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LOCAL GAUSSIAN FLUCTUATIONS IN THE AIRY AND DISCRETE PNG PROCESSES

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We prove that the Airy process, $\mathcal{A}(t)$, locally fluctuates like a Brownian motion. In the same spirit we also show that, in a certain scaling limit, the so-called discrete polynuclear growth process (PNG) behaves like a Brownian motion.

1. Introduction.

1.1. *The Airy process.* The central object of study in this paper is the local behavior of the Airy process, $t \to \mathcal{A}(t)$, $t \in \mathbb{R}$ [13]. The Airy process is a one-dimensional process with continuous paths [6, 13]. The interest in this process is mainly due to the fact that it is the limit of a number of processes appearing in the random matrix literature. One example is the top curve in Dyson's Brownian motion (see [3]), which, when appropriately rescaled, converges to the Airy process; see, for instance, [2] and [7]. Another example is the boundary of the north polar region in the Aztec diamond (see [4, 5] and [8]), a discrete process also converging to the Airy process [8]. A third example, the discrete polynuclear growth model (PNG) [7, 9], will be described in some detail in Section 1.3 where we also state a theorem about its local (in a certain sense) fluctuations.

A precise definition of $\mathcal{A}(t)$ goes as follows:

The extended Airy kernel [2, 10, 13] is defined by

$$A_{s,t}(x, y) = \begin{cases} \int_0^\infty e^{-z(s-t)} \operatorname{Ai}(x+z) \operatorname{Ai}(y+z)\, dz, & \text{if } s \geq t, \\ -\int_{-\infty}^0 e^{z(t-s)} \operatorname{Ai}(x+z) \operatorname{Ai}(y+z)\, dz, & \text{if } s < t, \end{cases} \tag{1.1}$$

where Ai is the Airy function. $A_{s,s}(x, y)$ is easily seen to be the ordinary Airy kernel [15]. Given $\xi_1, \dots, \xi_m \in \mathbb{R}$ and $t_1 < \cdots < t_m$ in $\mathbb{R}$, we define f on $\{t_1, \dots, t_m\} \times \mathbb{R}$ by

$$f(t_i, x) = \chi_{(\xi_i, \infty)}(x).$$

It is shown in [7] that

$$f^{1/2}(s, x) A_{s,t}(x, y) f^{1/2}(t, y)$$

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is the integral kernel of a trace class operator on $L^2(\{t_1,\dots,t_m\}\times\mathbb{R})$ where we have counting measure on $\{t_1,\dots,t_m\}$ and Lebesgue measure on $\mathbb{R}$. The Airy process, $t \to \mathcal{A}(t)$, is the stationary stochastic process with finite-dimensional distributions given by

$$\mathbb{P}[\mathcal{A}(t_1) \le \xi_1,\dots,\mathcal{A}(t_m) \le \xi_m] = \det(I - f^{1/2} A f^{1/2})_{L^2(\{t_1,\dots,t_m\}\times\mathbb{R})}.$$

The determinant in the right-hand side is a Fredholm determinant.

Our main theorem states that if we condition the Airy process to be at some given point at time t_1, it will then behave, on a local scale, like a Brownian motion.

THEOREM 1.1. *Let $\varepsilon > 0$ be small, $t_1 \in \mathbb{R}$ and $t_i = t_{i-1} + s_i\varepsilon$, $2 \le i \le m$, where $s_2,\dots,s_m > 0$. Also, let $p_1 \in \mathbb{R}$ and define the sets A_i, $i = 2,\dots,m$, by*

$$A_i = \{x \in \mathbb{R} | p_1 + a_i\sqrt{\varepsilon} \le x \le p_1 + b_i\sqrt{\varepsilon}\}$$

where a_i, b_i are given real numbers. It holds that

$$\mathbb{P}[\mathcal{A}(t_2) \in A_2,\dots,\mathcal{A}(t_m) \in A_m | \mathcal{A}(t_1) = p_1] = \int_{a_2}^{b_2} dx_2 \cdots \int_{a_m}^{b_m} dx_m \frac{1}{\sqrt{4\pi s_2}} e^{-x_2^2/(4s_2)} \prod_{i=3}^{m} \frac{1}{\sqrt{4\pi s_i}} e^{-(x_i - x_{i-1})^2/(4s_i)} + E,$$

where

$$|E| \le \sqrt{\varepsilon}\log\varepsilon^{-1} \prod_{i=2}^{m} (b_i - a_i) C_{p_1,s_2,\dots,s_m}.$$

Figure 1 describes the setup in the theorem.

REMARK 1. A couple of previous results about the Airy process are the following:

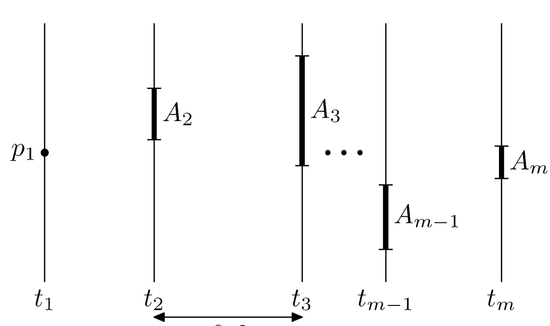

FIG. 1. *Conditioned that $\mathcal{A}(t_1) = p_1$, Theorem 1.1 gives the approximate probability for the process to move through the sets A_i. Note that $t_{i+1} - t_i \sim \varepsilon$ and $|A_i| \sim \sqrt{\varepsilon}$.*

In [13] it is shown that

$$\mathrm{Var}\big(\mathcal{A}(t) - \mathcal{A}(0)\big) = 2t + \mathcal{O}(t^2)$$

as $t \to 0$.

In [1] (see also [16]), the long-distance covariance asymptotics for the Airy process is calculated to be

$$\mathbb{E}[\mathcal{A}(t)\mathcal{A}(0)] - \mathbb{E}[\mathcal{A}(t)]\mathbb{E}[\mathcal{A}(0)] = t^{-2} + at^{-4} + \mathcal{O}(t^{-6})$$

as $t \to \infty$, where a is a known constant. This proves that $\mathcal{A}(t)$ is not a Markov process since this would imply exponential decay.

REMARK 2. Given Theorem 1.1, it is natural to ask the corresponding question about processes converging to the Airy process. Theorem 1.3 in Section 1.3 below provides such a result for the discrete polynuclear growth process.

1.2. *The extended Airy point process.* We now present another construction [7] of the Airy process that will help us in analyzing its local behavior.

Let $m \in \mathbb{Z}_+$ be arbitrary and let $t_1 < t_2 < \cdots < t_m$ be points in $\mathbb{R}$ which we shall think of as times. Define

$$E = \mathbb{R}_{t_1} \cup \mathbb{R}_{t_2} \cup \cdots \cup \mathbb{R}_{t_m}.$$

We shall refer to $\mathbb{R}_{t_j}$ as time line t_j. We define X to be the space of all locally finite countable configurations of points (or particles) in E. Locally finite means that, if $x = (x_1, x_2, \ldots) \in X$, then, for any bounded set $C \subset E$, it holds that $\#(C \cap x) < \infty$. Here $\#B$ represents the number of points in the set B. One can construct a σ-algebra on X from the cylinder sets: Let $B \subset E$ be any bounded Borel set and let $n \geq 0$. Define

$$C_n^B = \{x \in X : \#B = n\}$$

to be a cylinder set and Σ to be the minimal σ-algebra that contains all cylinder sets. One can now define probability measures on the space (X, Σ). The extended Airy point process is an example of such a measure and it will be described below.

For the sake of convenience, we will often denote the extended Airy kernel by $A(x, y)$ instead of $A_{t_i,t_j}(x, y)$ when it is clear that $x \in \mathbb{R}_{t_i}$ and $y \in \mathbb{R}_{t_j}$. Let $z_1, \ldots, z_k$ be points in E. The k-point correlation function is defined by

$$R(z_1, \ldots, z_k) = \det[A(z_i, z_j)]_{i,j=1}^k. \tag{1.2}$$

It is possible to show that these correlation functions determine a probability measure on (X, Σ), the extended Airy point process, by demanding that the following identity holds [14]:

$$\mathbb{E}\left[\prod_{i=1}^n \frac{\#B_i!}{(\#B_i - k_i)!}\right] = \int_{B_1^{k_1} \times \cdots \times B_n^{k_n}} R(z_1, \ldots, z_k)\, dz. \tag{1.3}$$

Here $B_1, \dots, B_n$ are disjoint Borel subsets of E and $k_i \in \mathbb{Z}_+$, $1 \le i \le n$, are such that $k_1 + \cdots + k_n = k$.

It is possible to show that, at each time line $\mathbb{R}_{t_i}$, there is almost surely a largest particle, $\lambda(t_i)$, and

$$(\lambda(t_1), \dots, \lambda(t_m)) = (\mathcal{A}(t_1), \dots, \mathcal{A}(t_m)) \tag{1.4}$$

in distribution [7]. It is through this representation that we are able to show that the Airy process behaves locally as a Brownian motion.

1.3. *Discrete polynuclear growth.* The second object of interest in this paper is the discrete polynuclear growth model (PNG) [7, 9]. It is defined by

$$h(x, t+1) = \max\bigl(h(x-1, t), h(x, t), h(x+1, t)\bigr) + \omega(x, t+1), \tag{1.5}$$

where $x \in \mathbb{Z}$, $t \in \mathbb{N}$, $h(x, 0) = 0\ \forall x \in \mathbb{Z}$ and $\omega(x, t+1) = 0$ if $|x| > t$ or if $t - x$ is even; otherwise $\omega(x, t+1)$ are independent geometric random variables with

$$\mathbb{P}[\omega(x, t+1) = m] = (1-q)q^m, \qquad 0 < q < 1. \tag{1.6}$$

It is convenient to extend the process to all $x \in \mathbb{R}$ by setting $h(x, t) = h(\lfloor x \rfloor, t)$. A description of this process using words and pictures goes as follows:

At time $t = 1$ a block of width 1 and height $\omega(0, 1)$ appears over the interval $[0, 1)$. This block then grows sideways one unit in both directions and at time $t = 2$ two blocks of width 1 and heights $\omega(-1, 2)$, $\omega(1, 2)$ are placed on top of it over the intervals $[-1, 0)$ and $[1, 2)$, respectively. These blocks now grow one unit in each direction disregarding overlaps. At time $t = 3$ three new blocks are placed over $[-2, -1)$, $[0, 1)$ and $[2, 3)$. This procedure goes on producing at each time the curve $h(x, t)$ that can be thought of as a growing interface. Figure 2 shows a realization for $t = 1, 2, 3$.

The process h is closely connected to a growth model, $G(M, N)$, studied in [6]. Let $w(i, j)$, $(i, j) \in \mathbb{Z}_+^2$, be independent random variables with distribution given by (1.6). Define

$$G(M, N) = \max_{\pi} \sum_{(i,j) \in \pi} w(i, j)$$

where the maximum is taken over all up/right paths from $(1, 1)$ to (M, N). One can think of $G(M, N)$ as a point-to-point last-passage time and

$$G_{pl}(N) = \max_{|K| < N} G(N + K, N - K)$$

as a point-to-line last-passage time. In [7] it is shown that

$$G(i, j) = h(i - j, i + j - 1).$$

The definition of G_{pl} therefore inspires the study of $K \to h(2K, 2N - 1)$, that is, the height curve at even sites at time $2N - 1$.

