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AN ALMOST SURE INVARIANCE PRINCIPLE FOR THE RANGE OF PLANAR RANDOM WALKS

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For a symmetric random walk in Z^2 with $2+\delta$ moments, we represent $|\mathcal{R}(n)|$, the cardinality of the range, in terms of an expansion involving the renormalized intersection local times of a Brownian motion. We show that for each $k \geq 1$

$$(\log n)^k\left[\frac{1}{n}|\mathcal{R}(n)| + \sum_{j=1}^{k}(-1)^j\left(\frac{1}{2\pi}\log n + c_X\right)^{-j}\gamma_{j,n}\right] \to 0 \qquad \text{a.s.,}$$

where W_t is a Brownian motion, $W_t^{(n)} = W_{nt}/\sqrt{n}$, $\gamma_{j,n}$ is the renormalized intersection local time at time 1 for $W^{(n)}$ and c_X is a constant depending on the distribution of the random walk.

1. Introduction. Let $S_n = X_1 + \cdots + X_n$ be a random walk in Z^2, where $X_1, X_2, \ldots$ are symmetric i.i.d. vectors in Z^2. We assume that the X_i have $2+\delta$ moments for some $\delta > 0$ and covariance matrix equal to the identity. We assume further that the random walk S_n is strongly aperiodic in the sense of Spitzer ([23], page 42). The range $\mathcal{R}(n)$ of the random walk S_n is the set of sites visited by the walk up to step n:

$$\mathcal{R}(n) = \{S_0, \ldots, S_{n-1}\}. \tag{1.1}$$

As usual, $|\mathcal{R}(n)|$ denotes the cardinality of the range up to step n.

Dvoretzky and Erdös [6] show that for nearest-neighbor symmetric random walks

$$\lim_{n\to\infty} \log n \frac{|\mathcal{R}(n)|}{n} = 2\pi \qquad \text{a.s.} \tag{1.2}$$

An error in [6] was corrected by Jain and Pruitt [11]. Le Gall [12] has obtained a central limit theorem for the second-order fluctuations of $|\mathcal{R}(n)|$:

$$(\log n)^2\left(\frac{|\mathcal{R}(n)| - E(|\mathcal{R}(n)|)}{n}\right) \xrightarrow{d} -(2\pi)^2\gamma_2(1) \tag{1.3}$$

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where $\xrightarrow{d}$ denotes convergence in law and $\gamma_2(t)$ is the second-order renormalized self-intersection local time for planar Brownian motion. See also [15].

In this paper we prove an a.s. asymptotic expansion for $|\mathcal{R}(n)|$ to any order of accuracy. In order to state our result we first introduce some notation. If $\{W_t; t \geq 0\}$ is a planar Brownian motion, we define the jth-order renormalized intersection local time for $\{W_t; t \geq 0\}$ as follows. $\gamma_1(t) = t$, $\alpha_{1,\varepsilon}(t) = t$ and for $k \geq 2$

$$\alpha_{k,\varepsilon}(t) = \int_{0 \leq t_1 \leq \cdots \leq t_k < t} \prod_{i=2}^{k} p_\varepsilon(W_{t_i} - W_{t_{i-1}})\, dt_1 \cdots dt_k, \tag{1.4}$$

$$\gamma_k(t) = \lim_{\varepsilon \to 0} \sum_{l=1}^{k} \binom{k-1}{l-1} (-u_\varepsilon)^{k-l} \alpha_{l,\varepsilon}(t), \tag{1.5}$$

where $p_t(x)$ is the density for W_t and

$$u_\varepsilon = \int_0^\infty e^{-t} p_{t+\varepsilon}(0)\, dt.$$

Renormalized self-intersection local time was originally studied by Varadhan [24] for its role in quantum field theory. In [21] we show that $\gamma_k(t)$ can be characterized as the continuous process of zero quadratic variation in the decomposition of a natural Dirichlet process. For further work on renormalized self-intersection local times see [3, 8, 14, 18, 20].

To motivate our result define the Wiener sausage of radius ε as

$$\mathcal{W}_\varepsilon(0,t) = \left\{ x \in R^2 \,\Big|\, \inf_{0 \leq s \leq t} |x - W_s| \leq \varepsilon \right\}. \tag{1.6}$$

Letting $m(\mathcal{W}_\varepsilon(0,t))$ denote the area of the Wiener sausage of radius ε, Le Gall [13] shows that for each $k \geq 1$

$$(\log n)^k \left[m(\mathcal{W}_{n^{-1/2}}(0,1)) + \sum_{j=1}^{k} (-1)^j \left(\frac{1}{2\pi} \log n + c \right)^{-j} \gamma_j(1) \right] \to 0 \qquad \text{a.s.}$$

as $n \to \infty$ where c is a finite constant. Using the heuristic which associates $\{S_{[nt]}/\sqrt{n}; 0 \leq t \leq 1\} \subseteq n^{-1/2}Z^2 \subseteq R^2$ with the Brownian motion $\{W_t; 0 \leq t \leq 1\}$, one would expect (note that space is scaled by $n^{-1/2}$) that $\frac{1}{n}|\mathcal{R}(n)|$ will be "close" to $m(\mathcal{W}_{n^{-1/2}}(0,1))$.

Our main result is the following theorem.

THEOREM 1. *Let $S_n = X_1 + \cdots + X_n$ be a symmetric, strongly aperiodic random walk in Z^2 with covariance matrix equal to the identity and with $2 + \delta$*

moments for some $\delta > 0$. *On a suitable probability space we can construct* $\{S_n; n \geq 1\}$ *and a planar Brownian motion* $\{W_t; t \geq 0\}$ *such that for each* $k \geq 1$

$$(1.7)\qquad (\log n)^k \Bigg[\frac{1}{n} |\mathcal{R}(n)| + \sum_{j=1}^{k} (-1)^j \left(\frac{1}{2\pi} \log n + c_X \right)^{-j} \gamma_j\big(1, W^{(n)}\big) \Bigg] \to 0 \qquad a.s.$$

where the random variables $\gamma_1(1, W^{(n)}), \gamma_2(1, W^{(n)}), \dots$ *are the renormalized self-intersection local times* (1.5) *with* $t = 1$ *for the Brownian motion* $\{W_t^{(n)} = W_{nt}/\sqrt{n}; t \geq 0\}$,

$$(1.8)\qquad c_X = \frac{1}{2\pi} \log(\pi^2/2) + \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{\phi(p) - 1 + |p|^2/2}{(1 - \phi(p))|p|^2/2} \, dp$$

is a finite constant and $\phi(p) = E(e^{ip \cdot X_1})$ *denotes the characteristic function of* X_1.

Note that the presence of the constant c_X shows that the heuristic mentioned before the statement of Theorem 1 does not completely capture the fine structure of $|\mathcal{R}(n)|$. (This can already be observed on the level of (1.3); see [15], (6.r).)

The case of two dimensions is the critical one. For dimensions 3 and higher there are almost sure invariance principles by Hamana [10] (for dimensions 4 and higher) and Bass and Kumagai [4] (for dimension 3) that say that the range, appropriately normalized, is close to a Brownian motion.

We begin our proof in Section 2 where we introduce renormalized intersection local times $\Gamma_{k,\lambda}(n)$ for our random walk. Let ζ be an independent exponential random variable of mean 1, and set $\zeta_\lambda = n$ when $(n-1)\lambda < \zeta \leq \lambda n$. Letting $|\mathcal{R}(\zeta_\lambda)|$ denote the cardinality of the range of our random walk killed at step ζ_λ, we derive an L^2 asymptotic expansion for $|\mathcal{R}(\zeta_\lambda)|$ in terms of the $\Gamma_{k,\lambda}(\zeta_\lambda)$ as $\lambda \to 0$. In Sections 3–5, on a suitable probability space, we construct $\{S_n; n \geq 1\}$ and a planar Brownian motion $\{W_t; t \geq 0\}$ and show that in the above L^2 asymptotic expansion for $|\mathcal{R}(\zeta_\lambda)|$ we can replace $\lambda \Gamma_{k,\lambda}(\zeta_\lambda)$ by $\gamma_k(\zeta, W^{(\lambda^{-1})})$, the renormalized intersection local times for the planar Brownian motion $\{W_t^{(\lambda^{-1})} = W_{\lambda^{-1}t}/\sqrt{\lambda^{-1}}; t \geq 0\}$. After some preliminaries on renormalized intersection local times for Brownian motion in Section 6, we show in Section 7 how our L^2 asymptotic expansion for $|\mathcal{R}(\zeta_\lambda)|$ leads to an a.s. asymptotic expansion. The proof of Theorem 1 is completed in Section 8 by showing how to replace the random time ζ_λ by fixed time. The Appendix derives some estimates used in this paper. Our methods obviously owe a great deal to Le Gall [13].

2. Range and random walk intersection local times. We first define the nonrenormalized random walk intersection local times for $k \geq 2$ by

$$\begin{aligned} I_k(n) &= \sum_{0 \leq i_1 \leq \cdots \leq i_k < n} \delta(S_{i_1}, S_{i_2}) \cdots \delta(S_{i_{k-1}}, S_{i_k}) \\ &= \sum_{x \in Z^2} \sum_{0 \leq i_1 \leq \cdots \leq i_k < n} \prod_{j=1}^{k} \delta(S_{i_j}, x) \end{aligned} \tag{2.1}$$

where

$$\delta(i, j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

is the usual Kronecker delta function. We set $I_1(n) = n$ so that also $I_1(n) = \sum_{x \in Z^2} \sum_{0 \leq i < n} \delta(S_i, x)$. [One might also take as a definition of the intersection local time the quantity $\sum_{0 < i_1 < \cdots < i_k < n} \delta(S_{i_1}, S_{i_2}) \cdots \delta(S_{i_{k-1}}, S_{i_k})$. The definition in (2.1) is more convenient for our purposes, and we see by (2.6) that either definition leads to the same value for $\Gamma_{k,\lambda}(n)$.]

Let $q_n(x)$ be the transition function for S_n and let

$$G_\lambda(x) = \sum_{j=0}^{\infty} e^{-j\lambda} q_j(x). \tag{2.2}$$

We will show in Lemma A.1 below that

$$g_\lambda := G_\lambda(0) = \frac{1}{2\pi} \log(1/\lambda) + c_X + O\big(\lambda^\delta \log(1/\lambda)\big) \qquad \text{as } \lambda \to 0, \tag{2.3}$$

where c_X is defined in (1.8). We show in (A.18) that for any $q > 1$

$$\sum_{x \in Z^2} (G_\lambda(x))^q = O(\lambda^{-1}) \qquad \text{as } \lambda \to 0. \tag{2.4}$$

Note also that

$$\sum_{x \in Z^2} G_\lambda(x) = \sum_{j=0}^{\infty} e^{-j\lambda} = \frac{1}{1 - e^{-\lambda}}. \tag{2.5}$$

We now define the renormalized random walk intersection local times by setting $\Gamma_{1,\lambda}(n) = I_1(n) = n$ and for $k \geq 2$

$$\begin{aligned} \Gamma_{k,\lambda}(n) &= \sum_{0 \leq i_1 \leq \cdots \leq i_k < n} \{\delta(S_{i_1}, S_{i_2}) - g_\lambda \delta(i_1, i_2)\} \\ &\qquad \cdots \{\delta(S_{i_{k-1}}, S_{i_k}) - g_\lambda \delta(i_{k-1}, i_k)\} \\ &= \sum_{j=1}^{k} \binom{k-1}{j-1} (-1)^{k-j} g_\lambda^{k-j} I_j(n). \end{aligned} \tag{2.6}$$

Let ζ be an independent exponential random variable of mean 1, and set $\zeta_\lambda = n$ when $(n-1)\lambda < \zeta \le \lambda n$. ζ_λ is then a geometric random variable with $P(\zeta_\lambda > n) = e^{-\lambda n}$. Note that $\zeta_{1/j} = n$ if $(n-1)/j < \zeta \le n/j$. By $\mathcal{R}(\zeta_\lambda)$ we mean the range of our random walk killed at step ζ_λ.

