The Annals of Probability
2008, Vol. 36, No. 2, 728–738
DOI: 10.1214/07-AOP338

LIMITING VELOCITY OF HIGH-DIMENSIONAL RANDOM WALK IN RANDOM ENVIRONMENT

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We show that random walk in uniformly elliptic i.i.d. environment in dimension ≥ 5 has at most one non zero limiting velocity. In particular this proves a law of large numbers in the distributionally symmetric case and establishes connections between different conjectures.

1. Introduction. Let $d \geq 1$. A random walk in random environment (RWRE) on $\mathbb{Z}^d$ is defined as follows: Let $\mathcal{M}^d$ denote the space of all probability measures on $\{\pm e_i\}_{i=1}^d$ and let $\Omega = (\mathcal{M}^d)^{\mathbb{Z}^d}$. An *environment* is a point $\omega \in \Omega$. Let P be a probability measure on Ω. For the purposes of this paper, we assume that P is an i.i.d. measure, that is,

$$P = Q^{\mathbb{Z}^d}$$

for some distribution Q on $\mathcal{M}^d$ and that P is *uniformly elliptic*, that is, there exists $\varepsilon > 0$ such that (s.t.) for every $e \in \{\pm e_i\}_{i=1}^d$,

$$Q(\{d : d(e) < \varepsilon\}) = 0.$$

For an environment $\omega \in \Omega$, the *random walk* on ω is a time-homogenous Markov chain with transition kernel

$$P_\omega(X_{n+1} = z + e | X_n = z) = \omega(z, e).$$

The *quenched law* P_ω^z is defined to be the law on $(\mathbb{Z}^d)^{\mathbb{N}}$ induced by the kernel P_ω and $P_\omega^z(X_0 = z) = 1$. We let $\mathbf{P} = P \otimes P_\omega^0$ be the joint law of the environment and the walk, and the *annealed* law is defined to be its marginal

$$\mathbb{P} = \int_\Omega P_\omega^0 \, dP(\omega).$$

We consider the limiting velocity

$$v = \lim_{n \to \infty} \frac{X_n}{n}.$$

Based on the work of Zerner [5] and Sznitman and Zerner [3], we know that v exists $\mathbb{P}$-a.s. Furthermore, there is a set A of size at most 2 such that almost surely $v \in A$.

Received October 2006; revised March 2007.
AMS 2000 subject classification. 60K37.
Key words and phrases. Random walk, random environment.

Zerner and Merkl [6] proved that in dimension 2 a 0–1 law holds and therefore the set A is of size 1, that is, a law of large numbers holds, in dimension 2 (see also [2] for a continuous version).

The main result of this paper is the following:

THEOREM 1.1. *For $d \geq 5$, there is at most one nonzero limiting velocity; that is, if $A = \{v_1, v_2\}$ with $v_1 \neq v_2$ and $v_1 \neq 0$, then $v_2 = 0$.*

Theorem 1.1 has the following immediate corollary:

COROLLARY 1.2. *For $d \geq 5$, if Q is distributionally symmetric, then the limiting velocity is an almost sure constant.*

REMARK ABOUT CONSTANTS. As is common in most of the RWRE literature, the value of the constant C may vary from line to line. In addition, C may implicitly depend on variables that are kept constant throughout the entire calculation, in particular the dimension d or the distribution Q.

2. Backward path—Construction.

In this section we describe the backward path, the main object studied in this paper. The backward path is, roughly speaking, a path of the RWRE from $-\infty$ through the origin to $+\infty$. Below we define it. In Section 3 we prove some basic facts about it. Note that the backward path appears, though implicitly, in [1] and [4].

Throughout the paper we are assuming, for contradiction, that two different nonzero limiting velocities v_1 and v_2 exist. Assume without loss of generality that $\langle \ell, v_1 \rangle > 0$ for $\ell = e_1$. We let A_ℓ be the event that the walk is transient in the direction ℓ, that is,

$$A_\ell = \left\{ \lim_{n\to\infty} \langle X_n, \ell \rangle = \infty \right\}.$$

By our assumptions, Q is a distribution on $\mathcal{M}^d$ s.t. both $\mathbf{P}(A_\ell)$ and $\mathbf{P}(A_{-\ell})$ are positive.

