1,\alpha_2)+\cdots+g_{\xi'}(\alpha_1,\dots,\alpha_k),$$

where random variables $g_{\xi'}(\alpha_1,\dots,\alpha_l)$ are Gaussian with variances $\xi'(q_l)-\xi'(q_{l-1})$ independent for different $1\le l\le k$ and different $(\alpha_1,\dots,\alpha_l)$. The field $(g'_{\xi'}(\alpha))$ on the right-hand side of (1.69) is again of the form

$$g'_{\xi'}(\alpha)=g'_{\xi'}(\alpha_1)+g'_{\xi'}(\alpha_1,\alpha_2)+\cdots+g'_{\xi'}(\alpha_1,\dots,\alpha_k),$$

and for each $l \le k$ the sequence $(g'_{\xi'}(\alpha_1,\dots,\alpha_l))_{\alpha_l\ge 1}$ is i.i.d. from distribution defined by the explicit change of density (equation (7) in [23]) which depends on $g'_{\xi'}(\alpha_1),\dots,g'_{\xi'}(\alpha_1,\dots,\alpha_{l-1})$, and these sequences are independent for different $(\alpha_1,\dots,\alpha_{l-1})$ conditionally on the sequences $(g'_{\xi'}(\alpha_1)),\dots,(g'_{\xi'}(\alpha_1,\dots,\alpha_{l-1}))$. This means that one can generate the process $(g'_{\xi'}(\alpha))$ recursively as follows. Let $v(\alpha_1,\dots,\alpha_l)$ be random variables uniform on $[0,1]$ independent for different $1 \le l \le k$ and different $(\alpha_1,\dots,\alpha_l)$. Then for $1 \le l \le k$ we can define

$$g'_{\xi'}(\alpha_1,\dots,\alpha_l) = Q_l(g'_{\xi'}(\alpha_1),\dots,g'_{\xi'}(\alpha_1,\dots,\alpha_{l-1}), v(\alpha_1,\dots,\alpha_l)), \tag{1.73}$$

where Q_l as a function of the last variable is the quantile transform of the distribution defined by the aforementioned change of density. Combining all the steps of the recursion we get

$$g'_{\xi'}(\alpha) = Q(v(\alpha_1),\dots,v(\alpha_1,\dots,\alpha_k)) \tag{1.74}$$

for some specific function Q. Equation (1.71) becomes

$$\bar\sigma(w,u,v) = \sum_{\alpha\in\mathbb{N}^k} I(u \in C_\alpha)\varphi(v(\alpha_1),\dots,v(\alpha_1,\dots,\alpha_k)), \tag{1.75}$$

where $\varphi = \mathrm{th}\circ Q$, and again, as in (1.71), we keep the dependence of (C_α) on w and $(v(\alpha_1,\dots,\alpha_l))$ on v implicit. Let us emphasize that the change of density that defines Q_l in (1.73) and, therefore, the functions Q,φ and $\bar\sigma$ are completely determined by the parameters of the distribution of one overlap in (1.67) which is the functional order parameter of the Parisi ansatz in the Sherrington–Kirkpatrick model. What seems to be the main (and only) difference in the Parisi ansatz for diluted models is that this function φ is allowed to be an arbitrary $(-1,1)$ valued function, which we will now explain.

The Parisi functional order parameter in the diluted models appears in the description of the free energy, and one can make the connection to the generic functional order parameter $\bar\sigma$ by comparing the Parisi formula for the free energy to the representation (1.23), (1.24). For example, in the notation of [22] where the order parameter was encoded by the Ruelle probability cascade weights (w_α) and associated random field $(x(\alpha))$ for $\alpha\in\mathbb{N}^k$, it is easy to see that in order for (1.23) to match the Parisi formula in [22], $\bar\sigma$ should be defined exactly as in (1.71),

$$\bar\sigma(w,u,v) = \sum_{\alpha\in\mathbb{N}^k} I(u\in C_\alpha)\,\mathrm{th}\,x(\alpha). \tag{1.76}$$

The only difference from (1.71) is how the random field $(x(\alpha))$ is generated compared to $(g'_{\xi'}(\alpha))$, and once we recall how $(x(\alpha))$ is generated according to the Parisi ansatz, we will realize that one can write exactly the same representation as (1.74),

$$x(\alpha) = Q(v(\alpha_1),\dots,v(\alpha_1,\dots,\alpha_k)), \tag{1.77}$$

only now Q is allowed to be arbitrary. The field $(x(\alpha))$ is customarily generated as follows. Let P_1 be the set of probability measures on $\mathbb{R}$, and by induction on $l \le k$ we define P_{l+1} as the set of probability measures on P_l. Let us fix $\eta \in \mathrm{P}_k$ (the basic parameter) and define a random sequence $(\eta(\alpha_1), \ldots, \eta(\alpha_1, \ldots, \alpha_{k-1}), x(\alpha_1, \ldots, \alpha_k))$ as follows. Given η, the sequence $(\eta(\alpha_1))_{\alpha_1 \ge 1}$ of elements of P_{k-1} is i.i.d. from distribution η. For $1 \le l \le k-1$, given all the elements $\eta(\alpha_1, \ldots, \alpha_s)$ for all values of the integers $\alpha_1, \ldots, \alpha_s$ and all $s \le l-1$, the sequence $(\eta(\alpha_1, \ldots, \alpha_l))_{\alpha_l \ge 1}$ of elements of P_{k-l} is i.i.d. from distribution $\eta(\alpha_1, \ldots, \alpha_{l-1})$, and these sequences are independent of each other for different values of $(\alpha_1, \ldots, \alpha_{l-1})$. Finally, given all the elements $\eta(\alpha_1, \ldots, \alpha_s)$ for all values of the integers $\alpha_1, \ldots, \alpha_s$ and all $s \le k-1$ the sequence $(x(\alpha_1, \ldots, \alpha_k))_{\alpha_k \ge 1}$ is i.i.d. on $\mathbb{R}$ with distribution $\eta(\alpha_1, \ldots, \alpha_{k-1})$ and these sequences are independent for different values of $(\alpha_1, \ldots, \alpha_{k-1})$. The process of generating x's can be represented schematically as

$$\eta \to \eta(\alpha_1) \to \cdots \to \eta(\alpha_1, \ldots, \alpha_{k-1}) \to x(\alpha_1, \ldots, \alpha_k). \tag{1.78}$$

Now, as above, let $v(\alpha_1, \ldots, \alpha_l)$ be random variables uniform on $[0, 1]$ independent for different $1 \le l \le k$ and different $(\alpha_1, \ldots, \alpha_l)$. First, random variables $(\eta(\alpha_1))_{\alpha_1 \ge 1}$ are i.i.d. from probability measure η on P_{k-1} and, therefore, can be generated as

$$\eta(\alpha_1) = Q_{k-1}(v(\alpha_1)) \tag{1.79}$$

for some function $Q_{k-1} : [0, 1] \to \mathrm{P}_{k-1}$. Next, random variables $(\eta(\alpha_1, \alpha_2))_{\alpha_2 \ge 1}$ are i.i.d. from probability measure $\eta(\alpha_1)$ on P_{k-2} and, therefore, can be generated as

$$\eta(\alpha_1, \alpha_2) = \tilde{Q}_{k-2}(\eta(\alpha_1), v(\alpha_1, \alpha_2))$$

for some function $\tilde{Q}_{k-2}(\eta(\alpha_1), \cdot) : [0, 1] \to \mathrm{P}_{k-2}$. Combining with (1.79), we can write

$$\eta(\alpha_1, \alpha_2) = Q_{k-2}(v(\alpha_1), v(\alpha_1, \alpha_2)) \tag{1.80}$$

for some function $Q_{k-2} : [0, 1]^2 \to \mathrm{P}_{k-2}$. We can continue this construction recursively and at the end we will get

$$x(\alpha_1, \ldots, \alpha_k) = Q(v(\alpha_1), \ldots, v(\alpha_1, \ldots, \alpha_k)) \tag{1.81}$$

for some function $Q : [0, 1]^k \to \mathbb{R}$, which is exactly (1.77). This representation gives some choice of Q for a given $\eta \in \mathrm{P}_k$, but any choice of Q corresponds to some η, which is obvious by reverse induction and identifying a function of uniform r.v. on $[0, 1]$ with the distribution on its image.

To summarize, the Parisi ansatz can be expressed in terms of $\bar{\sigma}$ by saying that equation (1.75) must hold for some choice of $(-1, 1)$ valued function φ. Of course, in general this statement should be understood in the limiting sense when the number $(k-1)$ of replica-symmetry breaking steps goes to infinity. Precise statement should be that in the diluted models any limiting distribution $\mu \in \mathcal{M}$ of the array (1.3) over a subsequence of (μ_N) can be approximated by the distribution of the array generated by $\bar{\sigma}(w, u, v)$ as in (1.75) for large enough k, some function $\varphi : [0, 1]^k \to (-1, 1)$ and some parameters $0 = m_1 < \cdots < m_k < 1$ of the distribution of weights (w_α) in the Ruelle probability cascades.

This formulation clarifies another statement of the physicists, namely, that multi-overlap $R^\infty_{1,\ldots,n}$ in (1.6) is the function of the overlaps $R^\infty_{l,l'}$ in (1.7) for $1 \le l < l' \le n$. According to (1.75) the choice of $u_1, \ldots, u_n$ corresponds to the choice of indices $\alpha^1, \ldots, \alpha^n \in \mathbb{N}^k$ so that

$$R^\infty_{1,\ldots,n} = \mathbb{E}\varphi(v(\alpha^1_1), \ldots, v(\alpha^1_1, \ldots, \alpha^1_k)) \cdots \varphi(v(\alpha^n_1), \ldots, v(\alpha^n_1, \ldots, \alpha^n_k)).$$

On the other hand, if we denote $\alpha^1 \wedge \alpha^2 = \min\{i : \alpha^1_i \ne \alpha^2_i\}$ and $\alpha^1 \wedge \alpha^2 = k + 1$ if $\alpha^1 = \alpha^2$, then the overlap takes finitely many values

$$\begin{aligned} R^\infty_{1,2} &= \mathbb{E}\varphi(v(\alpha^1_1), \ldots, v(\alpha^1_1, \ldots, \alpha^1_k))\varphi(v(\alpha^2_1), \ldots, v(\alpha^2_1, \ldots, \alpha^2_k)) \\ &= q_{\alpha^1 \wedge \alpha^2} \end{aligned}$$

for some $0 \le q_1 \le \cdots \le q_{k+1} \le 1$. This means that the values of the overlaps $(R^\infty_{l,l'})$ determine $(\alpha^l \wedge \alpha^{l'})$ for $1 \le l < l' \le n$. It is also clear that the multi-overlap $R^\infty_{1,\ldots,n}$ is the same for two sets of indices $(\alpha^1, \ldots, \alpha^n)$ and $(\beta^1, \ldots, \beta^n)$ for which $(\alpha^l \wedge \alpha^{l'}) = (\beta^{\rho(l)} \wedge \beta^{\rho(l')})$ for some permutation ρ the set $\{1, \ldots, n\}$. In this sense, given representation (1.75), the overlaps indeed determine the value of the multi-overlap. At the moment we have no idea how (1.75) can be proved, but it is helpful to have a point of view that formulates precisely the predictions of the Parisi ansatz.

While many technical details will be quite different, the main line of the arguments in the setting of the Sherrington–Kirkpatrick model in Section 3 will be parallel to the arguments in Section 2 for diluted models. A reader only interested in the Sherrington–Kirkpatrick model should read Lemma 2 before skipping to Section 3.

2. Diluted models.

2.1. *Properties of convergence.* Let us first record a simple consequence of the fact that the distribution of the array in (1.3) is the limit of the distribution of spins (σ^l_i) under the annealed product Gibbs measure. As usual, $\langle\cdot\rangle$ will denote the expectation with respect to the random Gibbs measure. Also, recall the definition of $\mathbb{E}'$ before Theorem 1.

LEMMA 1. *Let $h_1, \ldots, h_m : \{-1, +1\}^n \to [-K, K]$ be some bounded functions of n spins, and let h be a continuous function on $[-K, K]^m$. Let $\boldsymbol{\sigma} = (\sigma_i)_{1\le i\le n}$, and let $\mathbf{s} = (s^1_i)_{1\le i\le n}$ defined in (1.3) for some $\mu \in \mathcal{M}$. If μ_N converges* *to μ over subsequence (N_k), then*

$$\lim_{N_k\to\infty} \mathbb{E}h(\langle h_1(\boldsymbol{\sigma})\rangle, \dots, \langle h_m(\boldsymbol{\sigma})\rangle) = \mathbb{E}h(\mathbb{E}' h_1(\mathbf{s}), \dots, \mathbb{E}' h_m(\mathbf{s})). \tag{2.1}$$

PROOF. Since it is enough to prove this for polynomials h and since each h_l is a polynomial in its coordinates, this statement is simply a convergence of moments

$$\lim_{N_k\to\infty} \mathbb{E}\Big\langle \prod \sigma_i^l \Big\rangle = \mathbb{E} \prod s_i^l,$$

where the product is over a finite subset of indices (i, l). □

We will often use this lemma for random functions $h, (h_l)$ independent of all other randomness, simply by applying (2.1) conditionally on the randomness of these functions. Justifications of convergence will always be omitted because of their triviality.

