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METASTABILITY FOR GLAUBER DYNAMICS ON RANDOM GRAPHS[1]

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In this paper, we study metastable behaviour at low temperature of Glauber spin–flip dynamics on random graphs. We fix a large number of vertices and randomly allocate edges according to the configuration model with a prescribed degree distribution. Each vertex carries a spin that can point either up or down. Each spin interacts with a positive magnetic field, while spins at vertices that are connected by edges also interact with each other via a ferromagnetic pair potential. We start from the configuration where all spins point down, and allow spins to flip up or down according to a Metropolis dynamics at positive temperature. We are interested in the time it takes the system to reach the configuration where all spins point up. In order to achieve this transition, the system needs to create a sufficiently large droplet of up-spins, called critical droplet, which triggers the crossover.

In the limit as the temperature tends to zero, and subject to a certain *key hypothesis* implying metastable behaviour, the average crossover time follows the classical *Arrhenius law*, with an exponent and a prefactor that are controlled by the *energy* and the *entropy* of the critical droplet. The crossover time divided by its average is exponentially distributed. We study the scaling behaviour of the exponent as the number of vertices tends to infinity, deriving upper and lower bounds. We also identify a regime for the magnetic field and the pair potential in which the key hypothesis is satisfied. The critical droplets, representing the saddle points for the crossover, have a size that is of the order of the number of vertices. This is because the random graphs generated by the configuration model are expander graphs.

1. Introduction and main theorems. A physical system is in a *metastable state* when it remains locked for a very long time in a phase that is different from the one corresponding to thermodynamic equilibrium. The latter is referred to as the *stable state*. Classical examples are supersaturated vapours, supercooled liquids and ferromagnets in the hysteresis loop. The main three objects of interest for metastability are the transition time from the metastable state to the stable state,

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the gate of configurations the system has to cross in order to achieve the transition, and the tube of typical trajectories the system follows prior to and after the transition.

Metastability for interacting particle systems on *lattices* has been studied intensively in the past three decades. Various different approaches have been proposed. After initial work by Cassandro, Galves, Olivieri and Vares [9], Neves and Schonmann [24, 25], a powerful method—known as the *pathwise approach* to metastability based on large deviation theory—was developed in Olivieri and Scoppola [26, 27], Catoni and Cerf [10], Manzo, Nardi, Olivieri and Scoppola [21], Cirillo and Nardi [11], Cirillo, Nardi and Sohier [12]. This was successfully applied to low-temperature Ising and Blume-Capel models subject to Glauber spin–flip dynamics (in two and three dimensions, with isotropic, anisotropic and staggered interactions) in Kotecký and Olivieri [18–20], Cirillo and Olivieri [13], Ben Arous and Cerf [1], Nardi and Olivieri [23]. Later, another powerful method—known as the *potential-theoretic approach* to metastability based on the analogy between Markov processes and electric networks—was developed in Bovier, Eckhoff, Gayrard and Klein [4–7]. This was shown in Bovier and Manzo [8], Bovier, den Hollander and Spitoni [3] to lead to a considerable sharpening of earlier results. For other approaches to metastability, as well as further examples of metastable stochastic dynamics and relevant literature, we refer the reader to the monographs by Olivieri and Vares [28], Bovier and den Hollander [2].

Recently, there has been interest in the Ising model on *random graphs* (Dembo and Montanari [14], Dommers, Giardinà and van der Hofstad [16], Mossel and Sly [22]). The only results known to date about metastability subject to Glauber spin–flip dynamics are valid for r-regular random graphs (Dommers [15]). In the present paper, we investigate what can be said for more general degree distributions. Metastability is much more challenging on random graphs than on lattices. Moreover, we need to capture the metastable behaviour for a *generic realisation* of the random graph.

In Section 1.1, we define the Ising model on a random multigraph subject to Glauber spin–flip dynamics. We start from the configuration where all spins point down, and allow spins to flip up or down according to a Metropolis dynamics at positive temperature. We are interested in the time it takes the system to reach the configuration where all spins point up. In Section 1.2, we introduce certain geometric quantities that play a central role in the description of the metastable behaviour of the system, and state three general theorems that are valid under a certain key hypothesis. These theorems concern the average transition time, the distribution of the transition time and the gate of saddle point configurations for the crossover, all in the limit of low temperature. They involve certain key quantities associated with the random graph. Our goal is to study the scaling behaviour of these quantities as the size of the graph tends to infinity.

In Section 1.3, we describe four examples to which the three general theorems apply: three refer to regular lattices, while one refers to the Erdős–Rényi random graph. In Section 1.4, we recall the definition of the configuration model, which is an example of a random graph with a nontrivial geometric structure. In Section 1.5, we state our main metastability results for the latter. In Section 1.6, we place these results in their proper context and give an outline of the remainder of the paper.

1.1. *Ising model and Glauber dynamics.* Given a finite connected nonoriented multigraph $G = (V, E)$, let $\Omega = \{-1, +1\}^V$ be the set of configurations $\xi = \{\xi(v) \colon v \in V\}$ that assign to each vertex $v \in V$ a spin-value $\xi(v) \in \{-1, +1\}$. Two configurations that will be of particular interest to us are those where all spins point up, respectively, down:

$$\boxplus \equiv +1, \qquad \boxminus \equiv -1. \tag{1.1}$$

For $\beta \geq 0$, playing the role of *inverse temperature*, we define the Gibbs measure

$$\mu_\beta(\xi) = \frac{1}{Z_\beta} e^{-\beta \mathcal{H}(\xi)}, \qquad \xi \in \Omega, \tag{1.2}$$

where $\mathcal{H} \colon \Omega \to \mathbb{R}$ is the *Hamiltonian* that assigns an energy to each configuration given by

$$\mathcal{H}(\xi) = -\frac{J}{2} \sum_{(v,w) \in E} \xi(v)\xi(w) - \frac{h}{2} \sum_{v \in V} \xi(v), \qquad \xi \in \Omega, \tag{1.3}$$

with $J > 0$ the *ferromagnetic pair potential* and $h > 0$ the *magnetic field*. The first sum in the right-hand side of (1.3) runs over all nonoriented edges in E. Hence, if $v, w \in V$ have $k \in \mathbb{N}_0$ edges between them, then their joint contribution to the energy is $-k\frac{J}{2}\xi(v)\xi(w)$.

We write $\xi \sim \zeta$ if and only if ξ and ζ agree at all but one vertex. A transition from ξ to ζ corresponds to a flip of a single spin, and is referred to as an *allowed move*. Glauber spin–flip dynamics on Ω is the continuous-time Markov process $(\xi_t)_{t \geq 0}$ defined by the transition rates

$$c_\beta(\xi, \zeta) = \begin{cases} e^{-\beta[\mathcal{H}(\zeta) - \mathcal{H}(\xi)]_+}, & \xi \sim \zeta, \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

The Gibbs measure in (1.2) is the reversible equilibrium of this dynamics. We write $P_\xi^{G,\beta}$ to denote the law of $(\xi_t)_{t \geq 0}$ given $\xi_0 = \xi$, $\mathcal{L}^{G,\beta}$ to denote the associated generator, and $\lambda^{G,\beta}$ to denote the principal eigenvalue of $\mathcal{L}^{G,\beta}$. The upper indices G, β exhibit the dependence on the underlying graph G and the interaction strength β between neighbouring spins. For $A \subseteq \Omega$, we write

$$\tau_A = \inf\{t > 0 \colon \xi_t \in A, \exists 0 < s < t \colon \xi_s \neq \xi_0\} \tag{1.5}$$

to denote the first hitting time of the set A after the starting configuration is left.

1.2. *Metastability.* To describe the metastable behaviour of our dynamics, we need the following geometric definitions.

DEFINITION 1.1. (a) The communication height between two distinct configurations $\xi, \zeta \in \Omega$ is

$$\Phi(\xi, \zeta) = \min_{\gamma: \xi \to \zeta} \max_{\sigma \in \gamma} \mathcal{H}(\sigma), \tag{1.6}$$

where the minimum is taken over all paths $\gamma: \xi \to \zeta$ consisting of allowed moves only. The communication height between two nonempty disjoint sets $A, B \subset \Omega$ is

$$\Phi(A, B) = \min_{\xi \in A, \zeta \in B} \Phi(\xi, \zeta). \tag{1.7}$$

(b) The stability level of $\xi \in \Omega$ is

$$V_\xi = \min_{\substack{\zeta \in \Omega: \\ \mathcal{H}(\zeta) < \mathcal{H}(\xi)}} \Phi(\xi, \zeta) - \mathcal{H}(\xi). \tag{1.8}$$

(c) The set of stable configurations is

$$\Omega_{\mathrm{stab}} = \Big\{ \xi \in \Omega : \mathcal{H}(\xi) = \min_{\zeta \in \Omega} \mathcal{H}(\zeta) \Big\}. \tag{1.9}$$

(d) The set of metastable configurations is

$$\Omega_{\mathrm{meta}} = \Big\{ \xi \in \Omega \setminus \Omega_{\mathrm{stab}} : V_\xi = \max_{\zeta \in \Omega \setminus \Omega_{\mathrm{stab}}} V_\zeta \Big\}. \tag{1.10}$$

It is easy to check that $\Omega_{\mathrm{stab}} = \{\boxplus\}$ for all G because $J, h > 0$. For general G, however, Ω_{meta} is not a singleton, but we will be interested in those G for which the following *hypothesis* is satisfied:

$$\Omega_{\mathrm{meta}} = \{\boxminus\}. \tag{H}$$

The energy barrier between $\boxminus$ and $\boxplus$ is

$$\Gamma^\star = \Phi(\boxminus, \boxplus) - \mathcal{H}(\boxminus). \tag{1.11}$$

DEFINITION 1.2. Let $(\mathcal{P}^\star, \mathcal{C}^\star)$ be the unique maximal subset of $\Omega \times \Omega$ with the following properties (see Figure 1):

(1) $\forall \xi \in \mathcal{P}^\star \exists \xi' \in \mathcal{C}^\star : \xi \sim \xi'$,
$\forall \xi' \in \mathcal{C}^\star \exists \xi \in \mathcal{P}^\star : \xi' \sim \xi$.
(2) $\forall \xi \in \mathcal{P}^\star : \Phi(\xi, \boxminus) < \Phi(\xi, \boxplus)$.
(3) $\forall \xi \in \mathcal{C}^\star \exists \gamma : \xi \to \boxplus$:

   (i) $\max_{\zeta \in \gamma} \mathcal{H}(\zeta) - \mathcal{H}(\boxminus) \le \Gamma^\star$.
   (ii) $\gamma \cap \{\zeta \in \Omega : \Phi(\zeta, \boxminus) < \Phi(\zeta, \boxplus)\} = \varnothing$.