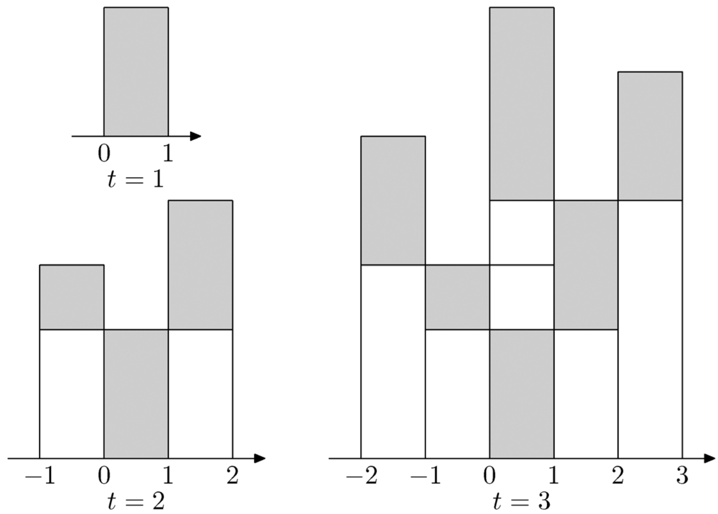

FIG. 2. *A sample of the discrete PNG process for $t = 1, 2, 3$. The shaded blocks represent the growth due to the random variables $\omega(x, t)$.*

In [7] the rescaled process, $t \to H_N(t)$, $t \in \mathbb{R}$, is, for appropriate t, defined by

$$dN^{1/3} H_N(t) = h\left(2\frac{1+\sqrt{q}}{1-\sqrt{q}} d^{-1} N^{2/3} t, 2N - 1\right) - \frac{2\sqrt{q}}{1-\sqrt{q}} N$$

and for the rest of $\mathbb{R}$ by the use of linear interpolation. The constant d is given by

$$d = \frac{(\sqrt{q})^{1/3}(1+\sqrt{q})^{1/3}}{1-\sqrt{q}}.$$

The main result about $H_N(t)$ in [7] is the following theorem:

THEOREM 1.2 [7]. *Let $\mathcal{A}(t)$ be the Airy process defined by its finite-dimensional distributions and let T be an arbitrary positive number. There is a continuous version of $\mathcal{A}(t)$ and*

$$H_N(t) \to \mathcal{A}(t) - t^2$$

as $N \to \infty$ in the weak-topology of probability measures on $C(-T, T)$.*

In particular this theorem shows that the fluctuations of h are of order $N^{1/3}$ and that nontrivial correlations in the transversal direction show up when looking at times t_i where $t_{i+1} - t_i \sim N^{2/3}$.

Motivated by Theorems 1.1 and 1.2, one could guess that h, on a time scale of order N^γ, $0 < \gamma < 2/3$, behaves like a Brownian motion. The theorem below shows that this is indeed the case.

Given some $m \in \mathbb{Z}_+$, set

$$K_1 = \frac{1+\sqrt{q}}{1-\sqrt{q}} d^{-1} N^{2/3} \tau_1,$$

$$K_{i+1} = K_i + \frac{1+\sqrt{q}}{1-\sqrt{q}} d^{-1} s_{i+1} N^{\gamma}, \qquad i = 1, \dots, m-1,$$

where $0 < \gamma < \frac{2}{3}$ and $\tau_1, s_i > 0$ are real numbers such that $K_i \in \mathbb{Z}$. Define

$$J_1 = \frac{2\sqrt{q}}{1-\sqrt{q}} N + \psi\, d N^{1/3} \in \mathbb{Z}_+$$

where ψ is any real number such that $J_1 \in \mathbb{Z}$.

THEOREM 1.3. *Define the sets A_i, $i = 2, \dots, m$, by*

$$A_i = \{ j \in Z_+ | j = J_1 + x_i\, d N^{\gamma/2}, a_i \le x_i \le b_i \}$$

where a_i, b_i are given real numbers. There exists $c > 0$ such that

$$\begin{aligned} &\mathbb{P}[h(2K_2, 2N-1) \in A_2, \dots, h(2K_m, 2N-1) \in A_m \\ &\qquad\qquad\qquad |h(2K_1, 2N-1) = J_1] \\ &= \int_{a_2}^{b_2} dx_2 \cdots \int_{a_m}^{b_m} dx_m \frac{1}{\sqrt{4\pi s_2}} e^{-x^2/(4s_2)} \prod_{i=3}^{m} \frac{1}{\sqrt{4\pi s_i}} e^{-(x_i - x_{i-1})^2/(4s_i)} + E, \end{aligned}$$

where

$$|E| \le N^{-c} \prod_{i=2}^{m} (b_i - a_i) C_{\psi_1, s_2, \dots, s_m}.$$

2. Proof of Theorem 1.1. The connection (1.4) shows that we can prove the theorem by studying the largest particle in the extended Airy point process at times $t_1, \dots, t_m$.

The appearance of C in formulae below should be interpreted as follows: There exists a positive constant which may depend on p_i, s_i, $i = 2, \dots, m$, validating the inequality to the left when inserted instead of C. Other error terms will typically also depend on p_i, s_i.

Set $J_1 = [p_1 - \delta_1, p_1] \subset \mathbb{R}_{t_1}$ and $J_i = [p_i - \sqrt{\varepsilon}\delta_i, p_i] \subset \mathbb{R}_{t_i}$, $2 \le i \le m$, where $\delta_i > 0$ and $p_i = p_{i-1} + y_i \sqrt{\varepsilon}$, $y_i \in \mathbb{R}$. We also set $I_i = (p_i, \infty)$, $i = 1, \dots, m$.

We will show that

$$\begin{aligned} &\lim_{\delta_1, \dots, \delta_m \to 0^+} \frac{1}{\delta_2 \cdots \delta_m} \frac{\mathbb{P}[\#J_1 \ge 1, \dots, \#J_m \ge 1, \#I_1 = \cdots = \#I_m = 0]}{\mathbb{P}[\#J_1 \ge 1, \#I_1 = 0]} \\ &\quad = \frac{1}{\sqrt{(4\pi)^{m-1} s_2 \cdots s_m}} e^{-y_2^2/(4s_2) - \cdots - y_m^2/(4s_m)} + \mathcal{O}(\sqrt{\varepsilon} \log \varepsilon), \end{aligned} \tag{2.1}$$

implying Theorem 1.1.

The first step is to show that the probabilities in the numerator and denominator above can be approximated by appropriate expected values.

For $k, n \in \mathbb{Z}_+$ we shall use the common notation

$$n^{[k]} = n(n-1)\cdots(n-k+1).$$

Let J be an interval on some time line and let χ_A be the indicator function for the event A. Since

$$\#J - \chi_{\{\#J\geq 1\}} = \begin{cases} k-1, & \#J = k \geq 2, \\ 0, & \#J = 0, 1, \end{cases}$$

$$\#J^{[2]} = \#J(\#J-1) = \begin{cases} k(k-1), & \#J = k \geq 2, \\ 0, & \#J = 0, 1, \end{cases}$$

it holds that

$$0 \leq \#J - \chi_{\{\#J\geq 1\}} \leq \#J^{[2]}. \tag{2.2}$$

This together with the following facts will be useful:

$$\begin{aligned} &\mathbb{P}[\#J_1 \geq 1, \dots, \#J_m \geq 1, \#I_1 = 0] \\ &\quad - \mathbb{P}[\#J_1 \geq 1, \dots, \#J_m \geq 1, \#I_1 = \cdots = \#I_m = 0] \\ &\quad = \mathbb{P}[\#J_1 \geq 1, \dots, \#J_m \geq 1, \#I_1 = 0, (\#I_2 = \cdots = \#I_m = 0)^c] \\ &\quad = \mathbb{P}\left[\#J_1 \geq 1, \dots, \#J_m \geq 1, \#I_1 = 0, \bigcup_{i=2}^m \{\#I_i \neq 0\}\right] \\ &\quad \leq \sum_{i=2}^m \mathbb{P}[\#J_1 \geq 1, \dots, \#J_m \geq 1, \#I_i \neq \#I_1]. \end{aligned} \tag{2.3}$$

We now express the probabilities in terms of expected values. If we set

$$T(J_i) = \#J_i - \chi_{\{\#J_i\geq 1\}}, \tag{2.4}$$

then

$$\begin{aligned} &\mathbb{P}[\#J_1 \geq 1, \dots, \#J_m \geq 1, \#I_1 = 0] \\ &\quad = \mathbb{E}\big[\big(\#J_1 - T(J_1)\big)\cdots\big(\#J_m - T(J_m)\big)\cdot \chi_{\{\#I_1=0\}}\big] \\ &\quad = \mathbb{E}\big[\big(\#J_1\cdots\#J_m + U(J_1, \dots, J_m)\big)\cdot \chi_{\{\#I_1=0\}}\big], \end{aligned}$$

where U is defined by the last equality. In view of (2.2) and (1.3) we get, for example,

$$\begin{aligned} \mathbb{E}[T(J_1)\cdot\#J_2\cdots\#J_m] &\leq \mathbb{E}\big[\#J_1^{[2]}\cdot\#J_2\cdots\#J_m\big] \\ &= \int_{J_1^2\times J_2\times\cdots\times J_m} R(x_1, x_2, \dots, x_{m+1})\,dx = \mathcal{O}(\delta_1^2\cdot\delta_2\cdots\delta_m). \end{aligned}$$

Since $U(J_1,\dots,J_m)$ is a sum of terms like this one [at least one $T(J_i)$], we see that

$$\lim_{\delta_1,\dots,\delta_m\to 0^+}\frac{1}{\delta_1\cdots\delta_m}\mathbb{E}\big[U(J_1,\dots,J_m)\cdot\chi_{\{\#I_1=0\}}\big]=0.$$

Repetition of this argument shows together with (2.3) that

$$\begin{aligned}
&\lim_{\delta_1,\dots,\delta_m\to 0^+}\frac{1}{\delta_1\cdots\delta_m}\mathbb{P}[\#J_1\geq 1,\dots,\#J_m\geq 1,\#I_1=\cdots=\#I_m=0]\\
&\quad=\lim_{\delta_1,\dots,\delta_m\to 0^+}\frac{1}{\delta_1\cdots\delta_m}\mathbb{E}\big[\#J_1\cdots\#J_m\cdot\chi_{\{\#I_1=0\}}\big]\\
&\qquad+\mathcal{O}\Bigg(\sum_{i=2}^{m}\lim_{\delta_1,\dots,\delta_m\to 0^+}\frac{1}{\delta_1\cdots\delta_m}\mathbb{E}\big[\#J_1\cdots\#J_m\cdot\chi_{\{\#I_i\neq\#I_1\}}\big]\Bigg)
\end{aligned}$$

and also that

$$\lim_{\delta_1\to 0^+}\frac{1}{\delta_1}\mathbb{P}[\#J_1\geq 1,\#I_1=0]=\lim_{\delta_1\to 0^+}\frac{1}{\delta_1}\mathbb{E}\big[\#J_1\cdot\chi_{\{\#I_1=0\}}\big].$$

Later it will be shown that

$$\lim_{\delta_1,\dots,\delta_m\to 0^+}\frac{1}{\delta_1\cdots\delta_m}\mathbb{E}\big[\#J_1\cdots\#J_m\cdot\chi_{\{\#I_i\neq\#I_1\}}\big]=\mathcal{O}\big(\sqrt{\varepsilon}\log\varepsilon\big),\tag{2.5}$$

but let us first be constructive.

We want to show that

$$\begin{aligned}
&\lim_{\delta_1,\dots,\delta_m\to 0^+}\frac{1}{\delta_1\cdots\delta_m}\mathbb{E}\big[\#J_1\cdots\#J_m\cdot\chi_{\{\#I_1=0\}}\big]\\
&\quad=\lim_{\delta_1\to 0^+}\frac{1}{\delta_1}\mathbb{E}\big[\#J_1\cdot\chi_{\{\#I_1=0\}}\big]\qquad\qquad(2.6)\\
&\qquad\times\frac{1}{\sqrt{(4\pi)^{m-1}s_2\cdots s_m}}e^{-y_2^2/(4s_2)-\cdots-y_m^2/(4s_m)}+\mathcal{O}\big(\sqrt{\varepsilon}\big).
\end{aligned}$$

To start with, we need to find a representation of the left-hand side of (2.6) that is suitable for analysis:

$$\begin{aligned}
\mathbb{E}\big[\#J_1\cdots\#J_m\cdot\chi_{\{\#I_1=0\}}\big]&=\mathbb{E}\Big[\#J_1\cdots\#J_m\cdot\lim_{\lambda\to\infty}e^{-\lambda\#I_1}\Big]\\
&=\mathbb{E}\Bigg[\#J_1\cdots\#J_m\cdot\lim_{\lambda\to\infty}\sum_{k=0}^{\infty}\frac{(e^{-\lambda}-1)^k}{k!}\#I_1^{[k]}\Bigg]\\
&=\mathbb{E}\Bigg[\#J_1\cdots\#J_m\cdot\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\#I_1^{[k]}\Bigg]\\
&=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\mathbb{E}\big[\#J_1\cdots\#J_m\cdot\#I_1^{[k]}\big].
\end{aligned}$$

In the second equality we have used the formula

$$e^{\lambda n} = \sum_{k=0}^{\infty} \frac{(e^{\lambda} - 1)^k}{k!} n^{[k]}. \tag{2.7}$$

In the fourth equality we take the sum out of the expectation. By Fubini's theorem we are allowed to do this since

$$\begin{aligned}\mathbb{E}\left[\#J_1 \cdots \#J_m \cdot \sum_{k=0}^{\infty} \frac{\#I_1^{[k]}}{k!}\right] &\le \mathbb{E}\left[\#J_1 \cdots \#J_m \cdot \sum_{k=0}^{\infty} \frac{\#I_1^{k}}{k!}\right] \\ &= \mathbb{E}[\#J_1 \cdots \#J_m \cdot e^{\#I_1}] \\ &\le \mathbb{E}[\#J_1^2 \cdots \#J_m^2]^{1/2} \mathbb{E}[e^{2\#I_1}]^{1/2} < \infty.\end{aligned}$$

In fact $\mathbb{E}[z^{\#I_1}]$ is an entire function in z [14].