We say that t is a regeneration time in the direction ℓ if:

1. $\langle X_s, \ell \rangle < \langle X_t, \ell \rangle$ for every $s < t$, and
2. $\langle X_s, \ell \rangle > \langle X_t, \ell \rangle$ for every $s > t$.

REMARK. Note that in the special case of ℓ being a coordinate vector this simple definition coincides with the more complex definition of a regeneration time from [3].

For every $L > 0$, let $\mathcal{K}_L = \{z | 0 \leq \langle z, \ell \rangle < L\}$.

Let t_1 be the first regeneration time (if one exists), let t_2 be the second (if exists), and so on. If t_{n+1} exists, let $L_n = \langle X_{t_{n+1}}, \ell \rangle - \langle X_{t_n}, \ell \rangle$, let

$$W_n : \mathcal{K}_{L_n} \to \mathcal{M}^d$$

be

$$W_n(z) = \omega(z + X_{t_n}),$$

let $u_n = t_{n+1} - t_n$ and let $K_n : [0, u_n] \to \mathbb{Z}^d$ be $K_n(t) = X_{t_n+t} - X_{t_n}$. We let S_n, the nth regeneration slab, be the ensemble $S_n = \{L_n, W_n, u_n, K_n\}$.

In [3] Sznitman and Zerner proved that on the event A_ℓ, almost surely there are infinitely many regeneration times, and, furthermore, that the regeneration slabs $\{S_i\}_{i=1}^{\infty}$ form an i.i.d. process. Let $\lambda = \lambda_\ell$ be the distribution of S_1 conditioned on A_ℓ.

We now construct an environment and a doubly infinite path in that environment. Let $\{S_n\}_{n\in\mathbb{Z}}$ be i.i.d. regeneration slabs sampled according to λ.

We now want to glue the regeneration slabs to each other. Let $Y_0 = 0$, and define, inductively, $Y_{n+1} = Y_n + K_n(u_n)$ for $n \geq 0$ and $Y_{n-1} = Y_n - K_{n-1}(u_{n-1})$ for $n \leq 0$. Almost surely $\mathbb{Z}^d$ is the disjoint union of the sets $Y_n + \mathcal{K}_{L_n}$. For every $z \in \mathbb{Z}^d$ let $n(z)$ be the unique n such that $z \in Y_n + \mathcal{K}_{L_n}$. Let ω be the environment

$$\omega(z) = W_{n(z)}\bigl(z - Y_{n(z)}\bigr).$$

Let $\mathcal{T} \subseteq \mathbb{Z}^d$ be

$$\mathcal{T} = \bigcup_{n=-\infty}^{\infty} (Y_n + K_n[0, u_n]).$$

Let μ be the joint distribution of ω and $\mathcal{T}$. $\mathcal{T}$ is called the *backward path in direction* ℓ. We let $\tilde{\mu}$ be the marginal distribution of ω in μ.

3. Backward path—Basic properties. In this section we prove two simple properties of the measure μ.

PROPOSITION 3.1. *There exists a coupling* $\tilde{P}$ *on* $\Omega \times \Omega \times \{0,1\}^{\mathbb{Z}^d}$ *with the distribution of* $\omega, \tilde{\omega}, \mathcal{T}$ *satisfying*:

1. ω *is distributed according to* P.
2. $(\tilde{\omega}, \mathcal{T})$ *is distributed according to* μ.
3. $\tilde{P}$*-almost surely*, $\omega(z) = \tilde{\omega}(z)$ *for every* $z \in \mathbb{Z}^d \setminus \mathcal{T}$.
4. ω *and* $\mathcal{T}$ *are independent.*

PROPOSITION 3.2. *Let* $\tilde{\omega}$ *be an environment sampled according to* $\tilde{\mu}$, *and let* $\{X_n\}$ *be a random walk on that environment. Then almost surely* $\{X_n\}$ *is transient in the direction* ℓ.

