The Annals of Probability
2013, Vol. 41, No. 3A, 1218–1242
DOI: 10.1214/11-AOP722

SHARP METASTABILITY THRESHOLD FOR AN ANISOTROPIC BOOTSTRAP PERCOLATION MODEL

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Bootstrap percolation models have been extensively studied during the two past decades. In this article, we study the following "anisotropic" bootstrap percolation model: the neighborhood of a point (m, n) is the set

$$\{(m+2, n), (m+1, n), (m, n+1), (m-1, n), (m-2, n), (m, n-1)\}.$$

At time 0, sites are occupied with probability p. At each time step, sites that are occupied remain occupied, while sites that are not occupied become occupied if and only if three of more sites in their neighborhood are occupied. We prove that it exhibits a sharp metastability threshold. This is the first mathematical proof of a sharp threshold for an anisotropic bootstrap percolation model.

1. Introduction.

1.1. *Statement of the theorem.* Bootstrap percolation models are interesting models for crack formation, clustering phenomena, metastability and dynamics of glasses. They also have been used to describe the phenomenon of jamming (see, for example, [30]), and they are a major ingredient in the study of so-called kinetically constrained models; see, for example, [14]. Other applications are in the theory of sandpiles [13] and in the theory of neural nets [2, 29]. Bootstrap percolation was introduced in [9] and has been an object of study for both physicists and mathematicians. For some of the earlier results, see, for example, [1, 7, 8, 10, 18, 26, 28, 31, 32].

The simplest model is the so-called *simple bootstrap percolation* on $\mathbb{Z}^2$. At time 0, sites of $\mathbb{Z}^2$ are occupied with probability $p \in (0, 1)$ independently of each other. At each time increment, sites become occupied if at least two of their nearest neighbors are occupied. The behavior of this model is now well-understood: the model exhibits a sharp metastability threshold. Nevertheless, slight modifications of the update rule provide challenging problems, and the sharp metastability threshold remains open in general. A few models have been solved, including simple bootstrap percolation and the modified bootstrap percolation in every dimension, and so-called balanced dynamics in two dimensions [3–6, 12, 20–22]. The

Received November 2010; revised September 2011.

[1]Supported in part by ANR Grant BLAN06-3-134462, the ERC AG CONFRA, as well as by the Swiss FNS.

MSC2010 subject classifications. 60K35, 83B43, 83C43.

Key words and phrases. Bootstrap percolation, sharp threshold, anisotropy, metastability.

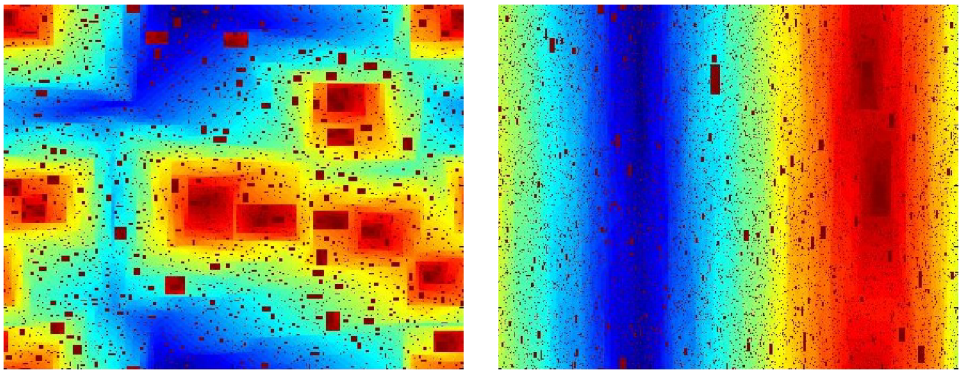

FIG. 1. *Left*: *An example of a simple bootstrap percolation's growth* (*red sites are the oldest*, *blue the youngest*). *Right*: *An example of the anisotropic model's growth*.

case of anisotropic dynamics, in which the neighborhood of a point is not invariant under a ninety-degree rotation in a lattice plane (even in two dimensions), has so far eluded mathematicians, and even the scale at which the metastability threshold occurs is not clear. (Note added in proof: For a general class of models in 3 dimensions, this scale has since been established in [33].)

In this article, we provide the first sharp metastability threshold for an anisotropic model. We consider the following model, first introduced in [15]. The neighborhood of a point (m, n) is the set

$$\{(m+2, n), (m+1, n), (m, n+1), (m-1, n), (m-2, n), (m, n-1)\}.$$

At time 0, sites are occupied with probability p. At each time step, sites that are occupied remain occupied, while sites that are not occupied become occupied if and only if three of more sites in their neighborhood are occupied. We are interested in the behavior (when the probability p goes to 0) of the (random) time T at which 0 becomes occupied. For earlier studies of two-dimensional anisotropic models, whose results, however, fall short of providing sharp results, we refer to [11, 15, 16, 24, 25, 27, 32, 34].

THEOREM 1.1. *Consider the dynamics described above*, *then*

$$\frac{1}{p}\left(\log\frac{1}{p}\right)^2 \log T \xrightarrow{(P)} \frac{1}{12} \qquad \textit{when } p \to 0.$$

This model and the simple bootstrap percolation have very different behavior, as illustrated in Figure 1.

Combined with techniques of [12] we believe that our proof paves the way toward a better understanding of general bootstrap percolation models. More directly, the following models fall immediately into the scope of the proof. Consider the neighborhood $\mathcal{N}_k$ of (m, n) defined by

$$\{(m+k, n), \ldots, (m+1, n), (m, n+1), (m-1, n), \ldots, (m-k, n), (m, n-1)\},$$

and assume that the site (m, n) becomes occupied as soon as $\mathcal{N}_k$ contains $k + 1$ occupied sites. Then the techniques developed in this article extend to this context, showing that

$$\frac{1}{p}\left(\log\frac{1}{p}\right)^2 \log T \xrightarrow{(P)} \frac{1}{4(k+1)} \qquad \text{when } p \to 0.$$

1.2. *Outline of the proof.* The time at which the origin becomes occupied is determined by the typical distance at which a "critical droplet" occurs; here, a critical droplet means a connected localized set of occupied sites, which after growing via the bootstrap rule spans a macroscopic proportion of the space. Furthermore, the typical distance of this critical droplet is connected to the probability for a critical droplet to be created. Such a droplet then keeps growing until it covers the whole lattice with high probability. In our case, the droplet will be created at a distance of order $\exp\frac{1}{6p}(\log\frac{1}{p})^2$. Determining this distance boils down to estimating how a rectangle, consisting of an occupied double vertical column of length $\varepsilon\frac{1}{p}\log\frac{1}{p}$, grows to a rectangle of size $1/p^2$ by $\frac{1}{3p}\log\frac{1}{p}$.

Obtaining an upper bound is usually the easiest part: one must identify "an almost optimal" way to create the critical droplet. This way follows a two-stage procedure. First, a vertical double line of height $\frac{\varepsilon}{p}\log\frac{1}{p}$ is created. Then, the rectangle grows to size $1/p^2$ by $\frac{1}{3p}\log\frac{1}{p}$. We mention that this step is quite different from the isotropic case. Indeed, after starting as a vertical double line, the droplet grows in a logarithmic manner; that is, it grows logarithmically faster in the horizontal than in the vertical direction. On the one hand, the computation of the integral determining the constant of the threshold is easier than in [20]. On the other hand, the growth mechanism is more intricate.

The lower bound is much harder: one must prove that our "optimal" way of spanning a rectangle of size $1/p^2$ by $\frac{1}{3p}\log\frac{1}{p}$ is indeed the best one. We combine existing technology with new arguments. The proof is based on Holroyd's notion of hierarchy applied to k-crossable rectangles containing internally filled sets (i.e., sets such that all their sites become eventually occupied when running the dynamics restricted to the sets). A large rectangle will be typically created by generations of smaller rectangles. These generations of smaller rectangles are organized in a tree structure which forms the hierarchy. In our context, the original notion must be altered in many different ways (see the proof).

For instance, one key argument in Holroyd's paper is the fact that hierarchies with many so-called "seeds" are unlikely to happen, implying that hierarchies corresponding to one small seed were the most likely ones. In our model, this is no longer true. There can be many seeds, and a new comparison scheme is needed. A second difficulty comes from the fact that there are stable sets that are not rectangles. We must use the notion of being k-crossed (see Section 3). Even though it is much easier to be k-crossed than to be internally filled, we can choose the free

parameter k to be large enough in order to get sharp enough estimates. We would like to mention a third difficulty. Proposition 21 of [20] estimates the probability that a rectangle R' becomes full knowing that a slightly smaller rectangle R is full. In this article, we need an analog of this proposition. However, the proof of Holroyd's Proposition uses the fact that the so-called "corner region" between the two rectangles is unimportant. In our case, this region matters, and we need to be more careful about the statement and the proof of the corresponding proposition.