In this section we prove the following lemma.

LEMMA 1. *For each* $k \ge 1$

$$\lim_{\lambda\to 0} \lambda g_\lambda^k \left(|\mathcal{R}(\zeta_\lambda)| - \sum_{j=1}^{k} (-1)^{j-1} g_\lambda^{-j} \Gamma_{j,\lambda}(\zeta_\lambda) \right) = 0 \qquad \textit{in } L^2. \tag{2.7}$$

PROOF. Define

$$T_x = \min\{n \ge 0 : S_n = x\},$$

the first hitting time to x. We will use the fact that

$$P(T_x < \zeta_\lambda) = \frac{G_\lambda(x)}{G_\lambda(0)}, \tag{2.8}$$

which follows from the strong Markov property:

$$\begin{aligned} G_\lambda(x) &= \sum_{j=0}^{\infty} e^{-j\lambda} P(S_j = x) \\ &= \sum_{j=0}^{\infty} \sum_{n=0}^{j} e^{-j\lambda} P(S_j = x, T_x = n) \\ &= \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} e^{-n\lambda} P(T_x = n) e^{-(j-n)\lambda} P(S_j = 0) \\ &= P(T_x < \zeta_\lambda) G_\lambda(0). \end{aligned} \tag{2.9}$$

To prove our lemma we square the expression inside the parentheses in (2.7) and then take expectations. We first show that

$$\begin{aligned} &E\big(|\mathcal{R}(\zeta_\lambda)|^2\big) \\ &\quad = 2\sum_{j=2}^{2k} (-1)^j g_\lambda^{-j} \sum_{x,y\in Z^2} G_\lambda(x)\big(G_\lambda(x-y)\big)^{j-1} + O\big(\lambda^{-2} g_\lambda^{-(2k+1)}\big). \end{aligned} \tag{2.10}$$

To this end we first note that

$$|\mathcal{R}(\zeta_\lambda)| = \sum_{x\in Z^2} \mathbb{1}_{\{T_x < \zeta_\lambda\}} \tag{2.11}$$

so that

$$E\big(|\mathcal{R}(\zeta_\lambda)|^2\big) = \sum_{x,y\in Z^2} P(T_x, T_y < \zeta_\lambda) \tag{2.12}$$
$$= \sum_{x\in Z^2} P(T_x < \zeta_\lambda) + 2 \sum_{x\neq y\in Z^2} P(T_x < T_y < \zeta_\lambda).$$

Using (2.8) we have that

$$\sum_{x\in Z^2} P(T_x < \zeta_\lambda) = \sum_{x\in Z^2} \frac{G_\lambda(x)}{g_\lambda} = \frac{1}{(1-e^{-\lambda})g_\lambda} = O(\lambda^{-1} g_\lambda^{-1}). \tag{2.13}$$

To evaluate $\sum_{x\neq y\in Z^2} P(T_x < T_y < \zeta_\lambda)$ we first introduce some notation. For any $u \neq v \in Z^2$ define inductively

$$\begin{aligned} A^1_{u,v} &= T_u,\\ A^2_{u,v} &= A^1_{u,v} + T_v \circ \theta_{A^1_{u,v}},\\ A^3_{u,v} &= A^2_{u,v} + T_u \circ \theta_{A^2_{u,v}},\\ A^{2k}_{u,v} &= A^{2k-1}_{u,v} + T_v \circ \theta_{A^{2k-1}_{u,v}},\\ A^{2k+1}_{u,v} &= A^{2k}_{u,v} + T_u \circ \theta_{A^{2k}_{u,v}}. \end{aligned} \tag{2.14}$$

We observe that for any $x \neq y$

$$\begin{aligned} &P(T_x < T_y < \zeta_\lambda)\\ &\quad = P(A^1_{x,y} < A^2_{x,y} < \zeta_\lambda) - P(T_y < A^1_{x,y} < A^2_{x,y} < \zeta_\lambda)\\ &\quad = P(A^2_{x,y} < \zeta_\lambda) - P(T_y < A^1_{x,y} < A^2_{x,y} < \zeta_\lambda) \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} &P(T_y < A^1_{x,y} < A^2_{x,y} < \zeta_\lambda)\\ &\quad = P(A^1_{y,x} < A^2_{y,x} < A^3_{y,x} < \zeta_\lambda) - P(T_x < A^1_{y,x} < A^2_{y,x} < A^3_{y,x} < \zeta_\lambda)\\ &\quad = P(A^3_{y,x} < \zeta_\lambda) - P(T_x < A^1_{y,x}; A^3_{y,x} < \zeta_\lambda). \end{aligned} \tag{2.16}$$

Proceeding inductively we find that

$$\begin{aligned} P(T_x < T_y < \zeta_\lambda) &= \sum_{j=1}^{k} P(A^{2j}_{x,y} < \zeta_\lambda) - \sum_{j=1}^{k} P(A^{2j+1}_{y,x} < \zeta_\lambda)\\ &\quad + P(T_x < A^1_{y,x}; A^{2k+1}_{y,x} < \zeta_\lambda). \end{aligned} \tag{2.17}$$

Using (2.8) and the strong Markov property we see that

$$
\begin{aligned}
P(T_x < T_y < \zeta_\lambda) &= \sum_{j=1}^{k} g_\lambda^{-2j} G_\lambda(x)\big(G_\lambda(y-x)\big)^{2j-1} \\
&\quad - \sum_{j=1}^{k} g_\lambda^{-(2j+1)} G_\lambda(y)\big(G_\lambda(x-y)\big)^{2j} \qquad (2.18)\\
&\quad + P(T_x < A_{y,x}^1;\, A_{y,x}^{2k+1} < \zeta_\lambda)
\end{aligned}
$$

and that

$$
\begin{aligned}
&P(T_x < A_{y,x}^1;\, A_{y,x}^{2k+1} < \zeta_\lambda) \\
&\qquad \le P(A_{x,y}^{2k+2} < \zeta_\lambda) = g_\lambda^{-(2k+2)} G_\lambda(x)\big(G_\lambda(y-x)\big)^{2k+1}.
\end{aligned} \tag{2.19}
$$

Equation (2.10) then follows using (2.3) and (2.4).

We next observe that

$$
\begin{aligned}
&E\big(I_n(\zeta_\lambda) I_m(\zeta_\lambda)\big) \\
&\quad = \sum_{x,y\in Z^2} E\left(\sum_{0\le i_1 \le \cdots \le i_n < \zeta_\lambda} \prod_{j=1}^{n} \delta(S_{i_j}, x) \sum_{0 \le l_1 \le \cdots \le l_m < \zeta_\lambda} \prod_{k=1}^{m} \delta(S_{l_k}, y) \right).
\end{aligned} \tag{2.20}
$$

We can bound the contribution from $x = y$ by

$$
(n+m)! \sum_{x\in Z^2} E\left(\sum_{0 \le i_1 \le \cdots \le i_{n+m} < \zeta_\lambda} \prod_{j=1}^{n+m} \delta(S_{i_j}, x) \right) \tag{2.21}
$$

$$
\begin{aligned}
&= (n+m)! \sum_{x\in Z^2} \sum_{0 \le i_1 \le \cdots \le i_{n+m} < \infty} E\left(\prod_{j=1}^{n+m} \delta(S_{i_j}, x) \right) e^{-\lambda i_{n+m}} \qquad (2.22)\\
&= (n+m)! \sum_{x\in Z^2} G_\lambda(x) G_\lambda^{n+m-1}(0).
\end{aligned}
$$

By (2.3) and (2.5) the contribution to (2.20) from $x = y$ is $O(\lambda^{-1} g_\lambda^{n+m})$, and by (2.6) such terms make a contribution to $E(\Gamma_{n,\lambda}(\zeta_\lambda)\Gamma_{m,\lambda}(\zeta_\lambda))$ which is $O(\lambda^{-1} g_\lambda^{n+m})$.

On the other hand

$$
\begin{aligned}
&\sum_{x\neq y\in Z^2} E\left(\sum_{0\le i_1 \le \cdots \le i_n < \zeta_\lambda} \prod_{j=1}^{n} \delta(S_{i_j}, x) \sum_{0 \le l_1 \le \cdots \le l_m < \zeta_\lambda} \prod_{k=1}^{m} \delta(S_{l_k}, y) \right) \\
&\quad = \sum_{x\neq y\in Z^2} \sum_{\pi} E\left(\sum_{0 \le i_1 \le \cdots \le i_{n+m} < \zeta_\lambda} \prod_{j=1}^{n+m} \delta(S_{i_j}, \pi(j)) \right),
\end{aligned} \tag{2.23}
$$

where the inner sum runs over all maps $\pi : \{1, 2, \dots, n+m\} \mapsto \{x, y\}$ such that $|\pi^{-1}(x)| = m$, $|\pi^{-1}(y)| = n$. Thus

$$\begin{aligned}
&\sum_{x\neq y\in Z^2} E\Biggl(\sum_{0\le i_1\le\cdots\le i_n<\zeta_\lambda}\prod_{j=1}^{n}\delta(S_{i_j},x)\sum_{0\le l_1\le\cdots\le l_m<\zeta_\lambda}\prod_{k=1}^{m}\delta(S_{l_k},y)\Biggr)\\
&\qquad = \sum_{x\neq y\in Z^2}\sum_{\pi}\sum_{0\le i_1\le\cdots\le i_{n+m}<\infty} E\Biggl(\prod_{j=1}^{n+m}\delta\bigl(S_{i_j},\pi(j)\bigr)\Biggr)e^{-\lambda i_{n+m}} \qquad (2.24)\\
&\qquad = \sum_{x\neq y\in Z^2}\sum_{\pi}\prod_{j=1}^{n+m} G_\lambda\bigl(\pi(j)-\pi(j-1)\bigr),
\end{aligned}$$

where $\pi(0) = 0$. When we look at the definition (2.6) of $\Gamma_{k,\lambda}(n)$ we see that the effect of replacing $I_n(\zeta_\lambda)I_m(\zeta_\lambda)$ in (2.22) by $\Gamma_{n,\lambda}(\zeta_\lambda)\Gamma_{m,\lambda}(\zeta_\lambda)$ is to eliminate all maps π in which $\pi(j) = \pi(j-1)$ for some j. For example, if $\pi(1) = x$ and $\pi(2) = x$, the contributions from the two terms in $\{\delta(S_{i_1}, S_{i_2}) - g_\lambda\delta(i_1, i_2)\}$ will cancel, but if $\pi(1) = x$ and $\pi(2) = y$, then there will be no contribution from $g_\lambda\delta(i_1, i_2)$.