Another simple property of convergence of spin distributions under the annealed Gibbs measure in diluted models is that adding or removing a finite number of terms to the Poisson number of terms $\pi(\alpha N)$ or $\pi(c_N)$ in (1.14) does not affect the limit of these distribution over any subsequence for which the limit exists. Let $(N_k)_{k\geq 1}$ be any such subsequence, and let n, m be fixed integers. In fact, it will be clear from the proof that one can let n, m grow with N_k, but we will not need this. Let H_N' be defined exactly as (1.14) only with $\pi(\alpha N) + n$ terms instead of $\pi(\alpha N)$ in the first sum and $\pi(c_N) + m$ instead of $\pi(c_N)$ in the perturbation term, and let $\langle\cdot\rangle'$ denote the corresponding Gibbs measure.

LEMMA 2. *For any bounded function h of finitely many spins in array (σ_i^l) we have*

$$\lim_{N\to\infty} |\mathbb{E}\langle h\rangle' - \mathbb{E}\langle h\rangle| = 0. \tag{2.2}$$

PROOF. For certainty, let us assume that $n, m \geq 0$ and $|h| \leq 1$. If we denote by $\langle\cdot\rangle_{i,j}$ the Gibbs average conditionally on $\pi(\alpha N) = i$ and $\pi(c_N) = j$, then

$$\mathbb{E}\langle h\rangle = \sum_{i,j\geq 0} \pi(\alpha N, i)\pi(c_N, j)\mathbb{E}\langle h\rangle_{i,j},$$

where from now on $\pi(\lambda, k) = \lambda^k e^{-\lambda}/k!$ and

$$\begin{aligned}
\mathbb{E}\langle h\rangle' &= \sum_{i,j\geq 0} \pi(\alpha N, i)\pi(c_N, j)\mathbb{E}\langle h\rangle_{i+n,j+m} \\
&= \sum_{i\geq n, j\geq m} \pi(\alpha N, i-n)\pi(c_N, j-m)\mathbb{E}\langle h\rangle_{i,j}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
|\mathbb{E}\langle h\rangle' - \mathbb{E}\langle h\rangle| \le{}& \sum_{i<n}\pi(\alpha N, i) + \sum_{j<m}\pi(c_N, j) \\
&+ \sum_{i\ge n, j\ge m} |\pi(\alpha N, i-n)\pi(c_N, j-m) - \pi(\alpha N, i)\pi(c_N, j)| \\
\le{}& \sum_{i<n}\pi(\alpha N, i) + \sum_{j<m}\pi(c_N, j) + \sum_{i\ge n}|\pi(\alpha N, i-n) - \pi(\alpha N, i)| \\
&+ \sum_{j\ge m}|\pi(c_N, j-m) - \pi(c_N, j)|.
\end{aligned}
$$

The first two sums obviously go to zero. One can see that the third sum goes to zero as follows. Poisson distribution with mean αN is concentrated inside the range

$$
\alpha N - \sqrt{N\log N} \le i \le \alpha N + \sqrt{N\log N}. \tag{2.3}
$$

If we write

$$
|\pi(\alpha N, i-n) - \pi(\alpha N, i)| = \pi(\alpha N, i)\left|1 - \frac{i!}{(i-n)!}(\alpha N)^{-n}\right|, \tag{2.4}
$$

then it remains to note that

$$
\frac{i!}{(i-n)!}(\alpha N)^{-n} = \frac{i(i-1)\cdots(i-n+1)}{(\alpha N)^n} \to 1
$$

uniformly inside the range (2.3). Similarly, the last sum goes to zero which finishes the proof. □

REMARK. Lemma 2 implies that (2.2) holds even if n is a random variable. We will use this observation in the case when H_N' is defined exactly as (1.14) only with $\pi(\alpha N + n)$ terms instead of $\pi(\alpha N)$. In fact, in this case one can write

$$
\mathbb{E}\langle h\rangle' = \sum_{i,j\ge 0}\pi(\alpha N + n, i)\pi(c_N, j)\mathbb{E}\langle h\rangle_{i,j+m}
$$

and instead of (2.4) use

$$
|\pi(\alpha N + n, i) - \pi(\alpha N, i)| = \pi(\alpha N, i)\left|1 - \left(1 + \frac{n}{\alpha N}\right)^i e^{-n}\right|
$$

and notice that again the last factor goes to zero uniformly over range (2.3). Similarly, one can have $\pi(c_N + n)$ instead of $\pi(c_N)$ terms in the perturbation Hamiltonian without affecting convergence.

Due to the perturbation term (1.13) the following important property of convergence holds.

LEMMA 3. *If μ_N converges to μ over subsequence (N_k) then it also converges to μ over subsequence $(N_k + n)$ for any $n \geq 1$.*

PROOF. We will show that the joint moments of spins converge to the same limit over subsequences that differ by a finite shift n. Let $h = \prod_{j \leq q} h_j$ where $h_j = \prod_{i \in C_j} \sigma_i^j$ over some finite sets of spin coordinates C_j. Let us denote by $\langle \cdot \rangle_N$ the Gibbs average with respect to the Hamiltonian (1.14) defined on N coordinates. We will show that

$$\lim_{N \to \infty} |\mathbb{E}\langle h \rangle_{N+n} - \mathbb{E}\langle h \rangle_N| = 0.$$

Let us rewrite $\mathbb{E}\langle h \rangle_{N+n}$ by treating the last n coordinates as cavity coordinates. Let us separate the $\pi(\alpha(N+n))$ terms in the first sum

$$\sum_{k \leq \pi(\alpha(N+n))} \theta_k(\sigma_{i_{1,k}}, \ldots, \sigma_{i_{p,k}}) \tag{2.5}$$

of the Hamiltonian $H_{N+n}(\boldsymbol{\sigma})$ in (1.14) into several groups:

(1) terms for k such that all indices $i_{1,k}, \ldots, i_{p,k} \leq N$;
For $1 \leq l \leq n$:
($2l$) terms with exactly one of indices $i_{1,k}, \ldots, i_{p,k}$ equal to $N + l$ and all others $\leq N$;
(3) terms with at least two of indices $i_{1,k}, \ldots, i_{p,k} \geq N$.

The probabilities that a term is of these three type are

$$p_1 = \left(\frac{N}{N+n}\right)^p, \qquad p_{2,l} = p\frac{1}{N+n}\left(\frac{N}{N+n}\right)^{p-1}, \qquad p_3 = 1 - p_1 - \sum_{l \leq n} p_{2,l}.$$

Therefore, the number of terms in these groups are independent Poisson random variables with means

$$\begin{aligned}
\alpha(N+n)p_1 &= \alpha(N+n-np) + O(N^{-1}), \\
\alpha(N+n)p_{2,l} &= \alpha p + O(N^{-1}), \\
\alpha(N+n)p_3 &= O(N^{-1}).
\end{aligned}$$

We can redefine the number of terms in each group to be exactly of means $\alpha(N + n - np)$, αp and 0 since asymptotically it does not affect $\mathbb{E}\langle h \rangle_{N+n}$ as in Lemma 2 or using assumption (1.12). Thus, if we write $\boldsymbol{\sigma} = (\boldsymbol{\rho}, \boldsymbol{\varepsilon}) \in \Sigma_{N+n}$ for the first N coordinates $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_N)$ and the last n cavity coordinates $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$, then (2.5) can be replaced with

$$\begin{aligned}
&\sum_{k \leq \pi(\alpha(N+n-np))} \theta_k(\rho_{i_{1,k}}, \ldots, \rho_{i_{p,k}}) \\
&\qquad + \sum_{l \leq n} \sum_{k \leq \pi_l(\alpha p)} \theta_{k,l}(\varepsilon_l, \rho_{i_{1,k,l}}, \ldots, \rho_{i_{p-1,k,l}}),
\end{aligned} \tag{2.6}$$

where indices $i_{1,k},\dots,i_{p,k}$ and $i_{1,k,l},\dots,i_{p-1,k,l}$ are all uniformly distributed on $\{1,\dots,N\}$. Let us now consider the perturbation term in (1.14)

$$\sum_{l\le\pi(c_{N+n})}\log \mathrm{Av}_\varepsilon \exp \sum_{k\le\hat\pi_l(\alpha p)} \hat\theta_{k,l}(\varepsilon,\sigma_{j_{1,k,l}},\dots,\sigma_{j_{p-1,k,l}}), \tag{2.7}$$

where $j_{1,k,l},\dots,j_{p-1,k,l}$ are uniformly distributed on $\{1,\dots,N+n\}$. Here, we used independent copies $\hat\pi_l$ and $\hat\theta_{k,l}$ since π_l and $\theta_{k,l}$ were already used in (2.6). The expected number of all such indices in (2.7) that belong to $\{N+1,\dots,N+n\}$ is $c_{N+n}\alpha p(p-1)n/(N+n)\to 0$ which means that with high probability all indices belong to $\{1,\dots,N\}$. As a result, asymptotically $\mathbb{E}\langle h\rangle_{N+n}$ will not be affected if we replace the perturbation term (2.7) with

$$\sum_{l\le\pi(c_{N+n})}\log \mathrm{Av}_\varepsilon \exp \sum_{k\le\hat\pi_l(\alpha p)} \hat\theta_{k,l}(\varepsilon,\rho_{j_{1,k,l}},\dots,\rho_{j_{p-1,k,l}}), \tag{2.8}$$

where $j_{1,k,l},\dots,j_{p-1,k,l}$ are uniformly distributed on $\{1,\dots,N\}$. Thus, we can assume from now on that $\mathbb{E}\langle h\rangle_{N+n}$ is computed with respect to the Hamiltonian which is the sum of (2.6) and (2.8). If $\langle\cdot\rangle'_N$ denotes the Gibbs average on Σ_N with respect to the Hamiltonian

$$\begin{aligned}-H'_N(\boldsymbol\rho) = &\sum_{k\le\pi(\alpha(N+n-np))}\theta_k(\rho_{i_{1,k}},\dots,\rho_{i_{p,k}})\\ &+\sum_{l\le\pi(c_{N+n})}\log \mathrm{Av}_\varepsilon \exp \sum_{k\le\hat\pi_l(\alpha p)} \hat\theta_{k,l}(\varepsilon,\rho_{j_{1,k,l}},\dots,\rho_{j_{p-1,k,l}}),\end{aligned}$$

then each factor in

$$\langle h\rangle_{N+n}=\prod_{j\le q}\langle h_j\rangle_{N+n}=\prod_{j\le q}\Big\langle\prod_{i\in C_j}\sigma_i\Big\rangle_{N+n}=\prod_{j\le q}\Big\langle\prod_{i\in C_j}\rho_i\Big\rangle_{N+n}$$

can be written as

$$\begin{aligned}\langle h_j\rangle_{N+n} &= \frac{\langle\prod_{i\in C_j}\rho_i\,\mathrm{Av}_\varepsilon\exp\sum_{l\le n}\sum_{k\le\pi_l(\alpha p)}\theta_{k,l}(\varepsilon_l,\rho_{i_{1,k,l}},\dots,\rho_{i_{p-1,k,l}})\rangle'_N}{\langle\mathrm{Av}_\varepsilon\exp\sum_{l\le n}\sum_{k\le\pi_l(\alpha p)}\theta_{k,l}(\varepsilon_l,\rho_{i_{1,k,l}},\dots,\rho_{i_{p-1,k,l}})\rangle'_N}\\ &=\Big\langle\prod_{i\in C_j}\rho_i\Big\rangle''_N,\end{aligned}$$

where $\langle\cdot\rangle''_N$ is the Gibbs average on Σ_N corresponding to the Hamiltonian

$$-H''_N(\boldsymbol\rho)=-H'_N(\boldsymbol\rho)+\sum_{l\le n}\log\mathrm{Av}_\varepsilon\exp\sum_{k\le\pi_l(\alpha p)}\theta_{k,l}(\varepsilon,\rho_{i_{1,k,l}},\dots,\rho_{i_{p-1,k,l}}).$$

But this Hamiltonian differs from the original Hamiltonian (1.14) only in that the first sum has $\pi(\alpha(N+n-np))$ terms instead of $\pi(\alpha N)$, and the perturbation term has $\pi(c_{N+n})+n$ terms instead of $\pi(c_N)$. Therefore, appealing to Lemma 2 and remark after it shows that $\mathbb{E}\langle h\rangle''_N$ is asymptotically equivalent to $\mathbb{E}\langle h\rangle_N$ and this finishes the proof. $\square$

2.2. *Lower bound.*

LEMMA 4. *There exists $\mu \in \mathcal{M}$ such that* $\lim_{N\to\infty} F_N \geq \mathcal{P}(\mu)$.