The upper bound together with the lower bound result in the sharp threshold. In a similar way as in ordinary bootstrap percolation Holroyd's approach refined the analysis of Aizenman and Lebowitz, here we refine the results of [15] and [34]. We find that the typical growth follows different "strategies" depending on which stage of growth we are in. The logarithmic growth into a critical rectangle is the main new qualitative insight of the paper. In [34], long vertical double lines were considered as critical droplets; before, Schonmann [26] had identified a single vertical line for the Duarte model as a possible critical droplet. The fact that these are not the optimal ones is the main new step toward the identification of the threshold, apart from the technical ways of proving it. Although the growth pattern is thus somewhat more complex, the computation of the threshold can still be performed.

1.3. *Notations*. Let $\mathbb{P}_p$ be the percolation measure with $p > 0$. The initial (random) set of occupied sites will always be denoted by K. We will denote by $\langle K \rangle$ the final configuration spanned by a set K. A set S (for instance a line) is said to be *occupied* if it contains one occupied site (i.e., $S \cap K \neq \varnothing$). It is *full* if all its sites are occupied (i.e., $S \subset K$). A set S is *internally filled* if $S \subset \langle K \cap S \rangle$. Note that this notation is nonstandard and corresponds to being internally spanned in the literature.

The neighborhood of 0 will be denoted by $\mathcal{N}$. Observe that the neighborhood of (m, n) is $(m, n) + \mathcal{N}$.

A *rectangle* $[a, b] \times [c, d]$ is the set of sites in $\mathbb{Z}^2$ included in the Euclidean rectangle $[a, b] \times [c, d]$. Note that a, b, c and d do not have to be integers. For a rectangle $R = [a, b] \times [c, d]$, we will usually denote by $(x(R), y(R)) = (b - a, d - c)$ the *dimensions* of the rectangle. When there is no possible confusion, we simply write (x, y). A *line* of the rectangle R is a set $\{(m, n) \in R : n = n_0\}$ for some n_0 fixed. A *column* is a set $\{(m, n) \in R : m = m_0\}$ for some m_0 fixed.

1.4. *Probabilistic tools*. There is a natural notion of increasing events in $\{0, 1\}^{\mathbb{Z}^2}$: an event A is *increasing* if for any pair of configurations $\omega \leq \omega'$—every occupied site in ω is occupied in ω'—such that ω is in A, then ω' is in A. Two important inequalities related to increasing events will be used in the proof: the first one is the so-called *FKG inequality*. Let A and B be two increasing events, then

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

The second is the *BK inequality*. We say that two events occur disjointly if for any $\omega \in A \cap B$, it is possible to find a set F so that $\omega_{|F} \in A$ and $\omega_{|F^c} \in B$ (the restriction means that the occupied sites of $\omega_{|F}$ are exactly the occupied sites of ω which are in F). We denote the disjoint occurrence by $A \circ B$ (we denote $A_1 \circ \cdots \circ A_n$ for n events occurring disjointly). Then

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

We refer the reader to the book [19] for proofs and a complete study of percolation models.

We will also use the following easy instance of Chernoff's inequality. For every $\varepsilon > 0$, there exists $p_0 > 0$ such that for every $p < p_0$ and $n \geq 1$, the probability of a binomial variable with parameters n and p being larger than εn is smaller than e^{-n}.

2. Upper bound of Theorem 1.1. A rectangle R is *horizontally traversable* if in each triplet of neighboring columns, there exists an occupied site. A rectangle is *north traversable* if for any (horizontal) line $\ell = \{(k, n), k \in \mathbb{Z}\}$, there exists a site $(m, n) \in R \cap \ell$ such that $\{(m+1, n), (m+2, n), (m, n+1), (m-1, n), (m-2, n)\}$ contains two occupied sites. It is *south traversable* if for any (horizontal) line ℓ, there exists a site $(m, n) \in R \cap \ell$ such that $\{(m-1, n), (m-2, n), (m, n-1), (m+1, n), (m+2, n)\}$ contains two occupied sites.

LEMMA 2.1. *Let $\varepsilon > 0$, then there exist $p_0, y_0 > 0$ satisfying the following: for any rectangle R with dimensions (x, y),*

$$\exp[-(1+\varepsilon)xe^{-3py}] \leq \mathbb{P}_p(R \text{ is horizontally traversable}) \leq \exp[-(1-\varepsilon)(x-2)e^{-3py}]$$

provided $y_0/p < y < 1/(y_0 p^2)$ and $p < p_0$.

PROOF. Let $u = 1 - (1-p)^y$ be the probability that there exists an occupied site in a column. Let A_i be the event that the ith column from the left is occupied. Then R is horizontally traversable if and only if the sequence $A_1, \ldots, A_x$ has no triple gap (meaning that there exists i such that A_i, A_{i+1} and A_{i+2} do not occur). This kind of event has been studied extensively; see [20]. It is elementary to prove that

$$\alpha(u)^{-x} \leq \mathbb{P}_p(R \text{ is horizontally traversable}) \leq \alpha(u)^{-(x-2)},$$

where $\alpha(u)$ is the positive root of the polynomial

$$X^3 - uX^2 - u(1-u)X - u(1-u)^2.$$

When py goes to infinity and $p^2 y$ goes to 0 (and therefore p goes to 0), u goes to 1 and

$$\log \alpha(u) \sim -(1-u)^3 \sim -e^{-3py}.$$

The result follows readily. □

LEMMA 2.2. *For every $\varepsilon > 0$, there exists $p_0, x_0 > 0$ satisfying the following property: for any rectangle R with dimensions (x, y),*

$$\exp[(1+\varepsilon)y\log(p^2x)] \le \mathbb{P}_p(R \text{ is north traversable})$$

provided $p < p_0$ and $x \le 1/(x_0p^2)$.

The same estimate holds for south traversability by symmetry under reflection.

PROOF. Let $n_0 \in \mathbb{N}$. Let v be the probability that one line is occupied. In other words, the probability that there exists a site (m, n_0) such that two elements of $(m-2, n_0)$, $(m-1, n_0)$, (m, n_0+1), $(m+1, n_0)$ and $(m+2, n_0)$ are occupied. If p^2x goes to 0, the probability that there is such a pair of sites is equivalent to the expected number of such pairs, giving

$$\log v \sim \log[(8x-16)p^2] \sim \log[p^2x].$$

(Here $8x-16$ is a bound for the number of such pairs.) Using the FKG inequality, we obtain

$$v^y \le \mathbb{P}_p(R \text{ is north traversable}).$$

Together with the asymptotics for v, the claim follows readily. □

For two rectangles $R_1 \subset R_2$, let $I(R_1, R_2)$ be the event that R_2 is internally filled whenever R_1 is full. This event depends only on $R_2 \setminus R_1$.

PROPOSITION 2.3. *Let $\varepsilon > 0$. Then there exist $p_0, k_0 > 0$ such that the following holds: for any rectangles $R_1 \subset R_2$ with dimensions (x_1, y_1) and (x_2, y_2),*

$$\exp\bigl(-(1+\varepsilon)[(x_2-x_1)e^{-3py_2} - (y_2-y_1)\log(p^2x_1)]\bigr) \le \mathbb{P}_p(I(R_1, R_2)),$$

providing $p < p_0$, $p^2x_1 < 1/k_0$ and $k_0/p < y_1 < 1/(k_0p^2)$.

Take two rectangles $R_1 \subset R_2$ such that $R_1 = [a_1, a_2] \times [b_1, b_2]$ and $R_2 = [c_1, c_2] \times [d_1, d_2]$. Define sets

$$R_\ell := [c_1, a_1] \times [d_1, d_2] \quad \text{and} \quad R_r := [a_2, c_2] \times [d_1, d_2],$$
$$R_t := [a_1, a_2] \times [b_2, d_2] \quad \text{and} \quad R_b := [a_1, a_2] \times [d_1, b_1],$$
$$H := R_2 \setminus \{(x, y) : x \in [a_1, a_2] \text{ or } y \in [b_1, b_2]\}.$$

The set H is the *corner region*; see Figure 2.