Thus, up to an error which is $O(\lambda^{-1}g_\lambda^{n+m})$ (which comes from $x = y$), we have

$$\begin{aligned}
&E\bigl(\Gamma_{n,\lambda}(\zeta_\lambda)\Gamma_{m,\lambda}(\zeta_\lambda)\bigr)\\
&\qquad = \begin{cases} 2\displaystyle\sum_{x,y\in Z^2} G_\lambda(x)\bigl(G_\lambda(y-x)\bigr)^{2n-1}, & \text{if } m = n,\\ \displaystyle\sum_{x,y\in Z^2} G_\lambda(x)\bigl(G_\lambda(y-x)\bigr)^{2n-1\pm1}, & \text{if } m = n\pm1,\\ 0, & \text{otherwise.}\end{cases} \qquad (2.25)
\end{aligned}$$

Consequently up to errors which are $O(\lambda^{-1}g_\lambda^{2k})$

$$\begin{aligned}
&E\Biggl(\Biggl\{\sum_{j=1}^{k}(-1)^{j-1}g_\lambda^{-j}\Gamma_{j,\lambda}(\zeta_\lambda)\Biggr\}^2\Biggr)\\
&\qquad = \sum_{n,m=1}^{k}(-1)^{n+m}g_\lambda^{-(n+m)}E\bigl(\Gamma_{n,\lambda}(\zeta_\lambda)\Gamma_{m,\lambda}(\zeta_\lambda)\bigr)\\
&\qquad = 2\sum_{n=1}^{k}g_\lambda^{-2n}\sum_{x,y\in Z^2}G_\lambda(x)\bigl(G_\lambda(y-x)\bigr)^{2n-1} \qquad (2.26)\\
&\qquad\quad - 2\sum_{n=2}^{k}g_\lambda^{-(2n-1)}\sum_{x,y\in Z^2}G_\lambda(x)\bigl(G_\lambda(y-x)\bigr)^{2n-2}\\
&\qquad = 2\sum_{j=2}^{2k}(-1)^j g_\lambda^{-j}\sum_{x,y\in Z^2}G_\lambda(x)\bigl(G_\lambda(x-y)\bigr)^{j-1}.
\end{aligned}$$

To handle the cross-product terms we define the random measure on Z_+^n

$$\Lambda_{n,y}(B) = \sum_{\{0\le i_1\le\cdots\le i_n<\zeta_\lambda\}\cap B} \prod_{j=1}^{n} \delta(S_{i_j}, y). \tag{2.27}$$

Using the notation $i_0 = 0$, $i_{n+1} = \zeta_\lambda$ we have

$$\begin{aligned} E\big(|\mathcal{R}(\zeta_\lambda)| I_n(\zeta_\lambda)\big) &= E \sum_{x,y\in Z^2} \sum_{0\le i_1\le\cdots\le i_n\le\zeta_\lambda} 1\!\!1_{(T_x<\zeta_\lambda)} \prod_{j=1}^{n} \delta(S_{i_j}, y) \\ &= \sum_{x,y\in Z^2} \sum_{j=0}^{n} E\big(\Lambda_{n,y}(\{i_j \le T_x < i_{j+1}\})\big). \end{aligned} \tag{2.28}$$

As above we have that

$$\begin{aligned} &\Lambda_{n,y}(\{i_j \le T_x < i_{j+1}\}) \\ &\quad = \Lambda_{n,y}\big(\{i_j + T_x\circ\theta_{i_j} < i_{j+1}\}\big) \\ &\qquad - \sum_{l=0}^{j-1} \Lambda_{n,y}\big(\{i_l \le T_x < i_{l+1}; i_j + T_x\circ\theta_{i_j} < i_{j+1}\}\big) \end{aligned} \tag{2.29}$$

and inductively we find that

$$\begin{aligned} &\sum_{x\ne y\in Z^2} \sum_{j=0}^{n} E\big(\Lambda_{n,y}(\{i_j \le T_x < i_{j+1}\})\big) \\ &\quad = \sum_{x\ne y\in Z^2} \sum_{m=1}^{n+1} (-1)^{m-1} \sum_{|A|=m} E\left(\Lambda_{n,y}\left(\bigcap_{j\in A}\{i_j + T_x\circ\theta_{i_j} < i_{j+1}\}\right)\right), \end{aligned} \tag{2.30}$$

where the inner sum runs over all nonempty $A \subseteq \{0, 1, \ldots, n\}$. Using (2.8) and the Markov property we see that

$$\begin{aligned} &\sum_{x\ne y\in Z^2} \sum_{j=0}^{n} E\big(\Lambda_{n,y}(\{i_j \le T_x < i_{j+1}\})\big) \\ &\quad = \sum_{x\ne y\in Z^2} \sum_{m=1}^{n+1} (-1)^{m-1} \sum_{|A|=m} g_\lambda^{-m} \prod_{j=1}^{n+m} G_\lambda\big(\sigma_A(j) - \sigma_A(j-1)\big), \end{aligned} \tag{2.31}$$

where $\sigma_A(0) = 0$ and $\sigma_A(j)$ is the jth element in the ordered set obtained by taking n y's and inserting, for each $l \in A$, an x between the lth and $(l+1)$st y. Estimating the contribution from $x = y$ we find that

$$E\big(|\mathcal{R}(\zeta_\lambda)|I_n(\zeta_\lambda)\big)$$

$$= \sum_{x,y\in Z^2} \sum_{m=1}^{n+1} (-1)^{m-1} \sum_{|A|=m} g_\lambda^{-m} \prod_{j=1}^{n+m} G_\lambda\big(\sigma_A(j) - \sigma_A(j-1)\big) \tag{2.32}$$

$$+ O\big(\lambda^{-2} g_\lambda^{-(2k+1)}\big).$$

Once again we see that the effect of replacing $I_n(\zeta_\lambda)$ in (2.32) by $\Gamma_{n,\lambda}(\zeta_\lambda)$ is to eliminate all sets A such that $\sigma_A(j) = \sigma_A(j-1)$ for some j. Thus we have

$$E\big(|\mathcal{R}(\zeta_\lambda)|\Gamma_{n,\lambda}(\zeta_\lambda)\big)$$

$$= 2(-1)^{n-1} \sum_{x,y\in Z^2} g_\lambda^{-n} G_\lambda(x)\big(G_\lambda(x-y)\big)^{2n-1}$$

$$+ (-1)^n \sum_{x,y\in Z^2} g_\lambda^{-(n-1)} G_\lambda(x)\big(G_\lambda(x-y)\big)^{2n-2} \tag{2.33}$$

$$+ (-1)^n \sum_{x,y\in Z^2} g_\lambda^{-(n+1)} G_\lambda(x)\big(G_\lambda(x-y)\big)^{2n}$$

$$+ O\big(\lambda^{-2} g_\lambda^{-(2k+1)}\big).$$

Consequently

$$E\left(|\mathcal{R}(\zeta_\lambda)| \sum_{n=1}^{k} (-1)^{n-1} g_\lambda^{-n} \Gamma_{n,\lambda}(\zeta_\lambda)\right)$$

$$= 2 \sum_{n=1}^{k} g_\lambda^{-2n} \sum_{x,y\in Z^2} G_\lambda(x)\big(G_\lambda(y-x)\big)^{2n-1}$$

$$- \sum_{n=2}^{k} g_\lambda^{-(2n-1)} \sum_{x,y\in Z^2} G_\lambda(x)\big(G_\lambda(y-x)\big)^{2n-2}$$

$$- \sum_{n=1}^{k} g_\lambda^{-(2n+1)} \sum_{x,y\in Z^2} G_\lambda(x)\big(G_\lambda(y-x)\big)^{2n} \tag{2.34}$$

$$+ O\big(\lambda^{-2} g_\lambda^{-(2k+1)}\big)$$

$$= 2 \sum_{j=2}^{2k} (-1)^j g_\lambda^{-j} \sum_{x,y\in Z^2} G_\lambda(x)\big(G_\lambda(x-y)\big)^{j-1}$$

$$+ O\big(\lambda^{-2} g_\lambda^{-(2k+1)}\big).$$

Our lemma then follows from (2.10), (2.26) and (2.34). $\square$

3. Strong approximation in L^2. As usual we let $\|X\|_p = (E|X|^p)^{1/p}$.

LEMMA 2. *Let X be an R^2-valued random vector with mean zero and covariance matrix equal to the identity I. Assume that for some $2 < p < 4$, $E|X|^p < \infty$. Given $n \geq 1$ one can construct on a suitable probability space two sequences of independent random vectors $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$, where each $X_i \stackrel{d}{=} X$ and the Y_i's are standard normal random vectors such that*

$$\left\| \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (X_i - Y_i) \right| \right\|_2 = O\big(n^{2/p - 2/p^2}\big).$$

PROOF. Let $x = n^{2/p-2/p^2}$. By (3.3) of [9] we can find a constant c_1 and such X_i and Y_i so that

$$P\left\{ \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (X_i - Y_i) \right| > x \right\} \leq c_1 n x^{-p} E|X|^p.$$

Write Z_n for $\max_{1 \leq k \leq n} |\sum_{i=1}^{k} (X_i - Y_i)|$. By Doob's inequality and Rosenthal's inequality [22],

$$\|Z_n\|_p \leq c_2 \left\| \sum_{i=1}^{n} (X_i - Y_i) \right\|_p \leq c_3 \sqrt{n}.$$

So using Hölder's inequality

$$\begin{aligned}
\|Z_n\|_2 &\leq x + \|Z_n \mathbb{1}_{(Z_n > x)}\|_2 \\
&\leq x + \|Z_n\|_p P(Z_n \geq x)^{1/2 - 1/p} \\
&\leq x + c_4 \sqrt{n} \left(\frac{n}{x^p} \right)^{1/2 - 1/p} \\
&= x + c_4 n^{1 - 1/p} x^{1 - p/2} \\
&= c_5 n^{2/p - 2/p^2}.
\end{aligned}$$

$\square$

Using the lemma we can readily construct two i.i.d. sequences $\{X_i\}_{i \geq 1}$ and $\{Y_i\}_{i \geq 1}$, where the X_i are equal in law to X and the Y_i are standard normal, such that for some constant $C > 0$ and any $m \geq 0$,

$$\left\| \max_{2^m \leq k < 2^{m+1}} \left| \sum_{i=2^m}^{k} (X_i - Y_i) \right| \right\|_2 \leq C (2^m)^{2/p - 2/p^2}.$$