Another technical issue we need to deal with is to prove that

$$\begin{aligned}&\lim_{\delta_1,\dots,\delta_m \to 0^+} \frac{1}{\delta_1 \cdots \delta_m} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mathbb{E}\big[\#J_1 \cdots \#J_m \cdot \#I_1^{[k]}\big] \\ &\quad = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lim_{\delta_1,\dots,\delta_m \to 0^+} \frac{1}{\delta_1 \cdots \delta_m} \mathbb{E}\big[\#J_1 \cdots \#J_m \cdot \#I_1^{[k]}\big] \\ &\quad = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{I_1^k} (\sqrt{\varepsilon})^{m-1} R(p_1, \dots, p_m, x_1, \dots, x_k)\, dx.\end{aligned}$$

Please recall definition (1.2) and note that the second equality is immediate from (1.3). Define $G_k(z_1, \dots, z_m)$, $z_i \in \mathbb{R}_{t_i}$, by

$$G_k(z_1, \dots, z_m) = \frac{(-1)^k}{k!} \int_{I_1^k} R(z_1, \dots, z_m, x_1, \dots, x_k)\, dx. \tag{2.8}$$

The identity sought for is

$$\begin{aligned}&\lim_{\delta_1,\dots,\delta_m \to 0^+} \frac{1}{\delta_1 \cdots \delta_m} \sum_{k=0}^{\infty} \int_{J_1 \times \cdots \times J_m} G_k(z_1, \dots, z_m)\, dz \\ &\qquad = \sum_{k=0}^{\infty} (\sqrt{\varepsilon})^{m-1} G_k(p_1, \dots, p_m).\end{aligned} \tag{2.9}$$

This will hold if for some neighborhood Ω of $(p_1, \dots, p_m)$ there exist constants $C_k > 0$ such that

$$|G_k(z_1, \dots, z_m)| \le C_k$$

if $(z_1, \dots, z_m) \in \Omega$ and

$$\sum_{k=0}^{\infty} C_k < \infty.$$

That this is indeed the case follows from calculations similar to the ones appearing in the proof of Lemma 2.2 which is given at the end of this section.

The following lemma can be found in [11]:

LEMMA 2.1. *Let* $\alpha > 0$, *then*

$$\int_{-\infty}^{\infty} e^{\alpha z} \operatorname{Ai}(x+z) \operatorname{Ai}(y+z)\, dz = \frac{1}{\sqrt{4\pi\alpha}} e^{-(x-y)^2/(4\alpha)-(\alpha/2)(x+y)+\alpha^3/12}.$$

In this section we call this function $\phi_\alpha(x, y)$ or simply $\phi(x, y)$ when it is clear what α is. From Lemma 2.1 and the definition of the Airy kernel, it follows that, for $s < t$

$$\begin{aligned} A_{s,t}(x, y) &= \int_0^{\infty} e^{z(t-s)} \operatorname{Ai}(x+z) \operatorname{Ai}(y+z)\, dz - \phi_{t-s}(x, y) \\ &=: \widetilde{A}_{s,t}(x, y) - \phi_{t-s}(x, y). \end{aligned}$$

For $s \geq t$ it is convenient to set $\widetilde{A}_{s,t}(x, y) = A_{s,t}(x, y)$.

LEMMA 2.2. *Suppose that* $1 \leq v \leq m$, $v \in \mathbb{Z}$. *Then, for some* C *depending on* $p_1, \dots, p_m$,

$$\begin{aligned} &(\sqrt{\varepsilon})^{m-1} \int_{I_v^k} R(p_1, \dots, p_m, x_1, \dots, x_k)\, dx \\ &\quad = (\sqrt{\varepsilon})^{m-1} \phi(p_1, p_2)\phi(p_2, p_3) \cdots \phi(p_{m-1}, p_m) \\ &\qquad \times \int_{I_1^k} R(p_1, x_1, \dots, x_k)\, dx + \sqrt{\varepsilon}\mathcal{O}\bigl((Ck)^{(k+m)/2}\bigr). \end{aligned} \tag{2.10}$$

Furthermore, if $v \geq 2$, *then*

$$\begin{aligned} &(\sqrt{\varepsilon})^{m-1} \int_{I_1} dx \int_{I_v} dy\, R(p_1, \dots, p_m, x, y) \\ &\quad = (\sqrt{\varepsilon})^{m-1} \phi(p_1, p_2)\phi(p_2, p_3) \cdots \phi(p_{m-1}, p_m) \\ &\qquad \times \left(\int_{I_1^2} R(p_1, x_1, x_2)\, dx + \int_{I_1} R(p_1, x)\, dx \right) + \mathcal{O}\bigl(\sqrt{\varepsilon} \log \varepsilon\bigr). \end{aligned} \tag{2.11}$$

From (2.10) we now get (2.6).

We turn now to (2.5). Clearly

$$\begin{aligned} &\mathbb{E}\bigl[\#J_1 \cdots \#J_m \cdot \chi_{\{\#I_i \neq \#I_1\}}\bigr] \\ &\quad \leq \mathbb{E}[\#J_1 \cdots \#J_m \cdot (\#I_i - \#I_1)^2] \\ &\quad = \mathbb{E}\bigl[\#J_1 \cdots \#J_m \cdot \bigl(\#I_i^{[2]} + \#I_1^{[2]} + \#I_i + \#I_1 - 2\#I_1\#I_i\bigr)\bigr]. \end{aligned}$$

We now obtain (2.5) since

$$(\sqrt{\varepsilon})^{m-1}\Big(\int_{I_i^2} R(p_1,\dots,p_m,x,y)\,dx\,dy + \int_{I_1^2} R(p_1,\dots,p_m,x,y)\,dx\,dy + \int_{I_i} R(p_1,\dots,p_m,x)\,dx + \int_{I_1} R(p_1,\dots,p_m,x)\,dx - 2\int_{I_1\times I_i} R(p_1,\dots,p_m,x,y)\,dx\,dy\Big) = \mathcal{O}(\sqrt{\varepsilon}\log\varepsilon) \tag{2.12}$$

by Lemma 2.2.

To get (2.1) we need one more result, namely that

$$\lim_{\delta_1\to 0^+}\frac{1}{\delta_1}\mathbb{E}[\#J_1\chi_{\{\#I_1=0\}}] > 0. \tag{2.13}$$

Let $F_2(s)$ be the Tracy–Widom distribution function corresponding to the largest eigenvalue in the Gaussian Unitary Ensemble (GUE) [15]. Then

$$\begin{aligned}\lim_{\delta_1\to 0^+}&\frac{1}{\delta_1}\mathbb{E}[\#J_1\chi_{\{\#I_1=0\}}]\\ &= \lim_{\delta_1\to 0^+}\frac{1}{\delta_1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{J_1}dx_0\int_{I_1^k}d^kx\,\det(A(x_i,x_j))_{0\le i,j\le k}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{I_1^k}\det(A(x_i,x_j))_{0\le i,j\le k}\,d^kx = F_2'(p_1),\end{aligned} \tag{2.14}$$

where in the last row $x_0 = p_1$. The last equality can be obtained by differentiating the corresponding equality for the distribution function $F_2(t)$ [15]; we omit the details here. The first equality has been shown above and the second is a special case of (2.9). Since $F_2'(s) > 0$ for all $s \in \mathbb{R}$ (see [15]), we obtain (2.13).

What is still left is to prove Lemma 2.2.

PROOF OF LEMMA 2.2. We start with (2.10). For $0 \le r \le m-1$ and $k \ge 1$, define $D_r(k)$ by

$$D_r(k) = (\sqrt{\varepsilon})^r\phi(p_1,p_2)\phi(p_2,p_3)\cdots\phi(p_r,p_{r+1})\int_{I_v^k}dx \times \begin{vmatrix} A(p_{r+1},p_1) & \sqrt{\varepsilon}A(p_{r+1},p_{r+2}) & \dots & \sqrt{\varepsilon}A(p_{r+1},p_m) & A(p_{r+1},x_j)\\ \vdots & \vdots & & \vdots & \vdots\\ A(p_m,p_1) & \sqrt{\varepsilon}A(p_m,p_{r+2}) & \dots & \sqrt{\varepsilon}A(p_m,p_m) & A(p_m,x_j)\\ A(x_i,p_1) & \sqrt{\varepsilon}A(x_i,p_{r+2}) & \dots & \sqrt{\varepsilon}A(x_i,p_m) & A(x_i,x_j)\end{vmatrix}.$$

In the determinant $1 \le i, j \le k$ and for $r = 0$ we set the empty product in front of the integral to 1. Please note that $D_0(k)$ is equal to the left-hand side in (2.10).

We let $\widetilde{D}_r(k)$ be almost the same as $D_r(k)$. The only difference is that we put in $\widetilde{A}(p_{r+1}, p_{r+2})$ in position $(1,2)$ in the matrix instead of $A(p_{r+1}, p_{r+2})$. By using induction we shall now prove that

$$D_0(k) = D_r(k) + \sqrt{\varepsilon}\mathcal{O}\big((Ck)^{(k+m)/2}\big) \tag{2.15}$$

for $0 \le r \le m-1$. Clearly (2.15) holds if $r = 0$. Suppose now that (2.15) holds for some r such that $0 \le r \le m-2$. By expanding the determinant in $D_r(k)$ along the first row we see that

$$D_r(k) = D_{r+1}(k) + \widetilde{D}_r(k). \tag{2.16}$$

What has to be proved is hence that

$$\widetilde{D}_r(k) = \sqrt{\varepsilon}\mathcal{O}\big((Ck)^{(k+m)/2}\big).$$

To do this, Hadamard's inequality will come in handy, but before we recall this inequality we present a lemma which will be frequently used from now on. The proof is readily obtained from Lemma 2.1 and the standard estimates (see [12]):

$$|\operatorname{Ai}(x)| \le C_M e^{-2|x|^{3/2}/3},$$
$$|\operatorname{Ai}'(x)| \le C_M \sqrt{|x|} e^{-2|x|^{3/2}/3}$$

that hold for $x \ge -M$. □

LEMMA 2.3. *Suppose that $s < t$ and $M > 0$. For $x, y \ge -M$ and any $\lambda > 0$ it holds that*

$$|A_{t,s}(x,y)| \le C_{M,\lambda} e^{-\lambda(x+y)},$$
$$A_{t,s}(x,y) = A_{t,t}(x,y) + \mathcal{O}(t-s)e^{-\lambda(x+y)},$$
$$A_{s,t}(x,y) = A_{t,t}(x,y) - \big(1+\mathcal{O}(t-s)\big)\frac{1}{\sqrt{4\pi(t-s)}}e^{-(x-y)^2/(4(t-s))} + \mathcal{O}(t-s)e^{-\lambda(x+y)}.$$

The errors depend only on M and λ. Moreover,

$$|A_{s,s}(x+\alpha, y) - A_{s,s}(x,y)| \le \alpha C_{M,\lambda} e^{-\lambda(x+y)}$$

for all $\alpha > 0$.

Let $B = (b_{i,j})_{1\le i,j\le n}$, $b_{i,j} \in \mathbb{R}$, be a matrix. Hadamard's inequality states that

$$|\det B| \le \left(\prod_{i=1}^{n}\sum_{j=1}^{n} b_{ji}^2\right)^{1/2}. \tag{2.17}$$

Below we find upper bounds for the equivalent to $\sum_{j=1}^n b_{ji}^2$ in the matrix appearing in $\widetilde{D}_r(k)$.