Both Proposition 3.1 and Proposition 3.2 follow from the fact that the $\tilde{\mu}$-environment around zero is similar to the P-environment around the location of the walker at a large regeneration time. More precisely, let $\omega, \{X_n\}$ be sampled according to $\mathbf{P}$ conditioned on the event $\forall_{n>0}(\langle X_n, \ell\rangle > 0) \cap A_\ell$, which is an event of positive probability. Let $t_1, t_2, \dots$ be the regeneration times. (Note that we conditioned on transience in the ℓ direction, and therefore infinitely many regeneration times exist.) Let ω_i be the environment defined by $\omega_i(z) = \omega(z + X_{t_i})$ and let $\mathcal{T}_i \subseteq \mathbb{Z}^d$ be defined as $\mathcal{T}_i = \{X_t - X_{t_i} | t \geq 0\}$.

For $X \in \mathbb{Z}^d$ let $\mathcal{H}(X)$ be the half-space

$$\mathcal{H}(X) = \{z \mid \langle z, \ell\rangle \geq \langle X, \ell\rangle\}.$$

LEMMA 3.3. *For every i, the distribution of*

$$\{-X_{t_i};\ \mathcal{T}_i \cap \mathcal{H}(-X_{t_i});\ \omega_i|_{\mathcal{H}(-X_{t_i})}\} \tag{1}$$

is the same as the distribution of

$$\{Y_{-i};\ \mathcal{T} \cap \mathcal{H}(Y_{-i});\ \tilde{\omega}|_{\mathcal{H}(Y_{-i})}\}. \tag{2}$$

PROOF. Let $\tilde{\mathbf{P}}$ be $\mathbf{P}$ conditioned on the event $\forall_{n>0}(\langle X_n, \ell\rangle > 0) \cap A_\ell$. By Theorem 1.4 of [3], the distribution of

$$\{\omega|_{\mathcal{H}(0)}, \{X_t | t \geq 0\}\}$$

according to $\tilde{\mathbf{P}}$ is the same as the distribution of

$$\{\tilde{\omega}|_{\mathcal{H}(0)}, \mathcal{T} \cap \mathcal{H}(0)\}$$

according to μ. The lemma now follows since the sequence $\{S_n\}_{n\in\mathbb{Z}}$ is i.i.d. □

We can now prove Propositions 3.1 and 3.2.

PROOF OF PROPOSITION 3.2. Let B be the event that the walk is transient in the direction of ℓ and never exits the half-space $\mathcal{H}(0)$, that is,

$$B = A_\ell \cap \{\forall_t X_t \in \mathcal{H}(0)\}.$$

For a configuration ω and $z \in \mathbb{Z}^d$, let

$$R_\omega(z) = P_\omega^z(B).$$

Note that $R_\omega(z)$ depends only on $\omega|_{\mathcal{H}(0)}$, so by the Markov property

$$\mathbf{P}_\omega^{X_0}(B | X_1, X_2, \dots, X_t) = R_\omega(X_t) \cdot \mathbf{1}_{X_1,\dots,X_t \in \mathcal{H}(0)}.$$

In addition, $B \in \sigma(X_1, X_2, \dots)$ and therefore almost surely

$$\lim_{t\to\infty} R_\omega(X_t) \geq \mathbf{1}_B.$$

In particular, $\tilde{\mathbf{P}}$-almost surely,

$$\lim_{t\to\infty} R_\omega(X_t) = 1,$$

and for the subsequence of regeneration times we get that $\tilde{\mathbf{P}}$-almost surely

$$\lim_{n\to\infty} R_\omega(X_{t_n}) = 1, \tag{3}$$

and using the bounded convergence theorem, for

$$R_n = \mathbf{E}_{\tilde{\mathbf{P}}}(R_\omega(X_{t_n}))$$

we get

$$\lim_{n\to\infty} R_n = 1. \tag{4}$$

Let $\{\tilde{\omega}, \mathcal{T}, \{Y_n\}\}$ be sampled according to μ and let X_n be a random walk on the environment $\tilde{\omega}$, which is independent of $\{\mathcal{T}, \{Y_n\}\}$ conditioned on $\tilde{\omega}$. Let B_N be the event

$$\lim_{n\to\infty} \langle X_n, \ell\rangle = \infty \quad \text{and} \quad \forall_n \langle X_n, \ell\rangle \geq \langle Y_{-N}, \ell\rangle.$$