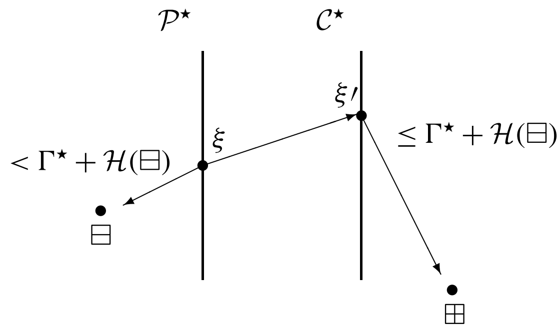

FIG. 1. *Schematic picture of the protocritical set $\mathcal{P}^\star$ and the critical set $\mathcal{C}^\star$.*

Think of $\mathcal{P}^\star$ as the set of configurations where the dynamics, on its way from $\boxminus$ to $\boxplus$, is "almost at the top", and of $\mathcal{C}^\star$ as the set of configurations where it is "at the top and capable of crossing over". We refer to $\mathcal{P}^\star$ as the *protocritical set* and to $\mathcal{C}^\star$ as the *critical set*. Uniqueness follows from the observation that if $(\mathcal{P}_1^\star, \mathcal{C}_1^\star)$ and $(\mathcal{P}_2^\star, \mathcal{C}_2^\star)$ both satisfy conditions (1)–(3), then so does $(\mathcal{P}_1^\star \cup \mathcal{P}_2^\star, \mathcal{C}_1^\star \cup \mathcal{C}_2^\star)$. Note that

$$\begin{aligned} &\mathcal{H}(\xi) < \Gamma^\star + \mathcal{H}(\boxminus) \qquad \forall \xi \in \mathcal{P}^\star, \\ &\mathcal{H}(\xi) = \Gamma^\star + \mathcal{H}(\boxminus) \qquad \forall \xi \in \mathcal{C}^\star. \end{aligned} \tag{1.12}$$

It is shown in Bovier and den Hollander ([2], Chapter 16) that *subject to hypothesis* (H) the following three theorems hold.

THEOREM 1.3. $\lim_{\beta\to\infty} P_{\boxminus}^{G,\beta}(\tau_{\mathcal{C}^\star} < \tau_{\boxplus} \mid \tau_{\boxplus} < \tau_{\boxminus}) = 1$.

THEOREM 1.4. *There exists a $K^\star \in (0, \infty)$ such that*

$$\lim_{\beta\to\infty} e^{-\beta\Gamma^\star} E_{\boxminus}^{G,\beta}(\tau_{\boxplus}) = K^\star. \tag{1.13}$$

THEOREM 1.5. (a) $\lim_{\beta\to\infty} \lambda^{G,\beta} E_{\boxminus}^{G,\beta}(\tau_{\boxplus}) = 1$.
(b) $\lim_{\beta\to\infty} P_{\boxminus}^{G,\beta}(\tau_{\boxplus}/E_{\boxminus}^{G,\beta}(\tau_{\boxplus}) > t) = e^{-t}$ *for all* $t \geq 0$.

The proofs of Theorems 1.3–1.5 in [2] do not rely on the details of the graph G, provided it is finite, connected and nonoriented (i.e., allowed moves are possible in both directions). For concrete choices of G, the task is to verify hypothesis (H) and to identify the triple (see Figure 2):

$$(\mathcal{C}^\star, \Gamma^\star, K^\star). \tag{1.14}$$

For lattice graphs this task has been carried out successfully (even for several classes of dynamics: see [2], Chapters 17–18). For random graphs, however, the

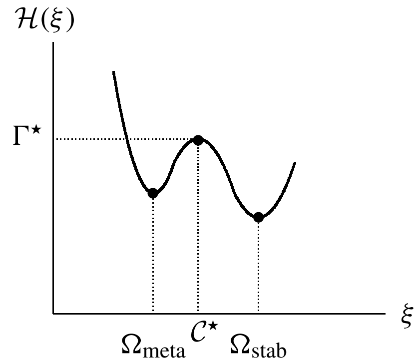

FIG. 2. *Schematic picture of* $\mathcal{H}$, Ω_{meta}, Ω_{stab} *and* $\mathcal{C}^\star$.

triplet in (1.14) is random, and describing it represents a *very serious challenge*. In what follows, we focus on a particular class of random graphs called the *configuration model*. But before doing so, we first summarise what is known in the literature.

1.3. *Examples of applications.*

1.3.1. *Torus.* If the underlying graph is a torus, then the computations needed to identify the critical set $\mathcal{C}^\star$ and the prefactor $K^\star$ simplify considerably. As shown in Bovier and den Hollander ([2], Chapter 17), for Glauber dynamics on a finite box $\Lambda \subset \mathbb{Z}^2$ (wrapped around to form a torus), the set $\mathcal{C}^\star$ consists of all $\ell_c \times (\ell_c - 1)$ quasi-squares (located anywhere in Λ in any of the two orientations) with an extra vertex attached to one of its longest sides, where $\ell_c = \lceil \frac{2J}{h} \rceil$ (the upper integer part of $\frac{2J}{h}$). Hypothesis (H) has been verified, and the exponent and the prefactor equal

$$\Gamma^\star = J(4\ell_c) - h\big(\ell_c(\ell_c - 1) + 1\big), \qquad K^\star = \frac{1}{|\Lambda|} \frac{1}{\frac{4}{3}(2\ell_c - 1)}. \tag{1.15}$$

Metastable behaviour occurs if and only if $\ell_c \in (1, \infty)$, and for reasons of parity it is assumed that $\frac{2J}{h} \notin \mathbb{N}$. Similar results apply for a torus in $\mathbb{Z}^3$.

1.3.2. *Hypercube.* For Glauber dynamics on the n-dimensional hypercube, Jovanovski [17] gives a complete description of the set $\mathcal{C}^\star$ and shows that for $0 \le \frac{h}{J} \le n$,

$$\Gamma_n^\star = \frac{1}{3}\left(2 - \frac{h}{J} + \left\lfloor \frac{h}{J} \right\rfloor\right)\left(2^{\lceil n-h \rceil} + 2\epsilon - 1\right) - \epsilon \tag{1.16}$$

and if in addition $\frac{h}{J} \notin \mathbb{N}$,

$$K_n^\star = 2^{-n}\left(1 + \frac{1}{n}\right) \tag{1.17}$$

for $\frac{h}{J} \geq n-2$, and

$$K_n^\star = \frac{3\lceil \frac{h}{J}\rceil!}{(1+\epsilon)n!2^n} \tag{1.18}$$

for $0 < \frac{h}{J} < n-2$, with $\epsilon = \lceil n-h\rceil \bmod 2$. Hypothesis (H) has been verified.

1.3.3. *Complete graph.* For Glauber dynamics on the complete graph K_n, it is easy to see that any monotone path from $\boxminus$ to $\boxplus$ is an optimal path. It is straightforward to show that $\mathcal{C}^\star = \{U \subseteq V : |U| = n^\star\}$ with $n^\star = \lceil \frac{1}{2}(n-1-\frac{h}{J})\rceil$, whenever $\frac{h}{J}$ is not an integer, and to compute

$$\Gamma_n^\star = n^\star\big(J\big(n-n^\star\big)-h\big), \qquad K_n^\star = \frac{1}{|\mathcal{C}^\star|}\frac{n}{n-n^\star}. \tag{1.19}$$

Metastable behaviour occurs for any value of h and J, provided n is large enough. Hypothesis (H) is also easy to confirm by observing that every configuration lies on some optimal path. Like the hypercube, K_n is an expander graph and consequently the communication height $\Gamma^\star$ grows at least linearly with the number of vertices (quadratically for K_n).

We can reduce the quadratic growth by introducing an interaction parameter that is inversely proportional to the size of the graph, for example, $J = \frac{J'}{n}$ for some constant $J' > 0$, with $h > 0$ fixed. It follows that

$$\Gamma_n^\star = n^\star\left(J'\left(\frac{n-n^\star}{n} - h\right)\right), \tag{1.20}$$

where this time $n^\star = \lceil \frac{n}{2}(1-\frac{h}{J'}) - \frac{1}{2}\rceil$, and $K_n^\star$ is the same as in (1.19). Metastable behaviour occurs if and only if $\frac{h}{J'} < 1 - \frac{1}{n}$.

1.3.4. *Erdős–Rényi random graph.* Sharp results of the above type become infeasible when the graph is random. The Erdős–Rényi random graph is the result of performing bond percolation on the complete graph, and is a toy model of a graph with a random geometry. Let $\mathrm{ER}_n(p)$ denotes the resulting random graph on n vertices with percolation parameter $p = f(n)/n$ for some $f(n)$ satisfying $\lim_{n\to\infty} f(n) = \infty$, the so-called *dense* Erdős–Rényi random graph. Then, as shown in the Appendix, metastable behaviour occurs for any $h, J > 0$, and

$$\lim_{n\to\infty} \frac{\Gamma_n^\star}{\frac{1}{4}Jnf(n)} = 1 \qquad \text{in distribution under the law of } \mathrm{ER}_n\big(f(n)/n\big), \tag{1.21}$$

which is accurate up to leading order. The computation of $\mathcal{C}_n^\star$ and $K_n^\star$, however, is a *formidable task*. The reason for this is that, while (1.21) allows for a small error in the energy, the set $\mathcal{C}_n^\star$ is made up of configurations that have *exactly* the critical energy $\Gamma_n^\star$.

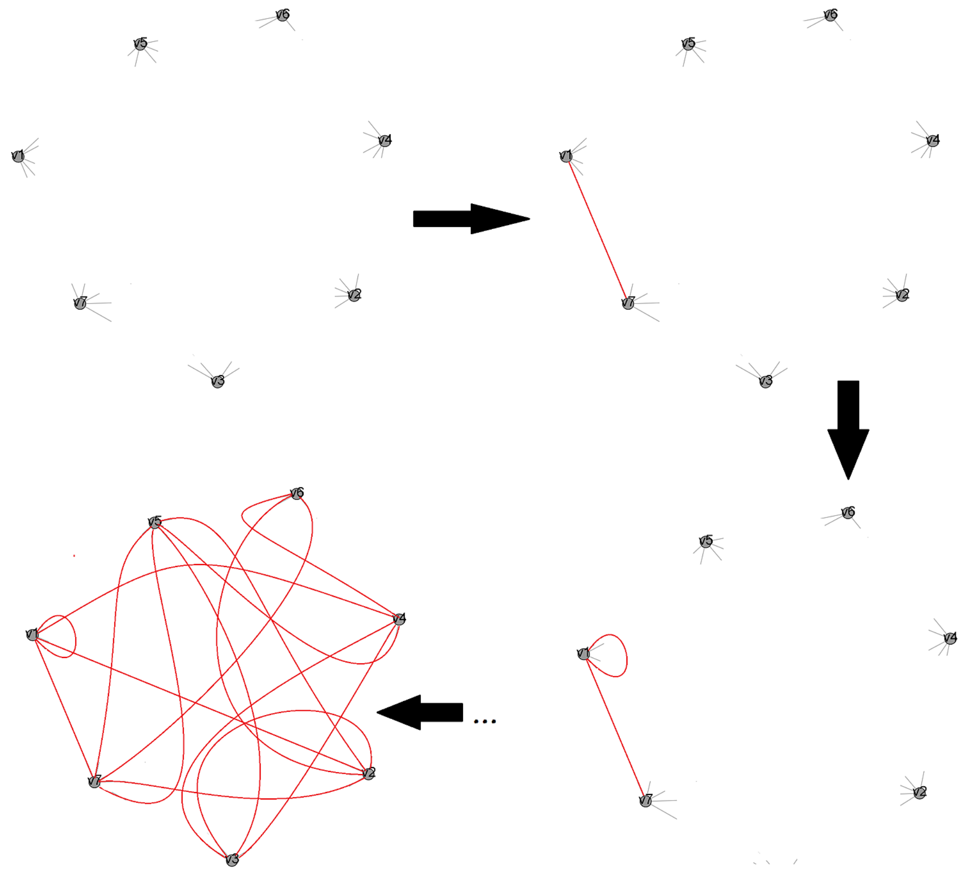

FIG. 3. *Illustration of the construction of* CM_n. *Three steps in the matching of stubs for* $n = 7$ *and degree sequence* $(5, 5, 4, 5, 5, 3, 5)$.

When $f(n) = \lambda$ for some constant $\lambda > 1$, the *sparse case*, an analysis similar to the one carried out in this paper can be used to obtain lower and upper bounds on the communication height. However, we have been unable to prove a convergence of the form in (1.21).