We see then that for any $2^m \le [nt] < 2^{m+1}$,

$$\left\| \sum_{i=1}^{[nt]} (X_i - Y_i) \right\|_2 \le \sum_{j=0}^{m} \left\| \max_{2^m \le k < 2^{m+1}} \left| \sum_{i=2^m}^{k} (X_i - Y_i) \right| \right\|_2,$$

which for some $D > 0$ is less than or equal to

$$\sum_{j=0}^{m} C(2^j)^{2/p-2/p^2} \le D(nt)^{2/p-2/p^2}.$$

Now choose a Brownian motion W such that for $m \ge 1$,

$$W(m) = \sum_{i=1}^{m} Y_j.$$

Noting that

$$\|W([mt]) - W(mt)\|_2 \le \left\| \sup_{0 \le s \le 1} |W(s)| \right\|_2 := M,$$

we see that for any $t > 0$

$$\begin{aligned} \left\| \frac{S([mt]) - W(mt)}{\sqrt{m}} \right\|_2 &\le D(mt)^{2/p-2/p^2} m^{-1/2} + M m^{-1/2} \\ &= O\big(m^{(2/p-2/p^2)-(1/2)}\big(t^{2/p-2/p^2} + 1\big)\big), \end{aligned} \tag{3.1}$$

where

$$S([mt]) = \sum_{i \le [mt]} X_i.$$

4. Spatial Hölder continuity for renormalized intersection local times. If $\{W_t; t \ge 0\}$ is a planar Brownian motion, set $\overline{\alpha}_{1,\varepsilon}(t) = t$ and for $k \ge 2$ and $x = (x_2, \dots, x_k) \in (R^2)^{k-1}$ let

$$\overline{\alpha}_{k,\varepsilon}(t, x) = \int_{0 \le t_1 \le \cdots \le t_k < t} \prod_{i=2}^{k} p_\varepsilon\big(W_{t_i} - W_{t_{i-1}} - x_i\big)\, dt_1 \cdots dt_k. \tag{4.1}$$

When $x_i \ne 0$ for all i and ζ is an independent exponential random variable with mean 1, the limit

$$\overline{\alpha}_k(\zeta, x) = \lim_{\varepsilon \to 0} \overline{\alpha}_{k,\varepsilon}(\zeta, x) \tag{4.2}$$

exists. When $x_i \ne 0$ for all i set

$$\overline{\gamma}_k(\zeta, x) = \sum_{A \subseteq \{2,\dots,k\}} (-1)^{|A|} \left(\prod_{i \in A} u^1(x_i) \right) \overline{\alpha}_{k-|A|}(\zeta, x_{A^c}), \tag{4.3}$$

where

$$u^1(y) = \int_0^\infty e^{-t} p_t(y)\,dt, \tag{4.4}$$

$p_t(x)$ is the density for W_t and $x_{A^c} = (x_{i_1}, \dots, x_{i_{k-|A|}})$ with $i_1 < i_2 < \cdots < i_{k-|A|}$ and $i_j \in \{2, \dots, k\} - A$ for each j, that is, the vector $(x_2, \dots, x_k)$ with all terms that have indices in A deleted. In [20] it is shown that for some $\bar{\delta} > 0$ and all m

$$E\big(|\overline{\gamma}_k(\zeta, x) - \overline{\gamma}_k(\zeta, y)|^m\big) \le C|x - y|^{\bar{\delta} m}. \tag{4.5}$$

As before, set $I_1(n) = n$ and for $k \ge 2$ and $x = (x_2, \dots, x_k) \in (Z^2)^{k-1}$ let

$$\overline{I_k}(n, x) = \sum_{0 \le i_1 \le \cdots \le i_k < n} \delta\big(S_{i_2} - S_{i_1} - x_2\big) \cdots \delta\big(S_{i_k} - S_{i_{k-1}} - x_k\big) \tag{4.6}$$

and for $x \in \sqrt{\lambda}(Z^2)^{k-1}$ let

$$\overline{\Gamma}_{k,\lambda}(n, x) = \sum_{A \subseteq \{2,\dots,k\}} (-1)^{|A|} \prod_{i \in A} G_\lambda\big(x_i/\sqrt{\lambda}\big) \overline{I}_{k-|A|}\big(n, x_{A^c}/\sqrt{\lambda}\big). \tag{4.7}$$

Note that $\Gamma_{k,\lambda}(n) = \overline{\Gamma}_{k,\lambda}(n, 0)$.

LEMMA 3. *For any $j \ge 1$ we can find some $\rho, \bar{\delta} > 0$ such that uniformly in $\lambda > 0$*

$$\sup_{|y| \le \lambda^\rho} E\big(|\lambda \overline{\Gamma}_{j,\lambda}(\zeta_\lambda, y) - \lambda \Gamma_{j,\lambda}(\zeta_\lambda)|^2\big) \le C\lambda^{\bar{\delta}}. \tag{4.8}$$

PROOF. We begin by considering

$$E\big(\overline{\Gamma}_{k,\lambda}(\zeta_\lambda, x^1)\overline{\Gamma}_{k,\lambda}(\zeta_\lambda, x^2)\big) \tag{4.9}$$

for $x^i \in (Z^2)^{k-1}$.

If h is a function which depends on the variable x, let

$$\mathcal{D}_x h = h(x) - h(0).$$

Let $\mathcal{S}$ be the set of all maps $s : \{1, 2, \dots, 2k\} \mapsto \{1, 2\}$ with $|s^{-1}(j)| = k$, $1 \le j \le 2$, and let $B_s = \{i | s(i) = s(i-1)\}$ and $c(i) = |\{j \le i | s(j) = s(i)\}|$.

Using the Markov property as in Lemma 5 of [20] we can then show that

$$\begin{aligned} &E\big(\overline{\Gamma}_{k,\lambda}(\zeta_\lambda, x^1)\overline{\Gamma}_{k,\lambda}(\zeta_\lambda, x^2)\big) \\ &\quad = \sum_{s \in \mathcal{S}} \left(\prod_{i \in B_s} G_\lambda\big(x^{s(i)}_{c(i)}/\sqrt{\lambda}\big)\right) \sum_{\substack{z_i \in Z^2 \\ i=1,2}} \left(\prod_{i \in B_s} \mathcal{D}_{x^{s(i)}_{c(i)}/\sqrt{\lambda}}\right) \\ &\qquad \times \prod_{i \in B_s^c} G_\lambda\left(z_{s(i)} + \sum_{j=2}^{c(i)} x^{s(i)}_j/\sqrt{\lambda} - \left(z_{s(i-1)} + \sum_{j=2}^{c(i-1)} x^{s(i-1)}_j/\sqrt{\lambda}\right)\right). \end{aligned} \tag{4.10}$$

Fix $s \in \mathcal{S}$ and note then that the corresponding summand will be 0 unless $x_{c(i)}^{s(i)} \neq 0$ for all $i \in B_s$. Note that by definition of B_s^c we necessarily have that the last line in (4.10) is of the form

$$G_\lambda(z_1) \prod_{i \in B_s^c, i \neq 1} G_\lambda(z_1 - z_2 + a_i), \tag{4.11}$$

where the a_i are linear combinations of x^1, x^2 but do not involve z_1, z_2. Then we observe that the effect of applying each $\mathcal{D}_{x_{c(i)}^{s(i)}/\sqrt{\lambda}}$ to the product on the last line of (4.10) is to generate a sum of several terms in each of which we have one factor of the form $\mathcal{D}_{x_{c(i)}^{s(i)}/\sqrt{\lambda}} G_\lambda$. Thus schematically we can write the contribution of such a term as

$$\left(\prod_{i \in B_s} G_\lambda\bigl(x_{c(i)}^{s(i)}/\sqrt{\lambda}\bigr) \right) \sum_{z_i \in Z^2, i=1,2} G_\lambda(z_1) \prod_{i \in B_s^c, i \neq 1} \Delta_{A_i} G_\lambda(z_1 - z_2 + a_i), \tag{4.12}$$

where each Δ_{A_i} is a product of k_i difference operators of the form $\Delta_{x_l^j/\sqrt{\lambda}}$, and we have $\sum_{i \in B_s^c} k_i = |B_s|$. If $B_s \neq \varnothing$ and if there is only one term in the last product on the right-hand side of (4.12), it is easily seen that the sum over z_2 gives 0. Thus the product contains at least two terms and then by Lemma A.2 we can see that for some $C < \infty$ and $\nu > 0$ independent of everything

$$\left| \sum_{z_i \in Z^2, i=1,2} G_\lambda(z_1) \prod_{i \in B_s^c, i \neq 1} \Delta_{A_i} G_\lambda(z_1 - z_2 + a_i) \right| \leq C \lambda^{-2} \prod_{i \in B_s} \bigl|x_{c(i)}^{s(i)}\bigr|^\nu. \tag{4.13}$$

With these results, we now turn to the bound (4.9). For ease of exposition we use y^i to denote the y in the ith factor; in the end we will set $y^i = y$. For ease of exposition we assume that y differs from 0 only in the υth coordinate, and we set $a = y_\upsilon$. (The general case is then easily handled.)

We again use Lemma 5 of [20] to expand

$$E\bigl(\bigl(\overline{\Gamma}_{k,\lambda}(\zeta_\lambda, y^1) - \Gamma_{k,\lambda}(\zeta_\lambda)\bigr)\bigl(\overline{\Gamma}_{k,\lambda}(\zeta_\lambda, y^2) - \Gamma_{k,\lambda}(\zeta_\lambda)\bigr)\bigr) \tag{4.14}$$

as a sum of many terms of the form

$$\begin{aligned} &\sum_{s \in \mathcal{S}} \left(\prod_{i=1}^2 \mathcal{D}_{y_\upsilon^i/\sqrt{\lambda}} \right) \left(\prod_{i \in B_s} G_\lambda\bigl(x_{c(i)}^{s(i)}/\sqrt{\lambda}\bigr) \right) \sum_{z_i \in Z^2, i=1,2} \left(\prod_{i \in B_s} \mathcal{D}_{x_{c(i)}^{s(i)}/\sqrt{\lambda}} \right) \\ &\quad \times \prod_{i \in B_s^c} G_\lambda\left(z_{s(i)} + \sum_{j=2}^{c(i)} x_j^{s(i)}/\sqrt{\lambda} - \left(z_{s(i-1)} + \sum_{j=2}^{c(i-1)} x_j^{s(i-1)}/\sqrt{\lambda} \right) \right), \end{aligned} \tag{4.15}$$

where now x^i is variously y^i or 0. For fixed $s \in \mathcal{S}$ we can expand the corresponding term as a sum of terms of the form

$$\left\{\left(\prod_{k\in F}\mathcal{D}_{y_v^k/\sqrt{\lambda}}\right)\left(\prod_{i\in B_s}G_\lambda\big(x_{c(i)}^{s(i)}/\sqrt{\lambda}\big)\right)\right\}$$

$$\times\sum_{z_i\in Z^2,i=1,2}\left(\prod_{k\in F^c}\mathcal{D}_{y_v^k/\sqrt{\lambda}}\right)\left(\prod_{i\in B_s}\mathcal{D}_{x_{c(i)}^{s(i)}/\sqrt{\lambda}}\right) \tag{4.16}$$

$$\times\prod_{i\in B_s^c}G_\lambda\left(z_{s(i)}+\sum_{j=2}^{c(i)}x_j^{s(i)}/\sqrt{\lambda}-\left(z_{s(i-1)}+\sum_{j=2}^{c(i-1)}x_j^{s(i-1)}/\sqrt{\lambda}\right)\right),$$

where F runs through the subsets of $\{1,2\}$. Note that the first line will be 0 unless for each $k\in F$ we have that $y_v^k = x_{c(i)}^{s(i)}$ for some $i\in B_s$. In particular

$$|F|\le|B_s|. \tag{4.17}$$

Using the fact that

$$G_\lambda(x)\le c\log(1/\lambda) \tag{4.18}$$

we can bound the first line of (4.16) by $(c\log(1/\lambda))^{|B_s|}$. As before [see in particular (4.13)], we can obtain the bound

$$\left|\sum_{z_i\in Z^2,i=1,2}\left(\prod_{k\in F^c}\mathcal{D}_{y_v^k/\sqrt{\lambda}}\right)\left(\prod_{i\in B_s}\mathcal{D}_{x_{c(i)}^{s(i)}/\sqrt{\lambda}}\right)\right.$$

$$\left.\times\prod_{i\in B_s^c}G_\lambda\left(z_{s(i)}+\sum_{j=2}^{c(i)}x_j^{s(i)}/\sqrt{\lambda}-\left(z_{s(i-1)}+\sum_{j=2}^{c(i-1)}x_j^{s(i-1)}/\sqrt{\lambda}\right)\right)\right| \tag{4.19}$$

$$\le c\lambda^{-2}\prod_{k\in F^c}|y_v^k|^\nu\prod_{i\in B_s}|x_{c(i)}^{s(i)}|^\nu.$$