Column 1:

$$\sum_{j=r+1}^{m} A^2(p_{r+1}, p_1) + \sum_{j=1}^{k} A^2(x_j, p_1) \le C(k+m).$$

Column 2:

$$\varepsilon\left(\widetilde{A}^2(p_{r+1}, p_{r+2}) + \sum_{j=r+2}^{m} A^2(p_j, p_{r+2}) + \sum_{j=1}^{k} A^2(x_j, p_{r+2})\right)$$

$$\le \varepsilon \begin{cases} C(k+m), & \text{if } v \ge r+2, \\ Cm + C\sum_{j=1}^{k}\bigl(\widetilde{A}(x_j, p_{r+2}) - \phi(x_j, p_{r+2})\bigr)^2, & \text{if } v < r+2. \end{cases}$$

Columns $3, \dots, m-r$ ($r+3 \le i \le m$):

$$\varepsilon\left(\sum_{j=r+1}^{m} A^2(p_j, p_i) + \sum_{j=1}^{k} A^2(x_j, p_i)\right) \le C(k+m).$$

Last k columns ($1 \le i \le k$):

$$\sum_{j=r+1}^{m} A^2(p_j, x_i) + \sum_{j=1}^{k} A^2(x_j, x_i)$$

$$\le \begin{cases} \sum_{j=r+1}^{v-1}\bigl(\widetilde{A}(p_j, x_i) - \phi(p_j, x_i)\bigr)^2 + Cke^{-2x_i}, & \text{if } v \ge r+2, \\ C(k+m)e^{-2x_i}, & \text{if } v < r+2. \end{cases}$$

Next we multiply everything together, take the square root and then integrate. Assume that $v < r+2$:

$$\int_{I_v^k}\Biggl[C(k+m)\varepsilon\left(C + C\sum_{j=1}^{k}\bigl(\widetilde{A}(x_j, p_{r+2}) - \phi(x_j, p_{r+2})\bigr)^2\right)$$

$$\times \bigl(C(m+k)\bigr)^{m-r-2}\bigl(C(k+m)\bigr)^k e^{-2(x_1+\cdots+x_k)}\Biggr]^{1/2} dx$$

$$\le \sqrt{\varepsilon}(Ck)^{(k+m)/2}\int_{I_v^k} e^{-(x_1+\cdots+x_k)}\left(1 + \sum_{j=1}^{k}\bigl(1 + \phi(x_j, p_{r+2})\bigr)\right) dx$$

$$\le \sqrt{\varepsilon}(Ck)^{(k+m)/2}.$$

The case $v \ge r+2$ can be treated similarly.

To obtain (2.10) it remains to show that

$$\int_{I_v^k} \det\begin{bmatrix} A(p_m,p_1) & A(p_m,x_j) \\ A(x_i,p_1) & A(x_i,x_j) \end{bmatrix}_{1\le i,j\le k} dx$$
$$= \int_{I_1^k} \det\begin{bmatrix} A(p_1,p_1) & A(p_1,x_j) \\ A(x_i,p_1) & A(x_i,x_j) \end{bmatrix}_{1\le i,j\le k} dx + \sqrt{\varepsilon}\mathcal{O}\big((Ck)^{(k+m)/2}\big).$$

This is quite easily achieved using Hadamard's inequality and Lemma 2.3. We do not present the details here but instead go on to prove (2.11).

The first part of the proof will be similar to the proof of (2.10) and the second part is an application of Lemma 2.4 below.

Let $D_r(2)$ and $\widetilde{D}_r(2)$ be as defined above with the exception that the variables x_1 and x_2 are now integrated over I_1 and I_v, respectively. By construction $D_0(2)$ equals the left-hand side in (2.11). If we can show that

$$\widetilde{D}_r(2) = \mathcal{O}\big(\sqrt{\varepsilon}\big), \tag{2.18}$$

then by the same argument as above

$$D_0(2) = D_{m-1}(2) + \mathcal{O}\big(\sqrt{\varepsilon}\big).$$

To see this we shall only need the trivial fact that

$$|\det B| \le \prod_{i=1}^{n}\sum_{j=1}^{n}|b_{ji}|,$$

where as before B is a real $n\times n$ matrix. Define B as the $(m+2-r)\times(m+2-r)$ matrix appearing in $\widetilde{D}_r(2)$. We now estimate the column sums

$$B_i := \sum_{j=1}^{n}|b_{ji}|.$$

Column 1:

$$B_1 = |A(p_{r+1},p_1)| + \cdots + |A(p_m,p_1)| + |A(x_1,p_1)| + |A(x_2,p_1)| \le Cm.$$

Column 2:

$$\begin{aligned} B_2 &= \sqrt{\varepsilon}\big(|\widetilde{A}(p_{r+1},p_{r+2})| + |A(p_{r+2},p_{r+2})| + \cdots + |A(p_m,p_{r+2})| \\ &\qquad\qquad + |A(x_1,p_{r+2})| + |A(x_2,p_{r+2})|\big) \\ &\le \sqrt{\varepsilon}\big(Cm + |A(x_1,p_{r+2})| + |A(x_2,p_{r+2})|\big). \end{aligned}$$

Middle columns (if any) ($r+3\le i\le m$):

$$B_i = \sqrt{\varepsilon}\big(|A(p_{r+1},p_i)| + \cdots + |A(p_m,p_i)| + |A(x_1,p_i)| + |A(x_2,p_i)|\big) \le Cm.$$

Last two columns:

$$\begin{aligned}
B_{m-r+1} &= |A(p_{r+1}, x_1)| + \cdots + |A(p_m, x_1)| \\
&\quad + |A(x_1, x_1)| + |A(x_2, x_1)| \le Cme^{-x_1}, \\
B_{m-r+2} &= |A(p_{r+1}, x_2)| + \cdots + |A(p_m, x_2)| \\
&\quad + |A(x_1, x_2)| + |A(x_2, x_2)| \\
&\le Ce^{-x_2} + \phi(x_1, x_2) + \sum_{k=r+1}^{v-1} \phi(p_k, x_2).
\end{aligned}$$

Consider the estimates above for B_2 and B_{m-r+2}. The function $A(x_2, p_{r+2})$ will contain a ϕ-function if and only if $v < r + 2$, but in this case the sum

$$\sum_{k=r+1}^{v-1} \phi(p_k, x_2)$$

is empty. This means that we do not get terms like

$$\phi(x_2, p_{r+2})\phi(p_k, x_2)$$

in the product $B_2 B_{m-r+2}$. Given this observation, it is easy to see that

$$\int_{I_1 \times I_v} B_2 B_{m-r+1} B_{m-r+2}\, dx = \mathcal{O}(\sqrt{\varepsilon})$$

and this proves (2.18).

The second part of the proof consists of showing that

$$\begin{aligned}
&\int_{I_1 \times I_v} \det \begin{bmatrix} A(p_m, p_1) & A(p_m, x_1) & A(p_m, x_2) \\ A(x_1, p_1) & A(x_1, x_1) & A(x_1, x_2) \\ A(x_2, p_1) & A(x_2, x_1) & A(x_2, x_2) \end{bmatrix} dx \\
&\quad = \int_{I_1^2} R(p_1, x_1, x_2)\, dx + \int_{I_1} R(p_1, x)\, dx + \mathcal{O}(\sqrt{\varepsilon} \log \varepsilon).
\end{aligned} \tag{2.19}$$

The left-hand side is equal to

$$\begin{aligned}
&\int_{I_1 \times I_v} \det \begin{bmatrix} A(p_m, p_1) & A(p_m, x_1) & A(p_m, x_2) \\ A(x_1, p_1) & A(x_1, x_1) & \widetilde{A}(x_1, x_2) \\ A(x_2, p_1) & A(x_2, x_1) & A(x_2, x_2) \end{bmatrix} dx \\
&\quad + \int_{I_1 \times I_v} \phi(x_1, x_2) \det \begin{bmatrix} A(p_m, p_1) & A(p_m, x_1) \\ A(x_2, p_1) & A(x_2, x_1) \end{bmatrix} dx.
\end{aligned}$$

In view of Lemma 2.3 and (2.21) in Lemma 2.4 below, we obtain (2.19).

LEMMA 2.4. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *has a continuous derivative and that* $g : \mathbb{R}^2 \to \mathbb{R}$ *has continuous first partial derivatives. Assume that*

$$|f(x)|, |f'(x)| \le Ce^{-x},$$
$$|g(x, y)|, |g'(x, y)| \le Ce^{-x-y}.$$

Then, for $1 \le i, j \le m$, *it holds that*

$$\int_{I_i} \frac{1}{\sqrt{4\pi\varepsilon}} e^{-(x-p_j)^2/(4\varepsilon)} f(x)\,dx = f(p_j)\int_{(p_i-p_j)/\sqrt{\varepsilon}}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-x^2/4}\,dx + \mathcal{O}(\sqrt{\varepsilon}), \tag{2.20}$$

$$\int_{I_i}\int_{I_j} \frac{1}{\sqrt{4\pi\varepsilon}} e^{-(x-y)^2/(4\varepsilon)} g(x,y)\,dx\,dy = \int_{I_i} g(x,x)\,dx + \mathcal{O}(\sqrt{\varepsilon}\log\varepsilon). \tag{2.21}$$

PROOF.

$$\int_{p_i}^{\infty} \frac{1}{\sqrt{4\pi\varepsilon}} e^{-(x-p_j)^2/(4\varepsilon)} f(x)\,dx = \left[z = \frac{x-p_j}{\sqrt{\varepsilon}}\right] = \int_{(p_i-p_j)/\sqrt{\varepsilon}}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-z^2/4} f(p_j + \sqrt{\varepsilon}z)\,dz.$$

By Taylor's theorem

$$f(p_j + \sqrt{\varepsilon}z) = f(p_j) + \sqrt{\varepsilon}zf'(p_j + \theta_\varepsilon(z)),$$

where $\theta_\varepsilon(z)$ is a number between 0 and $\sqrt{\varepsilon}z$. Since by assumption

$$|f'(p_j + \theta_\varepsilon(z))| \le Ce^{-p_j+\sqrt{\varepsilon}|z|},$$

we obtain (2.20):

$$\int_{p_i}^{\infty}\int_{p_j}^{\infty} \frac{1}{\sqrt{4\pi\varepsilon}} e^{-(x-y)^2/(4\varepsilon)} g(x,y)\,dx\,dy = \left[z = \frac{y-x}{\sqrt{\varepsilon}}\right] = \int_{p_i}^{\infty}\int_{(p_j-x)/\sqrt{\varepsilon}}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-z^2/4} g(x, x+\sqrt{\varepsilon}z)\,dx\,dz.$$

By Taylor's theorem

$$g(x, x+\sqrt{\varepsilon}z) = g(x,x) + \sqrt{\varepsilon}zg'(x, x+\theta_\varepsilon(x,z)),$$

where $\theta_\varepsilon(x,z)$ lies between 0 and $\sqrt{\varepsilon}z$. The error can be discarded since

$$\int_{p_i}^{\infty}\int_{(p_j-x)/\sqrt{\varepsilon}}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-z^2/4} |zg'(x, x+\theta_\varepsilon(x,z))|\,dx\,dz \le C\int_{p_i}^{\infty} dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi}} |z| e^{-z^2/4-2x+\sqrt{\varepsilon}|z|}\,dz \le C.$$

We now split the main term into two terms:

$$\begin{aligned}
&\int_{p_i}^{\infty} dx \int_{(p_j-x)/\sqrt{\varepsilon}}^{\infty} dz \frac{1}{\sqrt{4\pi}} e^{-z^2/4} g(x,x) \\
&\quad = \int_{p_i}^{p_i-\sqrt{\varepsilon\log\varepsilon}} dx \int_{(p_j-x)/\sqrt{\varepsilon}}^{\infty} dz \frac{1}{\sqrt{4\pi}} e^{-z^2/4} g(x,x) \\
&\qquad + \int_{p_i-\sqrt{\varepsilon\log\varepsilon}}^{\infty} dx \int_{(p_j-x)/\sqrt{\varepsilon}}^{\infty} dz \frac{1}{\sqrt{4\pi}} e^{-z^2/4} g(x,x) =: \int_1 + \int_2 .
\end{aligned}$$

We can estimate the first integral by

$$\left|\int_1\right| \le C \int_{p_i}^{p_i-\sqrt{\varepsilon\log\varepsilon}} dx \int_{-\infty}^{\infty} dz\, e^{-(z^2/4)-2x} \le -C\sqrt{\varepsilon}\log\varepsilon.$$

If $x \ge p_j - \sqrt{\varepsilon}\log\varepsilon$, then $\frac{p_j-x}{\sqrt{\varepsilon}} \le C\log\varepsilon$ and hence

$$\begin{aligned}
\int_{(p_j-x)/\sqrt{\varepsilon}}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-z^2/4}\, dz &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-z^2/4}\, dz + \int_{-\infty}^{(p_j-x)/\sqrt{\varepsilon}} \frac{1}{\sqrt{4\pi}} e^{-z^2/4} \\
&= 1 + \mathcal{O}\big(e^{-(\log\varepsilon)^2/4}\big).
\end{aligned}$$

We finally get

$$\begin{aligned}
\int_2 &= \int_{p_i-\sqrt{\varepsilon}\log\varepsilon}^{\infty} \big(1+\mathcal{O}\big(e^{-(\log\varepsilon)^2/4}\big)\big) g(x,x)\, dx \\
&= \int_{p_i}^{\infty} g(x,x)\, dx + \mathcal{O}\big(\sqrt{\varepsilon}\log\varepsilon\big).
\end{aligned}$$

This concludes the proof of the lemma. □

3. Theorem 1.3.

3.1. *Multilayer discrete PNG.* Before we give the proof of Theorem 1.3 we must present some preliminary results.

