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CLUSTER SIZE DISTRIBUTIONS OF EXTREME VALUES FOR THE POISSON–VORONOI TESSELLATION

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We consider the Voronoi tessellation based on a homogeneous Poisson point process in an Euclidean space. For a geometric characteristic of the cells (e.g., the inradius, the circumradius, the volume), we investigate the point process of the nuclei of the cells with large values. Conditions are obtained for the convergence in distribution of this point process of exceedances to a homogeneous compound Poisson point process. We provide a characterization of the asymptotic cluster size distribution which is based on the Palm version of the point process of exceedances. This characterization allows us to compute efficiently the values of the extremal index and the cluster size probabilities by simulation for various geometric characteristics. The extension to the Poisson–Delaunay tessellation is also discussed.

1. Introduction. *Stationary tessellations and the Poisson–Voronoi tessellation.* A tessellation m in $\mathbb{R}^d$, $d \geq 1$, endowed with its Euclidean norm $|\cdot|$, is a countable collection of nonempty convex compact subsets, called *cells*, with disjoint interiors which subdivides the space and such that the number of cells intersecting any bounded subset of $\mathbb{R}^d$ is finite. The set $\mathbf{T}$ of tessellations is endowed with the σ-field generated by the sets $\{\mathrm{m} \in \mathbf{T} : \bigcup_{C\in \mathrm{m}} \partial C \cap K = \varnothing\}$, where ∂K is the boundary of K for any compact set K in $\mathbb{R}^d$. By a random tessellation m, we mean a random variable with values in $\mathbf{T}$. For a complete account on random tessellations and their applications, we refer to the books [29, 32].

A tessellation m is said to be stationary if its distribution is invariant under translations of the cells. Given a fixed realization of a stationary tessellation m, we associate with each cell $C \in \mathrm{m}$, in a deterministic way, a point $z(C)$ which is called the *nucleus* of the cell, such that $z(C + x) = z(C) + x$ for all $x \in \mathbb{R}^d$. To describe the mean behavior of the tessellation, the notions of intensity and typical cell are introduced as follows. Let $A \subset \mathbb{R}^d$ be a Borel subset such that $\lambda_d(A) = 1$, where λ_d is the d-dimensional Lebesgue measure. The *intensity* of a stationary tessellation m is defined as

$$\gamma_{\mathrm{m}} := \mathbb{E}\big[\#\{C \in \mathrm{m} : z(C) \in A\}\big],$$

where $\#\mathcal{S}$ denotes the cardinality of any finite set $\mathcal{S}$. Thanks to the stationarity of m, the intensity does not depend on the choice of A. Without loss of generality, we assume that $\gamma_{\mathrm{m}} = 1$.

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The *typical cell* $\mathcal{C}$ of a stationary tessellation m is a random polytope with distribution given by

$$\mathbb{E}[f(\mathcal{C})] = \mathbb{E}\Big[\sum_{C\in \mathrm{m}: z(C)\in A} f\big(C - z(C)\big)\Big], \tag{1.1}$$

where $f : \mathcal{K}_d \to \mathbb{R}$ is any bounded measurable function on the family of convex bodies $\mathcal{K}_d$ (i.e., nonempty convex compact sets in $\mathbb{R}^d$), where $\mathcal{K}_d$ is endowed with the Hausdorff topology.

Let χ be a locally finite subset of $\mathbb{R}^d$. The Voronoi cell with nucleus $x \in \chi$ is the set of all sites $y \in \mathbb{R}^d$ whose distance from x is smaller or equal than the distances to all other points of χ, that is,

$$C_\chi(x) := \{y \in \mathbb{R}^d : |y - x| \le |y - x'|, x' \in \chi\}.$$

When $\chi = \eta$ is a homogeneous Poisson point process, the family $\mathrm{m} := \{C_\eta(x) : x \in \eta\}$ is the so-called *Poisson–Voronoi tessellation*. The intensity of such a tessellation equals the intensity of η. A consequence of the theorem of Slivnyak (see, e.g., Theorem 3.3.5 in [29]) shows that

$$\mathcal{C} \overset{\mathcal{D}}{=} C_{\eta\cup\{0\}}(0), \tag{1.2}$$

where $\overset{\mathcal{D}}{=}$ denotes the equality in distribution. The study of this typical cell in the literature includes mean values calculations [21], second-order properties [14] and distributional estimates [5, 22]. Voronoi tessellations are extensively used in many domains such as cellular biology [26], astrophysics [33], telecommunications [3] and finance [24]. For a complete account on Poisson–Voronoi tessellations and their applications, we refer to the book by Okabe et al. (see Chapter 5 in [23]).

Point process of exceedances for a stationary sequence of real random variables. Let $(X_n)_{n\in\mathbb{Z}}$ be a strictly stationary sequence of real random variables. Assume that for each $\tau > 0$ there exists a sequence of levels $(u_n(\tau))$ such that $\lim_{n\to\infty} n\mathbb{P}(X_1 > u_n(\tau)) = \tau$. The point process of time normalized exceedances is defined by $\phi_B(\tau) := n^{-1} \cdot \{i \in B : X_i > u_n(\tau)\}$ for any Borel set $B \subset W_n := [-n/2, n/2]$. If (X_n) satisfies a long range dependence condition [known as condition $\Delta(u_n(\tau))$] and if the point process $\phi_{W_n}(\tau)$ weakly converges to a point process in $[-1/2, 1/2]$, then the limiting point process is necessarily a homogeneous compound Poisson process with intensity $\nu \ge 0$ and limiting *cluster size distribution* π (see Corollary 3.3 in [15]). According to Leadbetter [19], the constant $\theta = \nu/\tau$ is referred to as the *extremal index*. It may be shown that $0 \le \theta \le 1$ and that the compound Poisson limit becomes Poisson when $\theta = 1$.

If $\lim_{n\to\infty} \mathbb{P}(\#\phi_{W_n}(\tau) = 0) = e^{-\theta\tau}$, then a necessary and sufficient condition for the convergence of ϕ_{W_n} is the convergence of the conditional distribution of $\#\phi_{B_n}$, with $B_n = [0, q_n]$, given that there is at least one exceedance of $u_n(\tau)$ among $X_1, \dots, X_{q_n}$, to the distribution $\pi = (\pi_k)_{k\ge 1}$, that is,

$$\lim_{n\to\infty} \mathbb{P}\big(\#\phi_{B_n}(\tau) = k | \#\phi_{B_n}(\tau) > 0\big) := \pi_k, \qquad k \ge 1, \tag{1.3}$$

where (q_n) is a $\Delta(u_n(\tau))$-separating sequence, with $\lim_{n\to\infty} q_n/n = 0$ (see Theorem 4.2 in [15]). This condition is known as the *blocks* characterization of the cluster size distribution π. Under additional mild conditions (see, e.g., [30]) the extremal index is equal to the reciprocal of the mean of π.

An equivalent condition to (1.3) is proposed in Theorem 4.1 in [28] (see also Theorem 2.5 in [25]) and is given by

$$\lim_{n\to\infty} \mathbb{P}\big(\#\phi_{B_n}(\tau) = k | X_0 > u_n(\tau)\big) := p'_k = \theta \sum_{m=k}^{\infty} \pi_m, \qquad k \geq 1. \tag{1.4}$$

In particular, we have $\theta = p'_1$. This second condition is useful to compute the values of the extremal index and the cluster size probabilities when the conditional distributions of the exceedances may be derived from the dynamics of $(X_n)_{n\in\mathbb{Z}}$, for example, for the regularly varying multivariate time series [4] or the Markov sequences [25]. This condition may be called the *runs* characterization of the cluster size distribution since the runs estimator of the extremal index is based on the following result:

$$\theta = \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{i=1}^{q_n} \{X_i \leq u_n(\tau)\} \middle| X_0 > u_n(\tau)\right).$$

The runs characterization is natural for a random object as a time series where the direction of time is used to design the dynamics of the series. Estimators of the extremal index and the cluster size distribution, based on the blocks and runs characterizations, are extensively investigated; see, for example, [27, 31].

However, we claim that it could also be useful to consider a new condition where the conditional event $\{X_0 > u_n(\tau)\}$ is not used as the starting point of the considered cluster, but as a part of this cluster. We therefore introduce a new discrete probability distribution $p = (p_k)_{k\geq 1}$ and the following condition:

$$\lim_{n\to\infty} \mathbb{P}\big(\#\phi_{C_n}(\tau) = k | X_0 > u_n(\tau)\big) := p_k, \qquad k \geq 1, \tag{1.5}$$

where $C_n = [-q_n/2, q_n/2]$. If p exists, an adaptation of our main result (see Theorem 4) shows that $p_k = \theta k \pi_k$ for $k \geq 1$, and therefore $\theta = \sum_{k=1}^{\infty} k^{-1} p_k$. Such a condition will be proposed for random tessellations for which there is no natural direction in the space $\mathbb{R}^d$. However, we think that our new condition could be fruitful for time series.

Point process of exceedances for a stationary tessellation. Let m be a stationary tessellation in $\mathbb{R}^d$. We consider a geometric characteristic $g : \mathcal{K}_d \to \mathbb{R}$ satisfying $g(C + x) = g(C)$ for all $C \in \mathcal{K}_d$ and $x \in \mathbb{R}^d$, and such that, for some $\tau_0 > 0$, there exists a sequence of thresholds $v_\rho(\tau_0)$ satisfying

$$\lim_{\rho\to\infty} \rho \mathbb{P}\big(g(\mathcal{C}) > v_\rho(\tau_0)\big) = \tau_0. \tag{1.6}$$

By Theorem 1.7.13 in [18], it is equivalent to the existence of sequences of thresholds $v_\rho(\tau)$ for any $\tau > 0$ satisfying $\lim_{\rho\to\infty} \rho \mathbb{P}(g(\mathcal{C}) > v_\rho(\tau)) = \tau$.

We observe only a part of the stationary tessellation m in the window $W_\rho := \rho^{1/d} \cdot [-1/2, 1/2]^d$, $\rho > 0$, and we are interested in the point process of exceedances $\Phi_{W_\rho}(\tau)$ where, for any Borel set $B \subset \mathbb{R}^d$, we let

$$\Phi_B(\tau) := \rho^{-1/d} \cdot \{z(C) : z(C) \in B, g(C) > v_\rho(\tau), C \in \mathrm{m}\}.$$

In this paper, we investigate the weak convergence of the point process $\Phi_{W_\rho}(\tau)$ in $[-1/2, 1/2]^d$ as ρ tends to infinity. In [10], a first result was obtained for geometric characteristics for which a short range dependence condition holds (equivalent to the so-called condition D' for stationary sequences of real random variables): it is shown that the point process $\Phi_{W_\rho}(\tau)$ weakly converges to a homogeneous Poisson point process with intensity τ. In this paper, we are interested in finding weaker conditions for other geometric characteristics such that the point process $\Phi_{W_\rho}(\tau)$ weakly converges to a homogeneous compound Poisson point process.

Let B_ρ be a subcube of W_ρ such that $\lim_{\rho\to\infty} \lambda_d(B_\rho)/\rho = 0$. Condition (1.3) for the tessellation m will be written in the following way:

$$\lim_{\rho\to\infty} \mathbb{P}\big(\#\Phi_{B_\rho}(\tau) = k | \#\Phi_{B_\rho}(\tau) > 0\big) = \pi_k, \qquad k \geq 1, \tag{1.7}$$

for a discrete probability distribution $\pi = (\pi_k)_{k\geq 1}$, which we also call the cluster size distribution. Additional assumptions on B_ρ will be necessary and will depend on the mixing properties of the tessellation. Condition (1.4) cannot be translated for stationary tessellations as explained previously. Condition (1.5) has to be modified since the cell which contains the origin (the Crofton cell) is not distributed as the typical cell. To overcome this difficulty, we consider a Palm version $\Phi^0_{\mathbb{R}^d}(\tau)$ of $\Phi_{\mathbb{R}^d}(\tau)$, that is, a point process whose distribution is given by the Palm distribution of $\Phi_{\mathbb{R}^d}(\tau)$ (see Sections 3.3 and 3.4 in [29] for a complete account on Palm theory). For any $B \subset \mathbb{R}^d$, we also let $\Phi^0_B(\tau) = \Phi^0_{\mathbb{R}^d}(\tau) \cap B$. An analogous version of Condition (1.5) in the context of random tessellations can be stated as follows:

$$\lim_{\rho\to\infty} \mathbb{P}\big(\#\Phi^0_{B_\rho}(\tau) = k\big) := p_k, \qquad k \geq 1, \tag{1.8}$$

for a discrete probability distribution $p = (p_k)_{k\geq 1}$.

In general, the distributions π and p cannot be made explicit. It is necessary to use simulations to compute approximate values of the probabilities π_k and p_k. The blocks method (1.7) competes with the Palm approach (1.8). The idea of the Palm approach is to consider clusters close to the origin given that the cell whose nucleus is the origin has an exceedance. Our approach provides better approximations of the extremal index and the cluster size distribution and requires less simulations. Indeed, we simulate only blocks that contain at least one exceedance (the one of the Crofton cell that contains the origin), while with the blocks approach, it is necessary to simulate a very large number of blocks (including those without any extreme value). More precisely, in our numerical illustrations in $\mathbb{R}^2$, we simulate tessellations only observed in the square $[-173, 173]^2$ to approximate

θ and $p = (p_k)_{k\geq 1}$ thanks to our Palm approach. A blocks approach would have required to simulate tessellations in the square $[-5.18 \cdot 10^{21}, 5.18 \cdot 10^{21}]$ for the same accuracy, which is practically impossible.

The point process $\Phi_{W_\rho}(\tau)$ can be derived from the marked point process $\{(z(C), \mathbb{I}_{g(C)>v_\rho(\tau)}) : C \in \mathrm{m}\}$. The theory of marked point processes is however not enough to investigate the convergence of the point process $\Phi_{W_\rho}(\tau)$ because additional properties of the structure of the random tessellation have to be used. In this paper, we only focus on the Poisson–Voronoi tessellation and we explain how our results can be extended to the Poisson–Delaunay tessellation.

Our paper is organized as follows. In Section 2, we give several preliminaries by introducing notation and conditions on our geometric characteristic. In Section 3, we investigate the convergence in distribution of the point process of exceedances to a homogeneous compound Poisson point process. This convergence is stated in our main result (Theorem 4). In Section 4, we give three examples and numerical illustrations. The extension to the Poisson–Delaunay tessellation is discussed in Section 5.

2. Preliminaries. In this section, we introduce some notation and conditions which will be used throughout the paper.

Notation.

- Let $x \in \mathbb{R}^d$ and let $A, B \subset \mathbb{R}^d$ be two subsets. We write $x + A := \{x + a : a \in A\}$, $A \oplus B := \{a + b : a \in A, b \in B\}$ and $A \ominus B := \{x \in \mathbb{R}^d : x + B \subset A\}$. Moreover, we denote the complement of A by $A^c := \mathbb{R}^d \setminus A$, and the set of points in B which do not belong to A by $B \setminus A := B \cap A^c$.
- For any $A, B \subset \mathbb{R}^d$, we denote the distance between A and B by $\delta(A, B) := \inf_{(a,b)\in A\times B} |a - b|$. If $a \in A$, we use the simpler notation $\delta(a, B) := \delta(\{a\}, B)$.
- For any k-tuple of points $x_1, \ldots, x_k \in \mathbb{R}^d$, we write $x_{1:k} := (x_1, \ldots, x_k)$. With a slight abuse of notation, we also write $\{x_{1:k}\} := \{x_1, \ldots, x_k\}$.
- We denote by $\mathcal{F}_{lf}$ the set of locally finite subsets in $\mathbb{R}^d$. This set is endowed with the σ-field induced by the so-called Fell topology on $\mathcal{F}_{lf}$ (see, e.g., page 563 in [29]).
- We denote by η a homogeneous Poisson point process in $\mathbb{R}^d$. Excepted in Section 5, we assume that the intensity of η is $\gamma_\eta = 1$.
- For any pair of functions $h_1, h_2 \colon \mathbb{R} \to \mathbb{R}$, we write $h_1(\rho) \underset{\rho\to\infty}{\sim} h_2(\rho)$ and $h_1(\rho) = O(h_2(\rho))$ to respectively mean that $h_1(\rho)/h_2(\rho) \to 1$ as $\rho \to \infty$ and $h_1(\rho)/h_2(\rho)$ is bounded for ρ large enough.