Our lemma then follows using (4.17) which implies that $|F^c|+|B_s|\ge 2$. □

5. Approximating intersection local times. The goal of this section is to prove the following lemma.

LEMMA 4. *We can find a Brownian motion such that for each $j\ge 1$ there exists $\beta>0$ such that*

$$\|\lambda\Gamma_{j,\lambda}(\zeta_\lambda)-\gamma_j(\zeta,\omega_{\lambda^{-1}})\|_2=O(\lambda^\beta). \tag{5.1}$$

PROOF. Let $f(x)$ be a smooth function on R^2, supported in the unit disc and with $\int f(x)\,dx=1$. We set $f_\varepsilon(x)=\frac{1}{\varepsilon^2}f(x/\varepsilon)$. On the one hand it is easy to see that if we set $\widetilde{u}^1(f_\tau) = \int u^1(x) f_\tau(x)\,dx$ and

$$\widetilde{\gamma}_k(\zeta, f_\tau) = \int \overline{\gamma_k}(\zeta, x) \prod_{i=2}^{k} f_\tau(x_i)\,dx_2 \cdots dx_k,$$

$$\widetilde{\alpha}_j(\zeta, f_\tau) = \int \overline{\alpha}_j(\zeta, x) \prod_{i=2}^{j} f_\tau(x_i)\,dx_2 \cdots dx_k,$$

we will have

$$\widetilde{\gamma}_k(\zeta, f_\tau) = \sum_{j=1}^{k} \binom{k-1}{j-1} \big(-\widetilde{u}^1(f_\tau)\big)^{k-j} \widetilde{\alpha}_j(\zeta, f_\tau) \tag{5.2}$$

and

$$\widetilde{\alpha}_j(t, f_\tau) = \int_{0 \le t_1 \le \cdots \le t_j < t} \prod_{i=2}^{j} f_\tau\big(W_{t_i} - W_{t_{i-1}}\big)\,dt_1 \cdots dt_j. \tag{5.3}$$

On the other hand it follows from (4.5) and Jensen's inequality that

$$\|\widetilde{\gamma}_k(\zeta, f_\tau) - \gamma_k(\zeta)\|_2 \le C\tau^{\bar{\delta}}. \tag{5.4}$$

If we set $\widetilde{G}_\lambda(f_\tau) = \sum_{x \in \sqrt{\lambda} Z^2} \lambda G_\lambda(x/\sqrt{\lambda}) f_\tau(x)$,

$$\widetilde{\Gamma}_{k,\lambda}(\zeta_\lambda, f_\tau) = \sum_{x_2,\dots,x_k \in \sqrt{\lambda} Z^2} \lambda^{k-1} \overline{\Gamma}_{k,\lambda}(\zeta_\lambda, x) \prod_{i=2}^{k} f_\tau(x_i)$$

and

$$\widetilde{I}_j(\zeta_\lambda, f_\tau) = \sum_{x_2,\dots,x_k \in \sqrt{\lambda} Z^2} \lambda^{k-1} \overline{I_j}\big(\zeta_\lambda, x/\sqrt{\lambda}\big) \prod_{i=2}^{j} f_\tau(x_i),$$

we similarly have

$$\widetilde{\Gamma}_{k,\lambda}(\zeta_\lambda, f_\tau) = \sum_{j=1}^{k} \binom{k-1}{j-1} \big(-\widetilde{G}_\lambda(f_\tau)\big)^{k-j} \widetilde{I}_j(\zeta_\lambda, f_\tau). \tag{5.5}$$

It then follows from (4.8) that with $\tau = \lambda^\rho$ for $\rho > 0$ small

$$\|\lambda\widetilde{\Gamma}_{k,\lambda}(\zeta_\lambda, f_\tau) - \lambda\Gamma_{k,\lambda}(\zeta_\lambda)\|_2 \le C\tau^{\bar{\delta}}. \tag{5.6}$$

To complete the proof of Lemma 4 it only remains to show that with $\tau = \lambda^\rho$ for $\rho > 0$ small

$$\|\lambda\widetilde{\Gamma}_{k,\lambda}(\zeta_\lambda, f_\tau) - \widetilde{\gamma}_k(\zeta, f_\tau, \omega_{\lambda^{-1}})\|_2 \le c\lambda^\zeta \tag{5.7}$$

for some $c < \infty$ and $\zeta > 0$. Note that

$$
\begin{aligned}
\lambda \widetilde{I}_j(\zeta_\lambda, f_\tau) &= \lambda^k \sum_{0\le t_1\le\cdots\le t_j<\zeta_\lambda} \prod_{i=2}^{j} f_\tau\bigl(\sqrt{\lambda}(S_{t_i}-S_{t_{i-1}})\bigr) \\
&= \lambda^k \sum_{0\le t_1\le\cdots\le t_j<\zeta/\lambda} \prod_{i=2}^{j} f_\tau\bigl(\sqrt{\lambda}(S_{t_i}-S_{t_{i-1}})\bigr) \\
&= \lambda^k \int_{0\le t_1\le\cdots\le t_j<\zeta/\lambda} \prod_{i=2}^{j} f_\tau\bigl(\sqrt{\lambda}(S_{[t_i]}-S_{[t_{i-1}]})\bigr)\,dt_1\cdots dt_j \\
&= \int_{0\le t_1\le\cdots\le t_j<\zeta} \prod_{i=2}^{j} f_\tau\bigl(\sqrt{\lambda}(S_{[t_i/\lambda]}-S_{[t_{i-1}/\lambda]})\bigr)\,dt_1\cdots dt_j.
\end{aligned} \tag{5.8}
$$

By (5.2)–(5.8) it suffices to show that for some $\delta' > 0$ and all sufficiently small τ, λ

$$
\widetilde{u}^1(f_\tau) = O\bigl(\log(1/|\tau|)\bigr), \qquad |\widetilde{G}_\lambda(f_\tau) - \widetilde{u}^1(f_\tau)| \le c\tau^{-3}\lambda^{\delta'}, \tag{5.9}
$$

and

$$
\begin{aligned}
\|\widetilde{\alpha}_k(\zeta, f_\tau, \omega_{\lambda^{-1}})\|_2 &\le c\tau^{-2(k-1)}, \\
\|\lambda \widetilde{I}_{k,\lambda}(\zeta_\lambda, f_\tau) - \widetilde{\alpha}_k(\zeta, f_\tau, \omega_{\lambda^{-1}})\|_2 &\le c\tau^{-2k+1}\lambda^{\delta'}.
\end{aligned} \tag{5.10}
$$

The first part of (5.9) follows from the fact that $u^1(x) = O(\log(1/|x|))$; see [13], (2.b). To prove the second part of (5.9), we note that $\sup_x |\nabla f_\tau(x)| \le c\tau^{-3}$, so

$$
\begin{aligned}
|\widetilde{G}_\lambda(f_\tau) &- \widetilde{u}^1(f_\tau)| \\
&= \left| \int_0^\infty e^{-t} E\bigl(f_\tau(\sqrt{\lambda} S_{[t/\lambda]}) - f_\tau(\sqrt{\lambda} W_{t/\lambda})\bigr)\,dt \right| \\
&\le c\tau^{-3} \int_0^\infty e^{-t} \bigl\|\sqrt{\lambda}(S_{[t/\lambda]} - W_{t/\lambda})\bigr\|_1\,dt.
\end{aligned} \tag{5.11}
$$

The second part of (5.9) then follows from the last inequality in Section 3.

The first part of (5.10) follows from the fact that $\sup_x |f_\tau(x)| \le c\tau^{-2}$, so that

$$
\|\widetilde{\alpha}_k(\zeta, f_\tau, \omega_{\lambda^{-1}})\|_2^2 \le c\tau^{-2(k-1)} \int_0^\infty e^{-t} t^n\,dt. \tag{5.12}
$$

To prove the second part of (5.10), we use the above bounds on $\sup_x |\nabla f_\tau(x)|$ and $\sup_x |f_\tau(x)|$ to see that

$$
\begin{aligned}
\|\lambda \widetilde{I}_{k,\lambda}(\zeta_\lambda, f_\tau) &- \widetilde{\alpha}_k(\zeta, f_\tau, \omega_{\lambda^{-1}})\|_2^2 \\
&\le c\tau^{-2k+1} \\
&\quad \times \sum_{j=1}^{k} \int_0^\infty e^{-t} \left(\int_{0\le t_1\le\cdots\le t_k<t} \bigl\|\sqrt{\lambda}(S_{[t_j/\lambda]} - W_{t_j/\lambda})\bigr\|_2^2\,dt_1\cdots dt_k \right) dt.
\end{aligned} \tag{5.13}
$$

The second part of (5.10) then follows from the last inequality in Section 3. □

6. Renormalized Brownian intersection local times. Recall the definition of $\gamma_k(t)$ given in (1.5). Note from [13], (2.b) that for some fixed constant c

$$u_\varepsilon = \int_0^\infty e^{-t} p_{t+\varepsilon}(0)\, dt = \frac{1}{2\pi} \log(1/\varepsilon) + c + O(\varepsilon). \tag{6.1}$$

In [20] we show that the limit in (1.5) exists a.s. and in all L^p spaces, and that $\gamma_k(t)$ is continuous in t. The rest of this section is basically contained in [13] but we point out that [20] came after [13] and resulted in some simplification.

For any given function $h : (0, \infty) \to R$ we set $\widehat{\gamma}_1(t, h) = t$ and for $k \geq 2$

$$\widehat{\gamma}_k(t, h) = \lim_{\varepsilon \to 0} \sum_{l=1}^{k} \binom{k-1}{l-1} (-h_\varepsilon)^{k-l} \alpha_{l,\varepsilon}(t), \tag{6.2}$$

where we write h_ε for $h(\varepsilon)$. In particular, $\gamma_k(t) = \widehat{\gamma}_k(t, u)$. Let $\mathcal{H}$ denote the set of functions h such that $\lim_{\varepsilon \to 0}(h_\varepsilon - u_\varepsilon)$ exists and is finite. In the next lemma we will see that the limit in (6.2) exists for all $h \in \mathcal{H}$.