How does one get a hand on the process h described in the Introduction? In [7] it is shown that h can be embedded as the top curve in a multilayer process given by a family of nonintersecting paths $\{h_i, 0 \le i < N\}$, $h = h_0$. It turns out (see [7]) that this multilayer process is an example of a discrete determinantal process.

THEOREM 3.1 ([7]). *Let $u, v \in \mathbb{Z}$ be such that $|u|, |v| < N$ and let $q = \alpha^2$. Set*

$$G(z,w) = (1-\alpha)^{2(v-u)} \frac{(1-\alpha/z)^{N+u}(1-\alpha w)^{N-v}}{(1-\alpha z)^{N-u}(1-\alpha/w)^{N+v}}$$

and

$$\widetilde{K}_N(2u, x; 2v, y) = \frac{1}{(2\pi i)^2} \int_{\gamma_{r_2}} \frac{dz}{z} \int_{\gamma_{r_1}} \frac{dw}{w} \frac{z}{z-w} G(z, w),$$

where γ_r *is the circle with radius* r *centered around the origin*, $\alpha < r_1 < r_2 < 1/\alpha$ *and* $x, y \in \mathbb{Z}$. *Furthermore, define*

$$\phi_{2u,2v}(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(y-x)\theta} G(e^{i\theta}, e^{i\theta})\, d\theta$$

for $u < v$ *and* $\phi_{2u,2v}(x, y) = 0$ *for* $u \geq v$. *Set*

$$K_N(2u, x; 2v, y) = \widetilde{K}_N(2u, x; 2v, y) - \phi_{2u,2v}(x, y).$$

Then,

$$\begin{aligned} &\mathbb{P}\big[(2u, x_j^{2u}) \in \{(2t, h_i(2t, 2N-1)); 0 \leq i < N, |t| < N\}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad |u| < N, 1 \leq j \leq k_u\big] \\ &\quad = \det(K_N(2u, x_i^{2u}; 2v, x_j^{2v}))_{|u|,|v|<N, 1 \leq i \leq k_u, 1 \leq j \leq k_v} \end{aligned}$$

for any $x_j^{2u} \in \mathbb{Z}$ *and any* $k_u \in \{0, \ldots, N\}$.

The asymptotic information about the kernel K_N needed to prove Theorem 1.3 is contained in two lemmas. The first can be extracted from [7], Chapter 4, and the proof of the second is provided at the end of this section. Please note that we make a slight redefinition of the function ϕ from the last section. However, for the purposes of this text ϕ acts as one and the same.

LEMMA 3.1. *Let* τ, τ' *be any real numbers such that*

$$u = \frac{1+\alpha}{1-\alpha} d^{-1} N^{2/3} \tau \in \mathbb{Z}_+,$$

$$v = \frac{1+\alpha}{1-\alpha} d^{-1} N^{2/3} \tau' \in \mathbb{Z}_+.$$

Let $x, y \in \mathbb{Z}_+$ *and define* x', y' *by*

$$x = 2\alpha(1-\alpha)^{-1} N + (x' - \tau^2)\, dN^{1/3},$$

$$y = 2\alpha(1-\alpha)^{-1} N + (y' - \tau'^2)\, dN^{1/3}.$$

For any $L \in \mathbb{R}$ *there exist positive constants*, c *and* C, *such that*

$$|\widetilde{K}_N(2u, x; 2v, y)| \leq C N^{-1/3} e^{-c(x'+y')}$$

if $x', y' \geq L$.

If $|x'|, |y'| \leq \log N$, *then there exists* $c > 0$ *such that*

$$dN^{1/3} \widetilde{K}_N(2u, x; 2v, y) = e^{((\tau^3 - \tau'^3)/3) + y'\tau' - x'\tau} \widetilde{A}(\tau, x'; \tau', y') + \mathcal{O}(N^{-c}).$$

LEMMA 3.2. *Let $x, y \in \mathbb{Z}_+$ and define x', y' by*

$$x = 2\alpha(1-\alpha)^{-1}N + x'\,dN^{1/3},$$
$$y = 2\alpha(1-\alpha)^{-1}N + y'\,dN^{1/3}.$$

Take $s > 0$, let $u \sim N^{2/3}$ and define v by

$$v = u + \frac{1+\alpha}{1-\alpha} d^{-1} s N^{\gamma}$$

where $0 < \gamma < \frac{2}{3}$. There exists a constant $C > 0$ such that

$$\phi_{2u,2v}(x, y) = \frac{1}{dN^{1/3}}\phi(x', y') + \phi_E(x', y')$$

where

$$\phi(x', y') = \frac{1}{\sqrt{4\pi s N^{\gamma-2/3}}} e^{-(x'-y')^2/(4sN^{\gamma-2/3})}$$

and

$$|\phi_E(x', y')| \le \begin{cases} CN^{-3\gamma/2}, \\ \dfrac{C}{N^{1/3}|x'-y'|N^{\gamma}}, \end{cases}$$

for all x, y.

3.2. *Proof of Theorem* 1.3. This proof is really a discrete analogue of the proof of Theorem 1.1. Unfortunately things are more involved in this case where $N^{\gamma-2/3}$ plays the role of ε.

Please recall that $J_1 = \mu N + \psi\, dN^{1/3}$, where $\mu = 2\alpha(1-\alpha)^{-1}$ and $q = \alpha^2$. Set $J_i = J_{i-1} + y_i\, dN^{\gamma/2} \in \mathbb{Z}$, $i = 2, \dots, m$, and

$$\widetilde{I}_i = \{z \in \mathbb{Z} | z > J_i\}.$$

Here the y_i's are arbitrary numbers such that $J_i \in \mathbb{Z}$. For later convenience we also define ψ_i, $i = 1, \dots, m$, by $J_i = \mu N + \psi_i\, dN^{1/3}$.

We will prove that

$$\begin{aligned}\mathbb{P}[\#J_2 = \cdots = \#J_m = 1, \#\widetilde{I}_2 = \cdots = \#\widetilde{I}_m = 0 | \#J_1 = 1, \#\widetilde{I}_1 = 0]\\ = \phi_{2K_1,2K_2}(J_1, J_2)\cdots\phi_{2K_{m-1},2K_m}(J_{m-1}, J_m)\\ + \mathcal{O}((N^{-\gamma/2})^{m-1}N^{-c}).\end{aligned}$$

This implies Theorem 1.3:

$$\begin{aligned}&\phi_{2K_1,2K_2}(J_1, J_2)\cdots\phi_{2K_{m-1},2K_m}(J_{m-1}, J_m)\\ &\quad= \frac{1}{\sqrt{4\pi s_2}} e^{-y_2^2/(4s_2)} \cdots \frac{1}{\sqrt{4\pi s_m}} e^{-y_m^2/(4s_m)} \frac{1}{(dN^{\gamma/2})^{m-1}}\big(1 + \mathcal{O}(N^{-c})\big)\end{aligned}$$

by Lemma 3.2. The sum of this function over the sets A_i is a Riemann sum that is well approximated by the integral in Theorem 1.3.

Define the finite integer intervals I_i, $1 \le i \le m$, by

$$I_i = \{z \in \mathbb{Z};\, J_i < z < \lfloor \mu N \rfloor + N\}.$$

The probability of finding a particle in $\widetilde{I}_i$ but outside of I_i is very small:

$$\begin{aligned}
\mathbb{P}[\#(\widetilde{I}_i \setminus I_i) \ge 1] &\le \sum_{x \in \widetilde{I}_i \setminus I_i} \mathbb{P}[\#x = 1] = \sum_{x \in \widetilde{I}_i \setminus I_i} K(x, x) \\
&= \sum_{k=0}^{\infty} K\biggl(\lfloor \mu N \rfloor + \biggl(\frac{1}{d} N^{2/3} + \frac{k}{dN^{1/3}}\biggr) dN^{1/3}, \\
&\qquad \lfloor \mu N \rfloor + \biggl(\frac{1}{d} N^{2/3} + \frac{k}{dN^{1/3}}\biggr) dN^{1/3}\biggr) \\
&\le C e^{-(1/d)N^{2/3}} \sum_{k=0}^{\infty} e^{-k/(dN^{1/3})} = \mathcal{O}(e^{-cN^{2/3}}).
\end{aligned}$$

This means that we can work with I_i instead of $\widetilde{I}_i$. We now proceed much like we did in the proof of Theorem 1.1. If we set

$$A = \{\#J_1 = 1, \dots, \#J_m = 1\},$$

then

$$\begin{aligned}
&\mathbb{P}[A, \#I_1 = \cdots = \#I_m = 0] + \mathbb{P}[A, \#I_1 = 0, (\#I_2 = \cdots = \#I_m = 0)^c] \\
&\quad = \mathbb{P}[A, \#I_1 = 0],
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{P}[A, \#I_1 = 0, (\#I_2 = \cdots = \#I_m = 0)^c] \\
&\quad = \mathbb{P}\Biggl[A, \#I_1 = 0, \bigcup_{i=2}^{m} \{\#I_i \ne 0\}\Biggr] \\
&\quad \le \sum_{i=2}^{m} \mathbb{P}[A, \#I_1 = 0, \#I_i \ne 0] \le \sum_{i=2}^{m} \mathbb{P}[A, \#I_i \ne \#I_1]
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}[A, \#I_i \ne \#I_1] &= \mathbb{E}\bigl[\chi_{\{\#J_1 = 1\}} \cdots \chi_{\{\#J_m = 1\}} \cdot \chi_{\{\#I_1 \ne \#I_i\}}\bigr] \\
&= \mathbb{E}\bigl[\#J_1 \cdots \#J_m \cdot \chi_{\{\#I_1 \ne \#I_i\}}\bigr] \\
&\le \mathbb{E}[\#J_1 \cdots \#J_m (\#I_1 - \#I_i)^2].
\end{aligned}$$

The second equality holds since the probability of finding two particles at the same place is zero.

We need to prove three things:

1. $\mathbb{P}[A, \#I_1 = 0] = \phi_{2K_1,2K_2}(J_1, J_2) \cdots \phi_{2K_{m-1},2K_m}(J_{m-1}, J_m)$
$$\times \mathbb{P}[\#J_1 = 1, \#I_1 = 0] + \mathcal{O}(N^{-1/3-c}(N^{-\gamma/2})^{m-1}).$$
2. $\mathbb{E}[\#J_1 \cdots \#J_m(\#I_1 - \#I_i)^2] = \mathcal{O}(N^{-1/3-c}(N^{-\gamma/2})^{m-1})$.
3. $\mathbb{P}[\#J_1 = 1, \#I_1 = 0] \geq CN^{-1/3}$.

Before giving the proofs we need some preliminaries.

When summing a function $f(x)$ over, say, I_1 we can write

$$\sum_{x \in I_1} f(x) = \sum_{l=1}^{T_1} f\left(\mu N + \left(\psi_1 + \frac{l}{dN^{1/3}}\right) dN^{1/3}\right),$$

where $T_1 \sim N$. The next lemma will be frequently used later on.

LEMMA 3.3. *There exist constants $C_1, C_2 > 0$ such that*

$$\sum_{k=1}^{\infty} \phi(k/N^{1/3}, x) N^{-1/3} \leq C_1$$

and

$$\sum_{k=1}^{N^2} \phi_E(k/N^{1/3}, x) \leq C_2 N^{-\gamma/2}$$

for any $x \in \mathbb{R}$.