Throughout the paper, we use c to signify a universal positive constant not depending on ρ but which may depend on other quantities. When required, we assume that ρ is sufficiently large.

Conditional independence. Let η be a homogeneous Poisson point process in $\mathbb{R}^d$ and let $W_\rho = \rho^{1/d} \cdot [-1/2, 1/2]^d$. We begin with a first lemma that characterizes the dependence structure of the Poisson–Voronoi tessellation induced by η.

Let $\varepsilon > 0$ be fixed. We partition $W_{\rho(1+\varepsilon)}$ into a set $\mathcal{V}_\rho$ of n_ρ^d subcubes of equal size, where $n_\rho := \lfloor (\log\rho)^{-(1+\varepsilon)/d} \cdot \rho^{1/d} \rfloor$. The fact that we consider a subdivision of $W_{\rho(1+\varepsilon)}$ instead of W_ρ allows us to deal with boundary effects. The subcubes are indexed by the set of $\mathbf{i} := (i_1, \dots, i_d) \in [1, n_\rho]^d$. With a slight abuse of notation, we identify a cube with its index. We now introduce a distance between subcubes $\mathbf{i}$ and $\mathbf{j}$ as $d(\mathbf{i}, \mathbf{j}) := \max_{1 \le s \le d} |i_s - j_s|$. Let $S(\mathbf{i}, r) = \{\mathbf{j} \in \mathcal{V}_\rho : d(\mathbf{i}, \mathbf{j}) \le r\}$ be the ball of subcubes of radius r around $\mathbf{i}$. If $\mathcal{I}$ and $\mathcal{J}$ are two sets of subcubes, we let

$$S(\mathcal{I}, r) = \bigcup_{\mathbf{i} \in \mathcal{I}} S(\mathbf{i}, r) \quad \text{and} \quad d(\mathcal{I}, \mathcal{J}) = \min_{\mathbf{i} \in \mathcal{I}, \mathbf{j} \in \mathcal{J}} d(\mathbf{i}, \mathbf{j}).$$

For any $A \subset \mathbb{R}^d$, we define

$$\mathcal{I}(A) = \{\mathbf{i} \in \mathcal{V}_\rho : \mathbf{i} \cap A \neq \varnothing\}.$$

Let $\chi \in \mathcal{F}_{lf}$. For any $x_{1:k} \in \chi^k$ and for any $v \ge 0$, we write $g^\chi(x_{1:k}) > v$ to specify that $g(C_\chi(x_j)) > v$ for any $1 \le j \le k$. In particular, we let $g^\chi(x) := g(C_\chi(x))$. Moreover, we introduce the following σ-algebra:

$$\Sigma_A^{\eta \cup \{x_{1:k}\}} := \sigma\left\{g^{\eta \cup \{x_{1:k}\}}(x) : x \in \left(\eta \cup \{x_{1:k}\}\right) \cap A\right\}. \tag{2.1}$$

Finally, to ensure several independence properties, we introduce the event

$$\mathscr{A}_\rho := \bigcap_{\mathbf{i} \in \mathcal{V}_\rho} \{\eta \cap \mathbf{i} \neq \varnothing\}.$$

The event $\mathscr{A}_\rho$ is extensively used in stochastic geometry to derive central limit theorems or to deal with extremes (see, e.g., [2, 10]). It will play a crucial role in the rest of the paper. The following lemma is the heart of our development and captures the idea of "local dependence."

LEMMA 1. *Let $x_1, \dots, x_k \in \mathbb{R}^d$, with $k \ge 0$, and let $A, B \subset W_\rho$. Then*

(i) *conditional on $\mathscr{A}_\rho$, the σ-fields $\Sigma_A^{\eta \cup \{x_{1:k}\}}$ and $\Sigma_B^{\eta \cup \{x_{1:k}\}}$ are independent when $d(\mathcal{I}(A), \mathcal{I}(B)) > D$, where $D := 4(\lfloor \sqrt{d} \rfloor + 1)$;*
(ii) *for any $\alpha > 0$, we have $\rho^\alpha \cdot \mathbb{P}(\mathscr{A}_\rho^c) \underset{\rho \to \infty}{\longrightarrow} 0$.*

PROOF. For the first assertion, we use the same arguments as in the proof of Proposition 3 in [2]. The main idea is to show that, conditional on the event $\mathscr{A}_\rho$, the σ-algebras $\Sigma_A^{\eta \cup \{x_{1:k}\}}$ and $\Sigma_B^{\eta \cup \{x_{1:k}\}}$ only depend on the set of points of η restricted to two disjoint neighborhoods of A and B. Since η is a Poisson point process, this will show that $\Sigma_A^{\eta \cup \{x_{1:k}\}}$ and $\Sigma_B^{\eta \cup \{x_{1:k}\}}$ are independent. To do it, let $x \in (\eta \cup \{x_{1:k}\}) \cap A$, with $x \in \mathbf{i}$ for some subcube $\mathbf{i} \in \mathcal{V}_\rho$. Let $\mathcal{N}(x)$ be the set of Voronoi neighbors of x, that is,

$$\mathcal{N}(x) = \left\{y \in \eta \cup \{x_{1:k}\} : C_{\eta \cup \{x_{1:k}\}}(x) \cap C_{\eta \cup \{x_{1:k}\}}(y) \neq \varnothing\right\}.$$

We show below that, under the event $\mathscr{A}_\rho$, the set of Voronoi neighbors $\mathcal{N}(x)$ is included in $S(\mathbf{i}, R)$, with

$$R = \frac{D}{2} = 2(\lfloor\sqrt{d}\rfloor + 1).$$

Indeed, let $y \in \mathcal{N}(x)$. Then there exists a ball containing x and y on its boundary, with no point of $\eta \cup \{x_{1:k}\}$ in its interior. Conditional on the event $\mathscr{A}_\rho$, the center of this ball belongs to some subcube $\mathbf{j} \in \mathcal{V}_\rho$. Moreover, since the diameter of $\mathbf{j}$ equals $\sqrt{d} \cdot (\lambda_d(\mathbf{i}))^{1/d}$ and since, conditional on the event $\mathscr{A}_\rho$, the subcube $\mathbf{j}$ contains at least one point of $\eta \cup \{x_{1:k}\}$, the radius of the ball is larger than $(\lfloor\sqrt{d}\rfloor + 1) \cdot (\lambda_d(\mathbf{i}))^{1/d}$. Hence

$$|x - y| \leq 2(\lfloor\sqrt{d}\rfloor + 1) \cdot (\lambda_d(\mathbf{i}))^{1/d}.$$

Therefore, $y \in \mathbf{k}$ for some subcube $\mathbf{k}$ such that $d(\mathbf{i}, \mathbf{k}) \leq 2(\lfloor\sqrt{d}\rfloor + 1)$, which proves that $\mathcal{N}(x) \subset S(\mathbf{i}, R)$.

Since $\mathcal{N}(x) \subset S(\mathbf{i}, R)$ for any point $x \in (\eta \cup \{x_{1:k}\}) \cap \mathbf{i}$, this shows that

$$\{g^{\eta\cup\{x_{1:k}\}}(x) : x \in (\eta \cup \{x_{1:k}\}) \cap \mathbf{i}\} \in \sigma(\eta \cap S(\mathbf{i}, R)).$$

Because $d(\mathcal{I}(A), \mathcal{I}(B)) > D$ implies that $S(\mathcal{I}(A), R)$ and $S(\mathcal{I}(B), R)$ are disjoint and because $\eta \cap S(\mathcal{I}(A), R)$ and $\eta \cap S(\mathcal{I}(B), R)$ are independent, the σ-algebras $\Sigma_A^{\eta\cup\{x_{1:k}\}}$ and $\Sigma_B^{\eta\cup\{x_{1:k}\}}$ are independent. This proves the first assertion of Lemma 1.

The second assertion comes from (3.3) and from the fact that

$$\mathbb{P}(\mathscr{A}_\rho^c) = \mathbb{P}\Bigg(\bigcup_{\mathbf{k}\in\mathcal{V}_\rho} \{\eta \cap \mathbf{k} = \varnothing\}\Bigg) \leq \#\mathcal{V}_\rho \cdot \mathbb{P}(\eta \cap \mathbf{i} = \varnothing) = n_\rho^d e^{-(1+\varepsilon)\rho/n_\rho^d}. \qquad \square$$

Condition on the geometric characteristic. To state our main theorem, we assume some condition on the geometric characteristic g, referred to as Condition (C).

CONDITION (C). *For any* $\tau > 0$, *there exists a constant c such that, for any* $(k-1)$*-tuple of points* $y_{2:k} \in \mathbb{R}^{d(k-1)}$, *for any* $z \in \{0, y_{2:k}\}$, *we have*

$$\mathbb{P}(g^{\eta\cup\{0,y_{2:k}\}}(z) > v_\rho(\tau)) \leq c \cdot \rho^{-1}, \tag{2.2}$$

where $v_\rho(\tau)$ *satisfies equation* (1.6), *with the convention* $\{y_{2:k}\} = \varnothing$ *when* $k = 1$.

In particular, Condition (C) is satisfied when $g^{\eta\cup\{0,y_{2:k}\}}(x) \leq g^{\eta\cup\{0\}}(x)$ for any $x \in \eta \cup \{0\}$ and for any $y_{2:k} \in \mathbb{R}^{d(k-1)}$, that is, when the geometric characteristic of a cell always decreases if new points are added to the point process η.

3. Weak convergence of the point process of exceedances for a Poisson–Voronoi tessellation. Let $\chi \in \mathcal{F}_{lf}$. For any $\rho > 0$ and $\tau > 0$, we denote by $\Phi^{\chi}(\tau)$ the point process of exceedances of the Voronoi tessellation induced by χ, that is,

$$\Phi^{\chi}(\tau) := \rho^{-1/d} \cdot \{x \in \chi : g^{\chi}(x) > v_{\rho}(\tau)\}.$$

Besides, for any $B \subset \mathbb{R}^d$, we write $\Phi^{\chi}_B(\tau) := \Phi^{\chi}(\tau) \cap (\rho^{-1/d} B)$.

For each $\tau > 0$, we denote by $\Phi^{\eta,0}(\tau)$ the Palm version of $\Phi^{\eta}(\tau)$. In particular, for any $B \subset \mathbb{R}^d$ we let $\Phi^{\eta,0}_B(\tau) := \Phi^{\eta,0}(\tau) \cap (\rho^{-1/d} B)$. We also associate two probabilities defined as follows:

$$\pi_{k,B}(\tau) := \mathbb{P}\big(\#\Phi^{\eta}_B(\tau) = k | \#\Phi^{\eta}_B(\tau) > 0\big)$$

and

$$p_{k,B}(\tau) := \mathbb{P}\big(\#\Phi^{\eta,0}_B(\tau) = k\big).$$

The quantity $\pi_{k,B}(\tau)$ is the probability that there are k exceedances in B conditional on the fact that there is at least an exceedance in B, whereas the quantity $p_{k,B}(\tau)$ is the probability that there are k exceedances in B conditional on the fact that the origin is a nucleus and that the cell with nucleus at the origin is an exceedance. Notice that these probabilities also depend on ρ.

3.1. *An explicit representation for $p_{k,B}$.* According to the theorem of Slivnyak, the Palm distribution of η is given by the distribution of $\eta \cup \{0\}$. As a consequence, the following lemma shows that for any $B \subset \mathbb{R}^d$, the distribution of $\#\Phi^{\eta,0}_B(\tau)$ is the same as the one of $\#\Phi^{\eta\cup\{0\}}_B(\tau)$ given that $g^{\eta\cup\{0\}}(0) > v_{\rho}(\tau)$.

LEMMA 2. *For any $B \subset \mathbb{R}^d$, $\rho > 0$ and $k \geq 1$, we have*

$$p_{k,B}(\tau) = \mathbb{P}\big(\#\Phi^{\eta\cup\{0\}}_B(\tau) = k | g^{\eta\cup\{0\}}(0) > v_{\rho}(\tau)\big).$$

PROOF. Since $p_{k,B}(\tau) = \mathbb{P}(\#\Phi^{\eta,0}(\tau) \cap (\rho^{-1/d} B) = k)$ and since the intensity of $\Phi^{\eta}(\tau)$ equals $\rho\,\mathbb{P}(g(\mathcal{C}) > v_{\rho}(\tau))$, we obtain for any Borel subset $A \subset \mathbb{R}^d$, with $\lambda_d(A) = 1$, that

$$p_{k,B}(\tau) = \frac{1}{\rho\,\mathbb{P}(g(\mathcal{C}) > v_{\rho}(\tau))} \mathbb{E}\Big[\sum_{z \in \Phi^{\eta}(\tau) \cap A} \mathbb{I}_{\#(\Phi^{\eta}(\tau) - z) \cap (\rho^{-1/d} B) = k}\Big]. \tag{3.1}$$

Moreover, it results from the Slivnyak–Mecke formula (e.g., Corollary 3.2.3 in [29]) that

$$\mathbb{E}\Big[\sum_{z \in \Phi^{\eta}(\tau) \cap A} \mathbb{I}_{\#(\Phi^{\eta}(\tau) - z) \cap (\rho^{-1/d} B) = k}\Big]$$
$$= \int_{\rho^{1/d} A} \mathbb{P}\big(\#\Phi^{\eta\cup\{x\} - x}_B(\tau) = k, g^{\eta\cup\{x\}}(x) > v_{\rho}(\tau)\big)\,\mathrm{d}x.$$

Thanks to the stationarity of η and because g is translation-invariant, the above integrand does not depend on x. By integrating over $x \in \rho^{1/d}A$ and by using the fact that $\lambda_d(\rho^{1/d}A) = \rho$, it follows that

$$\mathbb{E}\bigg[\sum_{z\in\Phi^\eta(\tau)\cap A} \mathbb{I}_{\#(\Phi^\eta(\tau)-z)\cap(\rho^{-1/d}B)=k}\bigg]$$
$$= \rho \cdot \mathbb{P}\big(\#\Phi_B^{\eta\cup\{0\}}(\tau) = k, g^{\eta\cup\{0\}}(0) > v_\rho(\tau)\big).$$

This together with (1.2) and (3.1) concludes the proof of Lemma 2. □

3.2. *A technical result.* In this section, we establish a technical proposition which will be the key ingredient to prove our main theorem. To state this proposition, we first give some notation. We denote by $q : \rho \mapsto q_\rho$ a generic function such that, for any $\alpha, \beta > 0$, we have simultaneously

$$q_\rho \cdot (\log \rho)^\alpha \cdot \rho^{-1} \underset{\rho\to\infty}{\longrightarrow} 0 \quad \text{and} \quad q_\rho^{-1} \cdot (\log \rho)^\beta \underset{\rho\to\infty}{\longrightarrow} 0. \tag{3.2}$$

We also consider for $\varepsilon > 0$, $\rho > 0$ the following integers:

$$\begin{aligned} n_\rho &:= \lfloor (\log \rho)^{-(1+\varepsilon)/d} \cdot \rho^{1/d} \rfloor \quad \text{and} \\ m_\rho &:= \lfloor q_\rho^{-1/d} \cdot (\log \rho)^{-(1+\varepsilon)/d} \cdot \rho^{1/d} \rfloor. \end{aligned} \tag{3.3}$$

Notice that n_ρ has already been defined on page 3296. We also define two cubes centered at 0 as follows:

$$C_\rho := \frac{((1+\varepsilon)\rho)^{1/d}}{n_\rho} \cdot [-D, D]^d \quad \text{and} \quad Q_\rho := \frac{\rho^{1/d}}{m_\rho} \cdot [-1/2, 1/2]^d, \tag{3.4}$$

where we recall that $D = 4(\lfloor\sqrt{d}\rfloor + 1)$ (see Lemma 1). Notice that for each subcube $\mathbf{i} \in \mathcal{V}_\rho$, we have $\lambda_d(\mathbf{i}) = (2D)^{-d} \cdot \lambda_d(C_\rho)$. Besides, $\lambda_d(C_\rho) = o(\lambda_d(Q_\rho))$, with

$$\lambda_d(C_\rho) \underset{\rho\to\infty}{\sim} (1+\varepsilon)(\log \rho)^{1+\varepsilon}(2D)^d \quad \text{and} \quad \lambda_d(Q_\rho) \underset{\rho\to\infty}{\sim} q_\rho(\log \rho)^{1+\varepsilon}. \tag{3.5}$$

We are now prepared to state our technical proposition.