1.4. *Configuration model.* In this section, we recall the construction of the random *multi-graph* known as the *configuration model* (illustrated in Figure 3). We refer to van der Hofstad ([29], Chapter 7) for further details.

Fix $n \in \mathbb{N}$, and let $V = \{v_1, \ldots, v_n\}$. With each vertex v_i we associate a *random degree* D_i, in such a way that $D_1, \ldots, D_n \in \mathbb{N}$ are i.i.d. with marginal probability distribution f conditional on the event $\{\sum_{i=1}^n D_i = \text{even}\}$. Consider a uniform matching of the elements in the set of *stubs* (also called half-edges), written

$$\{x_{i,j}\}_{1 \le i \le n, 1 \le j \le D_i}. \tag{1.22}$$

By erasing the second label of the stubs, we can associate with it a *multi-graph* CM_n satisfying the requirement that the degree of v_i is D_i for $1 \le i \le n$. The total number of edges is $\frac{1}{2}\sum_{i=1}^n D_i$.

Throughout the sequel, we use the symbol $\mathbb{P}_n$ to denote the law of the random multi-graph CM_n on n vertices generated by the configuration model. To avoid degeneracies, we assume that

$$d_{\min} = \min\{k \in \mathbb{N}: f(k) > 0\} \ge 3, \qquad d_{\mathrm{ave}} = \sum_{k\in\mathbb{N}} k f(k) < \infty, \tag{1.23}$$

that is, all degrees are at least three and the average degree is finite. In this case, the graph is connected *with high probability* (w.h.p.), that is, with a probability tending to 1 as $n \to \infty$ (see van der Hofstad [29]).

1.5. *Main theorems.* We are interested in proving hypothesis (H) and identifying the key quantities in (1.14) for $G = \mathrm{CM}_n$, which we henceforth denote by $(\mathcal{C}_n^\star, \Gamma_n^\star, K_n^\star)$, in the limit as $n \to \infty$.

Our first main theorem settles hypothesis (H) for small magnetic field.

THEOREM 1.6. *Suppose that the inequality in equation* (2.26) *holds. Then*

$$\lim_{n\to\infty} \mathbb{P}_n(\mathrm{CM}_n \textit{ satisfies } (\mathrm{H})) = 1. \tag{1.24}$$

Our second and third main theorem provide upper and lower bounds on $\Gamma_n^\star$. Label the vertices of the graph so that their degrees satisfy $d_1 \le \cdots \le d_n$. Let $\gamma : \boxminus \to \boxplus$ be the path that successively flips the vertices $v_1, \dots, v_n$ (in that order), and let $\ell_m = \sum_{i=1}^m d_i$.

THEOREM 1.7. *Define*

$$\bar{m} = \min\left\{1 \le m \le n : \ell_m\left(1 - \frac{\ell_m}{\ell_n}\right) \ge \ell_{m+1}\left(1 - \frac{\ell_{m+1}}{\ell_n}\right) - \frac{h}{J}\right\} < \frac{n}{2}. \tag{1.25}$$

Then, *w.h.p.*,

$$\Gamma_n^\star \le \Gamma_n^+, \qquad \Gamma_n^+ = J\ell_{\bar{m}}\left(1 - \frac{\ell_{\bar{m}}}{\ell_n}\right) - h\bar{m} \pm O(\ell_n^{3/4}). \tag{1.26}$$

For $0 < x \le \frac{1}{2}$ and $\delta > 1$, define (see Figure 4)

$$\begin{aligned} I_\delta(x) = \inf\{0 < y \le x :& \\ & 1 < x^{x(1-1/\delta)}(1-x)^{(1-x)(1-1/\delta)}(1-x-y)^{-(1-x-y)/2} \\ & \times (x-y)^{-(x-y)/2} y^{-y}\}. \end{aligned} \tag{1.27}$$

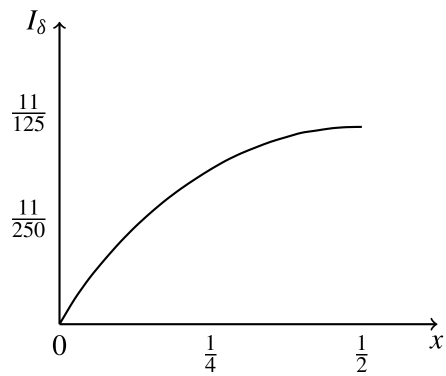

FIG. 4. *Plot of the function $I_\delta(x)$ for $\delta = 6$.*

THEOREM 1.8. *Define*

$$\tilde{m} = \min\left\{1 \le m \le n \colon \ell_m \ge \frac{1}{2}\ell_n\right\}. \tag{1.28}$$

Then, *w.h.p.*,

$$\Gamma_n^\star \ge \Gamma_n^-, \qquad \Gamma_n^- = J d_{\mathrm{ave}} I_{d_{\mathrm{ave}}}\left(\frac{1}{2}\right)n - h\tilde{m} - o(n). \tag{1.29}$$

COROLLARY 1.9. *Under hypothesis* (H) [*or the weaker version of* (H) *introduced in Section* 3], *Theorems* 1.7–1.8 *yield the following bounds on the crossover time* (*see Dommers* [15], *Proposition* 2.4):

$$\lim_{\beta\to\infty} P_{\boxminus}^{G,\beta}\left(e^{\Gamma_n^- - \varepsilon} \le \tau_{\boxplus} \le e^{\Gamma_n^+ + \varepsilon}\right) = 1. \tag{1.30}$$

In Corollary 4.1, we compute $\bar{m}$, $\ell_{\bar{m}}$, $\tilde{m}$ for two degree distributions: Dirac distributions and power-law distributions. It is clear that $\tilde{m} = \lceil \frac{1}{2}n \rceil$ for Dirac distributions.

The bounds we have found in Theorems 1.7–1.8 are tight in the limit of large degrees. Indeed, by the law of large numbers we have that

$$\ell_n \frac{\ell_{\bar{m}}}{\ell_n}\left(1 - \frac{\ell_{\bar{m}}}{\ell_n}\right) \le \frac{1}{4}\ell_n = \frac{1}{4} d_{\mathrm{ave}} n[1 + o(1)]. \tag{1.31}$$

Hence,

$$\frac{\Gamma_n^+}{\Gamma_n^-} = \frac{\frac{1}{4} d_{\mathrm{ave}}[1 + o(1)] - \frac{h}{J}\frac{\bar{m}}{n} + o(1)}{d_{\mathrm{ave}} I_{d_{\mathrm{ave}}}(\frac{1}{2}) - \frac{h}{J}\frac{\tilde{m}}{n} - o(1)}. \tag{1.32}$$

In the limit as $d_{\mathrm{ave}} \to \infty$, we have $I_{d_{\mathrm{ave}}}(\frac{1}{2}) \to \frac{1}{4}$, in which case (1.32) tends to 1.

1.6. *Discussion*. We close this introduction by discussing our main results.

1. The integer $\bar{m}$ in (1.25) has the following interpretation. The path $\gamma : \boxminus \to \boxplus$ is obtained by flipping (-1)-valued vertices to $(+1)$-valued vertices in order of increasing degree. We will see in the proof of Theorem 1.7 that, up to fluctuations of size $o(n)$, the energy along γ increases for the first $\bar{m}$ steps and decreases for the remaining $n - \bar{m}$ steps.

2. The integer $\tilde{m}$ in (1.28) has the following interpretation. In the proof of Theorem 1.8, to obtain our lower bound on $\Gamma_n^\star$ we consider configurations whose $(+1)$-valued vertices have total degree at most $\frac{1}{2}\ell_n$. The total number of $(+1)$-valued vertices in such configurations is at most $\tilde{m}$.

3. If we consider all sets on G that are of total degree $x\ell_n$ and share $y\ell_n$ edges with their complement, then $I_\delta(x)$ represents (a lower bound on) the least value for y such that the expected number of such sets is at least 1. In particular, for smaller values of y this expected number is exponentially small.

4. We believe that Theorem 1.6 holds as soon as

$$0 < h < (d_{\min} - 1)J, \tag{1.33}$$

that is, we believe that in the limit as $\beta \to \infty$ followed by $n \to \infty$ this choice of parameters corresponds to the *metastable regime* of our dynamics, that is, the regime where $(\boxminus, \boxplus)$ is a *metastable pair* in the sense of [2], Chapter 8.

5. The scaling behaviour of $\Gamma_n^\star$ as $n \to \infty$, as well as the geometry of $\mathcal{C}_n^\star$ are hard to capture. We can only offer some conjectures.

CONJECTURE 1.10. *There exists a* $\gamma^\star \in (0, \infty)$ *such that*

$$\lim_{n\to\infty} \mathbb{P}_n\big(|n^{-1}\Gamma_n^\star - \gamma^\star| > \delta\big) = 0 \qquad \forall \delta > 0. \tag{1.34}$$

CONJECTURE 1.11. *There exists a* $c^\star \in (0, 1)$ *such that*

$$\lim_{n\to\infty} \mathbb{P}_n\big(|n^{-1} \log|\mathcal{C}_n^\star| - c^\star| > \delta\big) = 0 \qquad \forall \delta > 0. \tag{1.35}$$

CONJECTURE 1.12. *There exists a* $\kappa^\star \in (1, \infty)$ *such that*

$$\lim_{n\to\infty} \mathbb{P}_n\big(||\mathcal{C}_n^\star|K_n^\star - \kappa^\star| > \delta\big) = 0 \qquad \forall \delta > 0. \tag{1.36}$$

As is clear from the results mentioned in Section 1.3, all three conjectures are true for the torus, the hypercube and the complete graph. This supports our belief that they should be true for a large class of random graphs as well.

6. In Section 4.1, we will give a *dynamical construction* of CM_n in which vertices are added one at a time and edges are relocated. This leads to a *random graph process* $(\mathrm{CM}_n)_{n\in\mathbb{N}}$ whose marginals respect the law of the configuration model. In

Section 5, we will show that this process is *tail trivial*, that is, all events in the tail sigma-algebra,

$$\mathcal{T} = \bigcap_{N\in\mathbb{N}} \sigma\Big(\bigcup_{n\geq N} \mathrm{CM}_n\Big) \tag{1.37}$$

have probability 0 or 1. Consequently, the associated communication height process $(\Gamma_n^\star)_{n\in\mathbb{N}}$ with $\Gamma_n^\star = \Gamma^\star(\mathrm{CM}_n)$ is tail trivial as well. In particular, both $\gamma_-^* = \liminf_{n\to\infty} n^{-1}\Gamma_n^\star$ and $\gamma_+^* = \limsup_{n\to\infty} n^{-1}\Gamma_n^\star$ exist and are constant a.s. Theorems 1.7–1.8 show that $0 < \gamma_-^* \leq \gamma_+^* < \infty$. Settling Conjecture 1.10 amounts to showing that $\gamma_-^* = \gamma_+^*$.