PROOF. We will obtain the lower bound using the well-known fact that

$$\lim_{N\to\infty} F_N \geq \liminf_{N\to\infty}\big((N+1)F_{N+1} - NF_N\big) = \liminf_{N\to\infty} \mathbb{E}\log\frac{Z_{N+1}}{Z_N}. \tag{2.9}$$

Suppose that this lower limit is achieved over subsequence (N_k), and let $\mu \in \mathcal{M}$ be a limit of (μ_N) over some subsubsequence of (N_k). Let $\sigma = \sigma_\mu$. The considerations will be very similar to the proof of Lemma 3. Let us consider $\mathbb{E}\log Z_{N+1}$, and let us start by separating the $\pi(\alpha(N+1))$ terms in the first sum in the Hamiltonian H_{N+1} in (1.14) into three groups: (1) terms for k such that all indices $i_{1,k},\dots,i_{p,k} \leq N$; (2) terms with exactly one of indices $i_{1,k},\dots,i_{p,k}$ equal to $N+1$; (3) terms with at least two of indices $i_{1,k},\dots,i_{p,k}$ equal to $N+1$. The probabilities that a term is of these three types are

$$p_1 = \left(\frac{N}{N+1}\right)^p, \qquad p_2 = p\frac{1}{N+1}\left(\frac{N}{N+1}\right)^{p-1}, \qquad p_3 = 1 - p_1 - p_2$$

correspondingly. Therefore, the number of terms in these three groups are independent Poisson random variables with means

$$\begin{aligned}
\alpha(N+1)p_1 &= \alpha(N-p+1) + O(N^{-1}),\\
\alpha(N+1)p_2 &= \alpha p + O(N^{-1}),\\
\alpha(N+1)p_3 &= O(N^{-1}).
\end{aligned}$$

For simplicity of notation, let us pretend that the number of terms in each group is exactly of means $\alpha(N-p), \alpha p$ and 0 since it will be clear from considerations below that asymptotically it does not affect the limit in (2.9). If we write $\boldsymbol{\sigma} = (\boldsymbol{\rho}, \varepsilon) \in \Sigma_{N+1}$ for $\boldsymbol{\rho} \in \Sigma_N$ and $\varepsilon \in \{-1,+1\}$, then we can write the first term in $H_{N+1}(\boldsymbol{\sigma})$ as

$$\sum_{k\leq\pi(\alpha(N-p+1))} \theta_k(\rho_{i_{1,k}},\dots,\rho_{i_{p,k}}) + \sum_{k\leq\pi(\alpha p)} \hat{\theta}_k(\varepsilon,\rho_{j_{1,k}},\dots,\rho_{j_{p-1,k}}), \tag{2.10}$$

where indices $i_{1,k},\dots,i_{p,k}$ and $j_{1,k},\dots,j_{p-1,k}$ are uniformly distributed on $\{1,\dots,N\}$. Similarly, we could split the $\pi(c_{N+1})$ terms in the perturbation Hamiltonian (1.13) into indices l for which all $i_{1,k,l},\dots,i_{p-1,k,l} \leq N$ and indices l for which at least one of these indices equals $N+1$. However, as in the proof of Lemma 3, since with high probability all these indices will be $\leq N$ and $|c_{N+1} - c_N| \to 0$, we can simply replace the perturbation term with

$$\sum_{l\leq\pi(c_N)} \log \mathrm{Av}_\varepsilon \exp \sum_{k\leq\pi_l(\alpha p)} \theta_{k,l}(\varepsilon,\rho_{i_{1,k,l}},\dots,\rho_{i_{p-1,k,l}}), \tag{2.11}$$

where $i_{1,k,l},\dots,i_{p-1,k,l}$ are uniformly distributed on $\{1,\dots,N\}$. Let $\langle\cdot\rangle'$ be the Gibbs average on Σ_N corresponding to the Hamiltonian

$$-H'_N(\boldsymbol{\rho}) = \sum_{k\le\pi(\alpha(N-p+1))} \theta_k(\rho_{i_{1,k}},\dots,\rho_{i_{p,k}}) + \sum_{l\le\pi(c_N)} \log \mathrm{Av}_\varepsilon \exp \sum_{k\le\pi_l(\alpha p)} \theta_{k,l}(\varepsilon,\rho_{i_{1,k,l}},\dots,\rho_{i_{p-1,k,l}})$$

and Z'_N be the corresponding partition function. Then

$$\mathbb{E}\log\frac{Z_{N+1}}{Z'_N} = \mathbb{E}\log\Big\langle \sum_{\varepsilon=\pm1}\exp\sum_{k\le\pi(\alpha p)} \hat{\theta}_k(\varepsilon,\rho_{j_{1,k}},\dots,\rho_{j_{p-1,k}})\Big\rangle'. \tag{2.12}$$

Conditionally on $\pi(\alpha p)$ and $(\hat{\theta}_k)$ and on the event that all indices $j_{1,k},\dots,j_{p-1,k}$ are different, Lemmas 1 and 2 imply that (2.12) converges to

$$\mathbb{E}\log\mathbb{E}'\sum_{\varepsilon=\pm1}\exp\sum_{k\le\pi(\alpha p)}\hat{\theta}_k(\varepsilon,s_{1,k},\dots,s_{p-1,k}).$$

For large N, indices $j_{1,k},\dots,j_{p-1,k}$ will all be different for all $k\le\pi(\alpha p)$ with high probability and, therefore, this convergence holds unconditionally. Similarly, one can analyze $\mathbb{E}\log(Z_N/Z'_N)$. Let us split the first sum in the definition of $-H_N(\boldsymbol{\rho})$ in (1.14) into two sums

$$\sum_{k\le\pi(\alpha(N-p+1))}\theta_k(\rho_{i_{1,k}},\dots,\rho_{i_{p,k}}) + \sum_{k\le\pi(\alpha(p-1))}\hat{\theta}_k(\rho_{j_{1,k}},\dots,\rho_{j_{p,k}}),$$

where indices $i_{1,k},\dots,i_{p,k}$ and $j_{1,k},\dots,j_{p,k}$ are uniformly distributed on $\{1,\dots,N\}$. Therefore,

$$\mathbb{E}\log\frac{Z_N}{Z'_N} = \mathbb{E}\log\Big\langle\exp\sum_{k\le\pi(\alpha(p-1))}\hat{\theta}_k(\rho_{j_{1,k}},\dots,\rho_{j_{p,k}})\Big\rangle'. \tag{2.13}$$

Again Lemmas 1 and 2 imply that this converges to

$$\mathbb{E}\log\mathbb{E}'\exp\sum_{k\le\pi(\alpha(p-1))}\hat{\theta}_k(s_{1,k},\dots,s_{p,k}),$$

and this finishes the proof of the lower bound. $\square$

If we knew that $\mu\in\mathcal{M}$ is the unique limit of the sequence (μ_N), this would finish the proof of the first half of Theorem 2, since $\lim_{N\to\infty}F_N = \lim_{N\to\infty}\mathbb{E}\log Z_{N+1}/Z_N$ when the limit on the right exists. However, the proof of the general case and the second half of Theorem 2 will require more work. Before we move to the upper bound, let us record one more consequence of the argument in Lemma 4. For $n\ge1$, let us define

$$\mathcal{P}_n(\mu) = \log 2 + \frac{1}{n}\mathbb{E}\log\mathbb{E}'\,\mathrm{Av}_\varepsilon\exp\sum_{i\le n}A_i(\varepsilon_i) - \frac{1}{n}\mathbb{E}\log\mathbb{E}'\exp\sum_{i\le n}B_i. \tag{2.14}$$

The following holds.

LEMMA 5. *For all $\mu \in \mathcal{M}$, $\mathcal{P}_n(\mu) = \mathcal{P}(\mu)$ for all $n \geq 1$.*

PROOF. We will only give a brief sketch since this will be proved for all $\mu \in \mathcal{M}_{\mathrm{inv}}$ in Lemma 7 below. What we showed in the proof of Lemma 4 is that if μ_N converges to μ over subsequence (N_k), then $\mathbb{E}\log Z_{N+1}/Z_N$ converges to $\mathcal{P}(\mu)$ over the same subsequence. Similarly, one can show that, given $n \geq 1$, over the same subsequence

$$\frac{1}{n}(\mathbb{E}\log Z_{N+n} - \mathbb{E}\log Z_N) \to \mathcal{P}_n(\mu).$$

The only difference is that we split the terms in the Hamiltonian $H_{N+n}(\boldsymbol{\sigma})$ into groups as in Lemma 3, that is, instead of group (2) we will have n groups each consisting of the terms with exactly one of the indices $i_{1,k}, \dots, i_{p,k}$ equal to $N+l$ for $l = 1, \dots, n$. On the other hand, if we write

$$\frac{1}{n}(\mathbb{E}\log Z_{N+n} - \mathbb{E}\log Z_N) = \frac{1}{n}\sum_{l=1}^{n}(\mathbb{E}\log Z_{N+l} - \mathbb{E}\log Z_{N+l-1}),$$

then repeating the proof of Lemma 4 one can show that for each term on the right-hand side

$$\lim_{N_k\to\infty}\mathbb{E}\log\frac{Z_{N_k+l}}{Z_{N_k+l-1}} = \mathcal{P}(\mu),$$

where instead of $\mu_{N_k} \to \mu$ one has to use that $\mu_{N_k+l-1} \to \mu$ which holds by Lemma 3. This finishes the proof. $\square$

2.3. *Upper bound and free energy.* Since the perturbation term in (1.14) does not affect the limit of free energy, we will now ignore it and consider free energy F_N defined for the original unperturbed Hamiltonian (1.8). Recall $A_i(\varepsilon)$ and B_i defined in (1.17) and (1.18).

LEMMA 6. *For any function $\sigma : [0,1]^4 \to \{-1,+1\}$ we have*

$$F_N \leq \log 2 + \frac{1}{N}\mathbb{E}\log\mathbb{E}'\operatorname{Av}_\varepsilon\exp\sum_{i\leq N}A_i(\varepsilon_i) - \frac{1}{N}\mathbb{E}\log\mathbb{E}'\exp\sum_{i\leq N}B_i. \tag{2.15}$$

REMARK. In general, this upper bound does not decouple and depends on N since all $s_{i,k,l}$ and $\hat{s}_{i,k,l}$ defined in (1.15) and (1.16) depend on the same variable u in the second coordinate. We will see that the proof of the upper bound (2.15) does not to work if one tries to replace u by independent copies u_i in the definition of $A_i(\varepsilon)$ and B_i. For $\sigma = \sigma_\mu$ for $\mu \in \mathcal{M}$, Lemma 5 implies that this upper bound does not depend on N and, thus, $F_N \le \mathcal{P}(\mu)$. Together with the lower bound of Lemma 4 this proves that

$$\lim_{N\to\infty} F_N = \inf_{\mu\in\mathcal{M}} \mathcal{P}(\mu).$$

To prove the second part of Theorem 2, we will show in Lemma 7 below that the invariance properties in (1.22) imply that $\mathcal{P}_n(\mu) = \mathcal{P}(\mu)$ for $\mu \in \mathcal{M}_{\mathrm{inv}}$ as well which will finish the proof of Theorem 2.