PROPOSITION 3. *Assume that g satisfies Condition (C). Then, for any $k \geq 1$, we have*

$$\begin{aligned} &k\mathbb{P}\big(\#\Phi_{Q_\rho}^\eta(\tau) = k\big) - \lambda_d(Q_\rho) \cdot \mathbb{P}\big(\#\Phi_{Q_\rho}^{\eta\cup\{0\}}(\tau) = k, g^{\eta\cup\{0\}}(0) > v_\rho(\tau)\big) \\ &\quad = o\big(\lambda_d(Q_\rho) \cdot \rho^{-1}\big). \end{aligned} \tag{3.6}$$

Notice that the previous proposition is trivial if we replace $o(\lambda_d(Q_\rho) \cdot \rho^{-1})$ by $O(\lambda_d(Q_\rho) \cdot \rho^{-1})$ in (3.6). Indeed, for each $k \geq 1$, we have

$$\mathbb{P}\big(\#\Phi_{Q_\rho}^\eta(\tau) = k\big) = O\big(\lambda_d(Q_\rho) \cdot \rho^{-1}\big)$$

and

$$\mathbb{P}\big(\#\Phi_{Q_\rho}^{\eta\cup\{0\}}(\tau)=k, g^{\eta\cup\{0\}}(0)>v_\rho(\tau)\big)=O\big(\rho^{-1}\big).$$

The main difficulty is to prove that the left-hand side in (3.6) is negligible compared to $\lambda_d(Q_\rho)\cdot\rho^{-1}$, which constitutes the main ingredient to prove Theorem 4.

Since q_ρ is any function such that (3.2) holds, we can take in practice $q_\rho = (\log\log\rho)^{\log\log\rho}$. Actually, we think that Proposition 3 remains true when $q_\rho = \log\rho$, which is slightly more efficient for simulating estimators of θ and p_k.

In what follows, we write $M_B^\chi := \max_{x\in\chi\cap B} g^\chi(x)$ for any $\chi\in\mathcal{F}_{lf}$ and for any $B\subset\mathbb{R}^d$. When $\chi\cap B=\varnothing$, we take $M_B^\chi := -\infty$.

PROOF. We split the proof into two cases: $k=1$ and $k\geq 2$. These two cases share similar arguments. We begin with the case $k=1$ because it is the simplest one.

Case $k=1$. To deal with, we first give an integral representation of the left-hand side of (3.6). Since η is stationary and since g is translation-invariant, we obtain from the Slivnyak–Mecke formula that

$$\mathbb{P}\big(\#\Phi_{Q_\rho}^{\eta}(\tau)=1\big)=\mathbb{E}\Big[\sum_{x\in\eta\cap Q_\rho}\mathbb{I}_{g^\eta(x)>v_\rho(\tau)}\mathbb{I}_{M^\eta_{Q_\rho\setminus\{x\}}\leq v_\rho(\tau)}\Big]=\int_{Q_\rho}p_x\,\mathrm{d}x,$$

where, for any $x\in\mathbb{R}^d$, we let

$$p_x := \mathbb{P}\big(g^{\eta\cup\{0\}}(0)>v_\rho(\tau), M^{\eta\cup\{0\}}_{(Q_\rho-x)\setminus\{0\}}\leq v_\rho(\tau)\big).$$

In particular, we have $\mathbb{P}(\#\Phi_{Q_\rho}^{\eta\cup\{0\}}(\tau)=1, g^{\eta\cup\{0\}}(0)>v_\rho(\tau))=p_0$. Integrating p_0 over Q_ρ, we obtain

$$\begin{aligned}&\mathbb{P}\big(\#\Phi_{Q_\rho}^{\eta}(\tau)=1\big)-\lambda_d(Q_\rho)\cdot\mathbb{P}\big(\#\Phi_{Q_\rho}^{\eta\cup\{0\}}(\tau)=1, g^{\eta\cup\{0\}}(0)>v_\rho(\tau)\big)\\&\quad=\int_{Q_\rho}(p_x-p_0)\,\mathrm{d}x.\end{aligned}\tag{3.7}$$

Now, we provide an upper bound for the integrand in (3.7). To do it, let $x\in Q_\rho$ be fixed. Then

$$\begin{aligned}|p_x-p_0|\leq{}&\mathbb{P}\big(g^{\eta\cup\{0\}}(0)>v_\rho(\tau), M^{\eta\cup\{0\}}_{Q_\rho\setminus(Q_\rho-x)}>v_\rho(\tau)\big)\\&+\mathbb{P}\big(g^{\eta\cup\{0\}}(0)>v_\rho(\tau), M^{\eta\cup\{0\}}_{(Q_\rho-x)\setminus Q_\rho}>v_\rho(\tau)\big).\end{aligned}\tag{3.8}$$

To bound the two probabilities of the right-hand side in the above equation, we proceed in a similar way. Each time, we discuss whether the events appearing in these probabilities are, or are not, independent conditional on the event $\mathscr{A}_\rho$. To deal with the first probability, we notice that

$$\big\{g^{\eta\cup\{0\}}(0)>v_\rho(\tau)\big\}\in\Sigma_{\{0\}}^{\eta\cup\{0\}}$$

and

$$\{M^{\eta\cup\{0\}}_{Q_\rho\setminus(Q_\rho-x)} > v_\rho(\tau)\} \in \Sigma^{\eta\cup\{0\}}_{Q_\rho\setminus(Q_\rho-x)}.$$

We now discuss two possibilities:

(i) If $d(\mathcal{I}(\{0\}), \mathcal{I}(Q_\rho \setminus (Q_\rho - x))) > D$ then, according to Lemma 1, conditional on $\mathscr{A}_\rho$ the events $\{g^{\eta\cup\{0\}}(0) > v_\rho(\tau)\}$ and $\{M^{\eta\cup\{0\}}_{Q_\rho\setminus(Q_\rho-x)} > v_\rho(\tau)\}$ are independent. Hence

$$\begin{aligned}
&\mathbb{P}\big(g^{\eta\cup\{0\}}(0) > v_\rho(\tau), M^{\eta\cup\{0\}}_{Q_\rho\setminus(Q_\rho-x)} > v_\rho(\tau) | \mathscr{A}_\rho\big)\mathbb{I}_{d(\mathcal{I}(\{0\}),\mathcal{I}(Q_\rho\setminus(Q_\rho-x)))>D}\\
&\quad\le \mathbb{P}\big(g^{\eta\cup\{0\}}(0) > v_\rho(\tau)|\mathscr{A}_\rho\big)\cdot\mathbb{P}\big(M^{\eta\cup\{0\}}_{Q_\rho\setminus(Q_\rho-x)} > v_\rho(\tau)|\mathscr{A}_\rho\big).
\end{aligned}$$

Besides, we know that $\mathbb{P}(g^{\eta\cup\{0\}}(0) > v_\rho(\tau)) \le c\cdot\rho^{-1}$. Moreover, it results from the Slivnyak–Mecke formula that

$$\begin{aligned}
&\mathbb{P}\big(M^{\eta\cup\{0\}}_{Q_\rho\setminus(Q_\rho-x)} > v_\rho(\tau)\big)\\
&\quad= \mathbb{P}\big(\exists z \in \big(Q_\rho \setminus (Q_\rho - x)\big) \cap \eta : g^{\eta\cup\{0\}}(z) > v_\rho(\tau)\big)\\
&\quad\le \mathbb{E}\Big[\sum_{z\in\eta\cap Q_\rho} \mathbb{I}_{g^{\eta\cup\{0\}}(z)>v_\rho(\tau)}\Big]\\
&\quad= \int_{Q_\rho} \mathbb{P}\big(g^{\eta\cup\{0,z\}}(z) > v_\rho(\tau)\big)\,\mathrm{d}z\\
&\quad\le c\cdot\lambda_d(Q_\rho)\rho^{-1},
\end{aligned}$$

where the last line is a consequence of Condition (C). This implies that

$$\begin{aligned}
&\mathbb{P}\big(g^{\eta\cup\{0\}}(0) > v_\rho(\tau), M^{\eta\cup\{0\}}_{Q_\rho\setminus(Q_\rho-x)} > v_\rho(\tau)\big)\mathbb{I}_{d(\mathcal{I}(\{0\}),\mathcal{I}(Q_\rho\setminus(Q_\rho-x)))>D}\\
&\quad\le c\cdot\lambda_d(Q_\rho)\rho^{-2}.
\end{aligned}$$

(ii) If $d(\mathcal{I}(\{0\}), \mathcal{I}(Q_\rho \setminus (Q_\rho - x))) \le D$, then $\delta(0, Q_\rho \setminus (Q_\rho - x)) \le c\cdot(\lambda_d(C_\rho))^{1/d}$, where $\delta(\cdot,\cdot)$ is defined on page 3295. Besides, we know that $\mathbb{P}(g^{\eta\cup\{0\}}(0) > v_\rho(\tau)) \le c\cdot\rho^{-1}$. This implies that

$$\begin{aligned}
&\mathbb{P}\big(g^{\eta\cup\{0\}}(0) > v_\rho(\tau), M^{\eta\cup\{0\}}_{Q_\rho\setminus(Q_\rho-x)} > v_\rho(\tau)\big)\mathbb{I}_{d(\mathcal{I}(\{0\}),\mathcal{I}(Q_\rho\setminus(Q_\rho-x)))\le D}\\
&\quad\le c\cdot\rho^{-1}\mathbb{I}_{\delta(0,Q_\rho\setminus(Q_\rho-x))\le c\cdot(\lambda_d(C_\rho))^{1/d}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{P}\big(g^{\eta\cup\{0\}}(0) > v_\rho(\tau), M^{\eta\cup\{0\}}_{Q_\rho\setminus(Q_\rho-x)} > v_\rho(\tau)\big)\\
&\quad\le c\cdot\lambda_d(Q_\rho)\rho^{-2} + c\cdot\rho^{-1}\mathbb{I}_{\delta(0,Q_\rho\setminus(Q_\rho-x))\le c\cdot(\lambda_d(C_\rho))^{1/d}}.
\end{aligned}$$

Proceeding along the same lines as above, we also get

$$\begin{aligned}&\mathbb{P}\big(g^{\eta\cup\{0\}}(0) > v_\rho(\tau), M^{\eta\cup\{0\}}_{(Q_\rho - x)\setminus Q_\rho} > v_\rho(\tau)\big)\\&\qquad \le c\cdot\lambda_d(Q_\rho)\rho^{-2} + c\cdot\rho^{-1}\mathbb{I}_{\delta(0,(Q_\rho-x)\setminus Q_\rho)\le c\cdot(\lambda_d(C_\rho))^{1/d}}.\end{aligned}$$

We can now conclude the case $k=1$. Indeed, by integrating over $x \in Q_\rho$, it follows from the above inequalities and (3.8) that

$$\begin{aligned}&\left|\int_{Q_\rho}(p_x - p_0)\,\mathrm{d}x\right|\\&\quad\le c\cdot\lambda_d(Q_\rho)^2\cdot\rho^{-2}\\&\qquad + c\cdot\rho^{-1}\cdot\lambda_d\big(\{x\in Q_\rho : \delta\big(0, Q_\rho\setminus(Q_\rho - x)\big)\le c\cdot\lambda_d(C_\rho)^{1/d}\}\big)\\&\qquad + c\cdot\rho^{-1}\cdot\lambda_d\big(\{x\in Q_\rho : \delta\big(0, (Q_\rho - x)\setminus Q_\rho\big)\le c\cdot\lambda_d(C_\rho)^{1/d}\}\big).\end{aligned}$$

We can easily prove that for each $x \in Q_\rho$, such that $\delta(0, Q_\rho\setminus(Q_\rho - x))$ is lower than $c\cdot\lambda_d(C_\rho)^{1/d}$, we have $x \in Q_\rho\setminus(Q_\rho \ominus c^{1/d}\cdot C_\rho)$. Hence

$$\begin{aligned}&\lambda_d\big(\{x\in Q_\rho : \delta\big(0, Q_\rho\setminus(Q_\rho - x)\big)\le c\cdot\lambda_d(C_\rho)^{1/d}\}\big)\\&\quad\le\lambda_d\big(Q_\rho\setminus\big(Q_\rho\ominus c^{1/d}\cdot C_\rho\big)\big) = O\big(\lambda_d(Q_\rho)^{(d-1)/d}\cdot\lambda_d(C_\rho)^{1/d}\big).\end{aligned}\tag{3.9}$$

Moreover, for ρ large enough, we have

$$\lambda_d\big(\{x\in Q_\rho : \delta\big(0, (Q_\rho - x)\setminus Q_\rho\big)\le c\cdot\lambda_d(C_\rho)^{1/d}\}\big) = 0\tag{3.10}$$

since for any $x\in\mathbb{R}^d$, we have $\delta(0,(Q_\rho - x)\setminus Q_\rho)\ge(\lambda_d(Q_\rho))^{1/d}$ and $\lambda_d(C_\rho) = o(\lambda_d(Q_\rho))$. According to (3.7), this shows that

$$\begin{aligned}&\big|\mathbb{P}\big(\#\Phi^\eta_{Q_\rho}(\tau)=1\big) - \lambda_d(Q_\rho)\cdot\mathbb{P}\big(\#\Phi^{\eta\cup\{0\}}_{Q_\rho}(\tau)=1, g^{\eta\cup\{0\}}(0) > v_\rho(\tau)\big)\big|\\&\quad\le c\cdot\lambda_d(Q_\rho)^2\cdot\rho^{-2} + c\cdot\rho^{-1}\lambda_d(Q_\rho)^{\frac{d-1}{d}}\lambda_d(C_\rho)^{1/d}.\end{aligned}$$

The right-hand side equals $o(\lambda_d(Q_\rho)\rho^{-1})$ because, according to (3.2) and (3.5), the quantities $\lambda_d(Q_\rho)\cdot\rho^{-1}$ and $\lambda_d(Q_\rho)^{-1/d}\cdot\lambda_d(C_\rho)^{1/d}$ converge to 0 as ρ goes to infinity. This concludes the proof of the case $k=1$.