7. It was shown by Dommers [15] that for the configuration model with $f = \delta_r$, $r \in \mathbb{N}\setminus\{1,2\}$, that is, for a random regular graph with degree r, there exist constants $0 < \gamma_-^\star(r) < \gamma_+^\star(r) < \infty$ such that

$$\lim_{n\to\infty}\lim_{\beta\to\infty} \mathbb{E}_n\big(P_{\boxminus}^{\mathrm{CM}_n}\big(e^{\beta n \gamma_-^\star(r)} \leq \tau_{\boxplus} \leq e^{\beta n \gamma_+^\star(r)}\big)\big) = 1, \tag{1.38}$$

provided $\frac{h}{J} \in (0, C_0\sqrt{r})$ for some constant $C_0 \in (0,\infty)$ that is small enough. Moreover, there exist constants $C_1 \in (0, \frac{1}{4}\sqrt{3})$ and $C_2 \in (0,\infty)$ (depending on C_0) such that

$$\begin{aligned} \gamma_-^\star(r) &\geq \frac{1}{4}Jr - C_1 J\sqrt{r}, \\ \gamma_+^\star(r) &\leq \frac{1}{4}Jr + C_2 J\sqrt{r}, \qquad r \in \mathbb{N}\setminus\{1,2\}. \end{aligned} \tag{1.39}$$

The result in (1.38) is derived without hypothesis (H), but it is shown that hypothesis (H) holds as soon as $r \geq 6$.

Outline. The rest of the paper is organised as follows. In Section 2, we prove that hypothesis (H) holds under certain constraints on the magnetic field h and the minimal degree of the graph $d_{\min}$. Section 3 gives an alternative to hypothesis (H), which holds for a broader range parameters, yet still permits us to claim our bounds on the crossover time. In Section 4, we prove our upper and lower bounds on $\Gamma_n^\star$ and give two examples for concrete degree distributions. Part of this proof depends on the *dynamical construction* of CM_n. In Section 5, we derive certain properties of this construction.

2. Proof of Theorem 1.6. This section gives a proof of hypothesis (H) under condition (2.26). This condition is formulated in terms of the function $I_\delta(x)$ defined in (1.27). First, we prove two properties of this function that will be useful later on. After that, we construct an explicit path from any configuration $\sigma \in \Omega\setminus\{\boxminus,\boxplus\}$ to a configuration with lower energy such that the energy of configurations on this path never exceed $\mathcal{H}(\sigma)$ by $\Gamma^\star$ or more, which proves (H). We start with the following remark about the configurations in Ω.

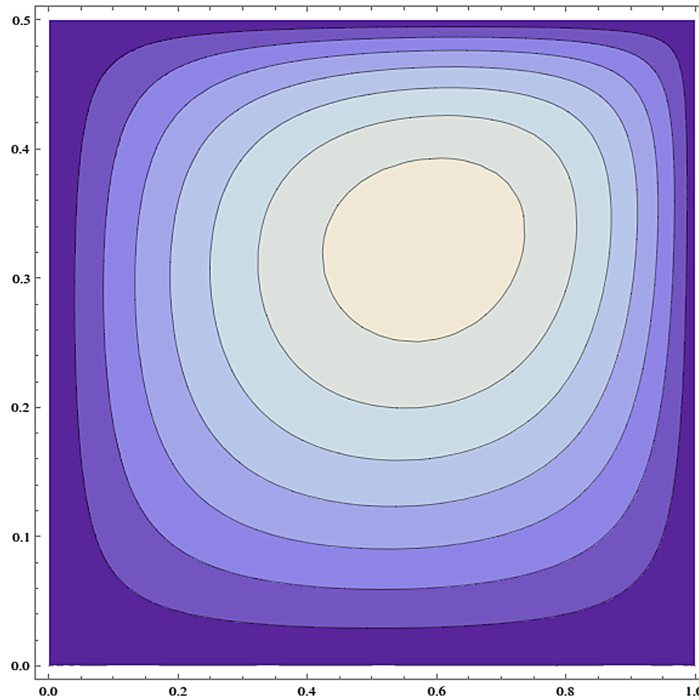

FIG. 5. *A contour plot of $\tilde{I}(x, w)$ for $x \in (0, 1)$ and $w \in (0, \frac{1}{2}]$. A lighter colour indicates a larger value.*

REMARK 2.1. A natural isomorphism between configurations and subsets of vertices of the underlying graph $G = (V, E)$ comes from identifying $\xi \in \Omega$ with the set $\{v \in V : \xi(v) = +1\}$. With this in mind, we denote by $\overline{\xi}$ the configuration corresponding to the complement of this set: $\{v \in V : \xi(v) = -1\}$. Furthermore, for $\zeta, \sigma \in \Omega$ we denote by $E(\zeta, \sigma) \subseteq E$ the set of all unoriented edges $\{(v, w) \in E : \zeta(v) = \sigma(w) = +1\}$. The main use of the last definition will be for $\sigma = \overline{\zeta}$: $E(\zeta, \overline{\zeta})$ is the edge boundary of the set $\{v \in V : \zeta(v) = +1\}$.

LEMMA 2.2. *For all $\delta \geq 2$ and $0 < x \leq \frac{1}{2}$, $I_\delta(x) \leq (1 - x) - (1 - x)^{2(1-1/\delta)}$.*

PROOF. The claim can be verified numerically. For $w \in (0, \frac{1}{2}]$, let $\tilde{y} = (1 - x) - (1 - x)^{2(1-w)}$. Figure 5 gives a contour plot of the function

$$\tilde{I}(x, w) = x^{x(1-w)}(1 - x)^{(1-x)(1-w)}(1 - x - \tilde{y})^{-(1-x-\tilde{y})/2} \times (x - \tilde{y})^{-(x-\tilde{y})/2}\tilde{y}^{-\tilde{y}}. \tag{2.1}$$

Note that $\tilde{I}(x, w) \geq 1$, which immediately implies Lemma 2.2 when we take $w = 1/\delta$. It is easy to verify that the boundary values corresponding to $x \downarrow 0$ and $x \uparrow 1$ result in $\tilde{I}(x, w) \downarrow 1$. $\square$

LEMMA 2.3. *The function* $x \to \frac{I_\delta(x)}{x}$ *is nonincreasing on* $(0, \frac{1}{2}]$.

PROOF. By definition of $I_\delta(x)$, the function

$$\hat{I}(x,z) = x^{x(1-1/\delta)}(1-x)^{(1-x)(1-1/\delta)}(1-x-xz)^{-(1-x-xz)/2} \times (x-xz)^{-(x-xz)/2}(xz)^{-xz} \tag{2.2}$$

satisfies $\hat{I}(x, \frac{I_\delta(x)}{x}) = 1$ for all $x \in (0, \frac{1}{2}]$. It will therefore suffice to show that

$$\frac{\partial}{\partial x}\bigg|_{z=\frac{I_\delta(x)}{x}} \hat{I}(x,z) \geq 0, \tag{2.3}$$

since this implies that, for ϵ sufficiently small, $\hat{I}(x+\epsilon, \frac{I_\delta(x)}{x}) \geq 1$, and hence that

$$\inf\{w \colon \hat{I}(x+\epsilon, w) \geq 1\} \leq \frac{I_\delta(x)}{x}, \tag{2.4}$$

and thus $\frac{I_\delta(x+\epsilon)}{x+\epsilon} \leq \frac{I_\delta(x)}{x}$. Observe that

$$\frac{\partial}{\partial x}\hat{I}(x,z) = \hat{I}(x,z)\left\{\log\left(\left(\frac{x}{1-x}\right)^{(1-1/\delta)}(1-x-xz)^{(1+z)/2} \times (x-xz)^{-(1-z)/2}(xz)^{-z}\right)\right\}. \tag{2.5}$$

For $z = \frac{I_\delta(x)}{x}$, $\hat{I}(x,z) = 1$ implies

$$\left(x^{x(1-1/\delta)}(1-x)^{(1-x)(1-1/\delta)}\right)^{\frac{1}{x}} = (1-x-xz)^{\frac{1}{2x}-(1+z)/2}(x-xz)^{(1-z)/2}(xz)^{z} \tag{2.6}$$

and hence

$$\frac{\partial}{\partial x}\hat{I}(x,z) = \hat{I}(x,z)\left\{\log\left(\left(\frac{x}{1-x}\right)^{(1-1/\delta)} \times \left(x^{x(1-1/\delta)}(1-x)^{(1-x)(1-1/\delta)}\right)^{-\frac{1}{x}}(1-x-xz)^{\frac{1}{2x}}\right)\right\}. \tag{2.7}$$

The term inside the logarithm in (2.7) simplifies to $(1-x)^{-\frac{1}{x}(1-1/\delta)}(1-x-xz)^{\frac{1}{2x}}$, which satisfies $(1-x)^{-\frac{1}{x}(1-1/\delta)}(1-x-xz)^{\frac{1}{2x}} \geq 1$ whenever $1-x-(1-x)^{2(1-1/\delta)} \geq xz = I_\delta(x)$. By Lemma 2.2, this is true for all $x \in (0, \frac{1}{2}]$, and so (2.3) follows. □

We can now proceed with the proof of hypothesis (H). Let $\sigma \in \Omega \setminus \{\boxminus, \boxplus\}$ be any configuration that satisfies $x = \ell_\sigma/\ell_n \leq \frac{1}{2}$, where $\ell_\sigma = \sum_{i\in\sigma} d_i$. We will construct a path from σ to some $\sigma' \in \Omega$ satisfying $\mathcal{H}(\sigma') < \mathcal{H}(\sigma)$ by removing one vertex at a time, obtaining a path $\sigma = \sigma_0, \dots, \sigma_m = \sigma'$. In particular, at step t we remove any vertex $v_t \in \sigma_{t-1}$ that minimises the quantity $|E(v_t, \sigma_{t-1} \setminus v_t)| - |E(v_t, \overline{\sigma_{t-1}})|$. It will follow that for every σ_i in this path, we have $|\mathcal{H}(\sigma_i) - \mathcal{H}(\sigma_0)| < \Gamma^\star$, which proves the claim of the theorem.

The probability that some configuration σ, chosen uniformly from all configurations in Ω with $\ell_\sigma = L$, has a boundary of size $|E(\sigma, \overline{\sigma})| = K$ equals

$$\begin{aligned}
&\binom{L}{K} K!(L-K-1)!!\binom{\ell_n - L}{K}(\ell_n - L - K - 1)!!/(\ell_n - 1)!! \\
&\quad\approx \frac{(L)!}{(L-K)!!}\frac{(\ell_n - L)!}{K!(\ell_n - L - K)!!}\frac{1}{\ell_n!!} \\
&\quad\approx L^L(\ell_n - L)^{(\ell_n - L)}(\ell_n - L - K)^{-(\ell_n - L - K)/2} \\
&\qquad\times (L-K)^{-(L-K)/2}K^{-K}\ell_n^{-\ell_n/2},
\end{aligned} \tag{2.8}$$

where the symbol $\approx$ stands for equality up to polynomial terms [here of order $O(n^2)$]. Let $x = L/\ell_n$ and $y = K/\ell_n$, so that the above expression becomes

$$\exp\big[\ell_n \log\big(x^x(1-x)^{(1-x)}(1-x-y)^{-(1-x-y)/2}(x-y)^{-(x-y)/2}y^{-y}\big)\big]. \tag{2.9}$$

Furthermore, if we define $\eta(x)$ by

$$\exp\big[\ell_n \log \eta(x)\big] = \big|\{U \subseteq V : \ell_U = \ell_n x\}\big|, \tag{2.10}$$

then the probability of there being any configuration of total degree L having a boundary size K is bounded from above by

$$\begin{aligned}
&\exp\big[\ell_n \log\big(\eta(x)x^x(1-x)^{(1-x)}(1-x-y)^{-(1-x-y)/2} \\
&\qquad\times (x-y)^{-(x-y)/2}y^{-y}\big)\big].
\end{aligned} \tag{2.11}$$

It is easy to see that, by using $\delta = d_{\min}$, the cardinality in the right-hand side of (2.10) is bounded from above by $\binom{\ell_n/\delta}{x\ell_n/\delta}$. Using Stirling's approximation for this term, and substituting in (2.11), we get

$$\begin{aligned}
&\mathbb{P}\big[\exists A \subseteq V : \ell_A = x\ell_n \text{ and } \big|E(A, \overline{A})\big| = y\ell_n\big] \\
&\quad\le \exp\big[\ell_n \log\big(x^{x(1-1/\delta)}(1-x)^{(1-x)(1-1/\delta)} \\
&\qquad\times (1-x-y)^{-(1-x-y)/2}(x-y)^{-(x-y)/2}y^{-y}\big)\big].
\end{aligned} \tag{2.12}$$

Recall the definition of I_δ from (1.27) and note that (2.12) is exponentially small for $y < I_\delta(x)$, and by a union bound it is exponentially small for all such y.