PROOF.

$$\begin{aligned}
&\sum_{k=1}^{\infty} \phi(k/N^{1/3}, x) N^{-1/3} \\
&\quad = \sum_{k=1}^{\infty} \phi\left(\frac{k - xN^{1/3}}{N^{1/3}}, 0\right) N^{-1/3} \\
&\quad \leq \sum_{k=-\infty}^{\infty} \phi\left(\frac{k - xN^{1/3}}{N^{1/3}}, 0\right) N^{-1/3} = [f := xN^{1/3} - \lfloor xN^{1/3} \rfloor] \\
&\quad = \sum_{k=-\infty}^{\infty} \phi\left(\frac{k - f}{N^{1/3}}, 0\right) N^{-1/3} \\
&\quad \leq \sum_{k=-\infty}^{0} \phi\left(\frac{k}{N^{1/3}}, 0\right) N^{-1/3} + \phi\left(\frac{1 - f}{N^{1/3}}, 0\right) N^{-1/3} + \sum_{k=2}^{\infty} \phi\left(\frac{k-1}{N^{1/3}}, 0\right) N^{-1/3} \\
&\quad \leq 2 \sum_{k=1}^{\infty} \phi\left(\frac{k}{N^{1/3}}, 0\right) N^{-1/3} + 2 \leq C
\end{aligned}$$

and

$$\sum_{k=1}^{N^2} \phi_E(k/N^{1/3}, x)$$
$$\le CN^{-\gamma} \sum_{k=1}^{xN^{1/3}-N^\gamma} \frac{1}{xN^{1/3}-k}$$
$$+ C \sum_{xN^{1/3}-N^\gamma}^{xN^{1/3}+N^\gamma} N^{-3\gamma/2} + CN^{-\gamma} \sum_{xN^{1/3}+N^\gamma}^{N^2} \frac{1}{k-xN^{1/3}}$$
$$\le CN^{-\gamma}\log N + CN^{-\gamma/2} + CN^{-\gamma}\log N \le CN^{-\gamma/2}. \qquad \square$$

We now turn to the proof of item 1. As in the proof of Theorem 1.1 we get

$$\mathbb{P}[A, \#I_1 = 0] = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mathbb{E}[\#J_1 \cdots \#J_m \#I_1^{[k]}]. \tag{3.1}$$

For $0 \le r \le m-1$ set

$$D_r(k) = \phi_{2K_1,2K_2}(J_1, J_2)\phi_{2K_2,2K_3}(J_2, J_3)\cdots\phi_{2K_r,2K_{r+1}}(J_r, J_{r+1}) \sum_{x_i \in I_1, 1\le i\le k} \times \begin{vmatrix} K(J_{r+1}, J_1) & K(J_{r+1}, J_{r+2}) & \dots & K(J_{r+1}, J_m) & K(J_{r+1}, x_j) \\ \vdots & \vdots & & \vdots & \vdots \\ K(J_m, J_1) & K(J_m, J_{r+2}) & \dots & K(J_m, J_m) & K(J_m, x_j) \\ K(x_i, J_1) & K(x_i, J_{r+2}) & \dots & K(x_i, J_m) & K(x_i, x_j) \end{vmatrix}.$$

The indicies i, j run from 1 to k and if $r = 0$ the (empty) product of ϕ-functions is to be interpreted as 1. Let $\widetilde{D}_r(k)$ be like $D_r(k)$ but having $\widetilde{K}(J_{r+1}, J_{r+2})$ in position $(1, 2)$ in the matrix. We want to show that

$$|D_0(k) - D_r(k)| \le N^{-1/3-c}(N^{-\gamma/2})^{m-1}(Ck)^{(k+m)/2}$$

which, by the induction argument in the proof of Theorem 1.1, follows if we can prove that

$$|\widetilde{D}_r(k)| \le N^{-1/3-c}(N^{-\gamma/2})^{m-1}(Ck)^{(k+m)/2}. \tag{3.2}$$

To show this we shall use Hadamard's inequality and therefore need to estimate sums of column elements squared (cf. with the proof of Theorem 1). Lemmas 3.1, 3.2 and 3.3 will be frequently used below.

Column 1:

$$\sum_{i=r+1}^{m} K^2(J_i, J_1) + \sum_{i=1}^{k} K^2(x_i, J_1) \le CN^{-2/3}(m+k).$$

Column 2:

$$\widetilde{K}^2(J_{r+1}, J_{r+2}) + \sum_{i=r+2}^{m} K^2(J_i, J_{r+2}) \le CN^{-2/3}m$$

and

$$\begin{aligned}\sum_{i=1}^{k} K^2(x_i, J_{r+2}) &\\ &\le CN^{-2/3}\sum_{i=1}^{k}[1+\phi(l_i/dN^{1/3}, \psi_1-\psi_{r+2})\\ &\qquad + N^{1/3}\phi_E(l_i/dN^{1/3}, \psi_1-\psi_{r+2})]^2.\end{aligned}$$

Columns $3, \dots, m-r$ ($r+3 \le j \le m$), if they exist:

$$\sum_{i=r+1}^{m} K^2(J_i, J_j) + \sum_{i=1}^{k} K^2(x_i, J_j) \le CN^{-\gamma}(k+m).$$

Last k columns ($1 \le j \le k$):

$$\sum_{i=r+1}^{m} K^2(J_i, x_j) + \sum_{i=1}^{k} K^2(x_i, x_j) \le C(k+m)N^{-2/3}e^{-cl_jN^{-1/3}}.$$

Using Hadamard's inequality, we get after some manipulations that

$$\begin{aligned}|\widetilde{D}_r(k)| \le &\sum_{l_1,\dots,l_k=1}^{T_1} N^{-2/3}(N^{-\gamma/2})^{m-2}(Ck)^{(k+m)/2}(N^{-1/3})^k \prod_{i=1}^{k} e^{-cl_iN^{-1/3}}\\ &\times \sum_{i=1}^{k}[1+\phi(l_i/dN^{1/3}, \psi_1-\psi_{r+2})\\ &\qquad + N^{1/3}|\phi_E(l_i/dN^{1/3}, \psi_1-\psi_{r+2})|].\end{aligned}$$

It follows from Lemma 3.3 that

$$\sum_{l_i=1}^{T_1} e^{-cl_iN^{-1/3}}\phi(l_i/dN^{1/3}, \psi_1-\psi_{r+2})N^{-1/3} \le C$$

and also that

$$\sum_{l_i=1}^{T_1} e^{-cl_iN^{-1/3}}|\phi_E(l_i/dN^{1/3}, \psi_1-\psi_{r+2})| \le CN^{-\gamma/2}.$$

From this we get (3.2).

To get 1 we also need to show that

$$\sum_{x_i \in I_1, 1\le i\le k} \det\begin{bmatrix} K(J_m, J_1) & K(J_m, x_j) \\ K(x_i, J_1) & K(x_i, x_j) \end{bmatrix}_{1\le i,j\le k}$$

$$= \sum_{x_i \in I_1, 1\le i\le k} \det\begin{bmatrix} K(J_1, J_1) & K(J_1, x_j) \\ K(x_i, J_1) & K(x_i, x_j) \end{bmatrix}_{1\le i,j\le k} + N^{-1/3-c}\mathcal{O}\big((Ck)^{(k+m)/2}\big). \tag{3.3}$$

Write

$$x_i = \mu N + \left(\psi_1 + \frac{l_i}{dN^{1/3}}\right) dN^{1/3}$$

and consider first the case $1 \le l_i \le N^{1/3}\log N$. From Lemmas 2.3 and 3.1 it is straightforward to deduce that if $z = x_i$ or $z = J_1$, then

$$K(J_m, z) = K(J_1, z) + \mathcal{O}(N^{-1/3-c}).$$

We now expand the determinant in the sum to the left in (3.3):

$$\begin{aligned}\det&\begin{bmatrix} K(J_m, J_1) & K(J_m, x_j) \\ K(x_i, J_1) & K(x_i, x_j) \end{bmatrix}_{1\le i,j\le k} \\ &= \det\begin{bmatrix} K(J_1, J_1) & K(J_1, x_j) \\ K(x_i, J_1) & K(x_i, x_j) \end{bmatrix}_{1\le i,j\le k} \\ &\quad + \mathcal{O}(N^{-1/3-c})\sum_{p=1}^{k} \det[K(x_i, J_1) K(x_i, x_j)]_{1\le i,j\le k, j\ne p} \\ &\quad + \mathcal{O}(N^{-1/3-c}) \det[K(x_i, x_j)]_{1\le i,j\le k}.\end{aligned}$$

We now use Hadamard's inequality to get

$$\begin{aligned}\sum_{l_i=1}^{N^{1/3}\log N} &|\det[K(x_i, J_1) K(x_i, x_j)]_{1\le i,j\le k, j\ne p}| \\ &\le \sum_{l_i=1}^{N^{1/3}\log N} (CkN^{-2/3})^{k/2} e^{-N^{-1/3}(l_1+\cdots+l_{p-1}+l_{p+1}+\cdots+l_k)} \\ &\le (Ck)^{k/2}\log N\end{aligned}$$

and

$$\begin{aligned}\sum_{l_i=1}^{N^{1/3}\log N} &|\det[K(x_i, x_j]_{1\le i,j\le k}| \\ &\le \sum_{l_i=1}^{N^{1/3}\log N} (CkN^{-2/3})^{k/2} e^{-N^{-1/3}(l_1+\cdots+l_k)} \le (Ck)^{k/2}.\end{aligned}$$

This takes care of the summation over $1 \le l_i \le N^{1/3} \log N$, $1 \le i \le m$. By using Hadamard's inequality once more, one readily shows that the contribution coming from the remaining terms in the sums in (3.3) is small enough to make (3.3) hold.

We now prove item 2. Note that

$$(\#I_1 - \#I_i)^2 = I_i^{[2]} + I_1^{[2]} + I_i + I_1 - 2I_i I_1.$$

By arguing as in the proof of item 1 above we obtain

$$\begin{aligned}\mathbb{E}\big[\#J_1 \cdots \#J_m \#I_u^{[k]}\big] &= \phi_{2K_1,2K_2}(J_1, J_2) \cdots \phi_{2K_{m-1},2K_m}(J_{m-1}, J_m) \\ &\quad \times \sum_{x_1,x_k \in I_u} \det \begin{bmatrix} K(J_m, J_1) & K(J_m, x_s) \\ K(x_r, J_1) & K(x_r, x_s) \end{bmatrix}_{1 \le r,s \le k} \\ &\quad + \mathcal{O}\big(N^{-1/3-c-\gamma(m-1)/2}\big),\end{aligned}$$

where $u = 1, i$ and $k = 1, 2$. One also gets

$$\begin{aligned}\mathbb{E}[\#J_1 \cdots \#J_m \#I_1 \#I_i] &= \phi_{2K_1,2K_2}(J_1, J_2) \cdots \phi_{2K_{m-1},2K_m}(J_{m-1}, J_m) \\ &\quad \times \sum_{x \in I_1, y \in I_i} \det \begin{bmatrix} K(J_m, J_1) & K(J_m, x) & K(J_m, y) \\ K(x, J_1) & K(x, x) & K(x, y) \\ K(y, J_1) & K(y, x) & K(y, y) \end{bmatrix} \\ &\quad + \mathcal{O}\big(N^{-1/3-c-\gamma(m-1)/2}\big).\end{aligned}$$

We omit the details. Using Lemma 3.1 and Lemma 2.3 one readily gets

$$\sum_{x_1,x_k \in I_i} \det \begin{bmatrix} K(J_m, J_1) & K(J_m, x_s) \\ K(x_r, J_1) & K(x_r, x_s) \end{bmatrix}_{1 \le r,s \le k} = \mathbb{E}\big[\#J_1 \#I_1^{[k]}\big] + \mathcal{O}(N^{-1/3-c})$$

for $k = 1, 2$ and

$$\sum_{x \in I_1, y \in I_i} \det \begin{bmatrix} K(J_m, J_1) & K(J_m, x) & K(J_m, y) \\ K(x, J_1) & K(x, x) & \widetilde{K}(x, y) \\ K(y, J_1) & K(y, x) & K(y, y) \end{bmatrix} = \mathbb{E}\big[\#J_1 \#I_1^{[2]}\big] + \mathcal{O}(N^{-1/3-c}).$$

We now see that item 2 follows if

$$\begin{aligned}&\sum_{x \in I_1, y \in I_i} \phi_{2K_1,2K_i}(x, y) \det \begin{bmatrix} K(J_m, J_1) & K(J_m, x) \\ K(y, J_1) & K(y, x) \end{bmatrix} \\ &\qquad = \mathbb{E}[\#J_1 \#I_1] + \mathcal{O}(N^{-1/3-c}).\end{aligned} \tag{3.4}$$

We shall prove this by showing that both sides are well approximated by integrals. On the integral containing the function ϕ we can then apply Lemma 2.4.