Case $k\ge 2$. To deal with the case $k\ge 2$, we also give an integral representation of the left-hand side of (3.6). According to the Slivnyak–Mecke formula, we have

$$\begin{aligned}\mathbb{P}\big(\#\Phi^\eta_{Q_\rho}(\tau)=k\big) &= \mathbb{E}\Big[\sum_{\{x_{1:k}\}\subset\eta\cap Q_\rho}\mathbb{I}_{g^\eta(x_{1:k})>v_\rho(\tau)}\mathbb{I}_{M^\eta_{Q_\rho\setminus\{x_{1:k}\}}\le v_\rho(\tau)}\Big]\\&= \frac{1}{k!}\int_{Q_\rho}\int_{(Q_\rho - x)^{k-1}} p_x(y_{2:k})\,\mathrm{d}y_{2:k}\,\mathrm{d}x,\end{aligned}$$

where, for any $x\in\mathbb{R}^d$ and for any $y_{2:k}\in\mathbb{R}^{(k-1)d}$, we write

$$p_x(y_{2:k}) := \mathbb{P}\big(g^{\eta\cup\{0,y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau), M^{\eta\cup\{0,y_{2:k}\}}_{(Q_\rho - x)\setminus\{0,y_{2:k}\}}\le v_\rho(\tau)\big).$$

In the same spirit as in the case $k=1$, we can show that

$$\begin{aligned}
&k\mathbb{P}\big(\#\Phi^{\eta}_{Q_\rho}(\tau)=k\big)-\lambda_d(Q_\rho)\cdot\mathbb{P}\big(\#\Phi^{\eta\cup\{0\}}_{Q_\rho}(\tau)=k,\, g^{\eta\cup\{0\}}(0)>v_\rho(\tau)\big)\\
&\quad=\frac{1}{(k-1)!}\int_{Q_\rho}\bigg(\int_{(Q_\rho-x)^{k-1}}p_x(y_{2:k})\,\mathrm{d}y_{2:k}-\int_{Q_\rho^{k-1}}p_0(y_{2:k})\,\mathrm{d}y_{2:k}\bigg)\,\mathrm{d}x.
\end{aligned}$$

Recall that the event $\{g^{\eta\cup\{0,y_{2:k}\}}(0,y_{2:k})>v_\rho(\tau)\}$ means that the cells with nucleus $0,y_2,\dots,y_k$ are simultaneously exceedances in the Voronoi tessellation associated with the point process $\eta\cup\{0,y_{2:k}\}$. As in the case $k=1$, we have to deal with the possibilities where $\{g^{\eta\cup\{0,y_{2:k}\}}(0,y_{2:k})>v_\rho(\tau)\}$ and $\{M^{\eta\cup\{0,y_{2:k}\}}_{(Q_\rho-x)\setminus\{0,y_{2:k}\}}\le v_\rho(\tau)\}$ are, or are not, conditionally independent. However, an additional problem compared to the case $k=1$ has also to be considered. Indeed, when the distance between each pair of nuclei in $0,y_2,\dots,y_k$ is large enough, conditional on the event $\mathscr{A}_\rho$, we also know that the events $\{g^{\eta\cup\{0,y_{2:k}\}}(0)>v_\rho(\tau)\}$, $\{g^{\eta\cup\{0,y_{2:k}\}}(y_2)>v_\rho(\tau)\}$, $\dots$, $\{g^{\eta\cup\{0,y_{2:k}\}}(y_k)>v_\rho(\tau)\}$ are independent (on the opposite, these events are not independent). To overcome this difficulty, we introduce, for each $1\le m\le k$, the following configuration of points:

$$E_m:=\big\{y_{2:k}\in\mathbb{R}^{(k-1)d}: S_0(y_{2:k})\text{ has } m \text{ connected components}\big\},$$

where, for any $y_{2:k}\in\mathbb{R}^{d(k-1)}$, the set $S_0(y_{2:k})\subset\mathbb{R}^d$ is defined as (see Figure 1)

$$S_0(y_{2:k}):=C_\rho\cup\bigcup_{j=2}^{k}(y_j+C_\rho).$$

Hence the set $S_0(y_{2:k})$ is a (finite) union of cubes, with volume $\lambda_d(C_\rho)$ centered at $0,y_2,\dots,y_k$. Splitting $\mathbb{R}^{(k-1)d}$ into the union of $E_1,\dots,E_k$, we have

$$\begin{aligned}
&k\mathbb{P}\big(\#\Phi^{\eta}_{Q_\rho}(\tau)=k\big)-\lambda_d(Q_\rho)\cdot\mathbb{P}\big(\#\Phi^{\eta\cup\{0\}}_{Q_\rho}(\tau)=k,\, g^{\eta\cup\{0\}}(0)>v_\rho(\tau)\big)\\
&\quad=\frac{1}{(k-1)!}\sum_{m=1}^{k}\int_{Q_\rho}P_x[m]\,\mathrm{d}x,
\end{aligned}$$

where, for any $x\in\mathbb{R}^d$, we write

$$P_x[m]:=\int_{(Q_\rho-x)^{k-1}\cap E_m}p_x(y_{2:k})\,\mathrm{d}y_{2:k}-\int_{Q_\rho^{k-1}\cap E_m}p_0(y_{2:k})\,\mathrm{d}y_{2:k}. \tag{3.11}$$

It is enough to prove that $\int_{Q_\rho}P_x[m]\,\mathrm{d}x=o(\lambda_d(Q_\rho)\rho^{-1})$ for any $1\le m\le k$.

Let us first give some heuristics for the approach. The main idea is to consider two subcases. The first one deals with the subcase $m=1$, that is, when $S_0(y_{2:k})$ is a connected space. In that subcase, as described above, the main difficulty is that the events considered in the probabilities $p_x(y_{2:k})$ and $p_0(y_{2:k})$ are not independent because the points $0,y_2,\dots,y_k$ are close to each other. However, we show that $\int_{Q_\rho}P_x[1]\,\mathrm{d}x$ is $o(\lambda_d(Q_\rho)\rho^{-1})$ because the sets $(Q_\rho-x)^{k-1}\cap E_1$ and $Q_\rho^{k-1}\cap$

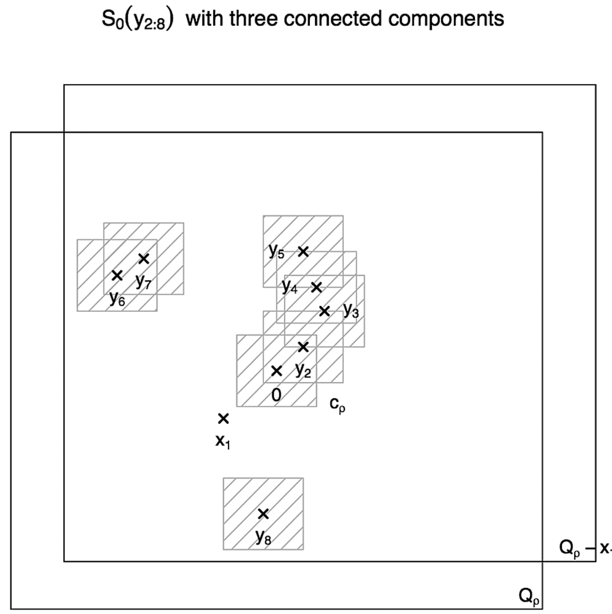

FIG. 1. *A configuration of points $y_{2:8}$, where $S_0(y_{2:k})$ has three connected components.*

E_1 are small in the sense of the Lebesgue measure. On the opposite, when $m \geq 2$, there exist m points among $0, y_2, \ldots, y_k$ which are far enough, in the sense that they belong to m different connected components in $S_0(y_{2:k})$. In that subcase, the probability of the event $\{g^{\eta \cup \{0, y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau)\}$ is lower than $c \cdot \rho^{-m}$ according to Lemma 1 and Condition (C). We now formally deal with the two subcases described above.

Subcase where the set $S_0(y_{2:k})$ has $m = 1$ connected component. In this subcase, we have $S_0(y_{2:k}) \subset (2k+1)C_\rho$. Let $x \in \mathbb{R}^d$ be fixed. First, we notice that

$$\begin{aligned} P_x[1] = & \int_{((Q_\rho - x)\cap Q_\rho)^{k-1} \cap E_1} \big(p_x(y_{2:k}) - p_0(y_{2:k})\big)\, \mathrm{d}y_{2:k} \\ & + \int_{(Q_\rho - x)^{k-1} \cap E_1} p_x(y_{2:k}) \mathbb{I}_{\exists j \leq k : y_j \in Q_\rho^c}\, \mathrm{d}y_{2:k} \\ & - \int_{Q_\rho^{k-1} \cap E_1} p_0(y_{2:k}) \mathbb{I}_{\exists j \leq k : y_j \in (Q_\rho - x)^c}\, \mathrm{d}y_{2:k}. \end{aligned}$$

Actually, the second integral in the above equation equals 0 for ρ large enough because

$$E_1 \cap \{y_{2:k} \in \mathbb{R}^{(k-1)d} : \exists j \leq k \text{ s.t. } y_j \in Q_\rho^c\} = \varnothing.$$

Indeed, if $y_{2:k} \in E_1$, we have $y_j \in (2k+1)C_\rho$ for any $1 \le j \le k$, so that $y_j \in Q_\rho$ for ρ large enough. Hence

$$
\begin{aligned}
P_x[1] = &\int_{((Q_\rho - x)\cap Q_\rho)^{k-1}\cap E_1} \big(p_x(y_{2:k}) - p_0(y_{2:k})\big)\,\mathrm{d}y_{2:k} \\
&- \int_{Q_\rho^{k-1}\cap E_1} p_0(y_{2:k})\mathbb{I}_{\exists j\le k: y_j \in (Q_\rho - x)^c}\,\mathrm{d}y_{2:k}.
\end{aligned}
\tag{3.12}
$$

We provide below bounds for the two terms considered in the right-hand side of the above equation.

Upper bound for the first term in (3.12) In the same spirit as (3.8), we get

$$
\begin{aligned}
&\left|\int_{((Q_\rho - x)\cap Q_\rho)^{k-1}\cap E_1} \big(p_x(y_{2:k}) - p_0(y_{2:k})\big)\,\mathrm{d}y_{2:k}\right| \\
&\quad\le \int_{((Q_\rho - x)\cap Q_\rho)^{k-1}\cap E_1} \mathbb{P}\big(g^{\eta\cup\{0,y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau), \\
&\qquad M^{\eta\cup\{0,y_{2:k}\}}_{Q_\rho\setminus(Q_\rho - x)} > v_\rho(\tau)\big)\,\mathrm{d}y_{2:k} \\
&\qquad + \int_{((Q_\rho - x)\cap Q_\rho)^{k-1}\cap E_1} \mathbb{P}\big(g^{\eta\cup\{0,y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau), \\
&\qquad M^{\eta\cup\{0,y_{2:k}\}}_{(Q_\rho - x)\setminus Q_\rho} > v_\rho(\tau)\big)\,\mathrm{d}y_{2:k}.
\end{aligned}
\tag{3.13}
$$

We begin with the first term of the right-hand side of (3.13). The second term will be dealt in a similar way. To do it, let $y_{2:k} \in ((Q_\rho - x)\cap Q_\rho)^{k-1}\cap E_1$ be fixed. As in the case $k = 1$, we consider two possibilities:

(i) If $d(\mathcal{I}(S_0(y_{2:k})), \mathcal{I}(Q_\rho \setminus (Q_\rho - x))) > D$ then, according to Lemma 1, conditional on $\mathscr{A}_\rho$, we know that the events $\{g^{\eta\cup\{0,y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau)\}$ and $\{M^{\eta\cup\{0,y_{2:k}\}}_{Q_\rho\setminus(Q_\rho - x)} > v_\rho(\tau)\}$ are independent. In the same spirit as in the case $k = 1$, we derive from the Condition (C) that

$$
\begin{aligned}
&\mathbb{P}\big(g^{\eta\cup\{0,y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau), M^{\eta\cup\{0,y_{2:k}\}}_{Q_\rho\setminus(Q_\rho - x)} > v_\rho(\tau)\big) \\
&\quad\times \mathbb{I}_{d(\mathcal{I}(S_0(y_{2:k})),\mathcal{I}(Q_\rho\setminus(Q_\rho - x)))>D} \\
&\quad\le c\cdot\lambda_d(Q_\rho)\rho^{-2}.
\end{aligned}
\tag{3.14}
$$

(ii) If not, we have $\delta(0, Q_\rho \setminus (Q_\rho - x)) \le c\cdot\lambda_d(C_\rho)^{1/d}$, which shows that

$$
\begin{aligned}
&\mathbb{P}\big(g^{\eta\cup\{0,y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau), M^{\eta\cup\{0,y_{2:k}\}}_{Q_\rho\setminus(Q_\rho - x)} > v_\rho(\tau)\big) \\
&\quad\times \mathbb{I}_{d(\mathcal{I}(S_0(y_{2:k})),\mathcal{I}(Q_\rho\setminus(Q_\rho - x)))\le D} \\
&\quad\le c\cdot\rho^{-1}\mathbb{I}_{\delta(0,Q_\rho\setminus(Q_\rho - x))\le c\cdot\lambda_d(C_\rho)^{1/d}}.
\end{aligned}
\tag{3.15}
$$

Integrating over $y_{2:k} \in ((Q_\rho - x) \cap Q_\rho)^{k-1} \cap E_1$, it follows from (3.14) and (3.15) that

$$\begin{aligned}
&\int_{((Q_\rho - x)\cap Q_\rho)^{k-1}\cap E_1} \mathbb{P}\big(g^{\eta\cup\{0,y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau),\\
&\quad M^{\eta\cup\{0,y_{2:k}\}}_{Q_\rho\setminus(Q_\rho - x)} > v_\rho(\tau)\big)\,\mathrm{d}y_{2:k}\\
&\quad \le c\cdot\lambda_{(k-1)d}(E_1)\cdot\big(\lambda_d(Q_\rho)\rho^{-2} + \rho^{-1}\mathbb{I}_{\delta(0,Q_\rho\setminus(Q_\rho - x))\le c\cdot\lambda_d(C_\rho)^{1/d}}\big)\\
&\quad \le c\cdot\lambda_d(C_\rho)^{k-1}\cdot\big(\lambda_d(Q_\rho)\cdot\rho^{-2} + \rho^{-1}\mathbb{I}_{\delta(0,Q_\rho\setminus(Q_\rho - x))\le c\cdot\lambda_d(C_\rho)^{1/d}}\big),
\end{aligned} \tag{3.16}$$

where the last line comes from the fact that $\lambda_{(k-1)d}(E_1) \le c\cdot\lambda_d(C_\rho)^{k-1}$.