LEMMA 5 (Renormalization lemma). *Let $h \in \mathcal{H}$. Then $\widehat{\gamma}_k(t, h)$ exists for all $k \geq 1$ and if $\bar{h} \in \mathcal{H}$ with* $\lim_{\varepsilon \to 0}(h_\varepsilon - \bar{h}_\varepsilon) = b$, *then for any $k \geq 1$*

$$\widehat{\gamma}_k(t, h) = \sum_{m=1}^{k} \binom{k-1}{m-1} (-b)^{k-m} \widehat{\gamma}_m(t, \bar{h}). \tag{6.3}$$

PROOF. Setting $b_\varepsilon = h_\varepsilon - \bar{h}_\varepsilon$ we have

$$\begin{aligned} &\sum_{l=1}^{k} \binom{k-1}{l-1} (-h_\varepsilon)^{k-l} \alpha_{l,\varepsilon}(t) \\ &\quad = \sum_{l=1}^{k} \binom{k-1}{l-1} (-\bar{h}_\varepsilon - b_\varepsilon)^{k-l} \alpha_{l,\varepsilon}(t) \\ &\quad = \sum_{l=1}^{k} \binom{k-1}{l-1} \sum_{j=0}^{k-l} \binom{k-l}{j} (-b_\varepsilon)^j (-\bar{h}_\varepsilon)^{(k-j)-l} \alpha_{l,\varepsilon}(t). \end{aligned} \tag{6.4}$$

Using

$$\binom{k-1}{l-1} \binom{k-l}{j} = \binom{k-1}{j} \binom{k-j-1}{l-1},$$

the last line in (6.4) becomes

$$\sum_{j=0}^{k-1} \binom{k-1}{j} (-b_\varepsilon)^j \sum_{l=1}^{k-j} \binom{k-j-1}{l-1} (-\bar{h}_\varepsilon)^{(k-j)-l} \alpha_{l,\varepsilon}(t). \tag{6.5}$$

Taking $\bar{h}_\varepsilon = u_\varepsilon$ then shows the existence of $\gamma_k(t,h)$. Returning to general $\bar{h} \in \mathcal{H}$ and now taking the $\varepsilon \to 0$ limit, we obtain

$$\begin{aligned}\widehat{\gamma}_k(t,h) &= \sum_{j=0}^{k-1} \binom{k-1}{j} (-b)^j \widehat{\gamma}_{k-j}(t,\bar{h}) \\ &= \sum_{m=1}^{k} \binom{k-1}{m-1} (-b)^{k-m} \widehat{\gamma}_m(t,\bar{h}),\end{aligned} \tag{6.6}$$

where the last line follows from the substitution $m = k - j$. □

Let $h \in \mathcal{H}$. We shall sometimes write $\widehat{\gamma}_k(t,h,\omega)$ for $\widehat{\gamma}_k(t,h)$ to emphasize its dependence on the path ω. We want to discuss how renormalized intersection local time changes with a time rescaling. Let $\omega_r(s) = r^{-1/2}\omega(rs)$. Then $\widehat{\gamma}_k(t,h,\omega_r)$ is the same as $\widehat{\gamma}_k(t,h)$ defined in terms of the Brownian motion $W_t^{(r)} = W_{rt}/\sqrt{r}$.

LEMMA 6 (Rescaling lemma). *Let $h \in \mathcal{H}$. Then for any $k \geq 1$*

$$\widehat{\gamma}_k(t,h,\omega_r) = r^{-1} \sum_{m=1}^{k} \binom{k-1}{m-1} \left(\frac{1}{2\pi}\log(1/r)\right)^{k-m} \widehat{\gamma}_m(rt,h,\omega). \tag{6.7}$$

PROOF. After replacing ω by ω_r the integral on the right-hand side of (6.2) is replaced by

$$\begin{aligned}&\int_{0\le t_1\le\cdots\le t_l<t} \prod_{i=2}^{l} p_\varepsilon\left(\frac{W_{rt_i} - W_{rt_{i-1}}}{\sqrt{r}}\right) dt_1 \cdots dt_l \\ &\quad = r^{-l} \int_{0\le t_1\le\cdots\le t_l<rt} \prod_{i=2}^{l} p_\varepsilon\left(\frac{W_{t_i} - W_{t_{i-1}}}{\sqrt{r}}\right) dt_1 \cdots dt_l \\ &\quad = r^{-1} \int_{0\le t_1\le\cdots\le t_l<rt} \prod_{i=2}^{l} p_{r\varepsilon}(W_{t_i} - W_{t_{i-1}})\, dt_1 \cdots dt_l.\end{aligned} \tag{6.8}$$

Abbreviating this last integral as $\alpha_{l,r\varepsilon}(rt,\omega)$, we have

$$\widehat{\gamma}_k(t,h,\omega_r) = r^{-1} \lim_{\varepsilon\to 0} \sum_{l=1}^{k} \binom{k-1}{l-1} (-h_\varepsilon)^{k-l} \alpha_{l,r\varepsilon}(rt,\omega). \tag{6.9}$$

Since $h \in \mathcal{H}$ it is easily seen that $\lim_{\varepsilon\to 0}(h_\varepsilon - h_{r\varepsilon}) = -\frac{1}{2\pi}\log(1/r)$ and our lemma then follows from Lemma 5. □

7. Range and Brownian intersection local times. In this section we prove the following theorem.

THEOREM 2. *For each* $k \geq 1$

$$g_\lambda^k\left(\lambda|\mathcal{R}(\zeta_\lambda)| - \sum_{j=1}^{k}(-1)^{j-1} g_\lambda^{-j}\gamma_j(\zeta, \omega_{\lambda^{-1}})\right) \to 0 \qquad a.s. \tag{7.1}$$

as $\lambda \to 0$.

PROOF. Using (5.1) together with Lemma 1 and its proof, we see that for some $M_k < \infty$

$$\left\| g_\lambda^{4k+1}\left(\lambda|\mathcal{R}(\zeta_\lambda)| - \sum_{j=1}^{4k}(-1)^{j-1} g_\lambda^{-j}\gamma_j(\zeta, \omega_{\lambda^{-1}})\right)\right\|_2^2 \leq M_k \tag{7.2}$$

for all $\lambda > 0$ sufficiently small.

We now follow [13]. With $\lambda_n = e^{-n^{1/2k}}$ we have that for any $\varepsilon > 0$

$$\begin{aligned} &\sum_{n=1}^{\infty} P\left\{ g_{\lambda_n}^k \left(\lambda_n|\mathcal{R}(\zeta_{\lambda_n})| - \sum_{j=1}^{4k}(-1)^{j-1} g_{\lambda_n}^{-j}\gamma_j(\zeta, \omega_{\lambda_n^{-1}})\right) \geq g_{\lambda_n}^{-1}\right\} \\ &\quad \leq \sum_{n=1}^{\infty} P\left\{ g_{\lambda_n}^{4k+1} \left(\lambda_n|\mathcal{R}(\zeta_{\lambda_n})| - \sum_{j=1}^{4k}(-1)^{j-1} g_{\lambda_n}^{-j}\gamma_j(\zeta, \omega_{\lambda_n^{-1}})\right) \geq g_{\lambda_n}^{3k}\right\} \\ &\quad \leq M_k \sum_{n=1}^{\infty} g_{\lambda_n}^{-6k} < \infty. \end{aligned} \tag{7.3}$$

Then by Borel–Cantelli

$$g_{\lambda_n}^k\left(\lambda_n|\mathcal{R}(\zeta_{\lambda_n})| - \sum_{j=1}^{4k}(-1)^{j-1} g_{\lambda_n}^{-j}\gamma_j(\zeta, \omega_{\lambda_n^{-1}})\right) \to 0 \qquad \text{a.s.} \tag{7.4}$$

Since for each $m \geq 1$ we have that $\gamma_j(\zeta, \omega_{\lambda_n^{-1}})$ is bounded in L^m uniformly in n, then by Chebyshev's inequality with m sufficiently large $P(\gamma_j(\zeta, \omega_{\lambda_n^{-1}}) > g_{\lambda_n})$ will be summable. So we may drop the terms for $j > k$ and we then have

$$g_{\lambda_n}^k\left(\lambda_n|\mathcal{R}(\zeta_{\lambda_n})| - \sum_{j=1}^{k}(-1)^{j-1} g_{\lambda_n}^{-j}\gamma_j(\zeta, \omega_{\lambda_n^{-1}})\right) \to 0 \qquad \text{a.s.} \tag{7.5}$$

Before continuing the proof of Theorem 2 we first prove the following lemma.

LEMMA 7. *For any* $k \geq 1$

$$\lim_{n\to 0} \sup_{\lambda_{n+1}\leq\lambda\leq\lambda_n} \left|\gamma_k(\zeta, \omega_{\lambda^{-1}}) - \gamma_k(\zeta, \omega_{\lambda_n^{-1}})\right| = 0 \qquad a.s. \tag{7.6}$$

PROOF. By (6.7) for any $k \geq 1$

$$\gamma_k(\zeta, \omega_{\lambda^{-1}}) = \frac{\lambda}{\lambda_n} \sum_{m=1}^{k} \binom{k-1}{m-1} \left(\frac{1}{2\pi} \log\left(\frac{\lambda}{\lambda_n}\right)\right)^{k-m} \gamma_m\left(\frac{\lambda_n}{\lambda}\zeta, \omega_{\lambda_n^{-1}}\right). \tag{7.7}$$

Hence for any $p \geq 1$

$$\begin{aligned}
&\left\| \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} \left|\gamma_k(\zeta, \omega_{\lambda^{-1}}) - \gamma_k(\zeta, \omega_{\lambda_n^{-1}})\right| \right\|_p \\
&\quad \leq \left\| \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} \left| \frac{\lambda}{\lambda_n} \gamma_k\left(\frac{\lambda_n}{\lambda}\zeta, \omega_{\lambda_n^{-1}}\right) - \gamma_k(\zeta, \omega_{\lambda_n^{-1}}) \right| \right\|_p \\
&\qquad + c \sum_{m=1}^{k-1} \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} \left(\frac{1}{2\pi} \log\left(\frac{\lambda}{\lambda_n}\right)\right)^{k-m} \left\| \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} \gamma_m\left(\frac{\lambda_n}{\lambda}\zeta, \omega_{\lambda_n^{-1}}\right) \right\|_p \\
&\quad = \left\| \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} \left| \frac{\lambda}{\lambda_n} \gamma_k\left(\frac{\lambda_n}{\lambda}\zeta\right) - \gamma_k(\zeta) \right| \right\|_p \\
&\qquad + c \sum_{m=1}^{k-1} \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} \left(\frac{1}{2\pi} \log\left(\frac{\lambda}{\lambda_n}\right)\right)^{k-m} \left\| \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} \gamma_m\left(\frac{\lambda_n}{\lambda}\zeta\right) \right\|_p .
\end{aligned} \tag{7.8}$$

It follows from (9.11) of [3] that for any $k \geq 1$ we can find $\beta > 0$ such that

$$\left\| \sup_{|t-s| \leq \delta, s,t \leq 1} |\gamma_k(s) - \gamma_k(t)| \right\|_p \leq c\delta^{\beta}. \tag{7.9}$$