PROOF. Let $\varepsilon > 0$. Set p_0 and $k_0 := \max(x_0, y_0)$ in such a way that Lemmata 2.1 and 2.2 apply. Let $R_1 \subset R_2$ two rectangles. If R_ℓ and R_r are horizontally

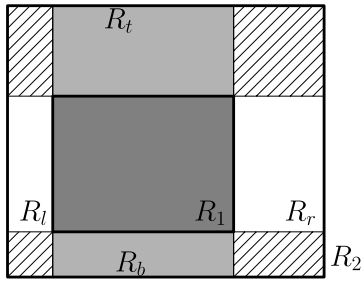

FIG. 2. *The rectangles R_ℓ and R_r are in light gray while R_t and R_b are in white. The corner region H is hatched.*

traversable while R_t and R_b are respectively north and south traversable, then R_2 is internally filled whenever R_1 is internally filled. Using the FKG inequality,

$$\begin{aligned}\mathbb{P}_p(I(R_1,R_2))&\\ &\geq \mathbb{P}_p(R_\ell \text{ hor. trav.})\mathbb{P}_p(R_t \text{ north trav.})\mathbb{P}_p(R_r \text{ hor. trav.})\mathbb{P}_p(R_b \text{ south trav.})\\ &\geq \exp\bigl(-(1+\varepsilon)[(x_2-x_1)e^{-3py_2}-(y_2-y_1)\log(p^2x_1)]\bigr),\end{aligned}$$

using Lemmata 2.1 and 2.2 (the conditions of these lemmata are satisfied). □

PROPOSITION 2.4 (Lower bound for the creation of a critical rectangle). *For any $\varepsilon>0$, there exists $p_0>0$ such that for $p<p_0$,*

$$\mathbb{P}_p([0,p^{-5}]^2 \textit{ is internally filled}) \geq \exp\biggl[-\Bigl(\frac{1}{6}+\varepsilon\Bigr)\frac{1}{p}\Bigl(\log\frac{1}{p}\Bigr)^2\biggr].$$

PROOF. Let $\varepsilon>0$. For any p small enough, consider the sequence of rectangles $(R_n^p)_{k_0\leq n\leq N}$

$$R_n^p := \bigl[0,p^{-1-3n/\log(1/p)}\bigr]\times[0,n/p],$$

where k_0 is defined in such a way that Proposition 2.3 applies with ε and $N := \frac{1}{3}\log\frac{1}{p}-\log k_0$. The following computation is straightforward, using Proposition 2.3:

$$\begin{aligned}\prod_{n=k_0}^{N}\mathbb{P}_p[I(R_n^p,R_{n+1}^p)]&\\ &\geq \exp\biggl[-(1+\varepsilon)\sum_{n=k_0}^{N}\Bigl((p^{-1-3(n+1)/\log(1/p)}-p^{-1-3n/\log(1/p)})e^{-3(n+1)}\\ &\qquad -\frac{1}{p}\log\bigl(p^2p^{-1-3n/\log(1/p)}\bigr)\Bigr)\biggr]\end{aligned}$$

$$= \exp\Bigg(-(1+\varepsilon)\frac{1}{p}\Big(\log\frac{1}{p}\Big)^2 \times\Bigg[\sum_{n=k_0}^{N}\frac{1-e^{-3}}{(\log(1/p))^2}+\sum_{n=k_0}^{N}\frac{1}{\log(1/p)}\Big(1-3\frac{n}{\log(1/p)}\Big)\Bigg]\Bigg).$$

The first sum goes to 0 as $O(1/\ln\frac{1}{p})$ while the second one is a Riemann sum converging to $\int_0^{1/3}(1-3y)\,dy=\frac{1}{6}$. The rectangle $[0,p^{-5}]^2$ is internally filled if all the following events occur (we include asymptotics when p goes to 0):

- E the event that $\{0,1\}\times[0,\varepsilon\frac{1}{p}\log\frac{1}{p}]$ is full, of probability $\exp[-2\varepsilon\frac{1}{p}(\log\frac{1}{p})^2]$;
- F the event that $R=[0,\varepsilon\frac{1}{p}\log\frac{1}{p}]\times[0,p^{-(1+\varepsilon)}]$ is horizontally traversable, of probability larger than

$$\big[1-(1-p)^{\varepsilon(1/p)\log(1/p)}\big]^{p^{-(1+\varepsilon)}}\geq\exp[-p^{\varepsilon-1-\varepsilon}]\geq\exp\Big[-\varepsilon\frac{1}{p}\Big(\log\frac{1}{p}\Big)^2\Big];$$

- G the intersection of $I(R_n^p,R_{n+1}^p)$ for $0\leq n\leq N-1$, of probability larger than $\exp[-(1+\varepsilon)(\frac{1}{6}+\varepsilon)\frac{1}{p}(\log\frac{1}{p})^2]$ using the computation above;
- H the event that $[0,p^{-2+\varepsilon}]\times[0,6\frac{1}{p}\log\frac{1}{p}]$ is north traversable, with probability larger than

$$\big(1-(1-p^2)^{p^{-2+\varepsilon}}\big)^{2(1/p)\log(1/p)}\approx(p^{\varepsilon})^{6(1/p)\log(1/p)}=\exp\Big[-6\varepsilon\frac{1}{p}\Big(\log\frac{1}{p}\Big)^2\Big];$$

- I the event that $[0,6\frac{1}{p}\log\frac{1}{p}]\times[0,p^{-5}]$ is horizontally traversable, with probability larger than $(1-(1-p)^{6(1/p)\log(1/p)})^{p^{-5}}$ and thus converging to 1;
- J the event that $[0,p^{-5}]^2$ is north traversable with probability larger than $[1-(1-p^2)^{p^{-5}}]^{p^{-5}}$ thus also converging to 1.

The FKG inequality gives

$$\mathbb{P}_p([0,p^{-5}]^2\text{ is int. filled})\geq\mathbb{P}_p(E\cap F\cap G\cap H\cap I\cap J)\geq\exp\Big[-(1+\varepsilon)\Big(\frac{1}{6}+10\varepsilon\Big)\frac{1}{p}\Big(\log\frac{1}{p}\Big)^2\Big]$$

when p is small enough. □

PROOF OF THE UPPER BOUND IN THEOREM 1.1. Let $\varepsilon>0$ and consider A to be the event that any line or column of length p^{-5} intersecting the box $[-L,L]^2$ where $L=\exp[(\frac{1}{12}+\varepsilon)\frac{1}{p}(\log\frac{1}{p})^2]$ contains two adjacent occupied sites. The probability of this event can be bounded from below.

$$\mathbb{P}_p(A)\geq[1-(1-p^2)^{p^{-5}/2}]^{8L^2}\approx\exp[8L^2e^{-p^{-5}/2}]\longrightarrow 1.$$

(The factor 8 is due to the number of possible segments of length p^{-5}.) Denote by B the event that there exists a translate of $[0, p^{-5}]^2$ which is included in $[-L, L]^2$ and internally filled. Applying Proposition 2.4 and dividing $[-L, L]^2$ into $(Lp^5)^2$ disjoint squares of size p^{-5}, one easily acquires

$$\begin{aligned}\mathbb{P}_p(B) &\geq 1 - \left[1 - e^{-(1/6+\varepsilon)(1/p)(\log(1/p))^2}\right]^{(Lp^5)^2} \\ &\approx 1 - \exp\left(-(Lp^5)^2 e^{-(1/6+\varepsilon)(1/p)(\log(1/p))^2}\right) \longrightarrow 1.\end{aligned}$$

Moreover, the occurrence of A and B implies that $\log T \leq (\frac{1}{12} + 2\varepsilon)\frac{1}{p}(\log \frac{1}{p})^2$ for p small enough. Indeed, a square of size p^{-5} is filled in less than p^{-10} steps. After the creation of this square, it only takes a number of steps of order $\exp[(\frac{1}{12} + \varepsilon)\frac{1}{p}(\log \frac{1}{p})^2]$ to progress and reach 0, thanks to the event A. The FKG inequality yields

$$\mathbb{P}_p\left[\log T \leq \left(\frac{1}{12} + 2\varepsilon\right)\frac{1}{p}\left(\log\frac{1}{p}\right)^2\right] \geq \mathbb{P}_p(E \cap F) \geq \mathbb{P}_p(E)\mathbb{P}_p(F) \to 1$$

which concludes the proof of the upper bound. □

3. Lower bound of Theorem 1.1.

3.1. *Crossed rectangles.* Two occupied points $x, y \in \mathbb{Z}^2$ are *connected* if $x \in y + \mathcal{N}$. A set is *connected* if there exists a path of occupied connected sites with end-points being x and y.

Two occupied points $x, y \in \mathbb{Z}^2$ are weakly connected if there exists $z \in \mathbb{Z}^2$ such that $x, y \in z + \mathcal{N}$. A set S is *weakly connected* if for any points $x, y \in S$, there exists a path of occupied weakly connected points with end-points x and y.

Let $k > 0$. A rectangle $[a, b] \times [c, d]$ is *k-vertically crossed* if for every $j \in [c, d - k]$, the final configuration in $[a, b] \times [j, j + k]$ knowing that $[a, b] \times [c, d] \setminus [a, b] \times [j, j + k]$ is full contains a connected path from top to bottom. A rectangle R is *k-crossed* (or simply crossed) if it is k-vertically crossed and horizontally traversable. Let $A_k(R_1, R_2)$ be the event that R_2 is k-vertically crossed whenever R_1 is full. Note that this event is contained in the event that R_ℓ and R_r are traversable, and R_t and R_b are k-vertically crossed.

LEMMA 3.1. *For every $\varepsilon > 0$, there exist p_0, $Q, k > 0$ satisfying the following property: for any rectangle R with dimensions (x, y),*

$$\mathbb{P}_p[R \text{ is } k\text{-vertically crossed}] \leq p^{-k} Q^y \exp[(1 - \varepsilon) y \log(p^2 x)] \qquad \text{when } \frac{1}{p} \leq x,$$

$$\mathbb{P}_p[R \text{ is } k\text{-vertically crossed}] \leq p^{-k} \exp[(1 - \varepsilon) y \log p] \qquad \text{when } x < \frac{1}{p},$$

providing $p < p_0$.