Proceeding along the same lines as above, we also obtain

$$\begin{aligned}
&\int_{((Q_\rho - x)\cap Q_\rho)^{k-1}\cap E_1} \mathbb{P}\big(g^{\eta\cup\{0,y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau), M^{\eta\cup\{0,y_{2:k}\}}_{(Q_\rho - x)\setminus Q_\rho} > v_\rho(\tau)\big)\,\mathrm{d}y_{2:k}\\
&\quad \le c\cdot\lambda_d(C_\rho)^{k-1}\cdot\big(\lambda_d(Q_\rho)\cdot\rho^{-2} + \rho^{-1}\mathbb{I}_{\delta(0,(Q_\rho - x)\setminus Q_\rho)\le c\cdot\lambda_d(C_\rho)^{1/d}}\big).
\end{aligned}$$

This together with (3.13) and (3.16) implies that

$$\begin{aligned}
&\left|\int_{((Q_\rho - x)\cap Q_\rho)^{k-1}\cap E_1} \big(p_x(y_{2:k}) - p_0(y_{2:k})\big)\,\mathrm{d}y_{2:k}\right|\\
&\quad \le c\cdot\lambda_d(C_\rho)^{k-1}\cdot\lambda_d(Q_\rho)\cdot\rho^{-2}\\
&\qquad + c\cdot\lambda_d(C_\rho)^{k-1}\cdot\rho^{-1}\cdot(\mathbb{I}_{\delta(0,Q_\rho\setminus(Q_\rho - x))\le c\cdot\lambda_d(C_\rho)^{1/d}}\\
&\qquad + \mathbb{I}_{\delta(0,(Q_\rho - x)\setminus Q_\rho))\le c\cdot\lambda_d(C_\rho)^{1/d}}).
\end{aligned}$$

This deals with the first term of the right-hand side in (3.12).

Upper bound for the second term in (3.12). To deal with this term, we notice that if $E_1 \cap \{y_{2:k} \in Q_\rho^{k-1} : \exists j \le k \text{ s.t. } y_j \in (Q_\rho - x)^c\} \ne \varnothing$, then $\delta(0, Q_\rho \setminus (Q_\rho - x)) \le c\cdot\lambda_d(C_\rho)^{1/d}$ because the diameter of $S_0(y_{2:k})$ is lower than $(2k+1)\cdot\lambda_d(C_\rho)^{1/d}$. Besides, since $p_0(y_{2:k}) \le c\cdot\rho^{-1}$ according to Condition (C), we obtain by integrating over $y_{2:k} \in Q_\rho^{k-1} \cap E_1$ that

$$\begin{aligned}
&\int_{Q_\rho^{k-1}\cap E_1} p_0(y_{2:k})\mathbb{I}_{\exists j\le k: y_j\in(Q_\rho - x)^c}\,\mathrm{d}y_{2:k}\\
&\quad \le c\cdot\lambda_d(C_\rho)^{k-1}\cdot\rho^{-1}\cdot\mathbb{I}_{\delta(0,Q_\rho\setminus(Q_\rho - x))\le c\cdot\lambda_d(C_\rho)^{1/d}}.
\end{aligned}$$

This deals with the second term of the right-hand side in (3.12).

By considering the two upper bounds discussed above and by integrating over $x \in Q_\rho$, we get

$$\begin{aligned}
&\left|\int_{Q_\rho} P_x[1]\,\mathrm{d}x\right|\\
&\quad \le c\cdot\lambda_d(C_\rho)^{k-1}\cdot\lambda_d(Q_\rho)^2\cdot\rho^{-2}
\end{aligned}$$

$$
\begin{aligned}
&+ c \cdot \lambda_d(C_\rho)^{k-1} \cdot \rho^{-1} \cdot \lambda_d\big(\{x \in Q_\rho : \delta\big(0, Q_\rho \setminus (Q_\rho - x)\big) \\
&\le c \cdot \lambda_d(C_\rho)^{1/d}\}\big) \\
&+ c \cdot \lambda_d(C_\rho)^{k-1} \cdot \rho^{-1} \cdot \lambda_d\big(\{x \in Q_\rho : \delta\big(0, (Q_\rho - x) \setminus Q_\rho\big) \\
&\le c \cdot \lambda_d(C_\rho)^{1/d}\}\big).
\end{aligned}
$$

According to (3.9), (3.10) and (3.2), we deduce that

$$
\begin{aligned}
\left|\int_{Q_\rho} P_x[1]\,\mathrm{d}x\right| &\le c \cdot \big(\lambda_d(C_\rho)^{k-1+1/d} \cdot \lambda_d(Q_\rho)^{-1/d}\big) \cdot \big(\lambda_d(Q_\rho) \cdot \rho^{-1}\big) \\
&= o\big(\lambda_d(Q_\rho) \cdot \rho^{-1}\big).
\end{aligned}
$$

This concludes the proof for the subcase $m = 1$.

Subcase where the set $S_0(y_{2:k})$ has m connected components with $m \ge 2$. Assume that $m \ge 2$ and $y_{2:k} \in E_m$.

First, we provide a uniform upper bound for $p_x(y_{2:k})$ which is independent of $y_2, \dots, y_k$, with $x \in Q_\rho$. Since $y_{2:k} \in E_m$, we can subdivide the set $S_0(y_{2:k})$ into its m connected components, say $C_1(y_{2:k}), \dots, C_m(y_{2:k})$. In particular, for each $1 \le l \le m$, the set $C_l(y_{2:k})$ is a connected space and is a finite union of l cubes with volume $\lambda(C_\rho)$ centered at l points among $0, y_2, \dots, y_k$. Let $J_l \subset \{1, \dots, k\}$ be the set of indices j of these points, that is, J_l is such that $C_l(y_{2:k}) = \bigcup_{j \in J_l}(y_j + C_\rho)$, with the convention $y_1 := 0$. Conditional on the event $\mathscr{A}_\rho$, the restrictions of the Voronoi tessellation to these connected components are independent because the distance between each pair of connected components is at least $\lambda_d(C_\rho)^{1/d}$. More precisely, for each $1 \le l < l' \le m$, we have

$$
\begin{aligned}
&\big\{g^{\eta \cup \{0, y_{2:k}\}}(y_{J_l}) > v_\rho(\tau)\big\} \in \Sigma^{\eta \cup \{0, y_{2:k}\}}_{\{y_{J_l}\}} \quad \text{and} \\
&d\big(\mathcal{I}(\{y_{J_l}\}), \mathcal{I}(\{y_{J_{l'}}\})\big) > D,
\end{aligned}
\tag{3.17}
$$

where we recall that $g^{\eta \cup \{0, y_{2:k}\}}(y_{J_l}) > v_\rho(\tau)$ means that $g^{\eta \cup \{0, y_{2:k}\}}(y) > v_\rho(\tau)$ for any $y \in J_l$.

We are now prepared to provide a uniform upper bound for $p_x(y_{2:k})$. Indeed, we first notice that

$$
\begin{aligned}
p_x(y_{2:k}) &\le \mathbb{P}\big(g^{\eta \cup \{0, y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau)\big) \\
&= \mathbb{P}\left(\bigcap_{l=1}^{m}\{g^{\eta \cup \{0, y_{2:k}\}}(y_{J_l}) > v_\rho(\tau)\}\right).
\end{aligned}
$$

According to (3.17) and Lemma 1(i), conditional on the event $\mathscr{A}_\rho$, the events $\{g^{\eta \cup \{0, y_{2:k}\}}(y_{J_l}) > v_\rho(\tau)\}$, $1 \le l \le m$, are independent, that is,

$$
\mathbb{P}\big(g^{\eta \cup \{0, y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau) | \mathscr{A}_\rho\big) = \prod_{l=1}^{m} \mathbb{P}\big(g^{\eta \cup \{0, y_{2:k}\}}(0, y_{J_l}) > v_\rho(\tau) | \mathscr{A}_\rho\big).
$$

Since g satisfies Condition (C), it follows from Lemma 1(ii) that there exists a constant $c > 0$ such that, for any $x \in Q_\rho$ and for any $y_{2:k} \in E_m$, we have $p_x(y_{2:k}) \leq c \cdot \rho^{-m}$.

Now, we derive an upper bound for $|\int_{Q_\rho} P_x[m]\,\mathrm{d}x|$. Indeed, by integrating over $y_{2:k}$, we obtain from (3.11) that

$$\begin{aligned}\big|P_x[m]\big| &\leq c \cdot \rho^{-m} \cdot \sup_{x \in Q_\rho} \lambda_{(k-1)d}\big((Q_\rho - x)^{k-1} \cap E_m\big) \\ &\leq c \cdot \rho^{-m} \cdot \lambda_d(Q_\rho)^{m-1} \cdot \lambda_d(C_\rho)^{k-m}.\end{aligned}$$

Integrating over $x \in Q_\rho$, we get

$$\left|\int_{Q_\rho} P_x[m]\,\mathrm{d}x\right| \leq c \cdot \rho^{-m} \cdot \lambda_d(Q_\rho)^m \cdot \lambda_d(C_\rho)^{k-m}.$$

Since $\rho^{-1} \cdot \lambda_d(Q_\rho) \cdot \lambda_d(C_\rho)^{(k-m)/(m-1)}$ converges to 0, we have $|\int_{Q_\rho} P_x[m]\,\mathrm{d}x| = o(\lambda_d(Q_\rho) \cdot \rho^{-1})$. This concludes the proof of Proposition 3 for any $k \geq 2$. $\square$

3.3. *Our main theorem.* Let g be a geometric characteristic such that (1.6) holds for some $\tau_0 > 0$. According to Leadbetter [19], we say that the extremal index $\theta \in (0, 1]$ of the Poisson–Voronoi tessellation exists if $\lim_{\rho\to\infty} \mathbb{P}(\#\Phi^\eta_{W_\rho}(\tau_0) = 0) = e^{-\theta\tau_0}$. We are now prepared to state our main theorem on the weak convergence of the point process $\Phi^\eta_{W_\rho}(\tau)$ for each $\tau > 0$.

THEOREM 4. *Let g be a geometric characteristic satisfying Condition* (C). *Assume that there exist $\tau_0 > 0$ such that* (1.6) *holds and $(a_k)_{k\geq 1}$ such that $\pi_{k,Q_\rho}(\tau_0) \leq a_k$ for any $k \geq 1$ and any $\rho > 0$, with $\sum_{k=1}^\infty a_k < \infty$.*

(i) *The following assertions are equivalent*:
 (A) *there exists $\theta \in (0, 1]$ such that $\lim_{\rho\to\infty} \mathbb{P}(\#\Phi^\eta_{W_\rho}(\tau_0) = 0) = e^{-\theta\tau_0}$ and the following limit exist $p_k := \lim_{\rho\to\infty} p_{k,Q_\rho}(\tau_0)$ for any $k \geq 1$;*
 (B) *for any $\tau > 0$, the point process $\Phi^\eta_{W_\rho}(\tau)$ converges to a homogeneous compound Poisson point process in $W := [-1/2, 1/2]^d$ with a positive intensity $\nu(\tau)$ and cluster size distributions $\pi_k := \lim_{\rho\to\infty} \pi_{k,Q_\rho}(\tau_0)$, with $k \geq 1$.*

(ii) *If one of the above assertions holds, we have $p_k = k\theta\pi_k$ for any $k \geq 1$ and $\theta = \sum_{k=1}^\infty k^{-1} p_k$.*

Our theorem provides a new characterization of the extremal index. Indeed, this index was previously interpreted as the reciprocal of the mean of the cluster size distribution π. Now, it can be viewed as the mean of the reciprocal of the Palm version of the cluster size. Besides, our new characterization: $\theta = \sum_{k=1}^\infty k^{-1} p_k$ will be extensively used in Section 4 to estimate the extremal indices for various geometric characteristics.

To prove Theorem 4, we associate with the point process $\Phi^{\eta}_{W_\rho}(\tau)$ its Laplace transform $\mathbb{L}_\rho$ defined as follows: for any continuous function $f : W \to \mathbb{R}_+$, we have

$$\mathbb{L}_\rho(f) := \mathbb{E}\Big[\exp\Big(-\sum_{y\in\Phi^{\eta}_{W_\rho}(\tau)} f(y)\Big)\Big].$$

It is well known that the weak convergence of $\Phi^{\eta}_{W_\rho}(\tau)$ is equivalent to the convergence of its Laplace transform for any positive and continuous function f. For a sequence of real random variables, the weak convergence of the point process of exceedances has been investigated in [15] and generalized to random fields on $\mathbb{N}^d_+$ in [12]. We use below the same type of approach. However, we have to take into account specific features of random tessellations in $\mathbb{R}^d$.

The first step consists in showing that exceedances over disjoint subcubes behave asymptotically as if they were independent. To do it, we divide W_ρ into m_ρ^d disjoint subcubes $B[l]$, $l = 1, \dots, m_\rho^d$, with the same volume as Q_ρ, where m_ρ is defined in (3.3).

LEMMA 5. (i) *For any measurable function* $f : W \to \mathbb{R}_+$, *we have*

$$\mathbb{L}_\rho(f) - \prod_{l=1}^{m_\rho^d} \mathbb{E}\Big[\exp\Big(-\sum_{y\in\Phi^{\eta}_{W_\rho}(\tau)\cap\rho^{-1/d}B[l]} f(y)\Big)\Big] \underset{\rho\to\infty}{\longrightarrow} 0.$$

(ii) *Moreover, we have*

$$\mathbb{P}\big(M^{\eta}_{W_\rho} \le v_\rho(\tau)\big) - \prod_{l=1}^{m_\rho^d} \mathbb{P}\big(M^{\eta}_{B[l]} \le v_\rho(\tau)\big) \underset{\rho\to\infty}{\longrightarrow} 0.$$

PROOF. We begin with the first assertion. To apply Lemma 1, we have to consider blocks whose distances from each others are large enough. To do it, we remove a stripe of size C_ρ from the boundary of $B[l]$. More precisely, for any $l \le m_\rho^d$, we define the block $B^\circ[l] := B[l] \ominus C_\rho$. Let

$$L_{\rho,l}(f) = \exp\Big(-\sum_{y\in\Phi^{\eta}_{W_\rho}(\tau)\cap\rho^{-1/d}B[l]} f(y)\Big)$$

and

$$L^\circ_{\rho,l}(f) = \exp\Big(-\sum_{y\in\Phi^{\eta}_{W_\rho}(\tau)\cap\rho^{-1/d}B^\circ[l]} f(y)\Big).$$

We write

$$\mathbb{L}_\rho(f) - \prod_{l=1}^{m_\rho^d} \mathbb{E}[L_{\rho,l}(f)] = \Delta L_{\rho,1}(f) + \Delta L_{\rho,2}(f) + \Delta L_{\rho,3}(f) + \Delta L_{\rho,4}(f),$$

where

$$\Delta L_{\rho,1}(f) = \mathbb{E}\Bigg[\prod_{l=1}^{m_\rho^d} L_{\rho,l}(f)\Bigg] - \mathbb{E}\Bigg[\prod_{l=1}^{m_\rho^d} L^\circ_{\rho,l}(f)\Bigg],$$

$$\Delta L_{\rho,2}(f) = \mathbb{E}\Bigg[\prod_{l=1}^{m_\rho^d} L^\circ_{\rho,l}(f)\Bigg] - \prod_{l=1}^{m_\rho^d} \mathbb{E}[L^\circ_{\rho,l}(f)],$$

$$\Delta L_{\rho,3}(f) = \prod_{l=1}^{m_\rho^d} \mathbb{E}[L^\circ_{\rho,l}(f)] - \prod_{l=1}^{m_\rho^d} \mathbb{E}[L_{\rho,l}(f)],$$

$$\Delta L_{\rho,4}(f) = \mathbb{E}\Bigg[\exp\Bigg(-\sum_{y\in\Phi^\eta_{W_\rho}(\tau)} f(y)\Bigg)\Bigg] - \mathbb{E}\Bigg[\prod_{l=1}^{m_\rho^d} L_{\rho,l}(f)\Bigg].$$

We prove below that each term converges to 0. For the third term, using the fact that $|\prod x_i - \prod y_i| \le \sum |x_i - y_i|$ for $0 \le x_i, y_i \le 1$ and the fact that $|\exp(-x) - \exp(-y)| \le |x - y|$ for all $x, y \ge 0$, we get

$$\begin{aligned}
\big|\Delta L_{\rho,3}(f)\big| &\le m_\rho^d \cdot \sup_{l\le m_\rho^d} \mathbb{E}\Bigg[\sum_{y\in\Phi^\eta_{W_\rho}(\tau)\cap\rho^{-1/d}(B[l]\setminus B^\circ[l])} f(y)\Bigg] \\
&\le m_\rho^d \cdot \sup_{l\le m_\rho^d} \mathbb{E}\Bigg[\sum_{x\in\eta\cap(B[l]\setminus B^\circ[l])} f\big(\rho^{-1/d}x\big)\mathbb{I}_{g^\eta(x)>v_\rho(\tau)}\Bigg] \\
&= c\cdot m_\rho^d \cdot \sup_{l\le m_\rho^d} \int_{B[l]\setminus B^\circ[l]} f\big(\rho^{-1/d}x\big)\mathbb{P}\big(g^{\eta\cup\{x\}}(x) > v_\rho(\tau)\big)\,\mathrm{d}x \\
&\le c\cdot m_\rho^d \cdot \lambda_d\big(B[l]\setminus B^\circ[l]\big)\cdot \mathbb{P}\big(g(\mathcal{C}) > v_\rho(\tau)\big),
\end{aligned}$$

where the third line comes from the Slivnyak–Mecke formula and where the fourth line comes from (1.1) and the fact that f is bounded because it is continuous on the compact set W. Since $m_\rho^d \underset{\rho\to\infty}{\sim} \rho\cdot q_\rho^{-1}\cdot(\log\rho)^{-(1+\varepsilon)}$ and

$$\lambda_d\big(B[l]\setminus B^\circ[l]\big) \le c\cdot q_\rho^{(d-1)/d}\cdot(\log\rho)^{(1+\varepsilon)},$$

we deduce that

$$\big|\Delta L_{\rho,3}(f)\big| = O\big(q_\rho^{-1/d}\big).$$

In the same spirit as above, we prove that $\Delta L_{\rho,1}(f)$ and $\Delta L_{\rho,4}(f)$ converges to 0.