By using Lemma 3.3 we get rid of the error term associated with ϕ_E:

$$\begin{aligned}&\sum_{l_1,l_2=1}^{N} \phi_E\left(\psi_1 + \frac{l_1}{dN^{1/3}}, \psi_i + \frac{l_2}{dN^{1/3}}\right) e^{-(l_1+l_2)/N^{1/3}} N^{-2/3} \\ &\qquad \le C \sum_{l_2=1}^{N} e^{-l_2/N^{1/3}} N^{-2/3-\gamma/2} \le C N^{-1/3-\gamma/2}.\end{aligned}$$

The following calculation, again using Lemma 3.3, shows that the main contribution to the sums in (3.4) comes from summing over $1 \le l_1, l_2 \le N^{1/3} \log N$:

$$\sum_{l_1=N^{1/3}\log N}^{N} \sum_{l_2=1}^{N} \phi(l_2/dN^{1/3}, \psi_1 - \psi_i + l_2/dN^{1/3}) e^{-(l_1+l_2)/N^{1/3}} N^{-2/3}$$

$$\le \sum_{l_1=N^{1/3}\log N}^{N} Ce^{-l_1/N^{1/3}} N^{-1/3} \le CN^{-1}.$$

We shall use Euler's summation formula for two variables:

LEMMA 3.4. *Let $f(x, y)$ be a function of two variables such that its partial derivatives up to second order are continuous in the rectangle*

$$\{(x, y) | a \le x \le b, c \le y \le d\}$$

where a, b, c, d are integers. Then

$$\begin{aligned}\sum_{a\le m\le b} \sum_{c\le n\le d} f(m, n) &= \int_a^b \int_c^d f(x, y)\, dx\, dy \\ &\quad + \int_a^b \int_c^d f_x(x, y)(x - \lfloor x \rfloor)\, dx\, dy \\ &\quad + \int_a^b \int_c^d f_y(x, y)(y - \lfloor y \rfloor)\, dx\, dy \\ &\quad + \int_a^b \int_c^d f_{xy}(x, y)(x - \lfloor x \rfloor)(y - \lfloor y \rfloor)\, dx\, dy.\end{aligned}$$

The case that we are interested in is when

$$f(x, y) = \phi\left(\psi_1 + \frac{x}{dN^{1/3}}, \psi_i + \frac{y}{dN^{1/3}}\right) g(x/dN^{1/3}, y/dN^{1/3}) N^{-1},$$

where

$$|g_x(x, y)|, |g_y(x, y)|, |g_x(x, y)| \le Ce^{-c(x+y)}.$$

We need to show that the integrals involving the absolute values of $f_x(x, y)$, $f_y(x, y)$ and $f_{xy}(x, y)$ are negligible. We only present the details for $|f_x(x, y)|$ here; the other terms are treated similarly:

$$\begin{aligned}&\int_1^{N^{1/3}\log N} \int_1^{N^{1/3}\log N} |f_x(x, y)|\, dx\, dy \\ &\quad \le (dN^{1/3})^2 \int_{\psi_1}^{\infty} \int_{\psi_i}^{\infty} \big|f_x\big((x - \psi_1)\, dN^{1/3}, (y - \psi_i)\, dN^{1/3}\big)\big|\, dx\, dy \\ &\quad \le CN^{-2/3} \int_{\psi_1}^{\infty} \int_{\psi_i}^{\infty} \big(|\phi_x(x, y)| + \phi(x, y)\big) e^{-c(x+y)}\, dx\, dy.\end{aligned}$$

By Lemma (2.4)

$$\int_{\psi_1}^{\infty}\int_{\psi_i}^{\infty}\phi(x,y)e^{-c(x+y)}\,dx\,dy \le C.$$

The remaining term demands some analysis:

$$\begin{aligned}
&\int_{\psi_1}^{\infty}\int_{\psi_i}^{\infty}|\phi_x(x,y)|e^{-c(x+y)}\,dx\,dy\\
&\quad=\int_{\psi_1}^{\infty}\int_{\psi_i}^{\infty}\frac{|x-y|}{2N^{\gamma-2/3}}\frac{1}{\sqrt{4\pi N^{\gamma-2/3}}}e^{-(x-y)^2/(4N^{\gamma-2/3})}e^{-c(x+y)}\,dx\,dy\\
&\quad=\int_{\psi_1}^{\infty}dx\biggl(\int_{\psi_i}^{x}+\int_{x}^{\infty}\biggr)\frac{|x-y|}{2N^{\gamma-2/3}}\frac{1}{\sqrt{4\pi N^{\gamma-2/3}}}\\
&\qquad\qquad\times e^{-(x-y)^2/(4N^{\gamma-2/3})}e^{-c(x+y)}\,dy,\\
&\int_{\psi_1}^{\infty}dx\int_{\psi_i}^{x}\frac{x-y}{2N^{\gamma-2/3}}\frac{1}{\sqrt{4\pi N^{\gamma-2/3}}}e^{-(x-y)^2/(4N^{\gamma-2/3})}e^{-c(x+y)}\,dy\\
&\quad=\int_{\psi_1}^{\infty}dx\biggl([\phi(x,y)e^{-c(x+y)}]_{\psi_i}^{x}+c\int_{\psi_i}^{x}\phi(x,y)e^{-c(x+y)}\,dy\biggr)\\
&\quad\le CN^{1/3-\gamma/2}+C\le CN^{1/3-\gamma/2}.
\end{aligned}$$

We can do the same calculation for the remaining integral. The $|f_x(x,y)|$ integral is hence $\mathcal{O}(N^{-1/3-\gamma/2})$ and the same goes for the $|f_y(x,y)|$ and $|f_{xy}(x,y)|$ integrals.

Set

$$A^{\tau_1}(x,y)=A(\tau_1,x+\tau_1^2;\tau_1,y+\tau_1^2).$$

Applying the above calculations to the left-hand side of (3.4) and using Lemmas 2.2–2.4 and 3.1, we obtain

$$\begin{aligned}
&\sum_{x\in I_1,y\in I_i}\phi_{2K_1,2K_i}(x,y)\det\begin{bmatrix}K(J_m,J_1) & K(J_m,x)\\ K(y,J_1) & K(y,x)\end{bmatrix}\\
&\quad=\sum_{l_1,l_2=1}^{N^{1/3}\log N}\frac{1}{dN^{1/3}}\phi(\psi_1+l_1/dN^{1/3},\psi_i+l_2/dN^{1/3})\biggl(\frac{1}{dN^{1/3}}\biggr)^2\\
&\qquad\times\begin{vmatrix}A^{\tau_1}(\psi_1,\psi_1) & e^{\tau_1 l_1/(dN^{1/3})}A^{\tau_1}(\psi_1,\psi_1+l_1/dN^{1/3})\\ e^{-\tau_1 l_2/(dN^{1/3})}A^{\tau_1}(\psi_1+l_2/dN^{1/3},\psi_1) & A^{\tau_1}(\psi_i+l_2/dN^{1/3},\psi_1+l_1/dN^{1/3})\end{vmatrix}\\
&\qquad+\mathcal{O}(N^{-1/3-c})\\
&\quad=\frac{1}{dN^{1/3}}\int_{\psi_1}^{\infty}\det\begin{bmatrix}A^{\tau_1}(\psi_1,\psi_1) & A^{\tau_1}(\psi_1,x)\\ A^{\tau_1}(x,\psi_1) & A^{\tau_1}(x,x)\end{bmatrix}dx+\mathcal{O}(N^{-1/3-c}).
\end{aligned}$$

We get the same expression for the right-hand side of (3.4) when applying Euler's summation formula. This concludes the proof of item 2.

Let $F_2(t)$ be the Tracy–Widom distribution function corresponding to the largest eigenvalue of the Gaussian Unitary Ensemble (GUE) [15]. That item 3 is true follows from the fact that $F_2'(t) > 0\ \forall t$ (see [15]) together with the next lemma.

LEMMA 3.5. *Let J_1 and I_1 be as above. It holds that*

$$\mathbb{P}[\#J_1 = 0, \#I_1 = 0] = \frac{1}{dN^{1/3}} F_2'(\psi_1 + \tau_1^2) + \mathcal{O}(N^{-2/3}).$$

PROOF. This will, again, be an exercise in using Hadamard's inequality. We have the following representation for F_2' [see the third equality in (2.14)]:

$$F_2'(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{(t,\infty)^k} \det(A(x_i, x_j))_{0 \le i,j \le k}\, d^k x \tag{3.5}$$

where $x_0 = t$. In three steps we will now show that

$$dN^{1/3} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{x_i \in I_1, 1 \le i \le k} \det(K(x_i, x_j))_{0 \le i,j \le k}$$

where $x_0 = J_1$ is well approximated by the right-hand side in (3.5). By (3.1) this will prove the lemma. In steps one and two we will use Lemma 3.1 to insert the kernel A instead of K. In the last step we show that we can change from summation to integration.

First we show that we can sum over $x_i = \mu N + (\psi_1 + l_i/dN^{1/3})\, dN^{1/3}$ where $1 \le l_i \le N^{1/3} \log N$, $1 \le i \le k$, instead of over I_1. By Hadamard's inequality and Lemma 3.1

$$\begin{aligned}
\det(K(x_i, x_j))_{0 \le i,j \le k} &\le \left(\prod_{j=0}^{k} \sum_{i=0}^{k} K^2(x_i, x_j) \right)^{1/2} \\
&\le \left(C(k+1) N^{-2/3} \prod_{j=1}^{k} C(k+1) N^{-2/3} e^{-cl_j/N^{1/3}} \right)^{1/2} \\
&\le N^{-1/3} \big(C(k+1)\big)^{(k+1)/2} \prod_{j=0}^{k} e^{-cl_j/N^{1/3}} N^{-1/3}.
\end{aligned}$$

We have that

$$\begin{aligned}
&\sum_{\substack{l_i=1 \\ 1 \le i \le k}}^{\infty} \prod_{j=1}^{k} e^{-cl_j/N^{1/3}} N^{-1/3} - \sum_{\substack{l_i=1 \\ 1 \le i \le k}}^{N^{1/3}\log N} \prod_{j=1}^{k} e^{-cl_j/N^{1/3}} N^{-1/3} \\
&\qquad \le k \sum_{l_1=1}^{N^{1/3}\log N} \sum_{\substack{l_i=1 \\ 2 \le i \le k}}^{\infty} \prod_{j=1}^{k} e^{-cl_j/N^{1/3}} N^{-1/3} \le kC^k N^{-1}.
\end{aligned}$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{k!} N^{-1/3} \big(C(k+1)\big)^{(k+1)/2} k N^{-1} \le C N^{-4/3},$$

we see that we can indeed restrict the summation.

In the second step we replace K by A. As before we shall use the notation $A^\tau(x, y) = A(x + \tau^2, y + \tau^2)$. For $1 \le l_i \le N^{1/3} \log N$ it holds by Lemma 3.1 that

$$\begin{aligned}&\det(K(x_i, x_j))_{0\le i,j\le k}\\ &\quad = \frac{1}{(dN^{1/3})^{k+1}} \det\big(A^{\tau_1}(l_i/dN^{1/3}, l_j/dN^{1/3}) + \mathcal{O}(N^{-c})\big)_{0\le i,j\le k}\end{aligned}$$

where we let $l_0 = \psi_1 dN^{1/3}$. If we expand the determinant in the right-hand side we get $(k+1)^2$ error terms of type

$$\frac{N^{-c}}{(dN^{1/3})^{k+1}} \det\big(A^{\tau_1}(l_i/dN^{1/3}, l_j/dN^{1/3}) + \mathcal{O}(N^{-c})\big)_{\substack{0\le i,j\le k\\ i\ne i_0, j\ne j_0}}.$$

An application of Hadamard's inequality together with Lemma 3.1 shows that the total error we get when changing from K to A^{τ_1} is of order $N^{-1/3-c}$. We omit the details.