PROOF. Let $\varepsilon > 0$ and set $k = \lfloor 1/\varepsilon \rfloor$. Consider first the rectangle $[0, x] \times [1, k]$ and the event that there exists a connected path in the final configuration knowing that $\mathbb{Z} \times (\mathbb{Z} \setminus [1, k])$ is full. $\square$

CLAIM. *In the initial configuration, there exist* $A_1, \dots, A_r$ $(r \leq k-1)$ *disjoint weakly connected sets such that* $n_1 + \cdots + n_r \geq k + r - 1$ *where* $n_i \geq 2$ *is the cardinality of* A_i.

PROOF. We prove this claim by induction. For $k = 2$, the only way to cross the rectangle is to have a weakly connected set of cardinality 2. We define A_1 to be this set. For $k \geq 3$, there are three cases:

Case 1: No sites become occupied after time 0: It implies that the crossing from bottom to top is present in the original configuration. Therefore, there exists a connected set crossing the strip in the original configuration. Moreover, this set is of cardinality at least k since it must contain one site in each line at least. Taking the connected subset of cardinality k to be A_1, we obtain the claim in this case.

Case 2: The first line or the last line intersects a full weakly connected set of cardinality 2: Assume that the first line intersects a weakly connected set S of cardinality 2. The rectangle $[2, k] \times [0, x]$ is $(k-1)$-vertically crossed. There exist disjoint sets $B_1, \dots, B_r$ satisfying the conditions of the claim. If these sets are disjoint from S, set $A_1 = S$, $A_2 = B_1, \dots, A_{r+1} = B_r$. If one set (say B_1) intersects S, we set $A_1 = B_1 \cup S$, $A_2 = B_2, \dots, A_r = B_r$. In any case the condition on the cardinality is satisfied.

Case 3: Remaining cases: There must exist three sites in the same neighborhood (we call this set S), spanning $(m, n) \in [0, x] \times [2, k-1]$ at time 1. The rectangles $[0, x] \times [1, n-1]$ and $[0, x] \times [n+1, k]$ are respectively $(n-1)$-vertically crossed and $(k-n)$-vertically crossed. If $n \notin \{2, k-2\}$, then one can use the induction hypothesis in both rectangles, and perform the same procedure as before. If $n = 2$, then apply the induction hypothesis for the rectangle above. The same reasoning still applies. Finally, if $n = k-2$, one can do the same with the rectangle below. $\square$

Let $C = C(k)$ be a universal constant bounding the number of possible weakly connected sets of cardinality less than k (up to translation). For any weakly connected set of cardinality $n > 1$, we have that the probability to find such a set in the rectangle $[0, x] \times [1, k]$ is bounded by $Cp^n(kx)$. We deduce using the BK inequality that

$$\begin{aligned} \mathbb{P}_p\big[[0, x] \times [0, k] \text{ } k\text{-vert. cross.}\big] &\leq \sum_{r=1}^{k} \Bigg(\sum_{n_1 + \cdots + n_r \leq k+r-1} \prod_{j=1}^{r} (kC) p^{n_j} x \Bigg) \\ &\leq \sum_{r=1}^{k} (k+r)^r (kC)^r p^{k-1} (px)^r. \end{aligned} \tag{3.1}$$

First assume $px > 1$. We find

$$\begin{aligned}\mathbb{P}_p\big[[0,x]\times[0,k]\ k\text{-vert. cross.}\big] &\le \sum_{r=1}^{k}(k+r)^r(kC)^r p^{2k-2}x^{k-1}\\ &\le (2k^3C)^k(p^2x)^{k-1}\end{aligned}$$

since $px > 1$. We find

$$\mathbb{P}_p\big[[0,x]\times[0,k]\ k\text{-vert. cross.}\big] \le Q^k \exp[-(1-\varepsilon)k\log(p^2x)]$$

with $Q = 2k^3C$.

Now, we divide the rectangle R into $\lfloor y/k\rfloor$ rectangles of height k. If R is vertically crossed, then all the rectangles are vertically crossed. Using the previous estimate, we obtain

$$\begin{aligned}\mathbb{P}_p[R \text{ is } k\text{-vertically crossed}] &\le \prod_{i=1}^{\lfloor y/k\rfloor}\mathbb{P}_p(R_i \text{ is } k\text{-vertically crossed})\\ &\le Q^{k\lfloor y/k\rfloor}\exp\left[(1-\varepsilon)k\left\lfloor\frac{y}{k}\right\rfloor\log(p^2x)\right].\end{aligned}$$

Using that the rectangle of height k is k-vertically crossed with probability larger than p^k, we obtain the result in this case.

If $xp < 1$, then we can bound the right-hand term of (3.1) by Cp^{k-1} and conclude the proof similarly.

Observe that when $x > 1/p$, the rectangle will grow in the vertical direction using ℓ disjoint weakly connected pairs of occupied sites (if it grows by ℓ lines). When $x < 1/p$, a rectangle will grow in the vertical direction using one big weakly connected set of ℓ occupied sites. From this point of view, the dynamics is very different from the simple bootstrap percolation.

REMARK 3.2. We have seen in the previous proof that being k-vertically crossed involves only sites included in weakly connected sets of cardinality two.

This remark will be fundamental in the following proof.

For two rectangles $R_1 \subset R_2$, define

$$W^p(R_1,R_2) = \frac{p}{(\log(1/p))^2}[(x_2-x_1)e^{-3py_2}] \qquad \text{if } \frac{1}{p^2} \le x_2,$$

$$W^p(R_1,R_2) = \frac{p}{(\log(1/p))^2} \times [(x_2-x_1)e^{-3py_2} - (y_2-y_1)\log(p^2x_2)] \qquad \text{if } \frac{1}{p} \le x_2 < \frac{1}{p^2},$$

$$W^p(R_1,R_2) = \frac{p}{(\log(1/p))^2}[(x_2-x_1)e^{-3py_2} - (y_2-y_1)\log p] \qquad \text{if } x_2 < \frac{1}{p}.$$

PROPOSITION 3.3. *Let $\varepsilon, T > 0$, there exist $p_0, Q, k > 0$ such that for any $p < p_0$ and any rectangles $R_1 \subset R_2$ with dimensions (x_1, y_1) and (x_2, y_2) satisfying*

$$\frac{T}{p}\log\frac{1}{p} \le y_2 \le \frac{1}{p}\log\frac{1}{p},$$

then

$$\mathbb{P}_p[A_k(R_1, R_2)] \le p^{-2k} Q^{y_2 - y_1} \exp\Big[-(1-\varepsilon)\frac{1}{p}\Big(\log\frac{1}{p}\Big)^2 W^p(R_1, R_2)\Big].$$

PROOF. Let $\varepsilon > 0$ and set p_0, Q, y_0, k given by Lemmata 3.1 and 2.1 applied with ε. Note that Q can be taken greater than 1. Consider two rectangles $R_1 \subset R_2$ satisfying the conditions of the proposition. We will use that $p^2 y_2$ goes to 0 and y_2 goes to infinity (in particular, $y_0 \le y_2 \le p^{-2}/y_0$). We treat the case $1/p \le x_2 < 1/p^2$; the other cases are similar.

First assume

$$\varepsilon(y_2 - y_1)\log(p^2 x_2) \le -(1-\varepsilon)(x_2 - x_1)e^{-3py_2}.$$

The event $A_k(R_1, R_2)$ is included in the events that R_t and R_b are k-vertically crossed (these two events are independent). Using Lemma 3.1, we easily deduce the claim via

$$\begin{aligned}\mathbb{P}_p[A_k(R_1, R_2)] &\le \exp[(1-\varepsilon)(y_2 - y_1)\log(p^2 x_2)] \\ &\le \exp\big[-(1-\varepsilon)^2\big((x_2 - x_1)e^{-3py_2} - (y_2 - y_1)\log(p^2 x_2)\big)\big].\end{aligned}$$

We now assume

$$\varepsilon(y_2 - y_1)\log(p^2 x_2) \ge -(1-\varepsilon)(x_2 - x_1)e^{-3py_2}.$$

Let Y be the number of vertical lines containing one occupied site of H weakly connected to another occupied site. We have

$$\begin{aligned}\mathbb{P}_p[A_k(R_1, R_2)] \le{}& \mathbb{P}_p[A_k(R_1, R_2) \text{ and } Y \le \varepsilon(x_2 - x_1)] \\ &+ \mathbb{P}_p[Y \ge \varepsilon(x_2 - x_1)].\end{aligned} \tag{3.2}$$