PROOF OF LEMMA 6. A proof by interpolation is a slight modification of the proof in [22]. For $t \in [0,1]$, let us define similarly to (1.17) and (1.18)

$$A_i^t(\varepsilon) = \sum_{k\le \pi_i((1-t)p\alpha)} \theta_{k,i}(\varepsilon, s_{i,k,1}, \ldots, s_{i,k,p-1}) \tag{2.16}$$

and

$$B_i^t = \sum_{k\le \pi_i(t(p-1)\alpha)} \hat{\theta}_{k,i}(\hat{s}_{i,k,1}, \ldots, \hat{s}_{i,k,p}). \tag{2.17}$$

Consider an interpolating Hamiltonian

$$-H_{N,t}(\boldsymbol{\sigma}) = \sum_{k\le \pi(t\alpha N)} \theta_k(\sigma_{i_{1,k}}, \ldots, \sigma_{i_{p,k}}) + \sum_{i\le N} A_i^t(\sigma_i) + \sum_{i\le N} B_i^t \tag{2.18}$$

and let

$$\varphi(t) = \frac{1}{N}\mathbb{E}\log\mathbb{E}' \sum_{\boldsymbol{\sigma}\in\Sigma_N} \exp(-H_{N,t}(\boldsymbol{\sigma})).$$

Since, clearly,

$$\varphi(1) = F_N + \frac{1}{N}\mathbb{E}\log\mathbb{E}'\exp\sum_{i\le N} B_i$$

and

$$\varphi(0) = \log 2 + \frac{1}{N}\mathbb{E}\log\mathbb{E}'\,\mathrm{Av}_\varepsilon \exp\sum_{i\le N} A_i(\varepsilon_i),$$

it remains to prove that $\varphi'(t) \le 0$. Let us consider the partition function

$$Z = \sum_{\boldsymbol{\sigma}\in\Sigma_N} \exp(-H_{N,t}(\boldsymbol{\sigma}))$$

and define

$$Z_m = Z|_{\pi(t\alpha N)=m}, \qquad Z_{i,m}^A = Z|_{\pi_i((1-t)p\alpha)=m} \quad \text{and} \quad Z_{i,m}^B = Z|_{\pi_i(t(p-1)\alpha)=m}.$$

If we denote the Poisson p.f. as $\pi(\lambda,k) = (\lambda^k/k!)e^{-\lambda}$, then

$$\mathbb{E}\log\mathbb{E}'Z = \sum_{m\ge 0} \pi(t\alpha N, m)\mathbb{E}\log\mathbb{E}' Z_m$$

and, for any $i \le N$,

$$\mathbb{E}\log\mathbb{E}'Z = \sum_{m\ge 0}\pi\big((1-t)p\alpha, m\big)\mathbb{E}\log\mathbb{E}'Z^A_{i,m}$$

and

$$\mathbb{E}\log\mathbb{E}'Z = \sum_{m\ge 0}\pi\big(t(p-1)\alpha, m\big)\mathbb{E}\log\mathbb{E}'Z^B_{i,m}.$$

Therefore, we can write

$$\begin{aligned}
\varphi'(t) &= \sum_{m\ge 0}\frac{\partial\pi(t\alpha N, m)}{\partial t}\frac{1}{N}\mathbb{E}\log\mathbb{E}'Z_m \\
&\quad + \sum_{i\le N}\sum_{m\ge 0}\frac{\partial\pi((1-t)p\alpha, m)}{\partial t}\frac{1}{N}\mathbb{E}\log\mathbb{E}'Z^A_{i,m} \\
&\quad + \sum_{i\le N}\sum_{m\ge 0}\frac{\partial\pi(t(p-1)\alpha, m)}{\partial t}\frac{1}{N}\mathbb{E}\log\mathbb{E}'Z^B_{i,m} \\
&= \alpha\sum_{m\ge 0}\big(\pi(t\alpha N, m-1)I(m\ge 1) - \pi(t\alpha N, m)\big)\mathbb{E}\log\mathbb{E}'Z_m \\
&\quad - p\alpha\frac{1}{N}\sum_{i\le N}\sum_{m\ge 0}\big(\pi\big((1-t)p\alpha, m-1\big)I(m\ge 1) \\
&\qquad\qquad - \pi\big((1-t)p\alpha, m\big)\big)\mathbb{E}\log\mathbb{E}'Z^A_{i,m} \\
&\quad + (p-1)\alpha\frac{1}{N}\sum_{i\le N}\sum_{m\ge 0}\big(\pi\big(t(p-1)\alpha, m-1\big)I(m\ge 1) \\
&\qquad\qquad - \pi\big(t(p-1)\alpha, m\big)\big)\mathbb{E}\log\mathbb{E}'Z^B_{i,m} \\
&= \alpha\sum_{m\ge 0}\pi(t\alpha N, m)\mathbb{E}\log(\mathbb{E}'Z_{m+1}/\mathbb{E}'Z_m) \\
&\quad - p\alpha\frac{1}{N}\sum_{i\le N}\sum_{m\ge 0}\pi\big((1-t)p\alpha, m\big)\mathbb{E}\log(\mathbb{E}'Z^A_{i,m+1}/\mathbb{E}'Z^A_{i,m}) \\
&\quad + (p-1)\alpha\frac{1}{N}\sum_{i\le N}\sum_{m\ge 0}\pi\big(t(p-1)\alpha, m\big)\mathbb{E}\log(\mathbb{E}'Z^B_{i,m+1}/\mathbb{E}'Z^B_{i,m}) \\
&= \alpha\mathbb{E}\log\frac{\mathbb{E}'Z_{+1}}{\mathbb{E}'Z} - p\alpha\frac{1}{N}\sum_{i\le N}\mathbb{E}\log\frac{\mathbb{E}'Z^A_{i,+1}}{\mathbb{E}'Z} + (p-1)\alpha\mathbb{E}\log\frac{\mathbb{E}'Z^B_{+1}}{\mathbb{E}'Z},
\end{aligned} \tag{2.19}$$

where Z_{+1}, $Z^A_{i,+1}$ and Z^B_{+1} contain one extra term in the Hamiltonian in the corresponding Poisson sum. Namely,

$$Z_{+1} = \sum_{\boldsymbol{\sigma}\in\Sigma_N} \exp\theta(\sigma_{i_1},\dots,\sigma_{i_p})\exp(-H_{N,t}(\boldsymbol{\sigma})),$$
$$Z^A_{i,+1} = \sum_{\boldsymbol{\sigma}\in\Sigma_N} \exp\theta(\sigma_i, s_1,\dots,s_{p-1})\exp(-H_{N,t}(\boldsymbol{\sigma})),$$
$$Z^B_{+1} = \sum_{\boldsymbol{\sigma}\in\Sigma_N} \exp\theta(s_1,\dots,s_p)\exp(-H_{N,t}(\boldsymbol{\sigma})),$$

where random function θ and indices $i_1,\dots,i_p$ uniform on $\{1,\dots,N\}$ are independent of the randomness of the Hamiltonian $H_{N,t}$. If, for a function f of $\boldsymbol{\sigma}$, u and (x), we denote by $\langle f\rangle_t$ the Gibbs average

$$\langle f\rangle_t = \frac{1}{\mathbb{E}'Z}\mathbb{E}'\sum_{\boldsymbol{\sigma}\in\Sigma_N} f\exp(-H_{N,t}(\boldsymbol{\sigma})),$$

then (2.19) can be rewritten as

$$\begin{aligned}
&\alpha\mathbb{E}\log\langle\exp\theta(\sigma_{i_1},\dots,\sigma_{i_p})\rangle_t\\
&\quad - p\alpha\frac{1}{N}\sum_{i\le N}\mathbb{E}\log\langle\exp\theta(\sigma_i,s_1,\dots,s_{p-1})\rangle_t\\
&\quad + (p-1)\alpha\mathbb{E}\log\langle\exp\theta(s_1,\dots,s_p)\rangle_t.
\end{aligned}\tag{2.20}$$

By assumptions (1.9) and (1.11) we can write

$$\begin{aligned}
\log\langle\exp\theta(\sigma_{i_1},\dots,\sigma_{i_p})\rangle_t &= \log a + \log\bigl(1 + b\langle f_1(\sigma_{i_1})\cdots f_p(\sigma_{i_p})\rangle_t\bigr)\\
&= \log a - \sum_{n\ge 1}\frac{(-b)^n}{n}\langle f_1(\sigma_{i_1})\cdots f_p(\sigma_{i_p})\rangle_t^n.
\end{aligned}$$

Using replicas $\boldsymbol{\sigma}^l$, u_l and (x^l), we can write

$$\langle f_1(\sigma_{i_1})\cdots f_p(\sigma_{i_p})\rangle_t^n = \biggl\langle\prod_{l\le n} f_1(\sigma^l_{i_1})\cdots f_p(\sigma^l_{i_p})\biggr\rangle_t$$

and thus

$$\frac{1}{N^p}\sum_{i_1,\dots,i_p\le N}\langle f_1(\sigma_{i_1})\cdots f_p(\sigma_{i_p})\rangle_t^n = \biggl\langle\prod_{j\le p} A_{j,n}\biggr\rangle_t,$$

where

$$A_{j,n} = A_{j,n}(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n) = \frac{1}{N}\sum_{i\le N}\prod_{l\le n} f_j(\sigma^l_i).$$

Denote by $\mathbb{E}_0$ the expectation in $f_1, \dots, f_p$. Since $f_1, \dots, f_p$ are i.i.d. and independent of the randomness in $\langle\cdot\rangle_t$,

$$\mathbb{E}_0\Big\langle \prod_{j\le p} A_{j,n}\Big\rangle_t = \Big\langle \mathbb{E}_0 \prod_{j\le p} A_{j,n}\Big\rangle_t = \langle B_n^p\rangle_t,$$

where $B_n = \mathbb{E}_0 A_{j,n}$. Therefore, since we also assumed that b is independent of $f_1, \dots, f_p$,

$$\mathbb{E}_0 \frac{1}{N^p} \sum_{i_1,\dots,i_p\le N} \log\langle \exp\theta(\sigma_{i_1},\dots,\sigma_{i_p})\rangle_t = \mathbb{E}_0 \log a - \sum_{n\ge 1} \frac{(-b)^n}{n}\langle B_n^p\rangle_t. \tag{2.21}$$

A similar analysis applies to the second term in (2.20),

$$\begin{aligned}
&\log\langle \exp\theta(\sigma_i, s_1, \dots, s_{p-1})\rangle_t\\
&\quad= \log a - \sum_{n\ge 1} \frac{(-b)^n}{n}\Big\langle f_p(\sigma_i)\prod_{j\le p-1} f_j(s_j)\Big\rangle_t^n\\
&\quad= \log a - \sum_{n\ge 1} \frac{(-b)^n}{n}\Big\langle \prod_{l\le n} f_p(\sigma_i^l)\prod_{l\le n}\prod_{j\le p-1} f_j(s_j^l)\Big\rangle_t,
\end{aligned}$$

where in the last equality we again used replicas $\boldsymbol{\sigma}^l, u_l$ and (x^l); for example, compared to (1.15), s_j^l is now defined by $s_j^l = \sigma(w, u_l, v_j, x_j^l)$. Thus,

$$\frac{1}{N}\sum_{i\le N} \log\langle \exp\theta(\sigma_i, s_1, \dots, s_{p-1})\rangle_t = \log a - \sum_{n\ge 1}\frac{(-b)^n}{n}\Big\langle A_{p,n}\prod_{j\le p-1}\prod_{l\le n} f_j(s_j^l)\Big\rangle_t.$$

[*Note*: It was crucial here that s_j^l do not depend on i through independent copies u_i rather than the same u. It is tempting to define the interpolation (2.18) by using independent u_i for $i \le N$ since this would make the upper bound in (2.15) decouple, but the proof would break down at this step.] In addition to $f_1, \dots, f_p$, let $\mathbb{E}_0$ also denote the expectation in (v_j) and (x_j^l) in s_j^l, but not in sequences (v), (x) in the randomness of $\langle\cdot\rangle_t$. Then,

$$\begin{aligned}
&\mathbb{E}_0 \frac{1}{N}\sum_{i\le N} \log\langle \exp\theta(\sigma_i, s_1, \dots, s_{p-1})\rangle_t\\
&\qquad= \mathbb{E}_0 \log a - \sum_{n\ge 1}\frac{(-b)^n}{n}\langle B_n (C_n)^{p-1}\rangle_t,
\end{aligned} \tag{2.22}$$

where

$$C_n = C_n(w, u_1, \dots, u_n) = \mathbb{E}_0 \prod_{l\le n} f_j(s_j^l) = \mathbb{E}_0\prod_{l\le n} f_j(\sigma(w, u_l, v_j, x_j^l))$$

obviously does not depend on j. Finally, in an absolutely similar manner

$$\mathbb{E}_0 \log\langle \exp\theta(s_1,\dots,s_p)\rangle_t = \mathbb{E}_0 \log a - \sum_{n\geq 1} \frac{(-b)^n}{n} \langle (C_n)^p\rangle_t. \tag{2.23}$$

Combining (2.21), (2.22) and (2.23) we see that (2.20) can be written as

$$-\alpha \sum_{n\geq 1} \frac{\mathbb{E}(-b)^n}{n} \mathbb{E}\langle B_n^p - pB_nC_n^{p-1} + (p-1)(C_n)^p\rangle_t \leq 0, \tag{2.24}$$

which holds true using condition (1.10) and the fact that $x^p - pxy^{p-1} + (p-1)y^p \geq 0$ for all $x, y \in \mathbb{R}$ for even $p \geq 2$. This finishes the proof of the upper bound. □

Before proving the invariance properties of Theorem 1 let us finish the proof of Theorem 2 by showing that for invariant measures $\mathcal{M}_{\mathrm{inv}}$ the upper bound decouples.