Suppose that after s vertices have been removed, we reach a configuration σ_s with $\mathcal{H}(\sigma_s) < \mathcal{H}(\sigma)$, such that for every vertex $v \in \sigma_s$ we have

$$\big|E(v, \sigma_s \setminus v)\big| + \frac{h}{J} > \big|E(v, \overline{\sigma_s})\big|. \tag{2.13}$$

In other words, equation (2.13) states that after removing s vertices we are at a configuration of lower energy, and removing any additional vertex leads to a configuration of higher energy. Note that if no such s exists, then we keep on removing vertices until $\boxminus$ has been reached. By the assumption that h is sufficiently small [by (1.29), it would suffice if $h < \frac{1}{2} J d_{\mathrm{ave}} I_{d_{\mathrm{ave}}}(\frac{1}{2}) \frac{n}{\tilde{m}}$, where $\tilde{m}$ was also defined in the aforementioned equation], w.h.p., every configuration σ of total degree $\ell_\sigma \le \frac{1}{2}\ell_n$ satisfies $\mathcal{H}(\sigma) > \mathcal{H}(\boxminus)$. If $v \in \sigma_s$ has no self-loops, then we have

$$\bigl|E(v, \sigma_s \setminus v)\bigr| = d_v - \bigl|E(v, \overline{\sigma_s})\bigr| \tag{2.14}$$

and thus the condition in (2.13) is satisfied when, for all $v \in \sigma_s$,

$$\frac{1}{2}\biggl(d_v + \frac{h}{J}\biggr) > \bigl|E(v, \overline{\sigma_s})\bigr|. \tag{2.15}$$

The total number of vertices with self-loops is w.h.p. of order $o(n)$, and so it will be evident from the bounds below that this assumption is immaterial. The second inequality in

$$\begin{aligned} \bigl|E(\sigma_s, \overline{\sigma_s})\bigr| &< \frac{1}{2}\sum_{v\in\sigma_s}\biggl(d_v + \frac{h}{J}\biggr) = \frac{1}{2}\Biggl(x\ell_n - \sum_{i=1}^{s} d_i + \frac{h(|\sigma| - s)}{J}\Biggr) \\ &\le \bigl|E(\sigma, \overline{\sigma})\bigr| \end{aligned} \tag{2.16}$$

holds whenever

$$x\ell_n - 2\bigl|E(\sigma, \overline{\sigma})\bigr| + \frac{h(|\sigma| - s)}{J} \le \sum_{i=1}^{s} d_i, \tag{2.17}$$

which in particular is true when we take the smallest s such that

$$\sum_{i=1}^{s} d_i \ge x\ell_n - 2I_\delta(x)\ell_n + \frac{h(|\sigma| - s)}{J}. \tag{2.18}$$

Furthermore, by removing s vertices, the change in the size of the boundary at step t is given by

$$\begin{aligned} \bigl|E(\sigma_t, \overline{\sigma_t})\bigr| - \bigl|E(\sigma, \overline{\sigma})\bigr| &= \sum_{i=1}^{t}\bigl(\bigl|E(v_i, \sigma_{i-1} \setminus v_i)\bigr| - \bigl|E(v_i, \overline{\sigma_{i-1}})\bigr|\bigr) \\ &\le \sum_{i=1}^{t}\biggl(d_i - 2\biggl\lceil d_i \frac{I_\delta(x)}{x}\biggr\rceil\biggr) \\ &\le \biggl(1 - 2\frac{I_\delta(x)}{x}\biggr)\sum_{i=1}^{t} d_i. \end{aligned} \tag{2.19}$$

The first inequality in (2.19) follows from the following observation: note that w.h.p. $|E(\sigma, \overline{\sigma})| \ge I_\delta(x)\ell_n$, and hence the "proportion" of the total degree of σ

that is paired with vertices in $\overline{\sigma}$ is at least $I_\delta(x)/x$. This implies that there must be some vertex v_i with a proportion of at least $I_\delta(x)/x$ of its degree connected with vertices in $\overline{\sigma}$. In other words, v_i shares at least $\lceil d_i \frac{I_\delta(x)}{x} \rceil$ edges with $\overline{\sigma}$.

By the definition of s, we see that $|E(\sigma, \overline{\sigma})| = |E(\sigma_s, \overline{\sigma_s})| + o(n)$ [when $d_s = o(n)$], and hence dropping the $o(n)$-term is of no consequence in the following computations. This implies that if t is such that $|E(\sigma_t, \overline{\sigma_t})| - |E(\sigma, \overline{\sigma})|$ is maximised, then we get [again, possibly after dropping a term of order $o(n)$]

$$\begin{aligned}
&\sum_{i=1}^{t} \bigl(\bigl|E(v_i, \sigma_{i-1})\bigr| - \bigl|E(v_i, \overline{\sigma_{i-1}})\bigr| \bigr) \\
&\qquad = \sum_{i=t+1}^{s} \bigl(\bigl|E(v_i, \overline{\sigma_{i-1}})\bigr| - \bigl|E(v_i, \sigma_{i-1})\bigr| \bigr).
\end{aligned} \tag{2.20}$$

Let m_t denote the left-hand side of (2.20) so that

$$\begin{aligned}
m_t &= \sum_{i=1}^{t} \bigl(d_i - 2\bigl|E(v_i, \overline{\sigma_{i-1}})\bigr| \bigr) = \sum_{i=t+1}^{s} \bigl(d_i - 2\bigl|E(v_i, \sigma_{i-1})\bigr| \bigr) \\
&= \sum_{i=1}^{s} \bigl(d_i - 2\bigl|E(v_i, \sigma_{i-1})\bigr| \bigr) - \sum_{i=1}^{t} \bigl(d_i - 2\bigl|E(v_i, \sigma_{i-1})\bigr| \bigr).
\end{aligned} \tag{2.21}$$

Hence,

$$\sum_{i=1}^{t} d_i \biggl(\frac{\sum_{i=1}^{t} (d_i - 2|E(v_i, \overline{\sigma_{i-1}})|)}{\sum_{i=1}^{t} d_i} + 1 \biggr) = \sum_{i=1}^{s} d_i - \sum_{i=t+1}^{s} 2\bigl|E(v_i, \sigma_{i-1})\bigr|, \tag{2.22}$$

and thus

$$\begin{aligned}
m_t &= \biggl(\frac{\sum_{i=1}^{t} (d_i - 2|E(v_i, \overline{\sigma_{i-1}})|)}{\sum_{i=1}^{t} d_i} \biggr) \biggl(\frac{\sum_{i=1}^{t} (d_i - 2|E(v_i, \overline{\sigma_{i-1}})|)}{\sum_{i=1}^{t} d_i} + 1 \biggr)^{-1} \\
&\quad \times \Biggl(\sum_{i=1}^{s} d_i - \sum_{i=t+1}^{s} 2\bigl|E(v_i, \sigma_{i-1})\bigr| \Biggr) \\
&\le \frac{1}{2} \biggl(1 - 2\frac{I_\delta(x)}{x} \biggr) \biggl(1 - \frac{I_\delta(x)}{x} \biggr)^{-1} \biggl(x\ell_n - 2I_\delta(x)\ell_n + \frac{h(|\sigma| - s)}{J} \biggr),
\end{aligned} \tag{2.23}$$

where for the last inequality we use (2.18)–(2.19) and the monotonicity of $y \to y(y+1)^{-1}$. From (1.29), using the fact that $nd_{\mathrm{ave}} = \ell_n + o(n)$, we get that $\mathcal{H}(\sigma_t) - \mathcal{H}(\sigma) < \Gamma^\star$ whenever

$$\begin{aligned}
&\frac{h}{J\ell_n} \biggl(2\tilde{m} + t + \bigl(|\sigma| - s\bigr) \biggl(\frac{x - 2I_\delta(x)}{x - I_\delta(x)} \biggr) \biggr) \\
&\qquad < 2I_{d_{\mathrm{ave}}} \biggl(\frac{1}{2} \biggr) - \bigl(x - 2I_\delta(x) \bigr)^2 \bigl(x - I_\delta(x) \bigr)^{-1}.
\end{aligned} \tag{2.24}$$

Note that if $x < 2I_\delta(x)$, then for sufficiently small h we can find a monotone downhill path to $\boxminus$. More precisely, $x < 2I_\delta(x)$ implies that the terms in the right-hand side of (2.19) become negative, and hence for $\frac{h}{J} < d_{\min}(\frac{2I_\delta(x)}{x} - 1)$ every step in our path is a downhill step. For $x \geq 2I_\delta(x)$, observe first that since the function $u \to \frac{(1-2u)^2}{(1-u)}$ is nonincreasing for $u \leq \frac{1}{2}$, by Lemma 2.3

$$\frac{(x - 2I_\delta(x))^2}{(x - I_\delta(x))} = x\frac{(1 - 2\frac{I_\delta(x)}{x})^2}{(1 - \frac{I_\delta(x)}{x})} \leq \frac{1}{2}\frac{(1 - 4I_\delta(\frac{1}{2}))^2}{(1 - 2I_\delta(\frac{1}{2}))}, \tag{2.25}$$

and thus a sufficient condition for (2.24) to hold is

$$\frac{h}{J}\left(\frac{1}{d_{\mathrm{ave}}} + \frac{1}{2}\right) < 2I_{d_{\mathrm{ave}}}\left(\frac{1}{2}\right) - \frac{1}{2}\left(1 - 4I_{d_{\min}}\left(\frac{1}{2}\right)\right)^2\left(1 - 2I_{d_{\min}}\left(\frac{1}{2}\right)\right)^{-1}. \tag{2.26}$$

Hence, we have a path $\sigma \to \sigma_s$ [or, when such an s satisfying (2.13) does not exist, a path $\sigma \to \boxminus$] with $\mathcal{H}(\sigma_s) < \mathcal{H}(\sigma)$ that never exceeds $\mathcal{H}(\sigma)$ by $\Gamma^\star$ or more, whenever h is sufficiently small and (2.24) holds. This proves the claim of the theorem for all configurations σ with $\ell_\sigma \leq \frac{1}{2}\ell_n$.

Note also that, for $\ell_\sigma > \frac{1}{2}\ell_n$, the same argument can be repeated by adding a vertex at each step, which will also come at a lower cost since at each step the magnetisation changes by $-h$.

3. An alternative to hypothesis (H). In this section, we give a weaker version of hypothesis (H), which nonetheless suffices as a prerequisite for Theorem 1.4. This weaker version can be verified for a parameter range that is larger than the one needed in Section 2.