Bound on the second term. Note that the probability α that a line contains one occupied site in H weakly connected with another occupied site behaves like $Cp^2(y_2 - y_1)$ (where C is universal) and therefore goes to 0 when p goes to 0. The probability of $Y \ge \varepsilon(x_2 - x_1)$ is bounded by the probability that a binomial variable with parameters $n = x_2 - x_1$ and $\alpha = Cp^2(y_2 - y_1)$ is larger than $\frac{\varepsilon}{3}(x_2 - x_1)$. Invoking Chernoff's inequality, we find

$$\mathbb{P}_p[Y \ge \varepsilon(x_2 - x_1)] \le \exp[-(x_2 - x_1)]$$

for p small enough. Since e^{-3py_2} converges to 0, we obtain for p small enough,

$$\begin{aligned}
&\mathbb{P}_p[Y \geq \varepsilon(x_2 - x_1)] \\
&\quad \leq \exp\left[-\frac{1-\varepsilon}{\varepsilon}(x_2 - x_1)e^{-3py_2}\right] \\
&\quad \leq \exp\left[-(1-\varepsilon)\left((x_2 - x_1)e^{-3py_2} - (y_2 - y_1)\log(p^2 x_2)\right)\right].
\end{aligned} \tag{3.3}$$

Bound on the first term. Let E be the event that R_t and R_b are k-vertically crossed and $Y \leq \varepsilon(x_2 - x_1)$. We know that

$$\begin{aligned}
&\mathbb{P}_p[A_k(R_1, R_2) \text{ and } Y \leq \varepsilon(x_2 - x_1)] \\
&\quad = \mathbb{P}_p[R_\ell \text{ and } R_r \text{ hor. trav.}|E]\mathbb{P}_p[E] \\
&\quad \leq \mathbb{P}_p[R_\ell \text{ and } R_r \text{ hor. trav.}|E]\mathbb{P}_p[R_t \text{ and } R_b \text{ are } k\text{-vertically crossed}].
\end{aligned} \tag{3.4}$$

We want to estimate the first term of the last line. Let Ω be the (random) set of all pairs of weakly connected occupied sites in H. Conditioning on E corresponds to determining the set Ω thanks to the remark preceding the proof. Let ω be a possible realization of Ω. Slice $R_\ell \cup R_r$ into m rectangles $R_1, \ldots, R_m$ (with widths $x^{(i)}$) such that:

- no element of ω intersects these rectangles;
- all the lines that do not intersect ω belong to a rectangle R_i;
- m is minimal for this property (note that $m \leq 2\varepsilon[x_2 - x_1]$).

For each of these rectangles, conditioning on $\{\Omega = \omega\}$ boils down to assuming that there are no full pairs in the corner region, which is a decreasing event, so that via the FKG inequality,

$$\mathbb{P}_p(R_i \text{ hor. trav.}|\Omega = \omega) \leq \mathbb{P}_p(R_i \text{ hor. trav.}) \leq \exp\left[-(1-\varepsilon)\left(x^{(i)} - 2\right)e^{-3py_2}\right].$$

Since $Y \leq \varepsilon(x_2 - x_1)$, we know that

$$x^{(1)} + \cdots + x^{(m)} \geq (1 - 3\varepsilon)(x_2 - x_1).$$

We obtain

$$\begin{aligned}
\mathbb{P}_p[R_\ell \text{ and } R_r \text{ are hor. trav.}|\Omega = \omega] &\leq \mathbb{P}_p[\text{rectangles } R_i \text{ are all hor. trav.}|\Omega = \omega] \\
&\leq \prod_{i=1}^{m} \exp -\left[(1-\varepsilon)\left(x^{(i)} - 2\right)e^{-3py_2}\right] \\
&\leq \exp[-(1-\varepsilon)(1-7\varepsilon)(x_2 - x_1)e^{-3py_2}].
\end{aligned}$$

By summing over all possible ω, we find

$$\mathbb{P}_p[R_\ell \text{ and } R_r \text{ hor. trav.}|E] \leq \exp[-(1-\varepsilon)(1-7\varepsilon)(x_2 - x_1)e^{-3py_2}]. \tag{3.5}$$

Using Lemma 3.1 and inequality (3.5), inequality (3.4) becomes

$$\begin{aligned}\mathbb{P}_p[A_k(R_1, R_2) \cap \{Y \le \varepsilon(x_2 - x_1)\}] \\ \le \exp[-(1-\varepsilon)^2(x_2 - x_1)e^{-3py_2}]p^{-2k}Q^{y_2-y_1} \\ \times \exp[-(1-\varepsilon)(y_2 - y_1)\log(p^2 y_2)].\end{aligned}$$

The claim follows by plugging the previous inequality and inequality (3.3) into inequality (3.2). □

3.2. *Hierarchy of a growth.* We define the notion of hierarchies, and the specific vocabulary associated to it. This notion is now well established. We slightly modify the definition, weakening the conditions imposed in [20].

- *Hierarchy, seed, normal vertex and splitter*: A *hierarchy* $\mathcal{H}$ is a tree with vertex degrees at most three with vertices v labeled by nonempty rectangles R_v such that the rectangle labeled by v contains the rectangles labeled by its descendants. If the number of descendants of a vertex is 0, it is a *seed*, and if it is one, it is a *normal* vertex [we denote by $u \mapsto v$ if u is a normal vertex of (unique) descendant v] and if it is two or more, it is a *splitter*. Let $N(\mathcal{H})$ be the number of vertices in the tree.
- *Precision of a hierarchy*: A hierarchy *of precision* t (with $t \ge 1$) is a hierarchy satisfying these additional conditions:

  (1) if w is a seed, then $y(R_w) < 2t$, if u is a normal vertex or a splitter, $y(R_u) \ge 2t$;
  (2) if u is a normal vertex with descendant v, then $y(R_u) - y(R_v) \le 2t$;
  (3) if u is a normal vertex with descendant v and v is a seed or a normal vertex, then $y(R_u) - y(R_v) > t$;
  (4) if u is a splitter with descendants $v_1, \dots, v_i$, there exists j such that $y(R_u) - y(R_{v_j}) > t$.

- *Occurrence of a hierarchy*: Let $k > 0$, a hierarchy *k-occurs* if all of the following events occur *disjointly*:

  (1) R_w is k-crossed for each seed w;
  (2) $A_k(R_v, R_u)$ occurs for each pair u and v such that u is normal;
  (3) R_u is the smallest rectangle containing $\langle R_{v_1} \cup \cdots \cup R_{v_j} \rangle$ for every splitter u ($v_1, \dots, v_i$ are the descendants of u).

REMARK 3.4. In the literature, the precision of a hierarchy is an element of $\mathbb{R}^2$. In our case, we do not need to control the x-coordinate.

REMARK 3.5. We can use the BK inequality to deduce that for any hierarchy $\mathcal{H}$ and $k \ge 1$,

$$\mathbb{P}_p[\mathcal{H}\ k\text{-occurs}] \le \prod_{v\text{seed}} \mathbb{P}_p[R_v\ k\text{-crossed}] \prod_{u \mapsto w} \mathbb{P}_p[A_k(R_w, R_u)].$$

The following lemmata are classical.

LEMMA 3.6 (Number of hierarchies). *Let $t \geq 1$, the number $N_t(R)$ of hierarchies of precision t for a rectangle R is bounded by*

$$N_t(R) \leq [x(R) + y(R)]^{c[y(R)/t]}$$

where c is a prescribed function.

PROOF. The proof is straightforward once we remark that the depth of the hierarchy is bounded by $y(R)/t$ (every two steps going down in the tree, the perimeter reduces by at least t). For a very similar proof, see [20]. □

LEMMA 3.7 (Disjoint spanning). *Let $\mathcal{S}$ be an internally filled and connected set of cardinality greater than 3, then there exist i disjoint nonempty connected sets $\mathcal{S}_1, \ldots, \mathcal{S}_i$ with $i \in \{2, 3\}$ such that*:

(i) *the strict inclusions $\mathcal{S}_1 \subset \mathcal{S}, \ldots, \mathcal{S}_i \subset \mathcal{S}$ hold*;
(ii) $\langle \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_i \rangle = \mathcal{S}$;
(iii) $\{\mathcal{S}_1$ *is internally filled*$\} \circ \cdots \circ \{\mathcal{S}_i$ *is internally filled*$\}$ *occurs*.

PROOF. Let K be finite, $\langle K \rangle$ may be constructed via the following algorithm: for each time step $t = 0, \ldots, \tau$, we find a collection of m_t connected sets $\mathcal{S}_1^t, \ldots, \mathcal{S}_{m_t}^t$ and corresponding sets of sites $K_1^t, \ldots, K_{m_t}^t$ with the following properties:

(i) $K_1^t, \ldots, K_{m_t}^t$ are pairwise disjoint;
(ii) $K_i^t \subset K$;
(iii) $\mathcal{S}_i^t = \langle K_i^t \rangle$ is connected;
(iv) if $i \neq j$, then we cannot have $\mathcal{S}_i^t \subset \mathcal{S}_j^t$;
(v) $K \subset \mathcal{S}^t \subset \langle K \rangle$ where

$$\mathcal{S}^t := \bigcup_{i=1}^{m_t} \mathcal{S}_i^t;$$

(vi) $\mathcal{S}^\tau = \langle K \rangle$.