LEMMA 7. *For all $\mu \in \mathcal{M}_{\mathrm{inv}}$, $\mathcal{P}_n(\mu) = \mathcal{P}(\mu)$ for all $n \geq 1$.*

PROOF. If we recall A_i defined in (1.26), then we can rewrite (2.14) as

$$\mathcal{P}_n(\mu) = \log 2 + \frac{1}{n}\mathbb{E}\log\mathbb{E}'\exp\sum_{i\leq n} A_i - \frac{1}{n}\mathbb{E}\log\mathbb{E}'\exp\sum_{i\leq n} B_i. \tag{2.25}$$

The result will follow if we show that for any $n \geq 1$,

$$\mathbb{E}\log\frac{\mathbb{E}'\exp\sum_{i\leq n+1} A_i}{\mathbb{E}'\exp\sum_{i\leq n} A_i} = \mathbb{E}\log\mathbb{E}'\exp A_{n+1} \tag{2.26}$$

and

$$\mathbb{E}\log\frac{\mathbb{E}'\exp\sum_{i\leq n+1} B_i}{\mathbb{E}'\exp\sum_{i\leq n} B_i} = \mathbb{E}\log\mathbb{E}'\exp B_{n+1}. \tag{2.27}$$

To prove this we will use the invariance properties (1.27) and (1.28). If in (1.28) we choose r to be a Poisson r.v. with mean $n(p-1)\alpha$, then it becomes

$$\mathbb{E}\prod_{l\leq q}\mathbb{E}'\prod_{i\in C_l} s_i = \mathbb{E}\frac{\prod_{l\leq q}\mathbb{E}'\prod_{i\in C_l} s_i \exp\sum_{i\leq n} B_i}{(\mathbb{E}'\exp\sum_{i\leq n} B_i)^q}. \tag{2.28}$$

We will only show how (1.27) implies (2.26) since the proof that (2.28) implies (2.27) is exactly the same. We only need to prove (2.26) conditionally on the Poisson r.v. $\pi_{n+1}(p\alpha)$ and functions $(\theta_{k,n+1})$ in the definition of A_{n+1},

$$\exp A_{n+1} = \mathrm{Av}_\varepsilon \exp \sum_{k\leq \pi_{n+1}(p\alpha)} \theta_k(\varepsilon, s_{1,n+1,k}, \dots, s_{p-1,n+1,k}), \tag{2.29}$$

since we can control these functions uniformly with high probability using condition (1.12). Approximating the logarithm by polynomials, in order to prove (2.26), it is enough to prove that

$$\mathbb{E}\biggl(\frac{\mathbb{E}'\exp A_{n+1}\exp\sum_{i\le n}A_i}{\mathbb{E}'\exp\sum_{i\le n}A_i}\biggr)^q = \mathbb{E}(\mathbb{E}'\exp A_{n+1})^q \tag{2.30}$$

for all $q \ge 1$. Condition (1.9) implies that the right-hand side of (2.29) is a polynomial of spins $(s_{j,n+1,k})$ for $k \le \pi_{n+1}(p\alpha)$ and $j \le p-1$, and, therefore, (2.30) is obviously implied by (1.27) if we simply enumerate spins $(s_{j,n+1,k})$ as spins (s_i) for $n+1 \le i \le m$ by choosing m large enough. Averaging over random $\pi_{n+1}(p\alpha)$ and $(\theta_{k,n+1})$ proves (2.30) and finishes the proof. □

Let us note that, similarly, (1.28) implies

$$\mathbb{E}\log\frac{\mathbb{E}'\exp\sum_{i\le n+1}\hat{\theta}_i(\hat{s}_{1,i},\dots,\hat{s}_{p,i})}{\mathbb{E}'\exp\sum_{i\le n}\hat{\theta}_i(\hat{s}_{1,i},\dots,\hat{s}_{p,i})} = \mathbb{E}\log\mathbb{E}'\exp\hat{\theta}_1(\hat{s}_{1,1},\dots,\hat{s}_{p,1}),$$

which obviously implies (1.25), that is,

$$\begin{aligned}\mathbb{E}\log\mathbb{E}'\exp B &= \mathbb{E}\log\mathbb{E}'\exp\sum_{k\le\pi((p-1)\alpha)}\theta_k(s_{1,k},\dots,s_{p,k})\\ &= (p-1)\alpha\mathbb{E}\log\mathbb{E}'\exp\theta(s_1,\dots,s_p).\end{aligned}$$

2.4. *Invariance and self-consistency equations.*

PROOF OF THEOREM 1. Let $h = \prod_{l\le q}h_l$ where $h_l = \prod_{j\in C_l}\sigma_j^l$. Consider $\mu \in \mathcal{M}$ which is a limit of μ_N over some subsequence (N_k). Using Lemma 3, the left-hand side of (1.22) is the limit of $\mathbb{E}\langle h\rangle_{N+n}$ over subsequence (N_k). The right-hand side of (1.22) will appear as a similar limit once we rewrite this joint moment of spins using cavity coordinates and "borrowing" some terms in the Gibbs measure from the Hamiltonian (1.14). The spins with coordinates $i \le n$ will play the role of cavity coordinates. Let us separate the $\pi(\alpha(N+n))$ terms in the first sum

$$\sum_{k\le\pi(\alpha(N+n))}\theta_k(\sigma_{i_{1,k}},\dots,\sigma_{i_{p,k}}) \tag{2.31}$$

in (1.14) in the Hamiltonian H_{N+n} into three groups:

(1) terms for k such that all indices $i_{1,k},\dots,i_{p,k} > n$;

For $1 \le j \le n$:

$(2j)$ terms with exactly one of indices $i_{1,k},\dots,i_{p,k}$ equal to j and all others $> n$;

(3) terms with at least two of indices $i_{1,k},\dots,i_{p,k} \le n$.

The probabilities that a term is of these three type are

$$p_1 = \left(\frac{N}{N+n}\right)^p, \qquad p_{2,j} = p\frac{1}{N+n}\left(\frac{N}{N+n}\right)^{p-1}, \qquad p_3 = 1 - p_1 - \sum_{l\le n} p_{2,l}.$$

Therefore, the number of terms in these groups are independent Poisson random variables with means

$$\begin{aligned}\alpha(N+n)p_1 &= \alpha(N+n-np) + O(N^{-1}),\\ \alpha(N+n)p_{2,j} &= \alpha p + O(N^{-1}),\\ \alpha(N+n)p_3 &= O(N^{-1}).\end{aligned}$$

We can redefine the number of terms in each group to be exactly of means $\alpha(N+n-np), \alpha p$ and 0 since asymptotically it does not affect $\mathbb{E}\langle h\rangle_{N+n}$. Thus, if we write $\boldsymbol{\sigma} = (\boldsymbol{\varepsilon}, \boldsymbol{\rho}) \in \Sigma_{N+n}$ for the first the first n cavity coordinates $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ and the last N coordinates $\boldsymbol{\rho} = (\rho_1, \dots, \rho_N)$, then (2.31) can be replaced with

$$\begin{aligned}&\sum_{k\le\pi(\alpha(N+n-np))} \theta_k(\rho_{i_{1,k}}, \dots, \rho_{i_{p,k}})\\ &\quad + \sum_{j\le n}\sum_{k\le\pi_j(\alpha p)} \theta_{k,j}(\varepsilon_j, \rho_{i_{1,k,j}}, \dots, \rho_{i_{p-1,k,j}}),\end{aligned} \tag{2.32}$$

where indices $i_{1,k}, \dots, i_{p,k}$ and $i_{1,k,j}, \dots, i_{p-1,k,j}$ are all uniformly distributed on $\{1, \dots, N\}$. Let us now consider the perturbation term in (1.14),

$$\sum_{l\le\pi(c_{N+n})} \log \mathrm{Av}_\varepsilon \exp \sum_{k\le\hat{\pi}_l(\alpha p)} \hat{\theta}_{k,l}(\varepsilon, \sigma_{j_{1,k,l}}, \dots, \sigma_{j_{p-1,k,l}}), \tag{2.33}$$

where $j_{1,k,l}, \dots, j_{p-1,k,l}$ are uniformly distributed on $\{1, \dots, N+n\}$. Here, we used independent copies $\hat{\pi}_l$ and $\hat{\theta}_{k,l}$ since π_j and $\theta_{k,j}$ were already used in (2.32). The expected number of these indices that belong to $\{1, \dots, n\}$ is $c_{N+n}\alpha p(p-1)n/N \to 0$ which means that with high probability all indices belong to $\{n+1, \dots, N+n\}$. As a result, asymptotically $\mathbb{E}\langle h\rangle_{N+n}$ will not be affected if we replace the perturbation term (2.33) with

$$\sum_{l\le\pi(c_{N+n})} \log \mathrm{Av}_\varepsilon \exp \sum_{k\le\hat{\pi}_l(\alpha p)} \hat{\theta}_{k,l}(\varepsilon, \rho_{j_{1,k,l}}, \dots, \rho_{j_{p-1,k,l}}), \tag{2.34}$$

where $j_{1,k,l}, \dots, j_{p-1,k,l}$ are uniformly distributed on $\{1, \dots, N\}$. Thus, we can assume from now on that $\mathbb{E}\langle h\rangle_{N+n}$ is computed with respect to the Hamiltonian which is the sum of (2.32) and (2.34). If $\langle\cdot\rangle'_N$ denotes the Gibbs average on Σ_N with respect to the Hamiltonian

$$\begin{aligned}-H'_N(\boldsymbol{\rho}) = &\sum_{k\le\pi(\alpha(N+n-np))} \theta_k(\rho_{i_{1,k}}, \dots, \rho_{i_{p,k}})\\ &+ \sum_{l\le\pi(c_{N+n})} \log \mathrm{Av}_\varepsilon \exp \sum_{k\le\hat{\pi}_l(\alpha p)} \hat{\theta}_{k,l}(\varepsilon, \rho_{j_{1,k,l}}, \dots, \rho_{j_{p-1,k,l}}),\end{aligned} \tag{2.35}$$

then we can write

$$\mathbb{E}\langle h\rangle_{N+n} = \mathbb{E}\frac{\prod_{l\le q} U_{N,l}}{V_N^q}, \tag{2.36}$$

where

$$U_{N,l} = \Bigl\langle \mathrm{Av}_\varepsilon\, h_l(\boldsymbol{\varepsilon},\boldsymbol{\rho}) \exp \sum_{j\le n}\sum_{k\le \pi_j(\alpha p)} \theta_{k,j}(\varepsilon_j, \rho_{i_{1,k,j}},\dots,\rho_{i_{p-1,k,j}})\Bigr\rangle_N'$$

and

$$V_N = \Bigl\langle \mathrm{Av}_\varepsilon \exp \sum_{j\le n}\sum_{k\le \pi_j(\alpha p)} \theta_{k,j}(\varepsilon_j, \rho_{i_{1,k,j}},\dots,\rho_{i_{p-1,k,j}})\Bigr\rangle_N'.$$

Finally, given $r \ge 1$, let us borrow r terms from the first sum in (2.35) by splitting the last r terms and replacing the first sum in (2.35) with

$$\sum_{k\le \pi(\alpha(N+n-np))-r} \theta_k(\rho_{i_{1,k}},\dots,\rho_{i_{p,k}}) + \sum_{k\le r}\hat\theta_k(\rho_{j_{1,k}},\dots,\rho_{j_{p,k}}).$$

Here we ignore the negligible event when $\pi(\alpha(N+n-np)) < r$. If we define

$$\begin{aligned} -H_N''(\boldsymbol{\rho}) = {} & \sum_{k\le \pi(\alpha(N+n-np))-r} \theta_k(\rho_{i_{1,k}},\dots,\rho_{i_{p,k}}) \\ & + \sum_{l\le \pi(c_{N+n})} \log \mathrm{Av}_\varepsilon \exp \sum_{k\le \hat\pi_l(\alpha p)} \hat\theta_{k,l}(\varepsilon, \rho_{j_{1,k,l}},\dots,\rho_{j_{p-1,k,l}}) \end{aligned} \tag{2.37}$$

and let $\langle\cdot\rangle_N''$ denote the Gibbs average on Σ_N with respect to this Hamiltonian then $U_{N,l}/V_N = U_{N,l}'/V_N'$ where

$$\begin{aligned} U_{N,l}' = \Bigl\langle & \mathrm{Av}_\varepsilon\, h_l(\boldsymbol{\varepsilon},\boldsymbol{\rho}) \exp \sum_{j\le n}\sum_{k\le \pi_j(\alpha p)} \theta_{k,j}(\varepsilon_j, \rho_{i_{1,k,j}},\dots,\rho_{i_{p-1,k,j}}) \\ & \times \exp\sum_{k\le r}\hat\theta_k(\rho_{j_{1,k}},\dots,\rho_{j_{p,k}})\Bigr\rangle_N'' \end{aligned}$$

and

$$\begin{aligned} V_N'' = \Bigl\langle & \mathrm{Av}_\varepsilon \exp \sum_{j\le n}\sum_{k\le \pi_j(\alpha p)} \theta_{k,j}(\varepsilon_j, \rho_{i_{1,k,j}},\dots,\rho_{i_{p-1,k,j}}) \\ & \times \exp\sum_{k\le r}\hat\theta_k(\rho_{j_{1,k}},\dots,\rho_{j_{p,k}})\Bigr\rangle_N''. \end{aligned}$$

By Lemma 2, the distribution of spins under the annealed Gibbs measure $\mathbb{E}\langle\cdot\rangle_N''$ corresponding to the Hamiltonian $H_N''(\boldsymbol{\rho})$ still converges to μ over the subsequence (N_k). Conditionally on $(\pi_j(\alpha p))$, $(\theta_{k,j})$, $(\hat\theta_k)$ and on the event that all indices $i_{1,k,j}, \ldots, i_{p-1,k,j}$ and $j_{1,k}, \ldots, j_{p,k}$ are different, Lemma 1 implies that the right-hand side of (2.36) converges over subsequence (N_k) to $\mathbb{E}\prod_{l\le q} U_l/V^q$ where (U_l) and V are defined in (1.20) and (1.21) only now conditionally on the above sequences. Since asymptotically all indices are different with high probability, the same convergence holds unconditionally, and this completes the proof. $\square$

3. Sherrington–Kirkpatrick model.

3.1. *Properties of convergence.* Of course, Lemma 1 still holds since it does not really depend on the model. However, the role of this lemma in the Sherrington–Kirkpatrick model will be played by the statement that we made at the beginning of the introduction which we now record for the reference.