We can repeat the arguments given in Section 2. But, instead of insisting that $\mathscr{V}_\sigma < \Gamma^\star$ for every configuration $\sigma \in \Omega$, we require that $\mathscr{V}_\sigma$ is bounded from above by our upper bound on $\Gamma^\star$, since this guarantees that our upper bound on the crossover time is still valid and (1.30) still holds (see Dommers [15], Lemma 5.3). Thus, it follows from the arguments leading to (2.24) that we only need the condition

$$\begin{aligned}\frac{h}{J\ell_n}&\left(\bar{m} + t + |\sigma| - s\left(\frac{x - 2I_\delta(x)}{x - I_\delta(x)}\right)\right)\\ &\leq 2\frac{\ell_{\bar{m}}}{\ell_n}\left(1 - \frac{\ell_{\bar{m}}}{\ell_n}\right) - \left(x - 2I_\delta(x)\right)^2\left(x - I_\delta(x)\right)^{-1}.\end{aligned} \tag{3.1}$$

For h sufficiently small, the ratio $\frac{\ell_{\bar{m}}}{\ell_n}$ can be made arbitrarily close to $\frac{1}{2}$, in which case the right-hand side of (3.1) becomes strictly positive. This implies that the inequality in (3.1) holds for any $\delta \geq 3$ whenever h is sufficiently small.

4. Proofs of Theorems 1.7 and 1.8.

4.1. *A dynamic construction of the configuration model.* Prior to giving the proof of Theorems 1.8 and 1.7, we introduce a *dynamical construction* of the CM graph. This will be used to obtain the upper bound in Theorem 1.7.

Let $V = \{v_i\}_{i=1}^n$ be a sequence of vertices with degrees $\{d_i\}_{i=1}^n$. In this section, we construct a graph $G = (V, E)$ with the same distribution as a graph generated through the configuration model algorithm, but in a *dynamical way*, as follows.

Suppose that ξ_m is a uniform random matching of the integers $\{1, \dots, 2m\}$, denoted by $\xi_m = \{(x_1, x_2), \dots, (x_{2m-1}, x_{2m}\}$, where the pairs are listed in the order they were created (which is not an important issue, so long as we agree on some labeling). Next, let u be uniform on $\{1, \dots, 2m, 2m+1\}$ and set $\xi_{m+1} = \xi_m \cup \{(2m+2, u)\}$ if $u = 2m+1$. Else if $u \neq 2m+1$, then w.l.o.g. $u = x_{2i-1}$ for some $i \le m$, and we set $\xi_{m+1} = \{\xi_m \setminus \{(x_{2i-1}, x_{2i})\}\} \cup \{(2m+2, x_{2i-1}), (2m+1, x_{2i})\}$. Then ξ_{m+1} is a uniform matching of the points $\{1, \dots, 2m, 2m+2\}$. It is now obvious how the construction of G follows from the given scheme.

4.2. *Energy estimates.* Label the vertices of the graph so that their degrees satisfy $d_1 \le \cdots \le d_n$. Let $\gamma : \boxminus \to \boxplus$ be the path that successively flips the vertices $v_1, \dots, v_n$ (in that order), and let $\ell_m = \sum_{i=1}^m d_i$. We show that, w.h.p., for every $1 \le m \le n$,

$$\mathcal{H}(\gamma_m) - \mathcal{H}(\boxminus) = J\ell_m\left(1 - \frac{\ell_m}{\ell_n}\right) - mh \pm O\big(\ell_n^{3/4}\big). \tag{4.1}$$

We are particularly interested in the maximum of (4.1) over all $1 \le m \le n$. To this avail, observe that the function defined by

$$g(x) = Jx(1-x) - h(x) \tag{4.2}$$

has at most one maximum for $x \in [0, 1]$ if $x \mapsto h(x)$ is nondecreasing. Thus, taking $x = \frac{\ell_m}{\ell_n}$ and $h(x) = hx\frac{m}{\ell_m}$, we see that our definition of $\bar{m}$ in Theorem 1.7 is justified. Furthermore, note the equivalent conditions

$$\begin{aligned} &\ell_m\left(1 - \frac{\ell_m}{\ell_n}\right) \ge \ell_{m+1}\left(1 - \frac{\ell_{m+1}}{\ell_n}\right) - \frac{h}{J} \quad \Longleftrightarrow \\ &\frac{h}{J} \ge d_{m+1}\left(1 - \frac{2\ell_m}{\ell_n}\right) - O\left(\frac{d_m^2}{\ell_n}\right), \end{aligned} \tag{4.3}$$

with the last term in (4.3) disappearing whenever $d_{m+1} = o(\sqrt{\ell_n})$. Note that (4.3) gives us an alternative formulation for $\bar{m}$ in the statement of Theorem 1.7, which we will use to compute $\bar{m}$ below.

4.3. *Two examples*. Two commonly studied degree distributions for the configuration model are the Dirac distribution

$$q_r(k) = \delta_r(k), \qquad k \in \mathbb{N}_0, \tag{4.4}$$

for some $r \in \mathbb{N}$ (i.e., the r-regular graph), and the power-law distribution

$$q_{\tau,\delta}(k) = \mathbb{P}[d_i = \delta + k] = \frac{(\delta+k)^{-\tau}}{\sum_{i\in\mathbb{N}_0}(\delta+i)^{-\tau}}, \qquad k \in \mathbb{N}_0, \tag{4.5}$$

for some exponent $\tau \in (2,\infty)$ and shift $\delta \in \mathbb{N}$.

For these degree distributions, we get the following corollary of Theorems 1.7–1.8.

COROLLARY 4.1. (a) *For the Dirac-distribution in* (4.4),

$$JrI_r\left(\frac{1}{2}\right)n - \frac{hn}{2} - o(n) \le \Gamma_n^\star \le \frac{Jr}{4}n\left(1 - \left(\frac{h}{Jr}\right)^2\right) \pm O\left(n^{3/4}\right). \tag{4.6}$$

(b) *For the power-law distribution distribution in* (4.5), $\bar{m}$ *and* $\ell_{\bar{m}}$ *are given by* (4.11) *and* (4.12).

(c) *For the power-law distribution distribution in* (4.5), $\tilde{m}$ *is given by* (4.15).

PROOF. (a) Straightforward.

(b) $\{d_i\}_{i=1}^n$ are i.i.d. with degree distribution $q_{\tau,\delta}$. Let $s_{\tau,\delta,k} = \sum_{i=1}^n \mathbf{1}\{d_i \le \delta + k\}$, and note that

$$\mathbb{E}[s_{\tau,\delta,k}] = n\left(1 - \frac{\xi_\tau(\delta+k+1)}{\xi_\tau(\delta)}\right) \tag{4.7}$$

with $\xi_\tau(a) = \sum_{i=a}^\infty i^{-\tau}$ for $a \ge 0$. We claim that, for k sufficiently small, $s_{\tau,\delta,k}$ is concentrated around its mean. Indeed, define $a_{\delta,k} = \sum_i \mathbf{1}\{d_i = \delta + k\}$, $k \in \mathbb{N}_0$, and note that for any i.i.d. sequence we have $a_{\delta,k} \stackrel{d}{=} \mathrm{Bin}(n, p_{\delta,k})$, where $p_{\delta,k} = \mathbb{P}[d_i = \delta + k]$. From Hoeffding's inequality, we get that

$$\mathbb{P}\left[|a_{\delta,k} - np_{\delta,k}| > n^{\frac{1}{2}+\frac{1}{6}}\right] \le \exp\left(-2n^{\frac{1}{3}}\right). \tag{4.8}$$

Hence, for any $k = O(n^{1/6})$,

$$\begin{aligned}\mathbb{P}\left[\left|s_{\tau,\delta,k} - \mathbb{E}[s_{\tau,\delta,k}]\right| > n^{\frac{1}{2}+\frac{1}{3}}\right] &\le \mathbb{P}\left[\bigcup_{m=0}^{k} |a_{\delta,m} - np_{\delta,m}| > n^{\frac{1}{2}+\frac{1}{6}}\right] \\ &\le n^{\frac{1}{6}} \exp\left(-2n^{\frac{1}{3}}\right).\end{aligned} \tag{4.9}$$

Note that if $p_{\delta,k} = q_{\tau,\delta}(k)$, then $\mathbb{E}[a_{\delta,k}] = n\frac{(\delta+k)^{-\tau}}{\xi_\tau(\delta)}$, and w.h.p. $\ell_n = n\frac{\xi_{\tau-1}(\delta)}{\xi_\tau(\delta)} + o(n)$. Hence, we define

$$\kappa = \min\left\{k \in \mathbb{N} \colon \frac{h}{J} \ge (\delta + k - 1)\left(1 - \left(\frac{\xi_{\tau-1}(\delta+k)}{\xi_{\tau-1}(\delta)}\right)\right)\right\} - 1, \tag{4.10}$$

which certainly satisfies $\kappa = o(n^{1/6})$. From (4.3) and the monotonicity of (4.2), it follows that w.h.p.

$$\begin{aligned}\bar{m} = n\bigg[\bigg(1 - \frac{\xi_\tau(\delta+\kappa)}{\xi_\tau(\delta)}\bigg) \\ + \min\bigg\{y \in [0,1]: \\ \frac{h}{J} \geq (\delta+\kappa)\bigg(1 - \bigg(\frac{\xi_{\tau-1}(\delta+\kappa)}{\xi_{\tau-1}(\delta)} + \frac{y\kappa\xi_\tau(\delta)}{\xi_{\tau-1}(\delta)}\bigg)\bigg)\bigg\}\bigg] + o(n)\end{aligned} \tag{4.11}$$

and

$$\frac{\ell_{\bar{m}}}{\ell_n} = \bigg(\frac{\xi_{\tau-1}(\delta+\kappa)}{\xi_{\tau-1}(\delta)} + \frac{y\kappa\xi_\tau(\delta)}{\xi_{\tau-1}(\delta)}\bigg) + o(1), \tag{4.12}$$

where y is the taken as the argument of the minimum in (4.11). Since we know ℓ_n up to $o(n)$, this also gives the value of $\ell_{\bar{m}}$.

(c) Note that $\tilde{m} = \sum_{i=0}^{\kappa} a_{\delta,i}$ where κ is the least integer such that

$$\delta a_{\delta,0} + (\delta+1)a_{\delta,0} + \cdots + (\delta+\kappa)a_{\delta,\kappa} \geq \frac{1}{2}\ell_n. \tag{4.13}$$

By the concentration results given above, we see that w.h.p.

$$\kappa = \min\bigg\{m \in \mathbb{N}: \frac{\xi_{\tau-1}(\delta) + \xi_{\tau-1}(\delta+m+1)}{\xi_\tau(\delta)} \geq \frac{1}{2}d_{\mathrm{ave}}\bigg\} \tag{4.14}$$

and

$$\tilde{m} = \frac{n}{\xi_\tau(\delta)}\sum_{i=0}^{\kappa}(\delta+i)^\tau + o(n). \tag{4.15}$$

□

PROOF OF THEOREM 1.7. Consider a sequence of matchings $\{\xi_1, \xi_2, \dots, \xi_{M/2}\}$ constructed in a dynamical way as outlined above, where M is some even integer. Let $0 \leq x \leq M$ be even, and define $z_{x,0} = x$ and $z_{x,t} = \sum_{i=1}^{x}\sum_{m=1}^{x} \mathbf{1}\{(i,m) \in \xi_{x/2+t}\}$. Then

$$z_{x,t+1} = z_{x,t} - 2\mathbf{1}\{Y_{t+1}\}, \tag{4.16}$$

where Y_t is the event that $x+2t-1$ and $x+2t$ are both paired with terms in $[x]$. Note that

$$\mathbb{P}[Y_{t+1}|\mathscr{G}_{x+t}] = \frac{z_{x,t}}{x+2+1}, \tag{4.17}$$

where $\mathscr{G}_{x+t}$ is the σ-algebra generated by $\{\xi_1, \dots, \xi_{x/2+t}\}$. Therefore,

$$\mathbb{E}[z_{x,t+1}|\mathscr{G}_{x+t}] = z_{x,t} - \frac{2z_{x,t}}{x+2t+1} = z_{x,t}\bigg(\frac{x+2t-1}{x+2t+1}\bigg) \tag{4.18}$$

and so

$$\mathbb{E}[z_{x,t+1}] = z_0 \prod_{j=0}^{t} \left(\frac{x+2j-1}{x+2j+1} \right) = x \left(\frac{x-1}{x+2t+1} \right). \tag{4.19}$$