Initially, the sets are just the individual sites of K: let K be enumerated as $K = \{x_1, \ldots, x_r\}$, and set $m_0 = r$ and $\mathcal{S}_i^0 = K_i^0 = \{x_i\}$, so that in particular $\mathcal{S}_0 = K$. Suppose that we have already constructed the sets $\mathcal{S}_1^t, \ldots, \mathcal{S}_{m_t}^t$, then:

(a) if there exist j sets $\mathcal{S}_{i_1}^t, \ldots, \mathcal{S}_{i_j}^t$ (with $2 \leq j \leq 3$) such that the spanned set is connected, set K' to be the union of the previous sets and S' the spanned set. We construct the state $(\mathcal{S}_1^{t+1}, K_1^{t+1}), \ldots, (\mathcal{S}_{m_{t+1}}^{t+1}, K_{m_{t+1}}^{t+1})$ at time $t+1$ as follows. From the list $(\mathcal{S}_1^t, K_1^t), \ldots, (\mathcal{S}_{m_t}^t, K_{m_t}^t)$ at time t, delete every pair $(\mathcal{S}_l^t, K_l^t)$ for which $\mathcal{S}_l^t \subset \mathcal{S}'$. Then add $(\mathcal{S}', K')$ to the list. Next increase t by 1 and return to step (a).

(b) else stop the algorithm and set $t = \tau$.

Properties (i)–(v) are obviously preserved by this procedure, and m_t is strictly decreasing with t, so the algorithm must eventually stop.

To affirm that property (vi) holds, observe that if $\langle K\rangle \setminus \mathcal{S}^\tau$ is nonempty, then there exists a site $y \in \langle K\rangle \setminus \mathcal{S}^\tau$ such that $y+\mathcal{N}$ contains 3 occupied sites in $\mathcal{S}^\tau$; otherwise y would not belong to $\langle K\rangle$ (since y does not belong to K). These neighbors must lie in at least two distinct sets $S_1, \dots, S_i$ since y is not in $\mathcal{S}^\tau$. Observe that the set spanned by $S_1, \dots, S_i$ is connected (any spanned site remains connected to the set that spanned it). Therefore, these sets are in an i-tuplet which corresponds to the case (a) of the algorithm (since y links the connected components). Therefore the algorithm should not have stopped at time τ.

Finally, to prove the lemma, note that we must have at least one time step (i.e., $\tau \geq 1$) since the cardinality of $\mathcal{S}$ is greater than N. Considering the last time step of the algorithm (from time $\tau - 1$ to time τ) and sets involved in the creation of $\mathcal{S}' = \mathcal{S}^\tau = \langle K\rangle$. These sets fulfill all of the required properties. □

To any connected set $\mathcal{S}$, one can associate the smallest rectangle, denoted $[\mathcal{S}]$, containing it. For any $k \geq 1$, if $\mathcal{S}$ is internally filled, then $[\mathcal{S}]$ is k-crossed.

PROPOSITION 3.8. *Let $k \geq 1$ and $t \geq 3$, and take any connected set $\mathcal{S}$ which is internally spanned, then some hierarchy of precision t with root-label $R_r = [\mathcal{S}]$ k-occurs.*

PROOF. The proof is an induction on the height (the y-dimension) of the rectangle. Let $\mathcal{S}$ be an internally filled connected set, and let $R = [\mathcal{S}]$. If $y(R) < 2t$, then the hierarchy with only one vertex r and $R_r = R$ k-occurs. Consider that $y(R) \geq 2t$, and assume that the proposition holds for any rectangle with height less than $y(R)$.

First observe that, using Lemma 3.7, there exist m_1 disjoint connected sets $\mathcal{S}_1^1, \dots, \mathcal{S}_{m_1}^1$ spanning $\mathcal{S}$ (with associated rectangles called $R_1^1 = [\mathcal{S}_1^1], \dots, R_{m_1}^1 = [\mathcal{S}_{m_1}^1]$). Assume that $R_1^t, \dots, R_{m_t}^t$ are defined; while one of the sets is of cardinality greater than 3, it is possible to define iteratively $R_1^{t+1} = [\mathcal{S}_1^{t+1}], \dots, R_{m_{t+1}}^{t+1} = [\mathcal{S}_{m_{t+1}}^{t+1}]$. This is obtained by harnessing Lemma 3.7 iteratively. Stop at the first time step, called T, for which the rectangle R' with smallest height satisfies $y(R) - y(R') \geq t$ (R' obviously exists since the height of R is greater than $2t$). Three possibilities can occur:

Case 1: $y(R) - y(R') \leq 2t$. Since R' is crossed, the induction hypothesis claims that there exists a hierarchy of precision t, called $\mathcal{H}'$, with root r' and root-label $R_{r'} = R'$. Furthermore, the event $A_k(R', R)$ occurs (since R is k-crossed), and it does not depend on the configuration inside R'. We construct $\mathcal{H}$ by adding the root r with label R to the hierarchy $\mathcal{H}'$. This hierarchy is indeed a hierarchy of

precision t, and it occurs since the event $A_k(R', R)$ is disjoint from the other events appearing in $\mathcal{H}'$.

Case 2: $y(R) - y(R') > 2t$ and $T = 1$ (the algorithm stopped at time 1). There exist m_1 rectangles $R_1, \dots, R_{m_1}$ corresponding to connected sets created by the algorithm at time $T = 1$. Moreover, R is the smallest rectangle containing $\langle R_1 \cup \cdots \cup R_{m_1} \rangle$. It is easy to see that events $\{R_i \text{ is } k\text{-crossed}\}$ occur disjointly due to the fact that sets $\mathcal{S}_i$ are disjoint. By the induction hypothesis, there exist m_1 hierarchies $\mathcal{H}_i$ $(i = 1, \dots, m_1)$ such that events in these hierarchies depend only on $\mathcal{S}_i$. The hierarchy created by adding the root r with label $R_r = R$ is a hierarchy of precision t because $y(R) - y(R_i) \geq t$ for some $i \leq m_1$ (the algorithm stopped at time 1). Moreover, the hierarchy k-occurs since R is the smallest rectangle containing $\langle R_1 \cup \cdots \cup R_{m_1} \rangle$ and hierarchies $\mathcal{H}_i$ $(i = 1, \dots, m_1)$ k-occur disjointly.

Case 3: $y(R) - y(R') > 2t$ and $T \geq 2$. Consider the rectangle R'' from which R' has been created and denote by $R_1, \dots, R_m$ its other "descendants" (set $R_1 = R'$). There exist hierarchies $\mathcal{H}_1, \dots, \mathcal{H}_m$ associated to $R_1, \dots, R_m$ which occur disjointly. Consider a root r with label R and a second vertex y with label R''. One can construct a hierarchy through the process of adding $m + 1$ additional edges (r, y) and (y, r_j) for $j = 1, \dots, m$ where r_j is the root of $\mathcal{H}_j$. This hierarchy k-occurs. It is therefore sufficient to check that it is a hierarchy of precision t. To do so, notice that $y(R'') - y(R) \leq t$ and $y(R'') - y(R_1) \geq t$ [since $y(R) - y(R'') \leq t$ and $y(R) - y(R') > 2t$]. □

3.3. *Proof of the upper bound.* We want to bound the probability of a hierarchy $\mathcal{H}$ with precision $\frac{T}{p} \log \frac{1}{p}$ to occur. Let

$$\Lambda_T^p = \inf \left\{ \sum_{n=0}^{N} W^p(R_n, R_{n+1}) : (R_n)_{0 \leq n \leq N} \in \mathfrak{D}_T^p \right\},$$

where $\mathfrak{D}_T^p$ denotes the set of finite increasing sequences of rectangles $R_0 \subset \cdots \subset R_N$ such that:

- $x_0 \leq p^{-1-2T}$;
- $y_0 \leq \frac{2T}{p} \log \frac{1}{p}$ and $y_N \geq \frac{1}{3p} \log \frac{1}{p}$;
- $y_{n+1} - y_n \leq \frac{T}{p} \log \frac{1}{p}$ for $n = 0, \dots, N - 1$.

Before estimating the probability of a hierarchy, we bound Λ_T^p when p and T go to 0.