LEMMA 8. *The joint distribution of spins* (σ_i^l) *and multi-overlaps* (1.5) *converges to the joint distribution of spins* (1.3) *and multi-overlaps* (1.6) *over any subsequence along which* μ_N *converges to* μ.

Lemma 2 also has a straightforward analog for the Sherrington–Kirkpatrick model. Let $\langle\cdot\rangle$ denote the Gibbs average with respect to the sum of an arbitrary Hamiltonian on Σ_N and a perturbation term (1.34), and let $\langle\cdot\rangle'$ denote the Gibbs average corresponding to the sum of the same arbitrary Hamiltonian and a perturbation as in (1.34), only with the number of terms replaced by $\pi(c_N)+n$ instead of $\pi(c_N)$ in the first sum and $\pi'(c_N)+m$ instead of $\pi'(c_N)$ in the second sum, for any finite $m, n \ge 1$. Then the following holds.

LEMMA 9. *For any bounded function h of finitely many spins, or finitely many multi-overlaps, we have*

$$\lim_{N\to\infty} |\mathbb{E}\langle h\rangle' - \mathbb{E}\langle h\rangle| = 0. \tag{3.1}$$

The proof is exactly the same as in Lemma 2. The role of the perturbation (1.34) will finally start becoming clear in the following exact analog of Lemma 3.

LEMMA 10. *If* μ_N *converges to* μ *over subsequence* (N_k), *then it also converges to* μ *over subsequence* (N_k+n) *for any* $n\ge 1$.

PROOF. We will show that the joint moments of spins converge to the same limit over subsequences that differ by a finite shift n. Let $h=\prod_{j\le q} h_j$ where $h_j=\prod_{i\in C_j}\sigma_i^j$ over some finite sets of spin coordinates C_j. Let us denote by $\langle\cdot\rangle_N$ the Gibbs average with respect to the Hamiltonian (1.14) defined on N coordinates. We will show that

$$\lim_{N\to\infty} |\mathbb{E}\langle h\rangle_{N+n} - \mathbb{E}\langle h\rangle_N| = 0.$$

Let us rewrite $\mathbb{E}\langle h\rangle_{N+n}$ by treating the last n coordinates as cavity coordinates. Let us write $\boldsymbol{\sigma} = (\boldsymbol{\rho}, \boldsymbol{\varepsilon}) \in \Sigma_{N+n}$ for the first N coordinates $\boldsymbol{\rho} = (\rho_1, \dots, \rho_N)$ and the last n cavity coordinates $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ and rewrite (1.30) as

$$-H_{N+n}(\boldsymbol{\rho}) + \sum_{i\le n} \varepsilon_i Z_i(\boldsymbol{\rho}) + \delta(\boldsymbol{\sigma}), \tag{3.2}$$

where we define (slightly abusing notations)

$$-H_{N+n}(\boldsymbol{\rho}) := \sum_{p\ge 1} \frac{\beta_p}{(N+n)^{(p-1)/2}} \sum_{1\le i_1,\dots,i_p\le N} g_{i_1,\dots,i_p} \rho_{i_1}\cdots\rho_{i_p}; \tag{3.3}$$

the term $\varepsilon_i Z_i(\boldsymbol{\rho})$ consists of all terms in (1.30) with only one factor ε_i from $\boldsymbol{\varepsilon}$ present, and the last term δ is the sum of terms with at least two factors in $\boldsymbol{\varepsilon}$. It is easy to check that

$$\mathbb{E} Z_i(\boldsymbol{\rho}^1) Z_i(\boldsymbol{\rho}^2) = \xi'(R(\boldsymbol{\rho}^1, \boldsymbol{\rho}^2)) + o_N(1)$$

uniformly over all $\boldsymbol{\rho}^1, \boldsymbol{\rho}^2$, and the covariance of $\delta(\boldsymbol{\sigma})$ is also of small order uniformly over $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2$. By the usual Gaussian interpolation one can therefore redefine the Hamiltonian $H_{N+n}(\boldsymbol{\sigma})$ by

$$-H_{N+n}(\boldsymbol{\sigma}) = -H_{N+n}(\boldsymbol{\rho}) + \sum_{i\le n} \varepsilon_i Z_i(\boldsymbol{\rho}), \tag{3.4}$$

where Gaussian processes $Z_i(\boldsymbol{\rho})$ have covariance $\xi'(R(\boldsymbol{\rho}^1, \boldsymbol{\rho}^2))$. We can replace the perturbation term $-H^p_{N+n}(\boldsymbol{\sigma})$ by

$$-H^p_N(\boldsymbol{\rho}) = \sum_{k\le \pi(c_N)} \log\operatorname{ch} G_{\xi',k}(\boldsymbol{\rho}) + \sum_{k\le \pi'(c_N)} G_{\theta,k}(\boldsymbol{\rho}) \tag{3.5}$$

without affecting $\mathbb{E}\langle h\rangle_{N+n}$ asymptotically, since by Lemma 9 we can slightly modify the Poisson number of terms using that $|c_{N+n} - c_N| \to 0$ and then replace $G_{\xi',i}(\boldsymbol{\sigma})$ and $G_{\theta,i}(\boldsymbol{\sigma})$ by $G_{\xi',i}(\boldsymbol{\rho})$ and $G_{\theta,i}(\boldsymbol{\rho})$ by interpolation using that $c_N = o(N)$. If $\langle\cdot\rangle'_N$ denotes the Gibbs average on Σ_N with respect to the Hamiltonian

$$-H'_N(\boldsymbol{\rho}) = -H_{N+n}(\boldsymbol{\rho}) - H^p_N(\boldsymbol{\rho}), \tag{3.6}$$

then each factor in

$$\langle h\rangle_{N+n} = \prod_{j\le q} \langle h_j\rangle_{N+n} = \prod_{j\le q} \Big\langle \prod_{i\in C_j} \sigma_i \Big\rangle_{N+n} = \prod_{j\le q} \Big\langle \prod_{i\in C_j} \rho_i \Big\rangle_{N+n}$$

(in the last equality we used that for large N all sets C_j will be on the first N coordinates) can be written as

$$\langle h_j\rangle_{N+n} = \frac{\langle \prod_{i\in C_j} \rho_i \operatorname{Av}_\varepsilon \exp \sum_{i\le n} \varepsilon_i Z_i(\boldsymbol{\rho})\rangle'_N}{\langle \operatorname{Av}_\varepsilon \exp \sum_{i\le n} \varepsilon_i Z_i(\boldsymbol{\rho})\rangle'_N} = \Big\langle \prod_{i\in C_j} \rho_i \Big\rangle''_N,$$

where $\langle\cdot\rangle_N''$ is the Gibbs average on Σ_N corresponding to the Hamiltonian

$$-H_N''(\boldsymbol{\rho}) = -H_N'(\boldsymbol{\rho}) + \sum_{i\le n} \log \operatorname{ch} Z_i(\boldsymbol{\rho}).$$

Thus, $\mathbb{E}\langle h\rangle_{N+n} = \mathbb{E}\langle h\rangle_N''$. Since $Z_i(\boldsymbol{\rho})$ are independent copies of $G_{\xi'}(\boldsymbol{\rho})$, in distribution

$$-H_N''(\boldsymbol{\rho}) = -H_{N+n}(\boldsymbol{\rho}) - H_N^{p,1}(\boldsymbol{\rho}),$$

where

$$-H_N^{p,1}(\boldsymbol{\rho}) = \sum_{k\le\pi(c_N)+n} \log \operatorname{ch} G_{\xi',k}(\boldsymbol{\rho}) + \sum_{k\le\pi'(c_N)} G_{\theta,k}(\boldsymbol{\rho}). \tag{3.7}$$

Let us now consider $\mathbb{E}\langle h\rangle_N$. It is easy to check that, in distribution, the Hamiltonian $H_N(\boldsymbol{\rho})$ can be related to the Hamiltonian $H_{N+n}(\boldsymbol{\rho})$ in (3.3) by

$$-H_N(\boldsymbol{\rho}) = -H_{N+n}(\boldsymbol{\rho}) + \sum_{i\le n} Y_i(\boldsymbol{\rho}), \tag{3.8}$$

where $(Y_i(\boldsymbol{\rho}))$ are independent Gaussian processes with covariance

$$\mathbb{E}Y_i(\boldsymbol{\rho}^1)Y_i(\boldsymbol{\rho}^2) = \theta(R(\boldsymbol{\rho}^1,\boldsymbol{\rho}^2)) + o_N(1).$$

Again, without affecting $\mathbb{E}\langle h\rangle_N$ asymptotically, one can assume that the covariance of $Y_i(\boldsymbol{\rho})$ is exactly $\theta(R(\boldsymbol{\rho}^1,\boldsymbol{\rho}^2))$ which means that they are independent copies of $G_\theta(\boldsymbol{\rho})$. Therefore, we can assume that $\mathbb{E}\langle h\rangle_N$ is taken with respect to the Hamiltonian

$$-H_N'''(\boldsymbol{\rho}) = -H_{N+n}(\boldsymbol{\rho}) - H_N^{p,2}(\boldsymbol{\rho}),$$

where

$$-H_N^{p,2}(\boldsymbol{\rho}) = \sum_{k\le\pi(c_N)} \log \operatorname{ch} G_{\xi',k}(\boldsymbol{\rho}) + \sum_{k\le\pi'(c_N)+n} G_{\theta,k}(\boldsymbol{\rho}). \tag{3.9}$$

Lemma 9 then implies that both perturbation terms (3.7) and (3.9) can be replaced by the original perturbation term (1.34) without affecting $\mathbb{E}\langle h\rangle_N''$ and $\mathbb{E}\langle h\rangle_N$ asymptotically and this finishes the proof. $\square$

3.2. *Lower bound.*

LEMMA 11. *There exists* $\mu\in\mathcal{M}$ *such that* $\lim_{N\to\infty} F_N \ge \mathcal{P}(\mu)$.

PROOF. We again use (2.9). Suppose that this lower limit is achieved over subsequence (N_k) and let $\mu\in\mathcal{M}$ be a limit of (μ_N) over some subsubsequence of (N_k). Let Z_N' and $\langle\cdot\rangle$ be the partition function and the Gibbs average on Σ_N

corresponding to the Hamiltonian H'_N defined in (3.6), and let us compute the limit of

$$\mathbb{E}\log\frac{Z_{N+1}}{Z'_N} - \mathbb{E}\log\frac{Z_N}{Z'_N}$$

along the above subsubsequence. Using (3.4) and (3.8) for $n = 1$ and the fact that, as in (3.5), the perturbation Hamiltonian $H^p_{N+1}(\boldsymbol{\sigma})$ in Z_{N+1} can be replaced by $H^p_N(\boldsymbol{\rho})$, the above limit is equal to the limit of

$$\log 2 + \mathbb{E}\log\langle \operatorname{ch} G_{\xi'}(\boldsymbol{\rho})\rangle - \mathbb{E}\log\langle \exp G_\theta(\boldsymbol{\rho})\rangle.$$

It remains to show that

$$\lim_{N\to\infty} \mathbb{E}\log\langle \operatorname{ch} G_{\xi'}(\boldsymbol{\rho})\rangle = \mathbb{E}\log\mathbb{E}' \operatorname{ch} G_{\xi'}(\bar{\sigma}_\mu(w,u,\cdot)) \tag{3.10}$$

and

$$\lim_{N\to\infty} \mathbb{E}\log\langle \exp G_\theta(\boldsymbol{\rho})\rangle = \mathbb{E}\log\mathbb{E}' \exp G_\theta(\bar{\sigma}_\mu(w,u,\cdot)), \tag{3.11}$$

where for simplicity of notations we will write limits for $N \to \infty$ rather than over the above subsubsequence. The proof of this is identical to Talagrand's proof of the Baffioni–Rosati theorem in [30]. First of all, if $\mathbb{E}_g$ denotes the expectation in the randomness of $G_{\xi'}(\boldsymbol{\rho})$ conditionally on the randomness in $\langle\cdot\rangle$, then standard Gaussian concentration implies that (see, e.g., Lemma 3 in [18])

$$\mathbb{P}_g\bigl(|\log\langle \operatorname{ch} G_{\xi'}(\boldsymbol{\rho})\rangle - \mathbb{E}_g\log\langle \operatorname{ch} G_{\xi'}(\boldsymbol{\rho})\rangle| \ge A\bigr) \le e^{-cA^2}$$

for some small enough constant c, and since

$$0 \le \mathbb{E}_g \log\langle \operatorname{ch} G_{\xi'}(\boldsymbol{\rho})\rangle \le \log\langle \mathbb{E}_g \operatorname{ch} G_{\xi'}(\boldsymbol{\rho})\rangle \le \xi'(1)/2$$

for large enough $A > 0$, we get

$$\mathbb{P}\bigl(|\log\langle \operatorname{ch} G_{\xi'}(\boldsymbol{\rho})\rangle| \ge A\bigr) \le e^{-cA^2}. \tag{3.12}$$