Then by Lemma 3.3

$$(\mu \otimes P^0_{\tilde{\omega}})(B_n) = R_n. \tag{5}$$

Remembering that

$$A_\ell = \bigcup_{n=1}^{\infty} B_n$$

we get from (5) that

$$(\mu \otimes P^0_{\tilde{\omega}})(A_\ell) = \lim_{n\to\infty} R_n = 1,$$

as desired. □

PROOF OF PROPOSITION 3.1. We define the coupling on every regeneration slab. Let $\tilde{\lambda}$ be the distribution on $\tilde{S} = \{L, W, \tilde{W}, u, K\}$ so that $\{L, \tilde{W}, u, K\}$ is distributed according to λ and W is defined as follows:

$$W(z) = \begin{cases} \tilde{W}(z), & \text{if } z \notin K([0, u]), \\ \psi(z), & \text{if } z \in K([0, u]), \end{cases}$$

where $\psi : \mathbb{Z}^d \to \mathcal{M}$ is sampled according to P, independently of $\{L, \tilde{W}, u, K\}$.

CLAIM 3.4. *Conditioned on L, the environment W is i.i.d. with marginal distribution Q, and independent of u and K.*

We now sample the environments and the path as we did in Section 2: Let $\{\tilde{S}_n\}_{n=-\infty}^{\infty}$ be i.i.d. regeneration slabs sampled according to $\tilde{\lambda}$. Let $Y_0 = 0$ and define, inductively, $Y_{n+1} = Y_n + K_n(u_n)$ for $n \geq 0$ and $Y_{n-1} = Y_n - K_{n-1}(u_{n-1})$ for $n \leq 0$. Almost surely $\mathbb{Z}^d$ is the disjoint union of the sets $Y_n + \mathcal{K}_{L_n}$. For every

$z \in \mathbb{Z}^d$ let $n(z)$ be the unique n such that $z \in Y_n + \mathcal{K}_{L_n}$. We let ω be the environment

$$\omega(z) = W_{n(z)}(z - Y_{n(z)}),$$

we let $\tilde{\omega}$ be the environment

$$\tilde{\omega}(z) = \tilde{W}_{n(z)}(z - Y_{n(z)}),$$

and take $\mathcal{T} \subseteq \mathbb{Z}^d$ to be

$$\mathcal{T} = \bigcup_{n=-\infty}^{\infty} (Y_n + K_n[0, u_n]).$$

Clearly, $\{\tilde{\omega}, \mathcal{T}\}$ is distributed according to μ and ω and $\tilde{\omega}$ agree on $\mathbb{Z}^d - \mathcal{T}$. Therefore all we need to show is that ω is distributed according to P and is independent of the path $\mathcal{T}$. This follows from Claim 3.4: conditioned on $\{u_n\}_{n=-\infty}^{\infty}$, W is P-distributed and independent of the path $\mathcal{T}$. Therefore it is P-distributed and independent of the path $\mathcal{T}$ as we integrate over $\{u_n\}_{n=-\infty}^{\infty}$. $\square$

PROOF OF CLAIM 3.4. It is sufficient to show that conditioned on L, for every finite set $J = \{x_i : i = 1, \ldots, k\}$ with $J \subseteq \mathcal{K}_L$, the distribution of $\{W(x_i)\}_{x_i \in J}$ is i.i.d. with marginal Q and independent of u and K. This will follow if we prove that for every finite set $J = \{x_i | i = 1, \ldots, k\}$ with $J \subseteq \mathcal{K}_L$, conditioned on L, on K and u and on the event $J \cap K[0, u] = \varnothing$, the distribution of $\{\tilde{W}(x_i)\}_{x_i \in J}$ is i.i.d. with marginal Q.

To this end, fix J and note that for every finite set U that is disjoint of J, the event $\{K[0, u] = U\}$ is independent of $\{\tilde{W}(x_i)\}_{x_i \in J}$. Therefore, conditioned on the event $\{K[0, u] = U\}$ (and thus implicitly conditioning on K and u), the distribution of $\{\tilde{W}(x_i)\}_{x_i \in J}$ is i.i.d. with marginal Q. By integrating with respect to U we get that $\{W(x_i)\}_{x_i \in J}$ is Q-distributed, and by the fact that it was Q-distributed conditioned on K and u we get the independence. $\square$

4. Intersection of paths. In this section we will see some interaction between the backward path and the path of an independent random walk.