Actually, this is proved for a renormalized intersection local time $\xi_k(t)$ where $\xi_k(t) = \lim_{x \to 0} \xi_k(t, x)$ and $\xi_k(t, x)$ differs from $\overline{\gamma_k}(t, x)$ defined in (4.3) in that $u^1(x)$ is replaced by $\pi^{-1} \log(1/|x|)$. Since $u^1(x) - \pi^{-1} \log(1/|x|) = c + O(|x|^2 \log |x|)$, see [13], (2.b), we obtain (7.9). Using (6.7) with $r = t^{-1}$ and (7.9) we find that

$$\begin{aligned}
&\left\| \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} \left| \gamma_k\left(\frac{\lambda_n}{\lambda} t\right) - \gamma_k(t) \right| \right\|_p \\
&\qquad \leq ct(\log t)^k \left| \frac{\lambda_n}{\lambda_{n+1}} - 1 \right|^{\beta} \leq ct(\log t)^k n^{-\beta'},
\end{aligned} \tag{7.10}$$

where we have used

$$\log \frac{\lambda_n}{\lambda_{n+1}} = O(n^{-1+1/2k}). \tag{7.11}$$

Hence

$$\left\| \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} \left| \gamma_k\left(\frac{\lambda_n}{\lambda}\zeta\right) - \gamma_k(\zeta) \right| \right\|_p \leq cn^{-\beta''}. \tag{7.12}$$

Using (7.8) and (7.12) now shows that

$$\Bigl\| \sup_{\lambda_{n+1}\le\lambda\le\lambda_n} \bigl|\gamma_k(\zeta,\omega_{\lambda^{-1}}) - \gamma_k\bigl(\zeta,\omega_{\lambda_n^{-1}}\bigr)\bigr| \Bigr\|_p \le cn^{-\beta''} \tag{7.13}$$

and our lemma then follows using Hölder's inequality for sufficiently large p and the Borel–Cantelli lemma. □

Continuing the proof of Theorem 2, by our choice of λ_n

$$\lim_{n\to 0} g^k_{\lambda_{n+1}} - g^k_{\lambda_n} = 0. \tag{7.14}$$

Together with (7.6) we have that a.s.

$$\begin{aligned}\lim_{n\to 0} \sup_{\lambda_{n+1}\le\lambda\le\lambda_n} \Biggl| &\sum_{j=1}^{k} (-1)^{j-1} g_\lambda^{k-j} \gamma_j(\zeta,\omega_{\lambda^{-1}}) \\ &- \sum_{j=1}^{k} (-1)^{j-1} g_{\lambda_n}^{k-j} \gamma_j\bigl(\zeta,\omega_{\lambda_n^{-1}}\bigr)\Biggr| = 0.\end{aligned} \tag{7.15}$$

Using the fact that $|\mathcal{R}(\zeta_\lambda)|$ and g_λ are monotone decreasing we have that

$$\begin{aligned}&\sup_{\lambda_{n+1}\le\lambda\le\lambda_n} \bigl|\lambda g_\lambda^k|\mathcal{R}(\zeta_\lambda)| - \lambda_n g^k_{\lambda_n}|\mathcal{R}(\zeta_{\lambda_n})|\bigr| \\
&\quad\le \bigl|\lambda_n g^k_{\lambda_{n+1}}|\mathcal{R}(\zeta_{\lambda_{n+1}})| - \lambda_{n+1} g^k_{\lambda_n}|\mathcal{R}(\zeta_{\lambda_n})|\bigr| \\
&\quad\le \bigl|\lambda_n g^k_{\lambda_{n+1}}|\mathcal{R}(\zeta_{\lambda_{n+1}})| - \lambda_{n+1} g^k_{\lambda_{n+1}}|\mathcal{R}(\zeta_{\lambda_{n+1}})|\bigr| \\
&\qquad + \bigl|\lambda_{n+1} g^k_{\lambda_{n+1}}|\mathcal{R}(\zeta_{\lambda_{n+1}})| - \lambda_n g^k_{\lambda_n}|\mathcal{R}(\zeta_{\lambda_n})|\bigr| \\
&\qquad + \bigl|\lambda_n g^k_{\lambda_n}|\mathcal{R}(\zeta_{\lambda_n})| - \lambda_{n+1} g^k_{\lambda_n}|\mathcal{R}(\zeta_{\lambda_n})|\bigr| \\
&\quad\le 2|\lambda_n - \lambda_{n+1}| g^k_{\lambda_{n+1}}|\mathcal{R}(\zeta_{\lambda_{n+1}})| \\
&\qquad + \bigl|\lambda_{n+1} g^k_{\lambda_{n+1}}|\mathcal{R}(\zeta_{\lambda_{n+1}})| - \lambda_n g^k_{\lambda_n}|\mathcal{R}(\zeta_{\lambda_n})|\bigr| \to 0 \qquad \text{a.s.}\end{aligned} \tag{7.16}$$

Here the first term on the right-hand side of (7.16) goes to 0 using the fact that

$$|\lambda_n - \lambda_{n+1}| = \bigl|1 - e^{n^{1/2k}-(n+1)^{1/2k}}\bigr|\lambda_n \le n^{-1+1/2k}\lambda_n \le 2n^{-1+1/2k}\lambda_{n+1},$$

$g^k_{\lambda_{n+1}} = (n+1)^{1/2}$, (7.5) and the discussion immediately preceding (7.5). The second term on the right-hand side of (7.16) goes to 0 using (7.15) and (7.5). Combining (7.5), (7.15) and (7.16) we have (7.1). □

8. Nonrandom times. In this section we complete the proof of Theorem 1. Recall that $\zeta_\lambda = n$ if $n-1 < \frac{1}{\lambda}\zeta \le n$. So $\zeta_\lambda = \lceil \frac{1}{\lambda}\zeta \rceil$ where $\lceil x \rceil$ denotes the smallest integer $m \ge x$. Hence (7.1) can be written as

$$g_\lambda^k \left(\lambda |\mathcal{R}(\lceil \zeta/\lambda \rceil)| - \sum_{j=1}^k (-1)^{j-1} g_\lambda^{-j} \gamma_j(\zeta, \omega_{\lambda^{-1}}) \right) \to 0 \qquad \text{a.s.} \tag{8.1}$$

If (Ω, P) denotes our probability space for $\{S_n; n \ge 1\}$ and $\{W_t; t \ge 0\}$, then the almost sure convergence in (8.1) is with respect to the measure $e^{-t}\,dt \times P$ on $R^1_+ \times \Omega$, where $\zeta(t, \omega) = t$. Hence by Fubini's theorem we have that for almost every $t > 0$

$$g_\lambda^k \left(\lambda |\mathcal{R}(\lceil t/\lambda \rceil)| - \sum_{j=1}^k (-1)^{j-1} g_\lambda^{-j} \gamma_j(t, \omega_{\lambda^{-1}}) \right) \to 0 \qquad \text{a.s.} \tag{8.2}$$

Fix a t_0 for which (8.2) holds and let λ run through the sequence t_0/n. Then (2.3) and (8.2) tell us that

$$(\log n)^k \left(\frac{t_0}{n} |\mathcal{R}(n)| + \sum_{j=1}^k (-g_{t_0/n})^{-j} \gamma_j(t_0, \omega_{n/t_0}) \right) \to 0 \qquad \text{a.s.} \tag{8.3}$$

Using (6.7) and writing $b_r = \frac{1}{2\pi} \log(1/r)$ we have that

$$\begin{aligned} &(\log n)^k \Bigg(\frac{t_0}{n} |\mathcal{R}(n)| \\ &\qquad + t_0 \sum_{j=1}^k (-g_{t_0/n})^{-j} \sum_{m=1}^j \binom{j-1}{m-1} b_{1/t_0}^{j-m} \gamma_m(1, \omega_n) \Bigg) \to 0 \qquad \text{a.s.} \end{aligned} \tag{8.4}$$

Then

$$\begin{aligned} &\sum_{j=1}^k (-g_{t_0/n})^{-j} \sum_{m=1}^j \binom{j-1}{m-1} b_{1/t_0}^{j-m} \gamma_m(1, \omega_n) \\ &\qquad = \sum_{m=1}^k \left(\sum_{j=m}^k \binom{j-1}{m-1} \left(\frac{-b_{1/t_0}}{g_{t_0/n}} \right)^{j-m} \right) (-g_{t_0/n})^{-m} \gamma_m(1, \omega_n). \end{aligned} \tag{8.5}$$

Now,

$$\sum_{j=m}^k \binom{j-1}{m-1} x^{j-m} = \sum_{i=0}^{k-m} \binom{i+m-1}{m-1} x^i = \left(\frac{1}{1-x} \right)^m + O(x^{k-m+1}). \tag{8.6}$$

By (7.9) with $\delta = 1$ we have that $\sup_{t \le 1} |\gamma_j(t, \omega)|$ is in L^p for each p and each $j \ge 1$. If we set $V_{j,\ell} = \sup_{t \le 1} |\gamma_j(t, \omega_{2^\ell})|$, we then have, taking p large enough, that

$$\sum_{\ell=1}^{\infty} P\big(V_{j,\ell} > \eta \log(2^\ell)\big) \le \sum_{\ell=1}^{\infty} \frac{EV_{j,\ell}^p}{(\eta \log 2^\ell)^p}$$

is summable for each η. Hence by Borel–Cantelli $V_{j,\ell}/\log(2^\ell) \to 0$ a.s. for each $j \ge 1$. Since by Lemma 6 we have for $2^\ell \le r < 2^{\ell+1}$ that $\gamma_k(1, \omega_r)$ is bounded by a linear combination of the $V_{j,\ell}$, $1 \le j \le k$, with coefficients that are bounded independently of r, we conclude

$$\gamma_j(1, \omega_n)/\log n \to 0 \qquad \text{a.s.}$$

Thus we can replace (8.5) up to errors which are $O(\log n)^{-k-1}$ by

$$\sum_{m=1}^{k} \left(\frac{-1}{g_{t_0/n} + b_{1/t_0}} \right)^m \gamma_m(1, \omega_n) = \sum_{m=1}^{k} (-g_{1/n})^{-m} \gamma_m(1, \omega_n) \tag{8.7}$$

since by (2.3) we have that $g_{t_0/n} + b_{1/t_0} = g_{1/n} + O(n^{-\delta})$.

Thus we obtain

$$(\log n)^k \left(\frac{1}{n} |\mathcal{R}(n)| + \sum_{j=1}^{k} (-g_{1/n})^{-j} \gamma_j(1, \omega_n) \right) \to 0 \qquad \text{a.s.} \tag{8.8}$$

This, together with (A.2), gives Theorem 1.

APPENDIX

Estimates for random walks. In this appendix we will obtain some estimates for strongly aperiodic planar random walks $S_n = \sum_{i=1}^n X_i$, where the X_i are symmetric, have the identity as covariance matrix and have $2 + \delta$ moments for some $\delta > 0$.

Let

$$G_\lambda(x) := \sum_{n=0}^{\infty} e^{-\lambda n} q_n(x).$$

If

$$\phi(p) = E(e^{ip \cdot X_1})$$

denotes the characteristic function of X_1, we have

$$G_\lambda(x) = \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{e^{ip\cdot x}}{1 - e^{-\lambda} \phi(p)}\, dp. \tag{A.1}$$

LEMMA A.1. *Let S_n be as above. Then*

$$G_\lambda(0) = \frac{1}{2\pi}\log(1/\lambda) + c_X + O\big(\lambda^\delta \log(1/\lambda)\big), \tag{A.2}$$

where

$$c_X = \frac{1}{2\pi}\log(\pi^2/2) + \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \frac{\phi(p) - 1 + |p|^2/2}{(1-\phi(p))|p|^2/2}\,dp \tag{A.3}$$

is a finite constant.