Therefore, if we denote $\log_A x = \max(-A, \min(\log x, A))$, then for large enough A,

$$|\mathbb{E}\log\langle \operatorname{ch} G_{\xi'}(\boldsymbol{\rho})\rangle - \mathbb{E}\log_A\langle \operatorname{ch} G_{\xi'}(\boldsymbol{\rho})\rangle| \le e^{-cA^2}. \tag{3.13}$$

Next, if we define $\operatorname{ch}_A x = \min(\operatorname{ch} x, \operatorname{ch} A)$, then using that

$$|\log_A x - \log_A y| \le e^A|x-y| \quad \text{and} \quad |\operatorname{ch} x - \operatorname{ch}_A x| \le \operatorname{ch} x\, I(|x| \ge A)$$

we can write

$$\begin{aligned}|\mathbb{E}\log_A\langle \operatorname{ch} G_{\xi'}(\boldsymbol{\rho})\rangle - \mathbb{E}\log_A\langle \operatorname{ch}_A G_{\xi'}(\boldsymbol{\rho})\rangle| &\le e^A\mathbb{E}\langle| \operatorname{ch} G_{\xi'}(\boldsymbol{\rho}) - \operatorname{ch}_A G_{\xi'}(\boldsymbol{\rho})|\rangle \\ &\le e^A\mathbb{E}\bigl\langle \operatorname{ch} G_{\xi'}(\boldsymbol{\rho}) I\bigl(|G_{\xi'}(\boldsymbol{\rho})| \ge A\bigr)\bigr\rangle.\end{aligned}$$

By Hölder's inequality we can bound this by

$$e^{A}(\mathbb{E}\langle\mathbb{E}_g \operatorname{ch}^2 G_{\xi'}(\boldsymbol{\rho})\rangle)^{1/2}\big(\mathbb{E}\big\langle\mathbb{P}_g\big(|G_{\xi'}(\boldsymbol{\rho})| \geq A\big)\big\rangle\big)^{1/2} \leq e^{-cA^2}$$

for large enough A since $\mathbb{P}_g(|G_{\xi'}(\boldsymbol{\rho})| \geq A) \leq e^{-cA^2}$. Combining with (3.13) proves that

$$|\mathbb{E}\log\langle\operatorname{ch} G_{\xi'}(\boldsymbol{\rho})\rangle - \mathbb{E}\log_A\langle\operatorname{ch}_A G_{\xi'}(\boldsymbol{\rho})\rangle| \leq e^{-cA^2}. \tag{3.14}$$

Approximating logarithm by polynomials on the interval $[e^{-A}, e^{A}]$ we can approximate $\mathbb{E}\log_A\langle\operatorname{ch}_A G_{\xi'}(\boldsymbol{\rho})\rangle$ by some linear combinations of the moments

$$\mathbb{E}\langle\operatorname{ch}_A G_{\xi'}(\boldsymbol{\rho})\rangle^q = \mathbb{E}\Big\langle\prod_{l\leq q}\operatorname{ch}_A G_{\xi'}(\boldsymbol{\rho}^l)\Big\rangle = \mathbb{E}\Big\langle\mathbb{E}_g\prod_{l\leq q}\operatorname{ch}_A G_{\xi'}(\boldsymbol{\rho}^l)\Big\rangle$$

for $q \geq 1$. Since

$$\mathbb{E}_g\prod_{l\leq q}\operatorname{ch}_A G_{\xi'}(\boldsymbol{\rho}^l) = F((R_{l,l'})_{l,l'\leq q}) \tag{3.15}$$

for some continuous bounded function F of the overlaps $(R_{l,l'})_{l,l'\leq q}$, Lemma 8 implies that

$$\lim_{N\to\infty}\mathbb{E}\Big\langle\mathbb{E}_g\prod_{l\leq q}\operatorname{ch}_A G_{\xi'}(\boldsymbol{\rho}^l)\Big\rangle = \mathbb{E}F((R^{\infty}_{l,l'})_{l,l'\leq q}).$$

Let us rewrite the right-hand side in terms of the process $G_{\xi'}$ in (1.37). Recall the definition of the processes in (1.37) and (1.38). If $\mathbb{E}_G$ is the expectation in the Gaussian randomness of these processes, then the definition of the function F in (3.15) implies that

$$\mathbb{E}F((R^{\infty}_{l,l'})_{l,l'\leq q}) = \mathbb{E}\mathbb{E}_G\prod_{l\leq q}\operatorname{ch}_A G_{\xi'}(\bar{\sigma}_\mu(w,u_l,\cdot)) = \mathbb{E}(\mathbb{E}'\operatorname{ch}_A G_{\xi'}(\bar{\sigma}_\mu(w,u,\cdot)))^q$$

and, therefore,

$$\lim_{N\to\infty}\mathbb{E}\log_A\langle\operatorname{ch}_A G_{\xi'}(\boldsymbol{\rho})\rangle = \mathbb{E}\log_A\mathbb{E}'\operatorname{ch}_A G_{\xi'}(\bar{\sigma}_\mu(w,u,\cdot)).$$

[Notice that this approximation by moments depended on functions of the overlaps only which justifies the comment leading to (1.51).] One can show similarly to (3.14) that

$$|\mathbb{E}\log\mathbb{E}'\operatorname{ch} G_{\xi'}(\bar{\sigma}_\mu(w,u,\cdot)) - \mathbb{E}\log_A\mathbb{E}'\operatorname{ch}_A G_{\xi'}(\bar{\sigma}_\mu(w,u,\cdot))| \leq e^{-cA^2}, \tag{3.16}$$

which finishes the proof of (3.10). Equation (3.11) is proved similarly. $\square$

3.3. *Upper bound and free energy.* Since the perturbation term in (1.14) does not affect the limit of free energy, we will now ignore it and consider free energy F_N defined for the original unperturbed Hamiltonian (1.30).

LEMMA 12. *For any function $\bar{\sigma}:[0,1]^3 \to [-1,+1]$ we have*

$$
\begin{aligned}
F_N \le \log 2 + \frac{1}{N}\mathbb{E}\log\mathbb{E}'\prod_{i\le N}\operatorname{ch} G_{\xi',i}(\bar{\sigma}(w,u,\cdot)) \\
-\frac{1}{N}\mathbb{E}\log\mathbb{E}'\exp\sum_{i\le N} G_{\theta,i}(\bar{\sigma}(w,u,\cdot)).
\end{aligned}
\tag{3.17}
$$

PROOF. This is proved by the Guerra type interpolation as in [14]. If, for $t \in [0,1]$, we consider the interpolating Hamiltonian

$$
\begin{aligned}
-H_{N,t}(\boldsymbol{\sigma}) = -\sqrt{t}H_N(\boldsymbol{\sigma}) + \sqrt{1-t}\sum_{i\le N}\sigma_i G_{\xi',i}(\bar{\sigma}(w,u,\cdot)) \\
+\sqrt{t}\sum_{i\le N} G_{\theta,i}(\bar{\sigma}(w,u,\cdot))
\end{aligned}
$$

and interpolating free energy

$$
\varphi(t) = \frac{1}{N}\mathbb{E}\log\mathbb{E}'\sum_{\boldsymbol{\sigma}\in\Sigma_N}\exp(-H_{N,t}(\boldsymbol{\sigma})),
$$

then to prove (3.17) it is enough to show that $\varphi'(t) \le 0$. This is done by the usual Gaussian integration by parts as in [14]. □

Before proving invariance properties of Theorem 3 let us finish the proof of Theorem 4 by showing that if we let

$$
\begin{aligned}
\mathcal{P}_n(\mu) = \log 2 + \frac{1}{n}\mathbb{E}\log\mathbb{E}'\prod_{i\le n}\operatorname{ch} G_{\xi',i}(\bar{\sigma}_\mu(w,u,\cdot)) \\
-\frac{1}{n}\mathbb{E}\log\mathbb{E}'\exp\sum_{i\le n} G_{\theta,i}(\bar{\sigma}_\mu(w,u,\cdot)),
\end{aligned}
$$

then the invariance of Theorem 3 implies the following.

LEMMA 13. *For all $\mu \in \mathcal{M}_{\mathrm{inv}}$, $\mathcal{P}_n(\mu) = \mathcal{P}(\mu)$ for all $n \ge 1$.*

PROOF. The result will follow if we show that for $\bar{\sigma} = \bar{\sigma}_\mu$ for any $n \ge 1$,

$$
\mathbb{E}\log\frac{\mathbb{E}'\prod_{i\le n+1}\operatorname{ch} G_{\xi',i}(\bar{\sigma}(w,u,\cdot))}{\mathbb{E}'\prod_{i\le n}\operatorname{ch} G_{\xi',i}(\bar{\sigma}(w,u,\cdot))} = \mathbb{E}\log\mathbb{E}'\operatorname{ch} G_{\xi',n+1}(\bar{\sigma}(w,u,\cdot))
\tag{3.18}
$$

and

$$\mathbb{E}\log\frac{\mathbb{E}'\exp\sum_{i\le n+1}G_{\theta,i}(\bar\sigma(w,u,\cdot))}{\mathbb{E}'\exp\sum_{i\le n}G_{\theta,i}(\bar\sigma(w,u,\cdot))}=\mathbb{E}\log\mathbb{E}'\exp G_{\theta,n+1}(\bar\sigma(w,u,\cdot)). \tag{3.19}$$

To prove this we will use invariance properties (1.44) and (1.45). Using truncation and Gaussian concentration as in Lemma 11, to prove (3.18) it is enough to show that

$$\mathbb{E}\biggl(\frac{\mathbb{E}'\operatorname{ch}_A G_{\xi',n+1}(\bar\sigma(w,u,\cdot))\prod_{i\le n}\operatorname{ch}G_{\xi',i}(\bar\sigma(w,u,\cdot))}{\mathbb{E}'\prod_{i\le n}\operatorname{ch}G_{\xi',i}(\bar\sigma(w,u,\cdot))}\biggr)^q$$
$$=\mathbb{E}(\mathbb{E}'\operatorname{ch}_A G_{\xi',n+1}(\bar\sigma(w,u,\cdot)))^q.$$

Using replicas as in (1.47), the left-hand side can be written as

$$\mathbb{E}\frac{\mathbb{E}'F\prod_{l\le q}\prod_{i\le n}\operatorname{ch}G_{\xi',i}(\bar\sigma(w,u_l,\cdot))}{(\mathbb{E}'\prod_{i\le n}\operatorname{ch}G_{\xi',i}(\bar\sigma(w,u,\cdot)))^q}, \tag{3.20}$$

where

$$F=F((R^\infty_{l,l'})_{l,l'\le q})=\mathbb{E}_G\prod_{l\le q}\operatorname{ch}_A G_{\xi',n+1}(\bar\sigma(w,u_l,\cdot))$$

is a bounded continuous function of the overlaps defined in (1.7). Approximating F by polynomials of overlaps and using (1.44) proves that (3.20) is equal to

$$\mathbb{E}F=\mathbb{E}\mathbb{E}_G\prod_{l\le q}\operatorname{ch}_A G_{\xi',n+1}(\bar\sigma(w,u_l,\cdot))=\mathbb{E}(\mathbb{E}'\operatorname{ch}_A G_{\xi',n+1}(\bar\sigma(w,u,\cdot)))^q,$$

and this finishes the proof of (3.18). Equation (3.19) is proved similarly using (1.45) instead. □