To compute the second moment, observe that

$$z_{x,t+1}^2 = z_{x,t}^2 - 4z_{x,t}\mathbf{1}\{Y_{t+1}\} + 4\mathbf{1}\{Y_{t+1}\}, \tag{4.20}$$

and so

$$\begin{aligned} \mathbb{E}[z_{x,t+1}^2 | \mathscr{G}_{x+t}] &= z_{x,t}^2 - \frac{4z_{x,t}^2}{x+2t+1} + \frac{4z_{x,t}}{x+2t+1} \\ &= z_{x,t}^2 \left(\frac{x+2t-3}{x+2t+1} \right) + \frac{4z_{x,t}}{x+2t+1}. \end{aligned} \tag{4.21}$$

Then

$$\begin{aligned} &\mathbb{E}[z_{x,t+1}^2] \\ &\quad = \mathbb{E}[z_{x,t}^2] \left(\frac{x+2t-3}{x+2t+1} \right) + \frac{4\mathbb{E}[z_{x,t}]}{x+2t+1} \\ &\quad = x^2 \prod_{i=0}^{t} \left(\frac{x+2i-3}{x+2i+1} \right) + \sum_{i=0}^{t} \frac{4\mathbb{E}[z_{x,i}]}{x+2i+1} \prod_{j=i+1}^{t} \left(\frac{x+2j-3}{x+2j+1} \right) \\ &\quad = \frac{x^2(x-3)(x-1)}{(x+2t+1)(x+2t-1)} \\ &\qquad + \sum_{i=0}^{t} \frac{4x(x-1)}{(x+2i+1)(x+2i-1)} \frac{(x+2i-1)(x+2i+1)}{(x+2t+1)(x+2t-1)} \\ &\quad = \frac{x^2(x-3)(x-1)}{(x+2t+1)(x+2t-1)} + \frac{4x(x-1)(t+1)}{(x+2t+1)(x+2t-1)} \\ &\quad = \frac{x(x-1)}{(x+2t+1)(x+2t-1)} \big(x(x-3) + 4(t+1) \big), \end{aligned} \tag{4.22}$$

while

$$\big(\mathbb{E}[z_{x,t+1}]\big)^2 = x^2 \left(\frac{x-1}{x+2t+1} \right)^2 \tag{4.23}$$

and so

$$\begin{aligned} &\mathbb{E}[z_{x,t+1}^2] - \big(\mathbb{E}[z_{x,t+1}]\big)^2 \\ &\qquad = \frac{x(x-1)}{x+2t+1} \left(\frac{x(x-3)+4(t+1)}{x+2t-1} - \frac{x(x-1)}{x+2t+1} \right). \end{aligned} \tag{4.24}$$

It follows that if we let $w_{x,t} = \frac{z_{x,t}}{x+2t}$, $t \geq 1$, then

$$\begin{aligned}
&\mathbb{E}\big[w_{x,t+1}^2\big] - \big(\mathbb{E}[w_{x,t+1}]\big)^2 \\
&\quad = \frac{\frac{x}{x+2(t+1)}\frac{(x-1)}{x+2(t+1)}}{x+2t+1}\left(\frac{4(t+1)}{x+2t-1} + \frac{x-3}{1+2\frac{t}{x}-\frac{1}{x}} - \frac{x-1}{1+2\frac{t}{x}+\frac{1}{x}}\right) \\
&\quad = \frac{\frac{x}{x+2(t+1)}\frac{(x-1)}{x+2(t+1)}}{x+2t+1}\left(\frac{4(t+1)}{x+2t-1} + \frac{-4\frac{t}{x}-4\frac{1}{x}}{(1+2\frac{t}{x}-\frac{1}{x})(1+2\frac{t}{x}+\frac{1}{x})}\right) \qquad (4.25) \\
&\quad = \frac{4\frac{x}{x+2(t+1)}\frac{(x-1)}{x+2(t+1)}}{x+2t+1}\frac{t+1}{x+2t-1}\left(1 - \frac{1}{1+2\frac{t}{x}+\frac{1}{x}}\right).
\end{aligned}$$

Observe also that for

$$\bar{z}_{x,t} = \sum_{i=1}^{x}\sum_{m=x+1}^{M} \mathbf{1}\big\{(i,m) \in \xi_{x/2+t}\big\} = x - z_{x,t} \qquad (4.26)$$

and

$$\bar{w}_{x,t} = \frac{\bar{z}_{x,t}}{x+2t} = \frac{x}{x+2t} - w_{x,t} \qquad (4.27)$$

the same variance calculations follow, so that $\mathbb{E}[\bar{w}_{x,t+1}^2] - (\mathbb{E}[\bar{w}_{x,t+1}])^2$ is also given by (4.25). For $\alpha \in (0,1)$ and $1 \leq i \leq k(\alpha)$ with $k(\alpha) = M/\lfloor M^\alpha \rfloor$, let $x_i = i\lfloor M^\alpha \rfloor$ and note that

$$\begin{aligned}
&\sum_{i=1}^{k(\alpha)}\big(\mathbb{E}\big[\bar{w}_{x,t+1}^2\big] - \big(\mathbb{E}[\bar{w}_{x,t+1}]\big)^2\big) \\
&\quad = \frac{1}{2}M^{-2}(M-1)^{-1}(M-3)^{-1}\sum_{i=1}^{k(\alpha)} x_i(x_i-1)(M-x_i)\left(1 - \frac{x_i}{M-1}\right) \qquad (4.28) \\
&\quad = O\big(M^{-\alpha}\big).
\end{aligned}$$

From Markov's inequality, we have that

$$\begin{aligned}
&\mathbb{P}\left[\exists i \text{ such that } \frac{|\bar{z}_{x_i,(M-x_i)/2} - \mathbb{E}[\bar{z}_{x_i,(M-x_i)/2}]|}{M} > M^{-\frac{\alpha}{3}}\right] \\
&\quad = \mathbb{P}\left[\exists i \text{ such that } \frac{|\bar{z}_{x_i,(M-x_i)/2} - \mathbb{E}[\bar{z}_{x_i,(M-x_i)/2}]|^2}{M^2} > M^{-\frac{2\alpha}{3}}\right] \qquad (4.29) \\
&\quad = \mathbb{P}\big[\exists i \text{ such that } \big|\bar{w}_{x_i,(M-x_i)/2} - \mathbb{E}[\bar{w}_{x_i,(M-x_i)/2}]\big|^2 > M^{-2\alpha/3}\big] \\
&\quad = O\big(M^{-\alpha/3}\big)
\end{aligned}$$

and thus we have that w.h.p. for every $1 \le i \le k(\alpha)$,

$$\left|\bar{z}_{x_i,(M-x_i)/2} - \mathbb{E}[\bar{z}_{x_i,(M-x_i)/2}]\right| = O\left(M^{1-\alpha/3}\right). \tag{4.30}$$

Now suppose that $x_i \le x \le x_{i+1}$. Then, clearly, $\bar{z}_{x_i,(M-x_i)/2} - M^\alpha \le \bar{z}_{x,(M-x)/2} \le \bar{z}_{x_i,(M-x_i)/2} + M^\alpha$, and via (4.19) and (4.26) we conclude that w.h.p. for every $1 \le x \le M$ we have

$$\begin{aligned}
&\left|\bar{z}_{x,(M-x)/2} - \mathbb{E}[\bar{z}_{x,(M-x)/2}]\right| \\
&\quad \le |\bar{z}_{x,(M-x)/2} - \bar{z}_{x_i,(M-x_i)/2}| + \left|\bar{z}_{x_i,(M-x_i)/2} - \mathbb{E}[\bar{z}_{x_i,(M-x_i)/2}]\right| \\
&\qquad + \left|\mathbb{E}[\bar{z}_{x_i,(M-x_i)/2}] - \mathbb{E}[\bar{z}_{x,(M-x)/2}]\right| \\
&\quad = |\bar{z}_{x,(M-x)/2} - \bar{z}_{x_i,(M-x_i)/2}| + \left|\bar{z}_{x_i,(M-x_i)/2} - x_i\left(\frac{M-x_i}{M-1}\right)\right| \\
&\qquad + \left|x_i\left(\frac{M-x_i}{M-1}\right) - x\left(\frac{M-x}{M-1}\right)\right| \\
&\quad \le M^\alpha + O\left(M^{1-\alpha/3}\right) + M^\alpha.
\end{aligned} \tag{4.31}$$

Now let γ_s be any configuration on the path $\gamma : \boxminus \to \boxplus$ defined above. Then w.h.p.

$$\begin{aligned}
&\mathcal{H}(\gamma_s) - \mathcal{H}(\boxminus) \\
&\quad = J\left|E(\gamma_s, \overline{\gamma_s})\right| - hs \\
&\quad = J\bar{z}_{\ell_{\gamma_s},(\ell_n-\ell_{\gamma_s})/2} - hs = J\ell_{\gamma_s}\left(1 - \frac{\ell_{\gamma_s}}{\ell_n}\right) - hs + O\left(\ell_n^{3/4}\right),
\end{aligned} \tag{4.32}$$

where the last line follows from (4.31) with $x = \ell_m$, $M = \ell_n$ and $\alpha = \frac{3}{4}$, and uses the fact that $\mathbb{E}[\bar{z}_{x,t}] = x(1 - \frac{x-1}{x+2t+1})$. By definition, this quantity is maximised when ℓ_{γ_s} is replaced by $\ell_{\bar{m}}$, from which the statement of the theorem follows. □

PROOF OF THEOREM 1.8. Setting $x = \frac{1}{2}$ in (2.11), we get that the probability of there being any configuration of total degree $\ell_n/2$ having a boundary size $y\ell_n$ is bounded from above by

$$\exp\left[\ell_n \log\left(\frac{1}{2}\eta\left(\frac{1}{2}\right)\left(\frac{1}{2} - y\right)^{-(\frac{1}{2}-y)} y^{-y}\right)\right]. \tag{4.33}$$

By the law of large numbers, w.h.p. we have that $n = \ell_n/d_{\text{ave}} + o(n)$. Combining this with $|\{U \subseteq V : \ell_U = \ell_n x\}| \le 2^n$, we get that $\eta(\frac{1}{2}) \le 2^{1/d_{\text{ave}}}$. It follows that if $y < I_{d_{\text{ave}}}(\frac{1}{2})$, then (4.33) decays exponentially. Hence, all configurations σ with $\ell_\sigma = \ell_n/2$ have, w.h.p., an energy at least

$$\mathcal{H}(\sigma) \ge J I_{d_{\text{ave}}}\left(\frac{1}{2}\right)\ell_n - h|\sigma| + \mathcal{H}(\boxminus), \tag{4.34}$$

and the lower bound on $\Gamma_n^\star$ in (1.29) follows. □

5. Tail properties of the dynamically constructed CM_n. In this section, we explore some properties of the dynamical construction of CM_n introduced in Section 4.1.