For $\Delta L_{\rho,2}(f)$, we notice that conditional on $\mathscr{A}_\rho$, the random variables considered in the expectations are independent. Then we have

$$\mathbb{E}\left[\prod_{l=1}^{m_\rho^d} L_{\rho,l}^\circ(f)\right] = \prod_{l=1}^{m_\rho^d} \mathbb{E}[L_{\rho,l}^\circ(f)|\mathscr{A}_\rho] \\ + \mathbb{P}(\mathscr{A}_\rho^c)\left(\mathbb{E}\left[\prod_{l=1}^{m_\rho^d} L_{\rho,l}^\circ(f)\middle|\mathscr{A}_\rho^c\right] - \prod_{l=1}^{m_\rho^d} \mathbb{E}[L_{\rho,l}^\circ(f)|\mathscr{A}_\rho]\right).$$

Moreover, noting the fact that

$$\mathbb{E}[L_{\rho,l}^\circ(f)] = \mathbb{E}[L_{\rho,l}^\circ(f)|\mathscr{A}_\rho] + \mathbb{P}(\mathscr{A}_\rho^c)(\mathbb{E}[L_{\rho,l}^\circ(f)|\mathscr{A}_\rho^c] - \mathbb{E}[L_{\rho,l}^\circ(f)|\mathscr{A}_\rho]),$$

we write

$$\prod_{l=1}^{m_\rho^d} \mathbb{E}[L_{\rho,l}^\circ(f)] := \prod_{l=1}^{m_\rho^d} \mathbb{E}[L_{\rho,l}^\circ(f)|\mathscr{A}_\rho] + m_\rho^d \mathbb{P}(\mathscr{A}_\rho^c) H_\rho(f).$$

The term $H_\rho(f)$ appearing in the above equation is such that $|H_\rho(f)| \le c$: this is a consequence of Lemma 1(ii) and the fact that $\mathbb{E}[L_{\rho,l}^\circ(f)|\mathscr{A}_\rho^c]$ and $\mathbb{E}[L_{\rho,l}^\circ(f)|\mathscr{A}_\rho]$ belong to the interval $[0,1]$. By applying again Lemma 1(ii), it follows that $\Delta L_{\rho,2}(f)$ converges to 0. We proceed in a similar way for the proof of the second assertion. $\square$

We now adapt two theorems due to Leadbetter, Lindgren and Rootzén in our context. The following result is an adaptation of Theorem 4.2 in [15] (resp., Proposition 4.2 in [12]) and gives sufficient conditions to derive the convergence of $\Phi_{W_\rho}^\eta(\tau)$ to a homogeneous compound Poisson point process.

PROPOSITION 6. *Assume that* $\mathbb{P}(\#\Phi_{W_\rho}^\eta(\tau_0) = 0) \underset{\rho\to\infty}{\longrightarrow} e^{-\nu}$ *for some* $\tau_0 > 0$ *and* $\nu > 0$. *If* $(\pi_{k,Q_\rho})_{k\ge1}$ *converges to a probability distribution* π *on* $\mathbb{N}_+$, *then* $\Phi_{W_\rho}^\eta(\tau_0)$ *converges in distribution to a homogeneous compound Poisson point process with intensity* ν *and limiting cluster size distribution* π.

The following result adapted from Theorem 5.1 in [15] (resp., Proposition 4.3. in [12]) shows that if $\Phi_{W_\rho}^\eta(\tau_0)$ has a limit for some $\tau_0 > 0$, it has a limit for all $\tau > 0$.

PROPOSITION 7. *Assume that* $\Phi_{W_\rho}^\eta(\tau_0)$ *converges to a homogeneous compound Poisson point process in* W *with intensity* $\nu > 0$ *and cluster size distribution*

π, for some $\tau_0 > 0$. Then $\Phi^{\eta}_{W_\rho}(\tau)$ converges to a homogeneous compound Poisson point process with intensity $\nu \cdot \tau/\tau_0$ and limiting cluster size distribution π, for each $\tau > 0$.

We do not give the proofs of Propositions 6 and 7 since they are readily obtained through [12] substituting Lemma 2.1 by our Lemma 5. We are now prepared to give a proof of Theorem 4.

PROOF OF THEOREM 4. Proof of (i). First we show that (A)⇒(B). By Lemma 5, we have

$$\mathbb{P}\big(M^{\eta}_{W_\rho} \le v_\rho(\tau_0)\big) = \big(\mathbb{P}\big(M^{\eta}_{Q_\rho} \le v_\rho(\tau_0)\big)\big)^{\rho\cdot(\lambda_d(Q_\rho))^{-1}} + o(1).$$

Since $\lim_{\rho\to\infty} \mathbb{P}(M^{\eta}_{W_\rho} \le v_\rho(\tau_0)) = e^{-\theta\tau_0}$ and since $\{M^{\eta}_{Q_\rho} \le v_\rho(\tau_0)\}$ if and only if $\{\#\Phi^{\eta}_{Q_\rho}(\tau_0) = 0\}$, it follows that

$$\mathbb{P}\big(\#\Phi^{\eta}_{Q_\rho}(\tau_0) > 0\big) \underset{\rho\to\infty}{\sim} \frac{\lambda_d(Q_\rho)}{\rho} \cdot \theta\tau_0.$$

This together with Proposition 3 implies that $\pi_k = \lim_{\rho\to\infty} \pi_{k,Q_\rho}(\tau_0)$ exists and $\pi_k = p_k/(k\cdot\theta)$ for any $k \ge 1$. Since $\pi_{k,Q_\rho}(\tau_0) \le a_k$ with $\sum_{k=1}^{\infty} a_k < \infty$, it follows from the dominated convergence theorem that $\pi := (\pi_k)_{k\ge 1}$ is a probability measure on $\mathbb{N}_+$. Applying Proposition 6, we deduce that $\Phi^{\eta}_{W_\rho}(\tau_0)$ converges to a homogeneous compound Poisson point process with intensity $\nu(\tau_0) := \theta\tau_0 > 0$ and cluster size distribution π. This together with Proposition 7 proves Assertion (B).

Second, we show that (B)⇒(A). The fact that the extremal index exists and is positive is a consequence of the fact that

$$\lim_{\rho\to\infty} \mathbb{P}\big(M^{\eta}_{W_\rho}(1) \le v_\rho(\tau_0)\big) = \lim_{\rho\to\infty} \mathbb{P}\big(\#\Phi^{\eta}_{W_\rho}(\tau_0) = 0\big) = e^{-\theta\tau_0},$$

where $\theta := \nu(\tau_0)/\tau_0 \in (0, 1]$. By applying Proposition 3, we show that the limit of $p_k := p_{k,Q_\rho}(\tau_0)$ exists and $p_k = k\theta\pi_k$. This proves Assertion (A).

Proof of (ii). The fact that $p_k = k\theta\pi_k$ is established above. Moreover, we have $\sum_{k=1}^{\infty} k^{-1} p_k = \theta \sum_{k=1}^{\infty} \pi_k = \theta$ since $\pi = (\pi_k)_{k\ge 1}$ is a probability measure. □

4. Numerical illustrations. *Layout*. In this section, we illustrate our main theorem throughout simulations for three geometric characteristics for which the value of the extremal index is known or can be conjectured. For sake of simplicity, we only do our simulations in the particular setting $d = 2$. We provide approximations of $p_1, \dots, p_9$ and of the extremal index by using the fact that $\theta = \sum_{k=1}^{\infty} k^{-1} p_k$ [see Theorem 4(ii)] and we compare this approximation to the theoretical value of θ.

For each geometric characteristic g, we proceed as follows. We take $\tau = 1$ and $\rho = \exp(100)$. In particular, the cube Q_ρ, as defined in (3.4), is approximatively

$$Q_\rho \simeq [-173, 173]^2,$$

by taking $q_\rho = (\log\log\rho)^{\log\log\rho} \simeq 1134$ and $\varepsilon = 0.01$. Then, we compute theoretically $v_\rho(1)$ so that $\rho \cdot \mathbb{P}(g(\mathcal{C}) > v_\rho(1)) \underset{\rho\to\infty}{\longrightarrow} 1$. We simulate 10,000 realizations of independent Poisson–Voronoi tessellations given that the typical cell is an exceedance, that is, $g^{\eta\cup\{0\}}(0) > v_\rho(1)$ (see Lemma 2). This sample of size 10,000 is divided into 100 subsamples of size 100. For each $1 \le i \le 100$ and for each $1 \le k \le 9$, we denote by $\hat{p}_k^{(i)}$ the empirical mean of p_k, that is, the mean number of realizations in which there exist exactly k Voronoi cells with nucleus in $Q_\rho \simeq [-173, 173]$ and such that the geometric characteristic is larger than $v_\rho(1)$.

We summarize our empirical results by box plots associated with the empirical values $(\hat{p}_k^{(i)})_{1\le i\le 100}$. For each geometric characteristic, we explain how we simulate a Poisson–Voronoi tessellation conditional on the fact that $g^{\eta\cup\{0\}}(0) > v_\rho(1)$.

4.1. *Inradius.* For any $x \in \eta \subset \mathbb{R}^d$, we define the so-called inradius of the Voronoi cell $C_\eta(x)$ as

$$r^\eta(x) := r\big(C_\eta(x)\big) := \sup\big\{r \ge 0 : B(x,r) \subset C_\eta(x)\big\},$$

where $B(x,r)$ is the ball centered at x with radius r. The Condition (C) is satisfied since $r^{\eta\cup\{y_{2:k}\}}(x) \le r^{\eta\cup\{0\}}(x)$ for any $x \in \eta \cup \{0\}$ and for any $y_{2:k} \in \mathbb{R}^{d(k-1)}$. The distribution of $r(\mathcal{C})$, where $r(\mathcal{C}) = r^{\eta\cup\{0\}}(0)$ is the typical inradius, is given by $\mathbb{P}(r(\mathcal{C}) > v) = \mathbb{P}(\eta \cap B(0,2v) \ne \varnothing) = e^{-2^d\kappa_d v^d}$ for each $v \ge 0$. Hence, for any $\tau > 0$, we have $\rho \cdot \mathbb{P}(r(\mathcal{C}) > v_\rho(\tau)) = \tau$, when

$$v_\rho(\tau) := 2^{-1}\kappa_d^{-1/d}\big(\log\big(\rho\tau^{-1}\big)\big)^{1/d}.$$

Moreover, it is proved in [7] that

$$\mathbb{P}\Big(\max_{x\in\eta\cap W_\rho} r^\eta(x) \le v_\rho(\tau)\Big) \underset{\rho\to\infty}{\longrightarrow} e^{-\tau}.$$

Actually, the convergence was established for a fixed window and for a Poisson point process such that the intensity goes to infinity. By scaling property of the Poisson point process, the result can be rewritten as above for a fixed intensity and for a window W_ρ as ρ goes to infinity. Therefore, we deduce that the extremal index of the inradius of a Poisson–Voronoi tessellation exists and is equal to $\theta = 1$. Actually, according to Theorem 2 in [10], the point process of exceedances $\Phi_{W_\rho}^\eta(\tau)$ converges to a simple Poisson point process of intensity τ in W. In particular, the distributions π and p are equal to the Dirac measure at 1.

Now, we explain how we evaluate by simulation the value of the extremal index and the distribution p when $d = 2$. It is known (see, e.g., [20]) that for each $v \ge 0$, we have

$$\big(\eta \cup \{0\} | r^{\eta\cup\{0\}}(0) = v\big) \overset{\mathcal{D}}{=} \eta_{B(0,2v)^c} \cup \big\{(2v)X_0\big\} \cup \{0\},$$

where $\eta_{B(0,2v)^c}$ is a Poisson point process of intensity measure $\mathbb{I}_{\{x\in B(0,2v)^c\}}\,\mathrm{d}x$ and where X_0 is a random point uniformly distributed on the boundary of $B(0,1)$.

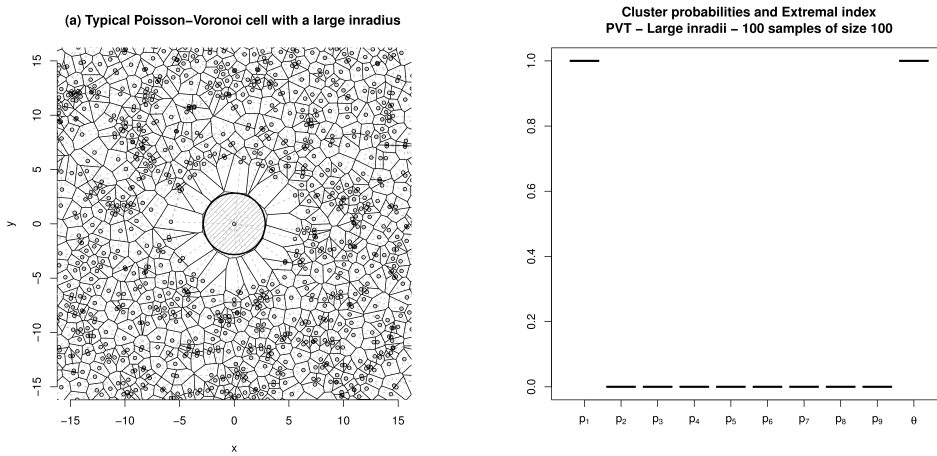

FIG. 2. *Large inradius for a Poisson–Voronoi tessellation.*

Hence, to simulate a Poisson–Voronoi tessellation provided that $r^{\eta\cup\{0\}}(0) > v_{\exp(100)}(1) \simeq 2.82$, we first simulate a random variable r with distribution given by $\mathbb{P}(r > v) = e^{-4\pi v^2}$, conditional on the fact that $r > 2.82$. Then we generate a Poisson–Voronoi tessellation associated with the point process $\eta_{B(0,2r)^c} \cup \{(2r)X_0\} \cup \{0\}$.