PROPOSITION 3.9. *We have*

$$\liminf_{T \to 0} \liminf_{p \to 0} \Lambda_T^p \geq \frac{1}{6}.$$

PROOF. Let $\varepsilon, p > 0$. Choose $0 < T < \varepsilon$ so that

$$\int_T^\infty \max\{1-3y, 0\}\, dy = \frac{1}{6} - \varepsilon.$$

For two rectangles R and R' with dimensions (x, y) [resp., (x', y')], define $x = p^{-X}$ and $y = \frac{Y}{p}(\log\frac{1}{p})$ where $X, Y \in \mathbb{R}_+$ [resp., $x' = p^{X'}$ and $y' = \frac{Y'}{p}(\log\frac{1}{p})$]. With these notations, we obtain

$$\begin{aligned} W^p(R, R') &= \frac{p}{(\log(1/p))^2}(p^{-X'} - p^{-X})e^{-3Y'\log(1/p)} \\ &\quad - (Y' - Y)\frac{\inf\{\log(p^2 p^{-X'}), \log p\}}{\log(1/p)} \\ &= \frac{(p^{1-X'+3Y'} - p^{1-X+3Y'})}{(\log(1/p))^2} + (Y' - Y)\inf\{2 - X', 1\}. \end{aligned} \tag{3.6}$$

Consider a sequence of rectangles (R_n^p) in $\mathfrak{D}_T^p$, then

$$\begin{aligned} \sum_{n=0}^N W^p(R_n^p, R_{n+1}^p) &= \sum_{n=0}^N \frac{(p^{1-X_{n+1}^p+3Y_{n+1}^p} - p^{1-X_n^p+3Y_{n+1}^p})}{(\log(1/p))^2} \\ &\quad + \sum_{n=0}^N (Y_{n+1}^p - Y_n^p)\inf\{2 - X_{n+1}^p, 1\}. \end{aligned}$$

First assume that there exists n such that $1 - X_{n+1}^p + 3Y_{n+1}^p < -2\varepsilon$ for some values of p going to 0. Since

$$-p^{1-X_m^p+3Y_{m+1}^p} + p^{1-X_m^p+3Y_m^p} \quad \text{and} \quad p^{1-X_{m+1}^p+3Y_{m+1}^p} - p^{1-X_m^p+3Y_{m+1}^p}$$

are positive for every m, the first sum is larger than

$$\frac{p^{1-X_{n+1}^p+3Y_{n+1}^p} - p^{1-X_0^p+3Y_1^p}}{(\log(1/p))^2} \geq \frac{p^{-2\varepsilon} - p^{3Y_1^p-\varepsilon}}{(\log(1/p))^2} \to \infty.$$

We can thus assume that for any sequence $\liminf \sum_{n=0}^{N-1}(1 - X_{n+1}^p + 3Y_{n+1}^p) \geq -2\varepsilon$. We deduce

$$\begin{aligned} &\liminf \sum_{n=0}^N (Y_{n+1}^p - Y_n^p)\inf\{2 - X_{n+1}^p, 1\} \\ &\quad \geq \liminf \sum_{n=0}^N (Y_{n+1}^p - Y_n^p)\max\{1 - 3Y_{n+1}^p - 2\varepsilon, 0\} \\ &\quad \geq \int_T^\infty \max\{1 - 3(y + 2T) - 2\varepsilon, 0\}\, dy = \frac{1}{6} - 9\varepsilon, \end{aligned}$$

where in the last line we used $Y^p_{n+1} \le Y^p_n + 2T$. The claim follows readily. □

The two following lemmata are easy (yet technical for the second) but fundamental in the proof of Proposition 3.12 below. They authorize us to control the probability of a hierarchy even though there could be many seeds (and even large seeds).

LEMMA 3.10. *Let $k \ge 1$, $p > 0$ and let R be a rectangle with dimensions (x, y). Then we have for any $a, b > 0$,*

$$\mathbb{P}_p[R \text{ } k\text{-crossed}] \le p^{-k}\, \mathbb{P}_p\big[A_k([0,a]\times[0,b],[0,a+x]\times[0,b+y])\big].$$

PROOF. Simply note that if the rectangle $[a, a+x]\times[b, b+y]$ is crossed, and $\{0\}\times[a+1, a+k]$ is full, then $A_k([0,a]\times[0,b],[0,a+x,b+y])$ occurs. The proof follows using the FKG inequality. □

Let $N(\mathcal{H})$ be the number of vertices in the hierarchy.

LEMMA 3.11. *Let $\varepsilon, T > 0$, there exist $p_0, Q, k > 0$ such that for $p < p_0$, we have the following: Let $\mathcal{H}$ be a hierarchy of precision $\frac{T}{p}\log\frac{1}{p}$ with root label R satisfying $\frac{1}{3p}\log\frac{1}{p} \le y(R) \le \frac{1}{p}\log\frac{1}{p}$, then there exists $N \ge 1$ and $R_0 \subset \cdots \subset R_N$ rectangles satisfying the following properties*:

- *R_N has dimensions larger than R;*
- *R_0 has dimensions*
$$\Big(\sum_{u \text{ seed}} x(R_u),\ \sum_{u \text{ seed}} y(R_u)\Big);$$
- *$y(R_{n+1}) - y(R_n) \le \frac{2T}{p}\log\frac{1}{p}$ for every $0 \le n \le N-1$;*
- *we have*
$$\prod_{v\mapsto w} \mathbb{P}_p[A_k(R_w, R_v)] \le \big(p^{-k}Q^{y(R)}\big)^{N(\mathcal{H})} \prod_{n=0}^{N} \exp\Big[-(1-\varepsilon)W^p(R_n, R_{n+1})\frac{1}{p}\Big(\log\frac{1}{p}\Big)^2\Big].$$

PROOF. Let $\varepsilon, T > 0$. Fix $p_0, Q, k > 0$ so that Proposition 3.3 applies with ε and T. Invoking Proposition 3.3 [note that $\frac{T}{p}\log\frac{1}{p} \le y(R_v) \le y(R)$ for a normal vertex], we know

$$\prod_{v\mapsto w} \mathbb{P}_p[A_k(R_w, R_v)] \le \prod_{v\mapsto w} Q^{y(R_w)-y(R_v)} \exp\Big[-(1-\varepsilon)W^p(R_w, R_v)\frac{1}{p}\Big(\log\frac{1}{p}\Big)^2\Big].$$

It is thus sufficient to find an increasing sequence of rectangles satisfying the three first conditions such that

$$\sum_{v \mapsto w} W^p(R_w, R_v) \geq \sum_{n=0}^{N} W^p(R_n, R_{n+1}) - kN_{\text{splitter}}(\mathcal{H}) \log p,$$

where N_{splitter} is the number of splitters in the hierarchy. This can be done by induction. If the root of the hierarchy is a seed, the result is obvious since the sums are empty.

If the root r of the hierarchy is a normal vertex, then we consider the hierarchy with root v being the only descendant of r. By induction there exists a sequence satisfying all the assumptions. By setting R_{N+1} with dimensions $x(R_N)+x(R_r)-x(R_v)$ and $y(R_N)+y(R_r)-y(R_v)$, we obtain from the decreasing properties of $W^p(\cdot,\cdot)$ that $W^p(R_N, R_{N+1}) \leq W^p(R_v, R_r)$. The claim follows readily.

If the root r of the hierarchy is a splitter, then we consider the hierarchies with roots $v_1, \ldots, v_i$ ($i \in \{2, 3\}$) being the descendant of r. There exist sequences $(R_n^{(i)})_{n \leq N_i}$ for each of these hierarchies. Consider the following sequence:

$$R_1^{(1)}, \ldots, R_{N_1}^{(1)}, R_{N_1}^{(1)} + R_1^{(2)}, \ldots, R_{N_1}^{(1)} + \cdots + R_{N_i}^{(i)}, \max\bigl(R, R_{N_1}^{(1)} + \cdots + R_{N_i}^{(i)}\bigr),$$

where $R + R'$ is any rectangle with dimensions being the sum of the dimensions of R and R', and $\max(R, R')$ is a rectangle with dimensions being the maximum of the dimensions of R and R'.

Since the dimensions of $R_{N_1}^{(1)} + \cdots + R_{N_i}^{(i)}$ can exceed those of R by at most 3 (some space is allowed when combining two or more sets: they are only weakly connected), we obtain via a simple computation that

$$W^p\bigl[R_{N_1}^{(1)} + \cdots + R_{N_i}^{(i)}, \max\bigl(R, R_{N_1}^{(1)} + \cdots + R_{N_i}^{(i)}\bigr)\bigr] \leq -k \log p.$$

We deduce that removing the last rectangle in the sequence costs at most $k \log p$. The sequence then satisfies all the required conditions. □

PROPOSITION 3.12. *Let $\varepsilon > 0$. Then there exist k, $p_0 > 0$ such that*

$$\mathbb{P}_p(\mathcal{S} \textit{ is internally filled}) \leq \exp\left[-\left(\frac{1}{6} - \varepsilon\right)\frac{1}{p}\left(\log\frac{1}{p}\right)^2\right]$$

for $p < p_0$ and $\mathcal{S}$ a connected set satisfying $\frac{1}{3p}\log\frac{1}{p} \leq y([\mathcal{S}]) \leq \frac{1}{p}\log\frac{1}{p}$.

PROOF. Choose T, $p_0 > 0$ such that $\Lambda_T^p \geq \frac{1}{6} - \varepsilon$ for any $p < p_0$. Consider a connected set $\mathcal{S}$ satisfying the conditions of the proposition, and set $R = [\mathcal{S}]$. First assume that $x(R) \geq p^{-5}$. Then a simple computation implies that

$$\begin{aligned}\mathbb{P}_p[\mathcal{S} \text{ internally filled}] &\leq \mathbb{P}_p[R \text{ hor. traversable}] \\ &\leq \exp\bigl[-(1-\varepsilon)p^{-5}e^{-3\log(1/p)}\bigr] \leq \exp[-p^{-2}],\end{aligned}$$

and the claim follows in this case. Therefore, we can assume that $x(R) \le p^{-5}$.