Let Q be a uniformly elliptic distribution so that $0 < \mathbf{P}(A_\ell) < 1$ and let $(\omega, \tilde{\omega}, \mathcal{T})$ be as in Proposition 3.1. Let z_0 be an arbitrary point in $\mathbb{Z}^d$, and let $\{X_i\}_{i=1}^{\infty}$ be a random walk on the configuration ω starting at z_0, such that:

1. $\{X_i\}$ is conditioned on the (positive probability) event that $\lim_{i\to\infty}\langle X_i, \ell\rangle = -\infty$.
2. Conditioned on ω, $\{X_i\}_{i=1}^{\infty}$ is independent of $\tilde{\omega}$ and $\mathcal{T}$.

The purpose of this section is the following easy lemma:

LEMMA 4.1. *Under the conditions stated above, almost surely there exist infinitely many values of i such that $X_i \in \mathcal{T}$.*

We will prove that almost surely there exists one such value of i. The proof that infinitely many exist is very similar but requires a little more care, and for the purpose of proving the main theorem of this paper one such i is sufficient.

PROOF. We need to show that

$$(\tilde{P} \otimes P_{\omega}^{z_0})\Big(\lim_{i\to\infty} \langle X_i, \ell\rangle = -\infty \text{ and } \forall_i (X_i \notin \mathcal{T})\Big) = 0. \tag{6}$$

In order to establish (6), let $\{Y_i\}_{i=1}^{\infty}$ be a random walk on the environment $\tilde{\omega}$, coupled to the rest of the probability space as follows:

Let

$$i_0 = \inf\{i : \omega(X_i) \neq \tilde{\omega}(X_i)\} \geq \inf\{i : X_i \in \mathcal{T}\}.$$

Now, for $i < i_0$, we define $Y_i = X_i$. For $i \geq i_0$, Y_i is determined based on Y_{i-1} according to $\tilde{\omega}(Y_{i-1})$ independently of X_i, ω and $\mathcal{T}$. Now, note that

$$\forall_i (X_i \notin \mathcal{T}) \quad \Longrightarrow \quad i_0 = \infty \quad \Longrightarrow \quad \forall_i (X_i = Y_i).$$

Therefore,

$$\Big(\lim_{i\to\infty} \langle X_i, \ell\rangle = -\infty \text{ and } \forall_i (X_i \notin \mathcal{T})\Big) \quad \Longrightarrow \quad \lim_{i\to\infty} \langle Y_i, \ell\rangle = -\infty.$$

The proof is concluded if we remember that by Proposition 3.2,

$$(\tilde{P} \otimes P_{\tilde{\omega}}^{z_0})\Big(\lim_{i\to\infty} \langle Y_i, \ell\rangle = -\infty\Big) = 0.$$

□

5. Proof of main theorem.

LEMMA 5.1. *Let $d \geq 5$, and assume that the set A of speeds contains two nonzero elements. Then there exists z_0 such that*

$$(\tilde{P} \otimes P_{\omega}^{z_0})\Big(\lim_{i\to\infty} \langle X_i, \ell\rangle = -\infty \textit{ and } \forall_i (X_i \notin \mathcal{T})\Big) > 0.$$

PROOF. Let

$$\tilde{\mathcal{T}} = \{X_i : i = 1, 2, \ldots\}.$$

We use the following claim whose proof is deferred:

CLAIM 5.2. *Let $\tilde{B}$ be the event that $\langle X_i, \ell\rangle < \langle X_0, \ell\rangle$ for all $i > 0$. Note that $\tilde{B}$ has positive probability. Also, let $\mathcal{T}' = \mathcal{T} \cap \{z : \langle z, \ell\rangle \leq 0\}$. Then, if A contains two distinct nonzero elements then*

$$\sum_{z\in\mathbb{Z}^d} \tilde{P}(z \in \mathcal{T}')^2 < \infty \tag{7}$$

and

$$\sum_{z\in\mathbb{Z}^d} \mathbb{P}^0(z \in \tilde{\mathcal{T}} | \tilde{B})^2 < \infty. \tag{8}$$