On the left part of Figure 2, we provide a simulation of a Poisson–Voronoi tessellation given that $r^{\eta\cup\{0\}}(0) > 2.82$. We notice that the typical cell has a shape which tends to be circular. Actually, such an observation is related to the D. G. Kendall's conjecture which claims that the shape of the typical Poisson–Voronoi cell in $\mathbb{R}^d$, given that the volume of the cell goes to infinity, tends a.s. to a ball in $\mathbb{R}^d$. Many results concerning typical cells with a large geometric characteristic can be found in [8] and [16]. On the left part of Figure 2, we also notice that there is no cell with a large inradius, excepted the typical cell. This confirms that the cluster of exceedances are of size 1, that is, $p_1 = 1$ and $\theta = 1$. The right part of Figure 2 provides the box plots of the empirical distributions. In particular, for all simulations, we notice that there is always exactly one cell with a large inradius.

4.2. *Reciprocal of the inradius.* In this example, we consider the large values of the reciprocal of the inradii for a Poisson–Voronoi tessellation in $\mathbb{R}^d$. Equivalently, this consists of the small values of the inradii. Since $\mathbb{P}(r(\mathcal{C})^{-1} > v) = 1 - e^{-2^d\kappa_d v^d}$, we have $\rho \cdot \mathbb{P}(r(\mathcal{C})^{-1} > v_\rho(\tau)) \underset{\rho\to\infty}{\longrightarrow} \tau$, when

$$v_\rho(\tau) := 2^{-1}(\kappa_d\rho)^{-1/d}\tau^{1/d}.$$

Note that Condition (C) is not satisfied. But a slight modification of the proof of Proposition 3 shows that Theorem 4 remains true if we replace the inequality (2.2)

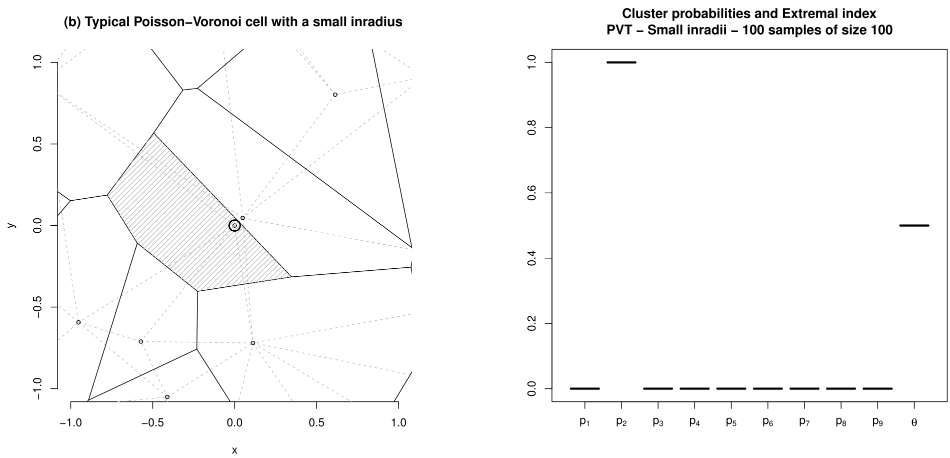

FIG. 3. *Small inradius for a Poisson–Voronoi tessellation.*

of Condition (C) by the inequality

$$\mathbb{P}\big(g^{\eta\cup\{0,y_{2:k}\}}(z) > v_\rho(\tau)\big) \le c \cdot \max\big\{\rho^{-1}, \mathbb{I}_{\delta(z,\{0,y_{2:k}\}\setminus\{z\})\le c\cdot\rho^{-1/d}}\big\},$$

which is satisfied for the reciprocal of the inradius.

Moreover, according to [7], we know that

$$\mathbb{P}\Big(\min_{x\in\eta\cap W_\rho} r^\eta(x) \ge v_\rho(\tau)\Big) \underset{\rho\to\infty}{\longrightarrow} e^{-\tau/2}.$$

We deduce that the extremal index of the reciprocal of the inradius of a Poisson–Voronoi tessellation exists and equals $\theta = 1/2$.

As in Section 4.1, we can easily simulate a Poisson–Voronoi tessellation in $\mathbb{R}^2$, conditional on the fact that $r^{\eta\cup\{0\}}(0) < v_{\exp(100)}(1) \simeq 5.44 \cdot 10^{-23}$. The left part of Figure 3 provides a realization of a Poisson–Voronoi tessellation when $r^{\eta\cup\{0\}}(0) < v_{\exp(4)}(1) \simeq 0.0381$ [here, we have taken the threshold $v_{\exp(4)}(1)$ instead of $v_{\exp(100)}(1)$ for convenience]. The fact that $\theta = 1/2$ can be explained by a trivial heuristic argument: if a cell minimizes the inradius, one of its neighbors has to do the same (see also the left part of Figure 3). Moreover, we can easily prove that the probability that there is more than one such a cell is negligible. Therefore, clusters are necessarily of size 2, that is, $p_2 = 1$. The right part of Figure 3 provides the box plots of the empirical distributions. In particular, for all simulations, we notice that there are always exactly two cells with a small inradius.

4.3. *Circumradius.* For $x \in \eta \subset \mathbb{R}^2$, we define the so-called circumradius of $C_\eta(x)$ as

$$R^\eta(x) := R\big(C_\eta(x)\big) := \inf\big\{r \ge 0 : B(x,r) \supset C_\eta(x)\big\}.$$

The Condition (C) is satisfied since $R^{\eta\cup\{y_{2:k}\}}(x) \le R^{\eta\cup\{0\}}(x)$ for any $x \in \eta \cup \{0\}$ and for any $y_{2:k} \in \mathbb{R}^{d(k-1)}$. According to Theorem 3 in [6], we know that

$$2ve^{-v} \le \mathbb{P}(\pi R(\mathcal{C})^2 > v) \le 4ve^{-v}$$

for each $v \ge 0.337$. Actually, simulations suggest that the upper bound above is the order of $\mathbb{P}(\pi R(\mathcal{C})^2 > v)$ as v goes to infinity (see Table 1 in [6]). If we assume that $\mathbb{P}(\pi R(\mathcal{C})^2 > v) \underset{\rho\to\infty}{\sim} ave^{-v}$ for some $2 \le a \le 4$, we have $\rho \cdot \mathbb{P}(R(\mathcal{C}) > v_\rho(\tau)) \underset{\rho\to\infty}{\longrightarrow} \tau$, when

$$v_\rho(\tau) := \pi^{-1/2}\bigl(\log(a\rho \log \rho\tau^{-1})\bigr)^{1/2}.$$

Thanks to (2.c) in [7], we know that

$$\mathbb{P}\Bigl(\max_{x\in\eta\cap W_\rho} R^{\eta}(x) \le v_\rho(\tau)\Bigr) \underset{\rho\to\infty}{\longrightarrow} e^{-\tau/a}.$$

Hence, provided that $\mathbb{P}(\pi R(\mathcal{C})^2 > v) \underset{\rho\to\infty}{\sim} ave^{-v}$, the extremal index of the maximum of circumradius of a planar Poisson–Voronoi tessellation exists and should be equal to $\theta = 1/a$.

Now, we explain how we evaluate by simulation the value of the extremal index and the distribution p. According to Lemma 1 in [13], we know that $R^{\eta\cup\{0\}}(0) > v$ if and only if there exists a disk of radius v containing the origin on its boundary and no particle inside. Without loss of generality, we can assume that the disk, that contains the origin on its boundary and no particle inside, has its center on the x-axis, since the Poisson point process is isotropic. Hence we proceed as follows. First, we simulate a random variable R_b, with distribution such that $\mathbb{P}(\pi R_b^2 > v) \underset{v\to\infty}{\sim} bve^{-v}$, with $b = 4$, given that $R_b > v_{\exp(100)}(1) \simeq 5.81$. We have taken $b = 4$ since we should have $\mathbb{P}(\pi R(\mathcal{C})^2 > v) \underset{v\to\infty}{\sim} 4ve^{-v}$ as suggested in the simulations in [6]. However, this choice is arbitrary and does not have influence on the final result since the conditional distribution of R_b does not depend on b for high thresholds. Then we generate a Voronoi tessellation induced by the point process $\eta_{B((R,0),R)^c} \cup \{0\}$, where $\eta_{B((R,0),R)^c}$ is a Poisson point process of intensity measure $\mathbb{I}_{x\in B((R,0),R)^c}\,\mathrm{d}x$.

On the left part of Figure 4, we provide a simulation of the Palm version of the Poisson–Voronoi tessellation, given that $R(\mathcal{C}) > v_{\exp(100)}(1) \simeq 5.81$. We notice that the typical cell is very elongated and that the same fact holds for a large number of its connected cells. In particular, the size of a cluster of exceedances is random. On the right part of Figure 4, we provide the box plots of the empirical distributions. This time, the empirical distributions of the cluster size probabilities are not degenerated for $k = 3, \dots, 9$, and their interquartile ranges are quite large for $k = 3, 4, 5$. We also notice that the empirical value of the extremal index is very concentrated around a value close to $1/4$. This confirms that if a exists, it should be close to 4.

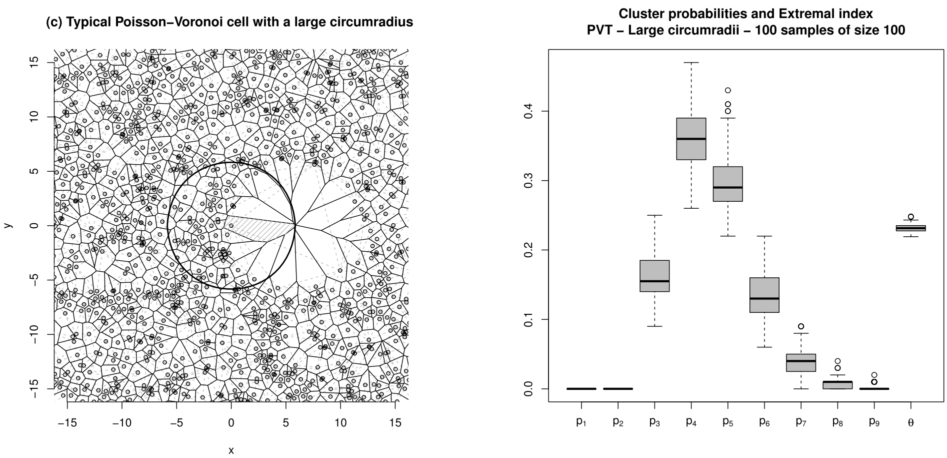

FIG. 4. *Large circumradius for a Poisson–Voronoi tessellation.*

5. The case of the Poisson–Delaunay tessellation. *The Poisson–Delaunay tessellation*. Let $\chi \in \mathcal{F}_{lf}$ be a locally finite subset of $\mathbb{R}^d$ such that each subset of size $n < d+1$ of points are affinely independent and no $d+2$ points lie on a sphere. If two points $x, y \in \chi$ are Voronoi neighbors, that is, $C_\chi(x) \cap C_\chi(y) \neq \varnothing$, we connect these two points by an edge. The family of these edges defines a partition of $\mathbb{R}^d$ into simplices which is the so-called Delaunay tessellation. Another useful characterization of the Delaunay tessellation is the following: a simplex associated with $d+1$ points of χ is a Delaunay simplex if and only if its circumball contains no point of χ in its interior. Delaunay tessellations are very popular structures in computational geometry [1] and are extensively used in many areas such as surface reconstruction [9] or mesh generation [11].

For each cell C of the Delaunay tessellation, the nucleus $z(C)$ is defined as the center of the circumball of C. The set of this nuclei is denoted by $Z(\chi)$. Besides, for each $z \in Z(\chi)$, we denote by $C(z)$ the Delaunay cell whose center of its circumball is z.

When $\chi = \eta$ is a homogeneous Poisson point process, the family of these cells is the so-called *Poisson–Delaunay tessellation*. If we denote by γ_η the intensity of η, then the intensity of the Poisson–Delaunay tessellation is $\gamma_{Z(\eta)} = \beta_d^{-1} \cdot \gamma_\eta$ (see, e.g., Theorem 10.2.8 and equation (10.31) in [29]), where

$$\beta_d := \frac{d^2(d+1)}{2^{d+1}\pi^{\frac{d-1}{2}}} \frac{\Gamma(\frac{d^2}{2})}{\Gamma(\frac{d^2+1}{2})} \left[\frac{\Gamma(\frac{d+1}{2})}{\Gamma(1+\frac{d}{2})} \right]^d.$$

In particular, if $d = 2$, we have $\beta_2 = 1/2$. In the rest of the paper, we assume that $\gamma_\eta = \beta_d$ to ensure that $\gamma_{Z(\eta)} = 1$.

The typical cell of a Poisson–Delaunay tessellation can be made explicit as follows. Let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ be the unit sphere of $\mathbb{R}^d$ and, for $u_{1:d+1} \in$

$(\mathbb{S}^{d-1})^{d+1}$, let $\Delta(u_{1:d+1})$ be the convex hull of $\{u_1,\dots,u_{d+1}\}$. According to Miles (see, e.g., Theorem 10.4.4 in [29]), for any bounded measurable function $f:\mathcal{K}_d\to\mathbb{R}$, we have

$$\begin{aligned}\mathbb{E}[f(\mathcal{C})] := a_d\gamma_\eta^d \int_{\mathbb{R}_+}\int_{\mathbb{S}^{d-1}}\cdots\int_{\mathbb{S}^{d-1}} a(u_{1:d+1}) r^{d^2-1} e^{-\gamma_\eta\kappa_d r^d} \\ \times f\big(\Delta(ru_{1:d+1})\big)\sigma(\mathrm{d}u_{1:d+1})\,\mathrm{d}r,\end{aligned}\tag{5.1}$$

where $a_d := \beta_d/(d+1)$ and $a(u_{1:d+1}) := \lambda_d(\Delta(u_{1:d+1}))$. The measure $\sigma(\mathrm{d}u)$ is the uniform distribution on $\mathbb{S}^{d-1}$ with normalization $\sigma(\mathbb{S}^{d-1}) = \omega_{d-1}$, where $\omega_{d-1} := d\kappa_d$ is the area of the unit sphere and $\sigma(\mathrm{d}u_{1:d+1}) := \bigotimes_{i=1}^{d+1}\sigma(\mathrm{d}u_i)$. Hence the typical cell $\mathcal{C}$ has the same distribution as the random closed set $\Delta(RU_{1:d+1})$, where $R \geq 0$ and $U_{1:d+1} \in (\mathbb{S}^{d-1})^{d+1}$ are two independent random variables whose the distributions are given by

$$\mathbb{P}(R \leq s) = \frac{d(\gamma_\eta\kappa_d)^d}{\Gamma(d)}\int_0^s r^{d^2-1}e^{-\gamma_\eta\kappa_d r^d}\,\mathrm{d}r$$

and

$$\mathbb{P}(U_{1:d+1}\in S) = \frac{a_d\Gamma(d)}{d\kappa_d^d}\int_{\mathbb{S}^{d-1}}\cdots\int_{\mathbb{S}^{d-1}} a(u_{1:d+1})\mathbb{I}_{u_{1:d+1}\in S}\sigma(\mathrm{d}u_{1:d+1})$$

for any $s \geq 0$ and for any Borel subset $S \subset (\mathbb{S}^{d-1})^{d+1}$.