Finally we want to go from summation to integration. To do this we shall use that

$$\begin{aligned}&\sum_{l_i=1}^{N^{1/3}\log N} A^{\tau_1}(l_i/dN^{1/3}, x) A^{\tau_1}(y, l_i/dN^{1/3})\\ &\qquad = dN^{1/3} \int_0^\infty A^{\tau_1}(z, x) A^{\tau_1}(y, z)\, dz + \mathcal{O}(e^{-x-y})\end{aligned} \tag{3.6}$$

and

$$\sum_{l_i=1}^{N^{1/3}\log N} A^{\tau_1}(l_i/dN^{1/3}, l_i/dN^{1/3}) = dN^{1/3} \int_0^\infty A^{\tau_1}(z, z)\, dz + \mathcal{O}(1). \tag{3.7}$$

This follows from the Euler–Maclaurin summation formula and Lemma 2.3. We will show that

$$\begin{aligned}&\sum_{\substack{l_i=1\\ 1\le i\le k}}^{N^{1/3}\log N} \frac{1}{(dN^{1/3})^{k+1}} \det\big(A^{\tau_1}(l_i/dN^{1/3}, l_j/dN^{1/3})\big)_{0\le i,j\le k}\\ &\qquad = \frac{1}{dN^{1/3}} \int_{(0,\infty)^k} \det(A^{\tau_1}(y_i, y_j))_{0\le i,j\le k}\, d^k y\\ &\qquad\quad + \mathcal{O}\big((Ck)^{(k+5)/2} N^{-2/3}\big),\end{aligned} \tag{3.8}$$

where $l_0 = dN^{1/3}\psi_1$ and $y_0 = \psi_1$. This will prove the lemma since

$$\sum_{k=1}^{\infty} \frac{1}{k!}(Ck)^{(k+5)/2} < \infty.$$

For $r = 0, \dots, k$ we set

$$D_r = \frac{1}{(dN^{1/3})^{k-r+1}} \det(A^{\tau_1}(z_i, z_j))_{0 \le i, j \le k},$$

where

$$z_i = \begin{cases} \psi_1, & i = 0, \\ y_i, & 1 \le i \le r, \\ l_i/dN^{1/3}, & r+1 \le i \le k. \end{cases}$$

Please note that D_0 is what we sum over in (3.8) and that D_k is what we integrate over. D_r should roughly be what we get after having changed summation over $l_1, \dots, l_r$ to integration over $y_1, \dots, y_r$. We can expand D_r in such a way that we get k^2 terms of type

$$\pm \frac{1}{(dN^{1/3})^{k-r+1}} A^{\tau_1}(z_{i_0}, l_{r+1}/dN^{1/3}) A^{\tau_1}(l_{r+1}/dN^{1/3}, z_{j_0}) \times \det(A^{\tau_1}(z_i, z_j))_{\substack{0 \le i, j \le k \\ i \ne r+1, i_0 \\ j \ne r+1, j_0}}$$

and one term

$$\frac{1}{(dN^{1/3})^{k-r+1}} A^{\tau_1}(l_{r+1}/dN^{1/3}, l_{r+1}/dN^{1/3}) \det(A^{\tau_1}(z_i, z_j))_{\substack{0 \le i, j \le k \\ i, j \ne r+1}}.$$

We now apply (3.6) and (3.7) and therefore need to deal with the corresponding errors:

$$\begin{aligned} &C(N^{1/3})^{k-r+1} e^{-z_{i_0} - z_{j_0}} \det(A^{\tau_1}(z_i, z_j))_{\substack{0 \le i, j \le k \\ i \ne i_0, r+1 \\ j \ne j_0, r+1}} \\ &\quad \le C(N^{1/3})^{k-r+1} e^{-z_{i_0} - z_{j_0}} \Biggl(\prod_{\substack{j=0 \\ j \ne j_0, r+1}}^{k} C(k-1) e^{-cz_j} \Biggr)^{1/2} \\ &\quad \le (N^{1/3})^{k-r+1} \bigl(C(k-1)\bigr)^{(k-1)/2} \prod_{\substack{j=1 \\ j \ne r+1}}^{k} e^{-cz_j}. \end{aligned}$$

Since

$$\int_{(0,\infty)^r} d^r x \sum_{\substack{l_i=1 \\ r+2\le i\le k}}^{N^{1/3}\log N} \prod_{\substack{j=1 \\ j\neq r+1}}^{k} e^{-cz_j} \le C^k (N^{1/3})^{k-(r+1)},$$

we find that the error from the k^2 terms of the first type is estimated by

$$k^2\big(C(k-1)\big)^{(k-1)/2} N^{-2/3}.$$

The error coming from the remaining term can be treated in the same way. Changing from summation over l_i to integration over y_i, $1\le i\le k$, hence results in an error estimated by

$$kk^2\big(C(k-1)\big)^{(k-1)/2} N^{-2/3} = (Ck)^{(k+5)/2} N^{-2/3}$$

as needed. □

PROOF OF LEMMA 3.2. By definition

$$\phi_{2u,2v}(x,y) = \frac{(1-\alpha)^{2(v-u)}}{2\pi} \int_{-\pi}^{\pi} e^{i(y-x)\theta + (u-v)\log(1+\alpha^2-2\alpha\cos\theta)}\, d\theta.$$

Define

$$g(\theta) = \log(1+\alpha^2 - 2\alpha\cos\theta)$$

in $[-\pi,\pi]$. This function is analytic in a neighborhood of zero and a Maclaurin expansion gives

$$g(\theta) = \log(1-\alpha)^2 + \frac{\alpha}{(1-\alpha)^2}\theta^2 + c_2\theta^4 + \mathcal{O}(\theta^6)$$

where $c_4 < 0$. It is easy to see that for any $\delta > 0$ there exists $\varepsilon > 0$ such that

$$g(\theta) \ge \log(1-\alpha)^2 + \varepsilon$$

if $|\theta| \ge \delta$. Hence

$$\left| \int_{|\theta|>\delta} \frac{(1-\alpha)^{2(v-u)}}{2\pi} e^{i(y-x)\theta + (u-v)g(\theta)}\, d\theta \right| \le \frac{1}{2\pi}\int_{\delta}^{\pi} e^{(u-v)\varepsilon}\, d\theta \sim e^{-\varepsilon N^{\gamma}}.$$

We expect that the main contribution to $\phi_{2u,2v}$ will be

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\delta}^{\delta} e^{i(y-x)\theta + (u-v)\alpha/(1-\alpha)^2\theta^2}\, d\theta \\
&\quad = \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{i(y'-x')dN^{1/3}\theta - sd^2N^{\gamma}\theta^2}\, d\theta = \big[t = \sqrt{s}\, dN^{1/3}\theta\big] \\
&\quad = \frac{1}{2\pi\sqrt{s}\, dN^{1/3}} \int_{-\delta\sqrt{s}dN^{1/3}}^{\delta\sqrt{s}\, dN^{1/3}} e^{i(y'-x')/\sqrt{s}t - N^{\gamma-2/3}t^2}\, dt
\end{aligned}$$

$$= \frac{1}{2\pi\sqrt{s}\,dN^{1/3}} \int_{-\infty}^{\infty} e^{i(y'-x')/\sqrt{s}t - N^{\gamma-2/3}t^2}\,dt + \mathcal{O}(e^{-N^{\gamma}})$$
$$= \frac{1}{dN^{1/3}} \frac{1}{\sqrt{4\pi s N^{\gamma-2/3}}} e^{-(x'-y')^2/(4sN^{\gamma-2/3})} + \mathcal{O}(e^{-N^{\gamma}}).$$

Below we will analyze the error. For simplicity we take $s = 1$.

Define $h(\theta)$ by

$$g(\theta) = \log(1-\alpha)^2 + \frac{\alpha}{(1-\alpha)^2}\big(\theta^2 + h(\theta)\big).$$

This means that

$$h(\theta) = \sum_{k=4}^{\infty} h_k \theta^k$$

where $h_4 < 0$. Note that h is even since g is and also that, for δ small enough, $h(\theta) < 0$ if $|\theta| \le \delta$. The error becomes

$$\mathrm{Err} = \left| \int_{-\delta}^{\delta} e^{i(y'-x')dN^{1/3}\theta} F(\theta)\,d\theta \right|,$$

where

$$F(\theta) = e^{-d^2N^{\gamma}\theta^2} - e^{-d^2N^{\gamma}\theta^2 - d^2N^{\gamma}h(\theta)}.$$

Next we integrate by parts:

$$\mathrm{Err} \le \left| \left[\frac{1}{i(y'-x')\,dN^{1/3}} e^{i(y'-x')\,dN^{1/3}\theta} F(\theta) \right]_{-\delta}^{\delta} \right|$$
$$+ \frac{1}{|y'-x'|dN^{1/3}} \left| \int_{-\delta}^{\delta} e^{i(y'-x')\,dN^{1/3}\theta} F'(\theta)\,d\theta \right|$$
$$\le \frac{3}{|y'-x'|\,dN^{1/3}} e^{-d^2N^{\gamma}\delta^2} + \frac{1}{|y'-x'|\,dN^{1/3}} \int_{-\delta}^{\delta} |F'(\theta)|\,d\theta.$$

The last integral will be easy to compute if we can find out where $F'(\theta)$ changes sign:

$$F'(\theta) = 2d^2N^{\gamma}\theta e^{-d^2N^{\gamma}(\theta^2 + h(\theta))}\left(1 + \frac{h'(\theta)}{2\theta} - e^{d^2N^{\gamma}h(\theta)}\right).$$

A point in $[-\delta, \delta] \setminus \{0\}$ where F' changes sign will satisfy

$$\frac{1}{d^2N^{\gamma}} = \frac{h(\theta)}{\log[1 + h'(\theta)/(2\theta)]} = \frac{\theta^2}{2} + \mathcal{O}(\theta^4).$$

This shows that if N is large, then F' has two zeros $\pm\theta_0$ in $[-\delta, \delta] \setminus \{0\}$. Moreover, θ_0 is of order $N^{-\gamma/2}$. Given this information, we check which sign F' has in different intervals and get

$$\begin{aligned}\int_{-\delta}^{\delta} |F'(\theta)|\, d\theta &= 2\int_0^{\delta} |F'(\theta)|\, d\theta \\ &= -\int_0^{\theta_0} F'(\theta)\, d\theta + \int_{\theta_0}^{\delta} F'(\theta)\, d\theta \\ &= F(0) - F(\theta_0) + F(\delta) - F(\theta_0) \\ &= \mathcal{O}(N^{-\gamma}).\end{aligned}$$

This almost finishes the proof of the second inequality in the lemma. We should not forget the exponentially small error terms that appeared above. They do not have the factor $|x' - y'|^{-1}$ in front of them. However, a couple of partial integrations can be used to take care of this obstacle.

The first inequality in the lemma follows from the following calculation:

$$\begin{aligned}\int_0^{\delta} |F(\theta)|\, d\theta &= [\theta = tN^{-\gamma/2}] \\ &= N^{-\gamma/2} \int_0^{N^{\gamma/2}\delta} e^{-d^2t^2 - d^2h(tN^{-\gamma/2})}\big(1 - e^{d^2N^{\gamma}h(tN^{-\gamma/2})}\big)\, dt \\ &\le N^{-\gamma/2} \int_0^{N^{\gamma/2}\delta} e^{-c_1t^2}(1 - e^{-c_2N^{-\gamma}t^4})\, dt \\ &\le N^{-\gamma/2} \int_1^{N^{\gamma/2}\delta} te^{-c_1t^2}(1 - e^{-c_2N^{-\gamma}t^4})\, dt + \mathcal{O}(N^{-3\gamma/2}).\end{aligned}$$

We now use partial integration:

$$\begin{aligned}&\int_1^{N^{\gamma/2}\delta} te^{-c_1t^2}(1 - e^{-c_2N^{-\gamma}t^4})\, dt \\ &\quad= \left[-\frac{1}{2c_1}e^{-c_1t^2}(1 - e^{-c_2N^{-\gamma}t^4})\right]_1^{N^{\gamma/2}\delta} \\ &\qquad + \frac{2c_2N^{-\gamma}}{c_1} \int_1^{N^{\gamma/2}\delta} e^{-c_1t^2}t^3e^{-c_2N^{-\gamma}t^4}\, dt \\ &\quad= \mathcal{O}(N^{-3\gamma/2}).\end{aligned}$$

This concludes the calculations in this section as well as in this paper. □

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