PROOF. We have

$$G_\lambda(0) = \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \frac{1}{1 - e^{-\lambda}\phi(p)}\,dp. \tag{A.4}$$

We intend to compare this with

$$\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \frac{1}{\lambda + |p|^2/2}\,dp$$

whose asymptotics are easier to compute. Indeed,

$$\begin{aligned} &\int_{[-\pi,\pi]^2} \frac{1}{\lambda + |p|^2/2}\,dp \\ &\qquad = \int_{D(0,\pi)} \frac{1}{\lambda + |p|^2/2}\,dp + \int_{[-\pi,\pi]^2 - D(0,\pi)} \frac{1}{\lambda + |p|^2/2}\,dp, \end{aligned} \tag{A.5}$$

where $D(0,\pi)$ is the disc centered at the origin of radius π. It is clear that

$$\int_{[-\pi,\pi]^2 - D(0,\pi)} \frac{1}{\lambda + |p|^2/2}\,dp = \int_{[-\pi,\pi]^2 - D(0,\pi)} \frac{1}{|p|^2/2}\,dp + O(\lambda). \tag{A.6}$$

On the other hand, using polar coordinates

$$\int_{D(0,\pi)} \frac{1}{\lambda + |p|^2/2}\,dp = 2\pi\big(\log(\lambda + \pi^2/2) - \log(\lambda)\big). \tag{A.7}$$

Thus

$$\begin{aligned} &\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \frac{1}{\lambda + |p|^2/2}\,dp \\ &\qquad = \frac{1}{2\pi}\log(1/\lambda) + \frac{1}{2\pi}\log(\pi^2/2) + O(\lambda). \end{aligned} \tag{A.8}$$

We then note that

$$\begin{aligned} &\int_{[-\pi,\pi]^2} \frac{1}{1 - e^{-\lambda}\phi(p)}\,dp - \int_{[-\pi,\pi]^2} \frac{1}{\lambda + |p|^2/2}\,dp \\ &\qquad = \int_{[-\pi,\pi]^2} \frac{(\lambda + |p|^2/2) - (1 - e^{-\lambda}\phi(p))}{(1 - e^{-\lambda}\phi(p))(\lambda + |p|^2/2)}\,dp \end{aligned} \tag{A.9}$$

$$
\begin{aligned}
&= \int_{[-\pi,\pi]^2} \frac{\phi(p) - 1 + |p|^2/2}{(1 - e^{-\lambda}\phi(p))(\lambda + |p|^2/2)}\, dp \\
&\quad - \lambda \int_{[-\pi,\pi]^2} \frac{\phi(p) - 1}{(1 - e^{-\lambda}\phi(p))(\lambda + |p|^2/2)}\, dp \\
&\quad + (e^{-\lambda} - 1 + \lambda) \int_{[-\pi,\pi]^2} \frac{\phi(p)}{(1 - e^{-\lambda}\phi(p))(\lambda + |p|^2/2)}\, dp.
\end{aligned}
$$

Since

$$
|e^{ip\cdot x} - 1 - ip \cdot x + (p \cdot x)^2/2| \le c(p \cdot x)^{2+\delta} \tag{A.10}
$$

for some $c < \infty$ we have by our assumptions that

$$
|\phi(p) - 1 + |p|^2/2| \le c' |p|^{2+\delta}. \tag{A.11}
$$

This implies that

$$
|\phi(p) - 1| \le c'' |p|^2 \tag{A.12}
$$

for $p \in [-\pi, \pi]^2$ and

$$
1 - e^{-\lambda}\phi(p) \ge \bar{c}(\lambda + |p|^2) \tag{A.13}
$$

for some $\bar{c} > 0$ and sufficiently small λ. Hence

$$
\begin{aligned}
&(e^{-\lambda} - 1 + \lambda) \int_{[-\pi,\pi]^2} \frac{|\phi(p)|}{(1 - e^{-\lambda}\phi(p))(\lambda + |p|^2/2)}\, dp \\
&\quad \le c\lambda^2 \int_{[-\pi,\pi]^2} \frac{1}{(\lambda + |p|^2)^2}\, dp \\
&\quad \le c\lambda \int_{[-\pi/\sqrt{\lambda},\pi\sqrt{\lambda}]^2} \frac{1}{(1 + |p|^2)^2}\, dp = O(\lambda)
\end{aligned} \tag{A.14}
$$

and

$$
\begin{aligned}
&\lambda \int_{[-\pi,\pi]^2} \frac{|\phi(p) - 1|}{(1 - e^{-\lambda}\phi(p))(\lambda + |p|^2/2)}\, dp \\
&\quad \le c\lambda \int_{[-\pi,\pi]^2} \frac{|p|^2}{(\lambda + |p|^2)^2}\, dp \\
&\quad \le c\lambda \int_{[-\pi/\sqrt{\lambda},\pi\sqrt{\lambda}]^2} \frac{|p|^2}{(1 + |p|^2)^2}\, dp = O\big(\lambda \log(1/\lambda)\big).
\end{aligned} \tag{A.15}
$$

Setting $f(p) = \phi(p) - 1 + |p|^2/2$ and using (A.11), we see that

$$
\int_{[-\pi,\pi]^2} \frac{|f(p)|}{|(1 - \phi(p))||p|^2/2}\, dp < \infty.
$$

Consider then

$$
\begin{aligned}
&\int_{[-\pi,\pi]^2} \frac{f(p)}{(1-e^{-\lambda}\phi(p))(\lambda+|p|^2/2)}\,dp \\
&\quad - \int_{[-\pi,\pi]^2} \frac{f(p)}{(1-\phi(p))|p|^2/2}\,dp \\
&\qquad = \int_{[-\pi,\pi]^2} \frac{f(p)}{(1-e^{-\lambda}\phi(p))(\lambda+|p|^2/2)}\,dp \\
&\qquad\quad - \int_{[-\pi,\pi]^2} \frac{f(p)}{(1-e^{-\lambda}\phi(p))|p|^2/2}\,dp \\
&\qquad\quad + \int_{[-\pi,\pi]^2} \frac{f(p)}{(1-e^{-\lambda}\phi(p))|p|^2/2}\,dp \\
&\qquad\quad - \int_{[-\pi,\pi]^2} \frac{f(p)}{(1-\phi(p))|p|^2/2}\,dp.
\end{aligned} \tag{A.16}
$$

We have

$$
\begin{aligned}
&\int_{[-\pi,\pi]^2} \frac{f(p)}{(1-e^{-\lambda}\phi(p))(\lambda+|p|^2/2)}\,dp \\
&\quad - \int_{[-\pi,\pi]^2} \frac{f(p)}{(1-e^{-\lambda}\phi(p))|p|^2/2}\,dp \\
&\quad = -\int_{[-\pi,\pi]^2} \frac{f(p)\lambda}{(1-e^{-\lambda}\phi(p))(\lambda+|p|^2/2)|p|^2/2}\,dp \\
&\quad = O\big(\lambda^{\delta}\log(1/\lambda)\big),
\end{aligned} \tag{A.17}
$$

and the last line in (A.16) can be bounded similarly. This completes the proof of Lemma A.1. $\square$

LEMMA A.2. *Let S_n be as above. For all $m \geq 1$*

$$\|G_\lambda\|_m = O(\lambda^{-1/m}) \qquad \text{as } \lambda \to 0 \tag{A.18}$$

and

$$\|G_\lambda - G_{\lambda'}\|_m = O\big(|\lambda-\lambda'|\big(\sqrt{\lambda\lambda'}\big)^{-1/m}\big) \qquad \text{as } \lambda \to 0. \tag{A.19}$$

For all $m \geq 2$ and $z \in Z^2$

$$\|\Delta_{z/\sqrt{\lambda}} G_\lambda\|_m \leq c'|z|^{2/m}\lambda^{-1/m}\big(\log(1/\lambda)\big)^{1-1/m} \tag{A.20}$$

and for any $0 < \beta < 1$

$$\|\Delta_{z/\sqrt{\lambda}} G_\lambda\|_m \leq c'|z|^{\beta/m}\lambda^{-1/m} \tag{A.21}$$

and

$$\left\| \left(\prod_{i=1}^{k} \Delta_{z_i/\sqrt{\lambda}} \right) G_\lambda \right\|_m \le c' \left(\prod_{i=1}^{k} |z_i|^{\beta/mk} \right) \lambda^{-1/m}. \tag{A.22}$$

PROOF. By [23], page 77, we know that $q_n(x) \le c_1/n$, where q_n is the transition probability for S_n. So

$$\|q_n\|_m^m = \sum_{x \in Z^2} q_n(x)^m \le c_1^{m-1} n^{-m+1} \sum_{z \in Z^2} q_n(x) = c_1^{m-1} n^{-m+1}.$$

Then

$$\|G_\lambda\|_m \le \sum_{n=0}^{\infty} e^{-\lambda n} \|q_n\|_m.$$

Substituting the above estimate for $\|q_n\|_m$ and breaking the sum into the sum over $n \le 1/\lambda$ and the sum over $n > 1/\lambda$, we easily obtain (A.18).

Equation (A.19) follows from (A.18) and the resolvent equation

$$G_\lambda - G_{\lambda'} = (\lambda' - \lambda) G_\lambda * G_{\lambda'}. \tag{A.23}$$

By Proposition 2.1 of [2], for each $\beta \in (0, 1]$ there exists a constant c_β such that

$$|q_n(x) - q_n(y)| \le c_\beta n^{-1} \big(|x - y|/\sqrt{n}\big)^\beta.$$

So for any fixed $w \in Z^2$

$$\begin{aligned}\|q_n(\cdot + w) - q_n(\cdot)\|_m^m &\le \|q_n(\cdot + w) - q_n\|_\infty^{m-1} \sum_{x \in Z^2} \big(q_n(x + w) q_n(x)\big) \\ &\le 2\big(c_\beta n^{-1} \big(|w|/\sqrt{n}\big)^\beta\big)^{m-1}.\end{aligned}$$

We take mth roots, substitute into

$$\|G_\lambda(\cdot + w) - G_\lambda(\cdot)\|_m \le \sum_{n=0}^{\infty} e^{-\lambda n} \|q_n(\cdot + w) - q_n(\cdot)\|_m,$$

break the sum into the sum over $n \le 1/\lambda$ and the sum over $n > 1/\lambda$, and let $w = z/\sqrt{\lambda}$ to obtain (A.21).

For (A.22) we note that for each j we can write $(\prod_{i=1}^{k} \Delta_{z_i/\sqrt{\lambda}}) G_\lambda$ as a sum of 2^{k-1} terms of the form $\Delta_{z_j/\sqrt{\lambda}} G_\lambda(z + b)$ for some b so that by (A.21)

$$\left\| \left(\prod_{i=1}^{k} \Delta_{z_i/\sqrt{\lambda}} \right) G_\lambda \right\|_m \le c' 2^{k-1} |z_j|^{\beta/m} \lambda^{-1/m}. \tag{A.24}$$

We have inequality (A.24) for $j = 1, \ldots, k$. If we take the product of these k inequalities and then take kth roots, we have (A.22). $\square$

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