The extremes of the Poisson–Delaunay tessellation. Let g be a geometric characteristic such that (1.6) holds. As for a Poisson–Voronoi tessellation, we consider the point process of normalized exceedances, say

$$\Psi^\eta(\tau) := \rho^{-1/d}\cdot\big\{z \in Z(\eta) : g\big(C(z)\big) > v_\rho(\tau)\big\}.$$

For any Borel subset $B \subset \mathbb{R}^d$, we write $\Psi^\eta_B(\tau) := \Psi^\eta(\tau)\cap(\rho^{-1/d}B)$. We also let $\Psi^{\eta,0}(\tau)$ be the Palm version of $\Psi^\eta(\tau)$ and $\Psi^{\eta,0}_B(\tau) = \Psi^{\eta,0}(\tau)\cap B$. In the rest of the paper, the quantity $p_{k,B}(\tau)$ refers to as the probability that there exist k exceedance cells in B conditional on the fact that the typical cell is an exceedance, that is, $p_{k,B}(\tau) := \mathbb{P}(\#\psi^{\eta,0}_B(\tau) = k)$. In the same spirit as Lemma 2, we provide below an explicit characterization of this probability.

PROPOSITION 8. *Let $\mathcal{A}$ be a Borel subset in $\mathcal{F}_{lf}$. Then*

$$\mathbb{P}\big(\Psi^{\eta,0}(\tau)\in\mathcal{A}\big) = \mathbb{P}\big(\Psi^{\eta_{\mathbb{R}^d\setminus B(0,R)}\cup\{RU_{1:d+1}\}}(\tau)\in\mathcal{A} \,|\, g\big(\Delta(RU_{1:d+1})\big) > v_\rho(\tau)\big).$$

Therefore, for any $B \subset \mathbb{R}^d$,

$$p_{k,B}(\tau) = \mathbb{P}\big(\#\Psi_B^{\eta_{\mathbb{R}^d\setminus B(0,R)}\cup\{RU_{1:d+1}\}}(\tau) = k \,|\, g\big(\Delta(RU_{1:d+1})\big) > v_\rho(\tau)\big).$$

PROOF. Let $A \subset \mathbb{R}^d$ be such that $\lambda_d(A) = 1$. Since the intensity of $\Psi^\eta(\tau)$ equals $\rho\,\mathbb{P}(g(\mathcal{C}) > v_\rho(\tau))$, it follows from the definition of the Palm distribution of the point process $\Psi^\eta(\tau)$, that

$$\begin{aligned}
&\mathbb{P}\big(\Psi^{\eta,0}(\tau) \in \mathcal{A}\big)\\
&\quad := \frac{1}{\rho\,\mathbb{P}(g(\mathcal{C}) > v_\rho(\tau))}\mathbb{E}\Big[\sum_{z\in\Psi^\eta(\tau)\cap A} \mathbb{I}_{(\Psi^\eta(\tau)-z)\in\mathcal{A}}\Big]\\
&\quad = \frac{1}{\rho\,\mathbb{P}(g(\mathcal{C}) > v_\rho(\tau))}\\
&\qquad \times \mathbb{E}\Big[\sum_{\{x_{1:d+1}\}\subset\eta} \mathbb{I}_{(\Psi^\eta(\tau)-\rho^{1/d}z(x_{1:d+1}))\in\mathcal{A}}\mathbb{I}_{z(x_{1:d+1})\in Z(\eta)\cap(\rho^{1/d}A)}\Big],
\end{aligned}$$

where $z(x_{1:d+1})$ is the center of the circumball of the simplex $\Delta(x_{1:d+1})$. According to the Slivnyak–Mecke formula and the Blaschke–Petkantschin formula (e.g., Theorem 7.3.1 in [29]), we have

$$\begin{aligned}
\mathbb{P}\big(\Psi^{\eta,0}(\tau) \in \mathcal{A}\big) = {}& \frac{\gamma_\eta^{d+1} d!}{\rho\,\mathbb{P}(g(\mathcal{C}) > v_\rho(\tau))(d+1)!}\int_{\rho^{1/d}A}\int_{\mathbb{R}_+}\int_{\mathbb{S}^{d-1}} r^{d^2-1}\\
&\times a(u_{1:d+1})\mathbb{P}\big(\big(\Psi^\eta(\tau) - \rho^{1/d}z\big) \in \mathcal{A}, \eta\cap B(z,r) = \varnothing\big)\\
&\times \mathbb{I}_{g(z+\Delta(ru_{1:d+1}))>v_\rho(\tau)}\sigma(\mathrm{d}u_{1:d+1})\,\mathrm{d}r\,\mathrm{d}z.
\end{aligned}$$

Since η is stationary and since g is translation-invariant, the integrand does not depend on z. Integrating over $z \in \rho^{1/d}A$, and using the fact that $\lambda_d(\rho^{1/d}A) = \rho$, we get

$$\begin{aligned}
&\mathbb{P}\big(\Psi^{\eta,0}(\tau) \in \mathcal{A}\big)\\
&\quad = \frac{\gamma_\eta^{d+1}\rho}{\rho\,\mathbb{P}(g(\mathcal{C}) > v_\rho(\tau))(d+1)}\int_{\mathbb{R}_+}\int_{(\mathbb{S}^{d-1})^{d+1}} r^{d^2-1}a(u_{1:d+1})\\
&\qquad \times \mathbb{P}\big(\Psi^\eta(\tau) \in \mathcal{A}, \eta\cap B(0,r) = \varnothing\big)\mathbb{I}_{g(\Delta(ru_{1:d+1}))>v_\rho(\tau)}\sigma(\mathrm{d}u_{1:d+1})\,\mathrm{d}r.
\end{aligned}$$

We give below an explicit representation for the integrand. Let $\eta_{B(0,r)}$ and $\eta_{\mathbb{R}^d\setminus B(0,r)}$ be two independent Poisson point processes with intensity measures $\gamma_\eta\mathbb{I}_{x\in B(0,R)}\,\mathrm{d}x$ and $\gamma_\eta\mathbb{I}_{x\in\mathbb{R}^d\setminus B(0,r)}\,\mathrm{d}x$, respectively. We know that

$$\eta \overset{\mathcal{D}}{=} \eta_{B(0,r)} \cup \eta_{\mathbb{R}^d\setminus B(0,r)}.$$

This gives

$$\begin{aligned}
&\mathbb{P}\big(\Psi^\eta(\tau) \in \mathcal{A}, \eta\cap B(0,r) = \varnothing\big)\\
&\quad = \mathbb{P}\big(\Psi^{\eta_{\mathbb{R}^d\setminus B(0,r)}\cup\{ru_{1:d+1}\}}(\tau) \in \mathcal{A}, \eta_{B(0,r)}\cap B(0,r) = \varnothing\big)\\
&\quad = e^{-\gamma_\eta\kappa_d r^d}\mathbb{P}\big(\Psi^{\eta_{\mathbb{R}^d\setminus B(0,r)}\cup\{ru_{1:d+1}\}}(\tau) \in \mathcal{A}\big).
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{P}\big(\Psi^{\eta,0}(\tau)\in\mathcal{A}\big)\\
&\quad=\frac{\gamma_\eta^{d+1}\rho}{\rho\,\mathbb{P}(g(\mathcal{C})>v_\rho(\tau))(d+1)}\int_{\mathbb{R}_+}\int_{(\mathbb{S}^{d-1})^{d+1}}r^{d^2-1}a(u_{1:d+1})\\
&\qquad\times e^{-\gamma_\eta\kappa_d r^d}\mathbb{P}\big(\Psi^{\eta_{\mathbb{R}^d\setminus B(0,r)}\cup\{ru_{1:d+1}\}}(\tau)\in\mathcal{A}\big)\\
&\qquad\times\mathbb{I}_{g(\Delta(ru_{1:d+1}))>v_\rho(\tau)}\sigma(\mathrm{d}u_{1:d+1})\,\mathrm{d}r.
\end{aligned}$$

Since $\gamma_\eta=(d+1)a_d$, we get

$$\begin{aligned}
&\mathbb{P}\big(\Psi^{\eta,0}(\tau)\in\mathcal{A}\big)\\
&\quad=\frac{a_d\gamma_\eta^{d}}{\mathbb{P}(g(\mathcal{C})>v_\rho(\tau))}\int_{\mathbb{R}_+}\int_{(\mathbb{S}^{d-1})^{d+1}}r^{d^2-1}a(u_{1:\mathrm{d}+1})\\
&\qquad\times e^{-\gamma_\eta\kappa_d r^d}\mathbb{P}\big(\Psi^{\eta_{\mathbb{R}^d\setminus B(0,r)}\cup\{ru_{1:d+1}\}}\in\mathcal{A}\big)\mathbb{I}_{g(\Delta(ru_{1:d+1}))>v_\rho(\tau)}\sigma(\mathrm{d}u_{1:d+1})\,\mathrm{d}r.
\end{aligned}$$

This proves the first equality in Proposition 8 since $\mathcal{C}\overset{\mathcal{D}}{=}\Delta(RU_{1:d+1})$. The second equality is a direct consequence of the first one. □

We think that Theorem 4 can be adapted in the context of a Poisson–Delaunay tessellation. To do it, we have to replace the point process $\Phi^\eta(\tau)$ by the point process $\Psi^\eta(\tau)$ and we have to use the characterization of the probability $p_{k,Q_\rho}(\tau)$ as described in the above proposition. We can easily extend Lemma 1 and adapt Condition (C) in the particular setting of a Poisson–Delaunay tessellation. However, the main difficulty to adapt Theorem 4 focuses on an analogous version of Proposition 3 since its proof seems very technical. We give below a numerical illustration which confirms that Theorem 4 should be true for a Poisson–Delaunay tessellation.

A numerical illustration. Let $\mathrm{m}_{\mathrm{PDT}}$ be a Poisson–Delaunay tessellation generated by a Poisson point process η in $\mathbb{R}^d$ with intensity $\gamma_\eta=\beta_d$. For each cell $C\in\mathrm{m}_{\mathrm{PDT}}$, we consider the so-called circumradius of C defined as

$$R(C):=\inf\{R\geq 0: C\subset B(z,R), z\in\mathbb{R}^d\}.$$

According to (5.1), the random variable $\kappa_d R(\mathcal{C})^d$ is Gamma distributed with parameters (d,β_d). A Taylor expansion of $\mathbb{P}(R(\mathcal{C})>v)$ as v goes to infinity (e.g., equation (3.14) in [10]), shows that $\rho\cdot\mathbb{P}(R(\mathcal{C})>v_\rho(\tau))\underset{\rho\to\infty}{\longrightarrow}\tau$, when

$$v_\rho(\tau):=(\kappa_d\beta_d)^{-1/d}\cdot\big(\log\big([(d-1)!]^{-1}\rho\log(\beta_d\rho)^{d-1}\tau^{-1}\big)\big)^{1/d}.$$

Moreover, with standard arguments, we can easily show that the maximum of circumradii of Delaunay cells $\max_{C\in\mathrm{m}_{\mathrm{PDT}}:z(C)\in W_\rho}R(C)$ has the same asymptotic behavior as the maximum of circumradii of the associated Voronoi cells

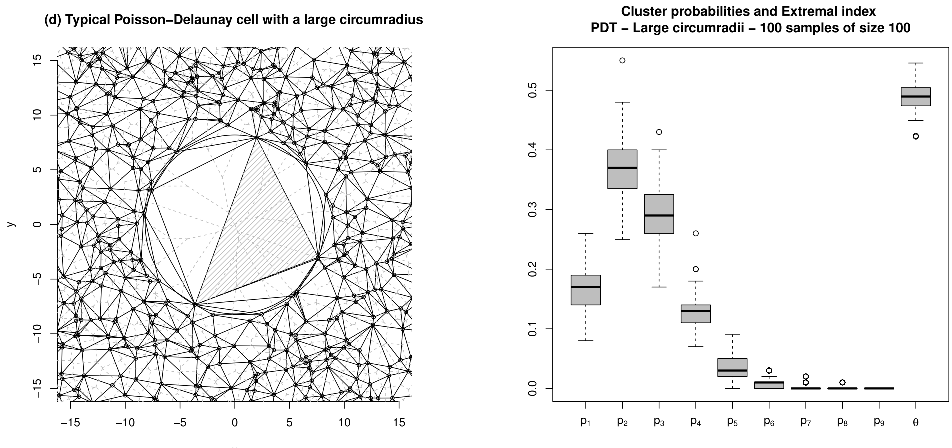

FIG. 5. *Large circumradius for a Poisson–Delaunay tessellation.*

$\max_{x\in\eta\cap W_\rho} R(C_\eta(x))$. Besides, according to (2c) in [10], we know that

$$\mathbb{P}\Big(\max_{x\in\eta\cap W_\rho} R\big(C_\eta(x)\big) \le (\kappa_d\beta_d)^{-1/d}\big(\log\big(\alpha_d\beta_d\rho\log(\beta_d\rho)^{d-1}\tau^{-1}\big)\big)^{1/d}\Big)$$
$$\underset{\rho\to\infty}{\longrightarrow} e^{-\tau},$$

where $\alpha_d := \frac{1}{d!}\big(\frac{\pi^{1/2}\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d+1}{2})}\big)^{d-1}$. It follows that

$$\mathbb{P}\Big(\max_{\substack{C\in\mathrm{m}_{\mathrm{PDT}}:\\ z(C)\in W_\rho}} R(C) \le v_\rho(\tau)\Big) \underset{\rho\to\infty}{\longrightarrow} e^{-\theta_d\tau},$$

where

$$\theta_d := \alpha_d\beta_d(d-1)! = \frac{(d^3+d^2)\Gamma(\frac{d^2}{2})\Gamma(\frac{d+1}{2})}{d\Gamma(\frac{d^2+1}{2})\Gamma(\frac{d+2}{2})2^{d+1}}.$$

In particular, when $d = 1, 2, 3$, the extremal index equals $\theta_1 = 1$, $\theta_2 = 1/2$ and $\theta_3 = 35/128$, respectively.

Now, we explain how we evaluate by simulation the value of the extremal index and the distribution p when $d = 2$. First, we simulate a random variable R such that πR^2 is Gamma distributed with parameters $(2, 1/2)$, given that $R > v_{\exp(100)}(1) \simeq 8.16$. Then we simulate a typical cell $\mathcal{C}$, with circumradius R, by using the method described in [17]. The Poisson–Delaunay tessellation which is generated is induced by the point process $\eta_{B(0,2R)^c} \cup \{0\}$, where $\eta_{B(0,2R)^c}$ is a Poisson point process with intensity measure $\mathbb{I}_{x\in B(0,2R)^c}\,\mathrm{d}x$ (see Proposition 8).

On the left part of Figure 5, we provide a simulation of the Palm version of the Poisson–Delaunay tessellation given that the typical cell has a circumradius larger than 8.16. The number of neighbors of the typical cells which are exceedances is random. The right part of Figure 5 provides the box plots of the empirical probabilities. Notice that these empirical distributions are not degenerated for $k = 1, \dots, 8$. Their interquartile ranges are not so important as for the circumradii of the Poisson–Voronoi tessellation, but the spread of the empirical distribution of the extremal index is larger. Besides, the empirical value of the extremal index is very concentrated around a value close to $1/2$, which is the theoretical value of θ.

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