By Proposition 3.1, $\mathcal{T}'$ and $\tilde{\mathcal{T}}$ are independent random sets and therefore so are $\mathcal{T}'$ and $\tilde{\mathcal{T}}|\tilde{B}$. Therefore,

$$\begin{aligned}(\tilde{E} \otimes E_\omega^{z_0})(|\mathcal{T}' \cap \tilde{\mathcal{T}}||\tilde{B}) &= \sum_{z\in\mathbb{Z}^d} \tilde{P}(z \in \mathcal{T}')\mathbb{P}^{z_0}(z \in \tilde{\mathcal{T}}|\tilde{B}) \\ &= \sum_{z\in\mathbb{Z}^d} \tilde{P}(z \in \mathcal{T}')\mathbb{P}^0(z - z_0 \in \tilde{\mathcal{T}}|\tilde{B}),\end{aligned}$$

with the last equality following from translation invariance of the annealed measure. Let

$$M = \sum_{z\in\mathbb{Z}^d} \tilde{P}(z \in \mathcal{T}')^2$$

and

$$\tilde{M} = \sum_{z\in\mathbb{Z}^d} \mathbb{P}^0(z \in \tilde{\mathcal{T}}|\tilde{B})^2,$$

let λ be so small that $\lambda M + \lambda\tilde{M} + \lambda^2 < 1$, and let R be so large that

$$\sum_{\|z\|>R} \tilde{P}(z \in \mathcal{T}')^2 < \lambda \quad \text{and} \quad \sum_{\|z\|>R} \mathbb{P}^0(z \in \tilde{\mathcal{T}}|\tilde{B})^2 < \lambda.$$

Taking z_0 such that $\|z_0\| > 2R$ and $\langle z_0, \ell\rangle < 0$ we get, using Cauchy–Schwarz, that

$$\begin{aligned}&(\tilde{E} \otimes E_\omega^{z_0})(|\mathcal{T}' \cap \tilde{\mathcal{T}}||\tilde{B}) \\ &\quad = \sum_{z\in\mathbb{Z}^d} \tilde{P}(z \in \mathcal{T}')\mathbb{P}^0(z - z_0 \in \tilde{\mathcal{T}}|\tilde{B}) \\ &\quad = \sum_{z\in B(0,R)} \tilde{P}(z \in \mathcal{T}')\mathbb{P}^0(z - z_0 \in \tilde{\mathcal{T}}|\tilde{B}) \\ &\qquad + \sum_{z\in B(z_0,R)} \tilde{P}(z \in \mathcal{T}')\mathbb{P}^0(z - z_0 \in \tilde{\mathcal{T}}|\tilde{B}) \\ &\qquad + \sum_{z\in\mathbb{Z}^d - B(0,R) - B(z_0,R)} \tilde{P}(z \in \mathcal{T}')\mathbb{P}^0(z - z_0 \in \tilde{\mathcal{T}}|\tilde{B}) \\ &\quad \le \lambda M + \lambda\tilde{M} + \lambda^2 < 1.\end{aligned}$$

Therefore $\tilde{P} \otimes P_\omega^{z_0}(\mathcal{T}' \cap \tilde{\mathcal{T}} = \varnothing | \tilde{B}) > 0$. $P_\omega^{z_0}(\tilde{B}) > 0$ and by the choice of z_0, conditioned on $\tilde{B}$, $\mathcal{T}' \cap \tilde{\mathcal{T}} = \varnothing$ if and only if $\mathcal{T} \cap \tilde{\mathcal{T}} = \varnothing$. Therefore $\mathcal{T} \cap \tilde{\mathcal{T}}$ is empty with positive probability. □

PROOF OF CLAIM 5.2. We will prove (7). Equation (8) follows from the exact same reasoning. First we get an upper bound on $\mu(Y_{-n} = z)$. The sequence $\{O_n = Y_{-n} - Y_{-n-1}\}$ is an i.i.d. sequence. Furthermore, due to ellipticity there exist d linearly independent vectors $v_1, \dots, v_d$ and $\varepsilon > 0$ such that for every $k = 1, \dots, d$, and every $\delta \in \{+1, -1\}$,

$$\mu(O_1 = 2v_1 + \delta v_k) > \varepsilon.$$

(v_1 is, approximately, in the direction of ℓ, while the others are, approximately, orthogonal to ℓ.)