5.1. *Trivial tail σ-algebra.* Let $V_n = (v_1, \dots, v_n)$ be the vertices with corresponding degree sequence

$$\vec{d}_n = (d_1, \dots, d_n), \tag{5.1}$$

and let $G_n = (V_n, E_n)$ and $G_{n'} = (V_n, E_{n'})$ be two independent configuration models with the same degree sequence $\vec{d}_n$. We will extend G_n and $G_{n'}$ to larger graphs, $G_{n+t} = (V_{n+t}, E_{n+t})$ and $G_{n+t'} = (V_{n+t}, E_{n+t'})$, respectively, with degree sequence

$$\vec{d}_n = (d_1, \dots, d_n, d_{n+1}, \dots d_{n+t}), \tag{5.2}$$

by utilising a pairing scheme similar to the one introduced in Section 4.1:

- If ξ_m is a uniform random matching of the integers $\{1, \dots, 2m\}$, denoted by $\xi_m = \{(x_1, x_2), \dots, (x_{2m-1}, x_{2m})\}$, then take u_1 to be uniform on $\{1, \dots, 2m\}$ and u_2 to be uniform on $\{1, \dots, 2m, 2m+1\}$. If $u_2 = 2m+1$, then set $\xi_{m+1} = \xi_m \cup \{(2m+1, 2m+2)\}$. Otherwise add to ξ_m the pairs $(2m+1, u_1)$ and $(2m+2, u_2)$ when $u_1 \neq u_2$, and when $u_1 = u_2$, only add to ξ_m the pair $(2m+2, u_2)$. In either case, if there are two remaining terms that are unpaired, then pair them to each other and add this pair to ξ_m. Needless to say, we also remove from ξ_m old pairs that were undone by the introduction of $2m+1$ and $2m+2$. Again, this construction leads to ξ_{m+1}, a uniform matching of the points $\{1, \dots, 2m+2\}$.

Now construct the coupled graphs $(G_{n+t}, G_{n+t'})$ by starting with $(G_n, G_{n'})$ and using the same uniform choice

$$\{u_i\}_{i=1}^{|\vec{d}_{n+t}|-|\vec{d}_n|} \tag{5.3}$$

to determine new edges in both graphs. Note that, under this scheme, every term (half-edge) $s > |\vec{d}_n|$ is paired with the same term in G_{n+t} as in $G_{n+t'}$. In other words, for all $1 \leq j \leq |\vec{d}_{n+t}|$ we have $(s, j) \in E_{n+t}$ if and only if $(s, j) \in E_{n+t'}$. For $s \leq |\vec{d}_n|$ and $1 \leq j \leq |\vec{d}_{n+t}|$, we have

$$\begin{aligned} &\mathbb{P}[\mathbf{1}_{\{(s,j)\in E_{n+t}\}} \neq \mathbf{1}_{\{(s,j)\in E_{n+t'}\}}] \\ &\quad \leq \mathbb{P}\Bigg[\bigcap_{i=1}^{|\vec{d}_{n+t}|-|\vec{d}_n|} \{u_i \neq s\}\Bigg] = \prod_{i=|\vec{d}_n|}^{|\vec{d}_{n+t}|} \left(1 - \frac{1}{i-1}\right) = \frac{|\vec{d}_n| - 2}{|\vec{d}_{n+t}| - 1}. \end{aligned} \tag{5.4}$$

Hence,

$$\mathbb{P}\Bigg[\bigcup_{s=1}^{|\vec{d}_n|} \{\mathbf{1}_{\{(s,j)\in E_{n+t}\}} \neq \mathbf{1}_{\{(s,j)\in E_{n+t'}\}}\}\Bigg] \leq \frac{|\vec{d}_n|(|\vec{d}_n| - 2)}{|\vec{d}_{n+t}| - 1}. \tag{5.5}$$

Thus, we conclude that $\mathbb{P}[G_{n+t} \neq G_{n+t'}] = O(\frac{1}{t})$.

We can now make the following standard argument to show that the process above has a trivial tail-sigma-algebra. Let $\mathscr{F}_t = \sigma(\xi_1, \dots, \xi_t)$ and $\mathscr{F}_t^+ = \sigma(\xi_{t+1}, \dots)$. The tail sigma-algebra is given by $\mathscr{T} = \bigcap_{n\in\mathbb{N}} \mathscr{F}_t^+$. For any $A \in \mathscr{T}$, there is a sequence of events $A_1, A_2, \dots$ such that

$$\lim_{t\to\infty} \mathbb{P}[A_t \triangle A] = 0, \tag{5.6}$$

and hence also

$$\lim_{t\to\infty} \mathbb{P}[A_t \cap A] = \mathbb{P}[A], \qquad \lim_{t\to\infty} \mathbb{P}[A_t] = \mathbb{P}[A]. \tag{5.7}$$

But, since $A \in \mathscr{F}_t^+$ for all t, it follows that $\mathbb{P}[A_t \cap A] = \mathbb{P}[A_t]\mathbb{P}[A]$, and hence $\mathbb{P}[A] = \mathbb{P}[A]^2$. This shows that $\mathscr{T}$ is a trivial sigma-algebra. Therefore, given $\{d_i\}_{i\in\mathbb{N}}$ (but also by the law of large numbers for i.i.d. sequences),

$$\limsup_{n\to\infty} \frac{\Gamma_n^\star}{n} = \gamma_+^*, \qquad \liminf_{n\to\infty} \frac{\Gamma_n^\star}{n} = \gamma_-^*, \tag{5.8}$$

for some $\gamma_+^*, \gamma_-^* \in \mathbb{R}$ with $\gamma_+^* \geq \gamma_-^*$.

5.2. *Oscillation bounds*. To show that $\gamma_+^* = \gamma_-^*$, it would suffice to prove Conjecture 1.10. By means of the following lemma, we will show in (5.13) that, for the dynamic construction given above, it is possible to obtain bounds on potential oscillations of $n \mapsto \Gamma_n^\star/n$.

LEMMA 5.1. *Let $G = (V, E)$ and $\tilde{G} = (\tilde{V}, \tilde{E})$ be two finite, connected graphs. Suppose that $|E \Delta \tilde{E}| \leq k$ under some labelling of the vertices in V and $\tilde{V}$ (i.e., a one-to-one map from the smaller to the larger of the sets V and $\tilde{V}$). Then*

$$\left|\Gamma^\star - \tilde{\Gamma}^\star\right| \leq Jk + h\big||\tilde{V}| - |V|\big|. \tag{5.9}$$

PROOF. Without loss of generality assume that $|\tilde{V}| \geq |V|$. Given the labelling of the vertices that satisfies the above condition, let $\gamma : \boxminus \to \boxplus$, denoted by $\gamma = (\gamma_1, \dots, \gamma_m)$, be an optimal path for the Glauber dynamics on G. Now let $\tilde{\gamma} : \tilde{\boxminus} \to \tilde{\boxplus}$ be the Glauber path of configurations on $\tilde{G}$, denoted by $\tilde{\gamma} = (\tilde{\gamma}_1, \dots, \tilde{\gamma}_m, \tilde{\gamma}_{m+1}, \dots, \tilde{\gamma}_{m+|\tilde{V}|-|V|})$, and defined by the following rule: whichever vertex $v \in V$ is flipped at step i in the path γ, flip the corresponding vertex $\tilde{v} \in \tilde{V}$ also at step i in $\tilde{\gamma}$. For steps $m+1, \dots, m+|\tilde{V}|-|V|$, flip the remaining -1 valued vertices in any arbitrary order. Then it follows that, for $1 \leq i \leq m$,

$$\tilde{\mathcal{H}}(\tilde{\gamma}_i) - \tilde{\mathcal{H}}(\tilde{\boxminus}) = J\big|\tilde{E}(\tilde{\gamma}_i, \overline{\tilde{\gamma}_i})\big| - h|\tilde{\gamma}_i| \leq J\big(\big|E(\gamma_i, \overline{\gamma_i})\big| + k\big) - h|\tilde{\gamma}_i|. \tag{5.10}$$

Similarly, for $m \leq i \leq m + |\tilde{V}| - |V|$, we have

$$\tilde{\mathcal{H}}(\tilde{\gamma}_i) - \tilde{\mathcal{H}}(\tilde{\boxminus}) \leq Jk - h\big(|\gamma_i| - |V|\big). \tag{5.11}$$

It follows that $\tilde{\Gamma}^\star \le \Gamma^\star + Jk - h(|\tilde{V}| - |V|)$. A similar argument gives $\Gamma^\star \le \tilde{\Gamma}^\star + Jk + h||\tilde{V}| - |V||$. $\square$

Now let $G = (V, E)$ and $\tilde{G} = (\tilde{V}, \tilde{E})$ be two configuration models and suppose w.l.o.g. that the total degree of the vertices in V is ℓ_V and the total degree of vertices in $\tilde{V}$ is $\ell_{\tilde{V}} \ge \ell_V$. Let G_t and $\tilde{G}_t$ be the extension of each these two graphs, obtained by adding vertices $\{v_1, \dots, v_t\}$ and $\{\tilde{v}_1, \dots, \tilde{v}_t\}$, both with the same degree sequence $\{d_1, \dots, d_t\}$. We will couple the construction leading to the two graphs G_t and $\tilde{G}_t$ in the following manner: for $1 \le i \le \sum_{k=1}^t d_k$, choose u_i uniformly as described above, and pair i with u_i in G_t. Let $\delta_i = (\ell_{\tilde{V}} - \ell_V)/[(\ell_{\tilde{V}} + i - 1)(\ell_V + i - 1)]$ and set $\tilde{u}_i = u_i$ with probability $1 - \delta_i$, and with probability δ_i independently and uniformly pick one of the remaining $(\ell_{\tilde{V}} - \ell_V)$ points. Then

$$\mathbb{E}\big[|E_t \Delta \tilde{E}_t|\big] \le |E \Delta \tilde{E}| + \sum_i \delta_i \le |E \Delta \tilde{E}| + 2(\ell_{\tilde{V}} - \ell_V). \tag{5.12}$$

Hence, from Markov's inequality and from Lemma 5.1, it follows that, w.h.p. and for any function $f(t)$ such that $\lim_{t\to\infty} f(t) = \infty$,

$$\big|\tilde{\Gamma}^\star{}_t - \Gamma^\star_t\big| \le J\big(|E \Delta \tilde{E}| + 2(\ell_{\tilde{V}} - \ell_V) + f(t)\big) - h\big(\big||\tilde{V}| - |V|\big|\big). \tag{5.13}$$

Hence, by this pairing scheme, we have that $\tilde{\Gamma}^\star_t / \Gamma^\star_t \to 1$ as $t \to \infty$.

APPENDIX

For the Erdős–Rényi random graph with n vertices and percolation parameter p, any configuration σ chosen uniformly from all configurations of size $|\sigma|$ satisfies ($\mathbb{E}$ denotes expectation w.r.t. bond percolation)

$$\mathbb{E}\big(\big|E(\sigma, \overline{\sigma})\big|\big) = \mu_{|\sigma|} \qquad \text{with } \mu_{|\sigma|} = p|\sigma|\big(n - |\sigma|\big). \tag{A.1}$$

Using Chernoff's inequality and a union bound, we can show that if $|\sigma| = \Theta(n)$, then

$$\mathcal{H}(\sigma) - \mathcal{H}(\boxminus) = J\mu_{|\sigma|}\big[1 \pm o(1)\big] - h|\sigma|. \tag{A.2}$$

Furthermore, any σ of size $|\sigma| \le [1 - o(1)]\frac{n}{2}$ has (modulo small fluctuations) a downhill path to $\boxminus$ (e.g., by flipping $+1$ spins in any arbitrary order), while every $|\sigma| \ge [1 + o(1)]\frac{n}{2}$ has a downhill path to $\boxplus$ (e.g., by flipping -1 spins in any arbitrary order). This proves hypothesis (H), and the claim in (1.21).

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