3.4. *Invariance and self-consistency equations.*

PROOFS OF THEOREMS 3 AND 5. Let $h=\prod_{l\le q}h_l$ where $h_l=\prod_{i\in C_l}\sigma_i^l$. Consider $\mu\in\mathcal{M}$ which is a limit of μ_N over some subsequence (N_k). By Lemma 10, the left-hand side of (1.41) is the limit of $\mathbb{E}\langle h\rangle_{N+n}$ over subsequence (N_k). The right-hand side of (1.41) will appear as a similar limit once we rewrite this joint moment of spins using cavity coordinates. The beginning of the proof will be identical to the proof of Lemma 10, only the spins with coordinates $i\le n$ will now play the role of cavity coordinates instead of spins with coordinates $N+1\le i\le N+n$. Let us write $\boldsymbol{\sigma}=(\boldsymbol{\varepsilon},\boldsymbol{\rho})\in\Sigma_{N+n}$ for the first n cavity coordinates $\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_n)$ and the last N coordinates $\boldsymbol{\rho}=(\rho_1,\ldots,\rho_N)$. Let us consider sequences of Gaussian processes $(Z_i(\boldsymbol{\rho}))$ and $(Y_i(\boldsymbol{\rho}))$ which are independent copies of $G_{\xi'}(\boldsymbol{\rho})$ and $G_\theta(\boldsymbol{\rho})$, correspondingly. First of all, we can replace the perturbation term $-H^p_{N+n}(\boldsymbol{\sigma})$ with

$$-H^p_N(\boldsymbol{\rho})=\sum_{k\le\pi(c_N)}\log\operatorname{ch}G_{\xi',k}(\boldsymbol{\rho})+\sum_{k\le\pi'(c_N)}G_{\theta,k}(\boldsymbol{\rho})+\sum_{k\le r}Y_k(\boldsymbol{\rho}) \tag{3.21}$$

for a fixed $r \geq 1$ without affecting $\mathbb{E}\langle h \rangle_{N+n}$ asymptotically, since by Lemma 9 we can slightly modify the Poisson number of terms, and then we can replace $G_{\xi',i}(\boldsymbol{\sigma})$ and $G_{\theta,i}(\boldsymbol{\sigma})$ with $G_{\xi',i}(\boldsymbol{\rho})$ and $G_{\theta,i}(\boldsymbol{\rho})$ by interpolation using that $c_N = o(N)$. Then, as in (3.4), we can redefine the Hamiltonian $-H_{N+n}(\boldsymbol{\sigma})$ by

$$-H_{N+n}(\boldsymbol{\sigma}) = -H_{N+n}(\boldsymbol{\rho}) + \sum_{i \leq n} \varepsilon_i Z_i(\boldsymbol{\rho}), \tag{3.22}$$

where $H_{N+n}(\boldsymbol{\rho})$ is defined in (3.3). Let $\langle \cdot \rangle$ denote the Gibbs average corresponding to the Hamiltonian

$$-H'_N(\boldsymbol{\rho}) = -H_{N+n}(\boldsymbol{\rho}) + \sum_{k \leq \pi(c_N)} \log \operatorname{ch} G_{\xi',k}(\boldsymbol{\rho}) + \sum_{k \leq \pi'(c_N)} G_{\theta,k}(\boldsymbol{\rho}). \tag{3.23}$$

Recalling the relationship (3.8) between $H_N(\boldsymbol{\rho})$ and $H_{N+n}(\boldsymbol{\rho})$, let us note that Lemma 9 implies, as in the proof of Lemma 10, that the joint distribution of spins μ'_N corresponding to the Hamiltonian (3.23) converges to the same limits (over subsequences) as the original sequence μ_N. Let us write the function $h_l(\boldsymbol{\sigma})$ in terms of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\rho}$ as

$$h_l(\boldsymbol{\sigma}) = \prod_{i \in C_l} \sigma_i = \prod_{i \in C_l^1} \sigma_i \prod_{i \in C_l^2} \sigma_i = \prod_{i \in C_l^1} \varepsilon_i \prod_{i \in C_l^2} \rho_i,$$

where we will abuse the notations and still write C_l^2 to denote the set of coordinates ρ_i corresponding to the original coordinates σ_{n+i}. Then we can write

$$\mathbb{E}\langle h \rangle_{N+n} = \mathbb{E} \frac{\prod_{l \leq q} U_{N,l}}{V_N^q}, \tag{3.24}$$

where

$$U_{N,l} = \left\langle \operatorname{Av}_\varepsilon \prod_{i \in C_l^1} \varepsilon_i \exp \sum_{i \leq n} \varepsilon_i Z_i(\boldsymbol{\rho}) \prod_{i \in C_l^2} \rho_i \exp \sum_{k \leq r} Y_k(\boldsymbol{\rho}) \right\rangle$$

and

$$V_N = \left\langle \operatorname{Av}_\varepsilon \exp \sum_{i \leq n} \varepsilon_i Z_i(\boldsymbol{\rho}) \exp \sum_{k \leq r} Y_k(\boldsymbol{\rho}) \right\rangle = \langle \exp X(\boldsymbol{\rho}) \rangle,$$

where we introduced

$$X(\boldsymbol{\rho}) = \sum_{i \leq n} \log \operatorname{ch} Z_i(\boldsymbol{\rho}) + \sum_{k \leq r} Y_k(\boldsymbol{\rho}).$$

It remains to show that

$$\lim_{N \to \infty} \mathbb{E} \frac{\prod_{l \leq q} U_{N,l}}{V_N^q} = \mathbb{E} \frac{\prod_{l \leq q} U_l}{V^q}, \tag{3.25}$$

where

$$U_l = \mathbb{E}' \operatorname{Av}_\varepsilon \prod_{i \in C_l^1} \varepsilon_i \exp \sum_{i \le n} \varepsilon_i G_{\xi',i}(\bar\sigma(w,u,\cdot)) \prod_{i \in C_l^2} \bar\sigma_i \exp \sum_{k \le r} G_{\theta,k}(\bar\sigma(w,u,\cdot))$$

for $\bar\sigma_i = \bar\sigma(w,u,v_i)$ and

$$V = \mathbb{E}' \operatorname{Av}_\varepsilon \exp \sum_{i \le n} \varepsilon_i G_{\xi',i}(\bar\sigma(w,u,\cdot)) \exp \sum_{k \le r} G_{\theta,k}(\bar\sigma(w,u,\cdot)),$$

which is, of course, the same equation as (1.41). [The proof that (1.57) implies (1.61) is exactly the same of the proof of (3.25).] The proof of (3.25) is nearly identical to the proof of (3.10) using truncation and Gaussian concentration, only instead of approximating a truncated version of $\log x$ by polynomials we now need to approximate a truncated version of $1/x$ by polynomials. If we denote

$$Y = \log V_N = \log\langle \exp X(\boldsymbol\rho)\rangle,$$

then, as in (3.12), one can show that for large enough $A > 0$

$$\mathbb{P}(|Y| \ge A) \le e^{-cA^2}. \tag{3.26}$$

For $A > 0$ let $(x)_A = \max(-A, \min(x, A))$ so that

$$|\exp(-qx) - \exp(-q(x)_A)| \le \max(e^{-qA}, \exp(-qx)) I(|x| > A).$$

If we denote $Z = \prod_{l \le q} U_{N,l}$, then, obviously, $\mathbb{E}Z^2 \le L$ for some large enough $L > 0$ that depends on q, n, r and function ξ, and (3.26) implies that

$$\begin{aligned} &|\mathbb{E}Z \exp(-qY) - \mathbb{E}Z \exp(-q(Y)_A)| \\ &\quad \le \mathbb{E}|Z| \max(e^{-qA}, \exp(-qY)) I(|Y| > A) \le e^{-cA^2} \end{aligned} \tag{3.27}$$

for large enough A. Next, let $\exp_A x = \max(e^{-A}, \min(\exp x, e^A))$, and let $Y' = \log\langle \exp_A X(\boldsymbol\rho)\rangle$. Since for all $x, y \in \mathbb{R}$

$$|\exp(-q(x)_A) - \exp(-q(y)_A)| \le q e^{(q+1)A} |\exp x - \exp y|,$$

we get

$$|\exp(-q(Y)_A) - \exp(-q(Y')_A)| \le q e^{(q+1)A} \langle |\exp X(\boldsymbol\rho) - \exp_A X(\boldsymbol\rho)|\rangle.$$

Next, since for all $x \in \mathbb{R}$

$$|\exp x - \exp_A x| \le \max(e^{-A}, \exp x) I(|x| \ge A),$$

we obtain the following bound:

$$|\exp(-q(Y)_A - \exp(-q(Y')_A)| \le q e^{(q+1)A} \langle \max(e^{-A}, \exp X(\boldsymbol\rho)) I\big(|X(\boldsymbol\rho)| \ge A\big)\rangle.$$

It is easy to see that $\mathbb{P}(|X(\boldsymbol{\rho})| \geq A) \leq e^{-cA^2}$ for large enough A, and using Hölder's inequality,

$$
\begin{aligned}
&|\mathbb{E}Z\exp(-q(Y)_A) - \mathbb{E}Z\exp(-q(Y')_A)| \\
&\quad \leq qe^{(q+1)A}(\mathbb{E}Z^2)^{1/2}(\mathbb{E}\langle\max(e^{-4A}, \exp 4X(\boldsymbol{\rho}))\rangle)^{1/4} \\
&\qquad \times \bigl(\mathbb{E}\langle I\bigl(|X(\boldsymbol{\rho})| \geq A\bigr)\rangle\bigr)^{1/4} \\
&\quad \leq e^{-cA^2}.
\end{aligned}
$$

Combining this with (3.27) we prove that

$$
|\mathbb{E}Z\exp(-qY) - \mathbb{E}Z\exp(-q(Y')_A)| \leq e^{-cA^2}
$$

for large enough A. We can now approximate $\exp(-q(Y')_A) = \langle\exp_A X(\boldsymbol{\rho})\rangle^{-q}$ uniformly by polynomials of $\langle\exp_A X(\boldsymbol{\rho})\rangle$, and therefore $\mathbb{E}Z\exp(-q(Y')_A)$ can be approximated by a linear combination of terms

$$
\mathbb{E}\prod_{l\leq q} U_{N,l}\langle\exp_A X(\boldsymbol{\rho})\rangle^s. \tag{3.28}
$$

If we write the product of the Gibbs averages using replicas and take expectation with respect to the Gaussian processes $(X_i(\boldsymbol{\rho}))$ and $(Y_i(\boldsymbol{\rho}))$ inside the Gibbs average, we will get a Gibbs average of some bounded continuous function of finitely many overlaps in addition to the spin terms $\prod_{i\in C_l^2}\rho_i$ that appear in the definition of $U_{N,l}$. Observe that if, from the beginning, we chose $m = n$, then factors $\prod_{i\in C_l^2}\rho_i$ would not be present, which means that the linear combination of (3.28) gives an approximation of $\mathbb{E}\langle h\rangle_{N+n}$ (and thus $\mathbb{E}\langle h\rangle_N$) by the annealed Gibbs average of some functions of overlaps only. In particular, this proves Theorem 5. In the general case, Lemma 8 implies that (3.28) converges to $\mathbb{E}\prod_{l\leq q} U_l(V_A)^s$ where

$$
V_A = \mathbb{E}'\exp_A\biggl(\sum_{i\leq n}\log\operatorname{ch} G_{\xi',i}(\bar{\sigma}(w,u,\cdot)) + \sum_{k\leq r} G_{\theta,k}(\bar{\sigma}(w,u,\cdot))\biggr).
$$

Since the same truncation and approximation arguments can be carried out in parallel for the right-hand side of (3.25), this proves (3.25) and finishes the proof of Theorem 3. □

PROOF OF THEOREM 7. Using Gaussian integration by parts and invariance in (1.61), (1.64) can be rewritten as

$$
1 - t^2\bigl(\mathbb{E}(R_{1,2}^\infty)^{2p} - 2\mathbb{E}(R_{1,2}^\infty)^p(R_{1,3}^\infty)^p + (\mathbb{E}(R_{1,2}^\infty)^p)^2\bigr),
$$

and the second term disappears whenever the Ghirlanda–Guerra identities (1.58) hold. On the other hand, if (1.64) is uniformly bounded for all $t > 0$, then for any bounded continuous function F of multi-overlaps (1.6) on n replicas,

$$\mathbb{E}\frac{FG_p(\bar\sigma(w,u_1,\cdot))\exp t\sum_{l\le n}G_p(\bar\sigma(w,u_l,\cdot))}{(\mathbb{E}'\exp tG_p(\bar\sigma(w,u,\cdot)))^n} - \mathbb{E}F\mathbb{E}\frac{G_p(\bar\sigma(w,u,\cdot))\exp tG_p(\bar\sigma(w,u,\cdot))}{\mathbb{E}'\exp tG_p(\bar\sigma(w,u,\cdot))} \tag{3.29}$$

is also uniformly bounded by invariance in (1.61) and Hölder's inequality. Using Gaussian integration by parts and invariance in (1.61), this is equal to

$$t\left(\sum_{l=2}^{n}\mathbb{E}F(R_{1,l}^{\infty})^p - n\mathbb{E}F(R_{1,n+1}^{\infty})^p + \mathbb{E}F\mathbb{E}(R_{1,2}^{\infty})^p\right), \tag{3.30}$$

which can be bounded only if (1.58) holds. □

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DEPARTMENT OF MATHEMATICS
TEXAS A&M UNIVERSITY
MAILSTOP 3386, ROOM 209
COLLEGE STATION, TEXAS 77843
USA
E-MAIL: panchenk@math.tamu.edu