Let

$$A = \{2v_1 + \delta v_k \mid k = 1, \dots, d;\ \delta \in \{+1, -1\}\}$$

and let $p = \mu(O_1 \in A)$. Fix n, and let $E^{(n)}$ be the event that at least $\pi_n = \lceil \frac{1}{2} pn \rceil$ of the O_i's, $i = 1, \dots, n$, are in A. For every subset H of $\{1, \dots, n\}$ of size π_n, let $E_H^{(n)}$ be the event that the elements of H are the smallest π_n numbers i such that $O_i \in A$. Then from heat kernel estimates for bounded i.i.d. random walks in Z^d we get that for every $z \in \mathbb{Z}^d$,

$$\mu\left(\sum_{i \in H} O_i = z \,\middle|\, E_H^{(n)}\right) < Cn^{-d/2}.$$

Conditioned on $E_H^{(n)}$,

$$\sum_{i \in H} O_i \quad \text{and} \quad \sum_{i \notin H} O_i$$

are independent, so remembering that $Y_{-n} = \sum_{i=1}^n O_i$, we get that

$$\mu(Y_{-n} = z | E_H^{(n)}) < Cn^{-d/2}.$$

The events

$$\{E_H^{(n)} | H \subseteq [1, n]\}$$

are mutually exclusive and

$$\mu\left(\bigcup_H E_H^{(n)}\right) > 1 - e^{-Cn}.$$

Therefore, for every n and $z \in \mathbb{Z}^d$,

$$\mu(Y_{-n} = z) < Cn^{-d/2}. \tag{9}$$

Now, for every n and $z \in \mathbb{Z}^d$, let $Q(z,n)$ be the probability that z is visited during the nth regeneration, that is, between Y_{1-n} and Y_{-n}. The nth regeneration is independent of Y_{1-n}, so

$$Q(z,n|Y_{1-n}) = Q(z - Y_{1-n}, 0).$$

The fact that the speed of the walk in direction ℓ is positive yields

$$\sum_{z\in\mathbb{Z}^d} Q(z,0) \le E(\tau_2 - \tau_1) < \infty. \tag{10}$$

From (9) we get that

$$\sum_{z\in\mathbb{Z}^d} [\mu(Y_{-n} = z)]^2 \le Cn^{-d/2}.$$

Combined with (10) and remembering that Young's inequality for convolution says that $\|f \star g\|_2 \le \|f\|_2\|g\|_1$ for all f and g (and noting that the next regeneration slab is independent of Y_{1-n}, and thus the result is a convolution), we get

$$\sum_{z\in\mathbb{Z}^d} [Q(z,n)]^2 \le Cn^{-d/2}$$

or

$$\sqrt{\sum_{z\in\mathbb{Z}^d} [Q(z,n)]^2} \le Cn^{-d/4}. \tag{11}$$

Noting that

$$\mu(z \in \mathcal{T}') = \sum_{n=1}^{\infty} Q(z,n),$$

(11) and the triangle inequality tell us that

$$\sqrt{\sum_{z\in\mathbb{Z}^d} [\mu(z \in \mathcal{T}')]^2} \le C\sum_{n=1}^{\infty} n^{-d/4}.$$

So for $d \ge 5$

$$\sum_{z\in\mathbb{Z}^d} [\mu(z \in \mathcal{T}')]^2 < \infty$$

as desired. $\square$

PROOF OF THEOREM 1.1. The theorem follows immediately from Lemma 4.1 and Lemma 5.1. $\square$

Acknowledgments. I thank G. Y. Amir, I. Benjamini, M. Biskup, N. Gantert, S. Sheffield and S. Starr for useful discussions. I thank O. Zeitouni for many important comments on a previous version of the paper. An anonymous referee is gratefully acknowledged for many important comments.

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