Since $\mathcal{S}$ is internally filled, there exists a hierarchy $\mathcal{H}$ of precision $\frac{T}{p}\log\frac{1}{p}$ with root label R that occurs (Proposition 3.8). Moreover, the number of possible hierarchies is bounded by

$$N_t(R) \le \left[p^{-5} + \frac{1}{p}\log\frac{1}{p}\right]^{c[(1/p)\log(1/p)/((T/p)\log(1/p))]} \le p^{-6c(1/T)}$$

when p is small enough (Lemma 3.6). We deduce

$$\mathbb{P}_p(R \text{ is crossed}) \le p^{-6c(1/T)} \max\left\{\mathbb{P}_p[\mathcal{H} \text{ occurs}] \,:\, \mathcal{H} \text{ of precision } \frac{T}{p}\log\frac{1}{p} \text{ and root } R\right\}.$$

Bounding the probability of $\mathcal{S}$ being internally filled boils down to estimating the probability for a hierarchy to occur.

CLAIM. *The probability that a hierarchy $\mathcal{H}$ of precision $\frac{T}{p}\log\frac{1}{p}$ with root label R is k-occurring is smaller than* $\exp[-(\frac{1}{6}-3\varepsilon)\frac{1}{p}(\log\frac{1}{p})^2]$.

PROOF. Let $\mathcal{H}$ be a hierarchy. First assume that there exists one seed with root label R' satisfying $x(R') \ge p^{-1-2T}$. Then the probability that this seed is horizontally traversable is smaller than $\exp -p^{-1-T}$ when p is small enough (same computation as usual). The claim follows easily in this case.

We now assume that $x(R') \le p^{-1-2T}$ for every seed of the hierarchy. Using Lemma 3.11, there exist rectangles $R_0 \subset \cdots \subset R_N$ satisfying the conditions of the lemma such that

$$\prod_{v\mapsto w} \mathbb{P}_p[A_k(R_u, R_v)] \le \left(p^{-k}Q^{y(R)}\right)^{N(\mathcal{H})} \prod_{n=0}^{N} \exp\left[-(1-\varepsilon)W^p(R_n, R_{n+1})\frac{1}{p}\left(\log\frac{1}{p}\right)^2\right].$$

Using Lemma 3.10, we can transform this expression into

$$\prod_{u \text{ seed}} \mathbb{P}_p[R_u \text{ crossed}] \le p^{-kN(\mathcal{H})} \prod_{n=1}^{N_{\text{seed}}(\mathcal{H})} \mathbb{P}_p[A_k(\tilde{R}_i, \tilde{R}_{i+1})],$$

where $N_{\text{seed}}(\mathcal{H}) \le N(\mathcal{H})$ is the number of seeds of $\mathcal{H}$ and

$$\tilde{R}_i = [0, x(R_{u_1}) + \cdots + x(R_{u_i})] \times [0, y(R_{u_1}) + \cdots + y(R_{u_i})].$$

(In the previous formula, we have indexed the seeds by $u_1, \dots, u_{\tilde{N}}$.) We conclude that

$$\begin{aligned}
&\mathbb{P}_p[\mathcal{H} \text{ occurs}] \\
&\quad \le \prod_{u \text{ seed}} \mathbb{P}_p[R_u \text{ crossed}] \prod_{u \mapsto v} \mathbb{P}_p[A_k(R_u, R_v)] \\
&\quad \le \left(p^{-2k} Q^{y(R)}\right)^{N(\mathcal{H})} \\
&\qquad \times \exp\left\{-(1+\varepsilon)\left[\sum_{n=0}^{\tilde{N}} W^p(\tilde{R}_i, \tilde{R}_{i+1}) + \sum_{n=0}^{N} W^p(R_i, R_{i+1})\right]\frac{1}{p}\left(\log\frac{1}{p}\right)^2\right\} \\
&\quad \le e^{\varepsilon(1/p)(\log(1/p))^2} \exp\left[-(1-\varepsilon)\left(\frac{1}{6}-\varepsilon\right)\frac{1}{p}\left(\log\frac{1}{p}\right)^2\right]
\end{aligned}$$

since the sequence $\tilde{R}_0, \dots, \tilde{R}_{\tilde{N}}, R_0, \dots, R_N$ is in $\mathfrak{D}_T^p$ for p small enough (we have excluded the case where the seeds are too large), the number of vertices is uniformly bounded by $3^{2/T}$ when p goes to 0 and $y(R) \le \frac{1}{p}\log\frac{1}{p}$. □

We can now conclude by the following computation:

$$\begin{aligned}
\mathbb{P}_p(\mathcal{S} \text{ internally filled}) &\le p^{-kc(1/T)} \exp\left(-\left(\frac{1}{6}-3\varepsilon\right)\frac{1}{p}\left(\log\frac{1}{p}\right)^2\right) \\
&\le \exp\left[-\left(\frac{1}{6}-4\varepsilon\right)\frac{1}{p}\left(\log\frac{1}{p}\right)^2\right]
\end{aligned}$$

for p small enough. □

PROOF OF LOWER BOUND IN THEOREM 1.1. Let $p_0, k > 0$ be such that

$$\mathbb{P}_p(\mathcal{S} \text{ internally filled}) \le e^{-(1/6-\varepsilon)(1/p)(\log(1/p))^2}$$

for any $p < p_0$ and any connected set $\mathcal{S}$ such that

$$\frac{1}{3p}\log\frac{1}{p} \le y([\mathcal{S}]) \le \frac{1}{p}\log\frac{1}{p}.$$

Let E be the event that the origin is spanned by the configuration in $[-\frac{1}{p}\log\frac{1}{p}, \frac{1}{p}\log\frac{1}{p}]^2$, the probability of this event goes to 0. Indeed, two cases are possible. Either the origin is occupied, which occurs with probability p going to 0, or the origin is not occupied at time 0. In this case, there must exist a neighborhood containing 3 occupied sites at distance less than $\frac{1}{p}\log\frac{1}{p}$ to the origin. This probability goes to 0 when p goes to 0.

Let F be the event that no rectangle in $[-L, L]^2$ of perimeter between $\frac{1}{3p}\log\frac{1}{p}$ and $\frac{1}{p}\log\frac{1}{p}$ is crossed, where

$$L = \exp\left[\left(\frac{1}{12}-\varepsilon\right)\frac{1}{p}\left(\log\frac{1}{p}\right)^2\right].$$

In this case, Lemma 3.7 implies that no connected set $\mathcal{S} \subset [-L, L]^2$ such that $y([\mathcal{S}]) \geq \frac{1}{p}\log\frac{1}{p}$ is crossed. Indeed, if it was the case, we could construct a connected set $\mathcal{S}$ with $\frac{1}{3p}\log\frac{1}{p} \leq y([\mathcal{S}]) \leq \frac{1}{p}\log\frac{1}{p}$ which is internally filled using Lemma 3.7.

It is easy to see that if E and F hold, we must have $\log T \geq (\frac{1}{12}-\varepsilon)\frac{1}{p}(\log\frac{1}{p})^2$. Indeed, the information must necessarily come from outside the box $[-L, L]^2$. Then

$$\begin{aligned}\mathbb{P}_p&\left[\log T \leq \left(\frac{1}{12}-\varepsilon\right)\frac{1}{p}\left(\log\frac{1}{p}\right)^2\right]\\ &\leq \mathbb{P}_p(E)+\mathbb{P}_p(F)\\ &\leq \mathbb{P}_p(E)+p^{-8}e^{2(1/12-\varepsilon)(1/p)(\log(1/p))^2}e^{-(1/6-\varepsilon)(1/p)(\log(1/p))^2},\end{aligned}$$

where p^{-8} bounds the number of possible dimensions of $[\mathcal{S}]$ and $\exp[2(\frac{1}{12}-\varepsilon)\frac{1}{p}(\log\frac{1}{p})^2]$ the number of possible locations for its bottom-left corner. When p goes to 0, the right-hand side converges to 0 and the lower bound follows. □

REMARK 3.13. Recently, improvements of estimates for the threshold of simple bootstrap percolation have been proved [17, 23]. It is an interesting question to try to improve the estimates in our case. We mention that a major difficulty will come from the fact that "small" seeds are not excluded in the hierarchy growth.

REMARK 3.14. Our approach gives a much improved upper bound, compared to [24, 25], for the semi-oriented bootstrap percolation. But at this point we do not have a sharp threshold result, due to the lack of a corresponding argument for the lower bound.

Acknowledgments. The first author would like to thank Stanislav Smirnov for his constant support. The second author is grateful to FOM for its longstanding support of the Mark Kac seminar; this gave rise to a visit of the first author to the Netherlands, which is when our collaboration started.

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