

## NUMERICAL SIMULATION OF QUADRATIC BSDES

BY JEAN-FRANÇOIS CHASSAGNEUX<sup>1</sup> AND ADRIEN RICHOU

*Imperial College London and Université de Bordeaux*

This article deals with the numerical approximation of Markovian backward stochastic differential equations (BSDEs) with generators of quadratic growth with respect to  $z$  and bounded terminal conditions. We first study a slight modification of the classical dynamic programming equation arising from the time-discretization of BSDEs. By using a linearization argument and BMO martingales tools, we obtain a comparison theorem, a priori estimates and stability results for the solution of this scheme. Then we provide a control on the time-discretization error of order  $\frac{1}{2} - \varepsilon$  for all  $\varepsilon > 0$ . In the last part, we give a fully implementable algorithm for quadratic BSDEs based on quantization and illustrate our convergence results with numerical examples.

**1. Introduction.** In this paper, we are interested in the numerical approximation of solutions to a special class of backward stochastic differential equations (BSDEs for short in the sequel). Let us recall that solving a BSDE consists in finding an adapted couple  $(Y, Z)$  satisfying the equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

where  $W$  is a  $d$ -dimensional Brownian motion on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . We denote by  $(\mathcal{F}_t)_{0 \leq t \leq T}$  the Brownian filtration. In their seminal paper [38], Pardoux and Peng prove the existence of a unique solution  $(Y, Z)$  to this equation for a given square integrable terminal condition  $\xi$  and a Lipschitz random driver  $f$ . Many extensions to this Lipschitz setting have been considered. In particular, the class of BSDE, with generators of quadratic growth with respect to the variable  $z$ , has received a lot of attention in recent years. These equations arise, by example, in the context of utility optimization problems with exponential utility functions, or alternatively in questions related to risk minimization for the entropic risk measure (see, e.g., [27, 36, 41] among many other references). Existence and uniqueness of solution for such BSDEs has been first proved by Kobylanski [34]. Since then, many authors worked on this question. When the terminal condition is bounded, we refer to [7, 34, 35, 42], and, in the unbounded case, we refer to [3, 8, 9, 19, 20].

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We will focus here on the numerical approximation of the so-called “quadratic BSDE” in a Markovian setting, namely

$$(1.1) \quad X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s,$$

$$(1.2) \quad Y_t = g(X_T) + \int_t^T f(X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.$$

Throughout this paper, we assume that the functions  $b: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ ,  $\sigma: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  are  $K$ -Lipschitz continuous functions and the function  $g$  is a bounded  $K$ -Lipschitz continuous function, for a positive constant  $K$ . We also assume that the function  $f: \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$  is a  $K$ -Lipschitz continuous function with respect to  $x$  and  $y$ , that is,

$$|f(x_1, y_1, z) - f(x_2, y_2, z)| \leq K(|x_1 - x_2| + |y_1 - y_2|)$$

for all  $y_1, y_2 \in \mathbb{R}$ ,  $x_1, x_2 \in \mathbb{R}^d$  and  $z \in \mathbb{R}^{1 \times d}$ , and a  $L$ -locally Lipschitz continuous function with respect to  $z$ : for all  $x \in \mathbb{R}^d$ ,  $y \in \mathbb{R}$ ,  $z, z' \in \mathbb{R}^{1 \times d}$ ,

$$|f(x, y, z) - f(x, y, z')| \leq L(1 + |z| + |z'|)|z - z'|,$$

where  $L$  is a positive constant. Moreover  $f$  is bounded with respect to  $x$ : for all  $x \in \mathbb{R}^d$ ,  $y \in \mathbb{R}$ ,  $z \in \mathbb{R}^{1 \times d}$ ,

$$|f(x, y, z)| \leq L(1 + |y| + |z|^2).$$

Let us notice that all convergence results obtained in this paper do not need extra assumptions on  $b, \sigma, f$  and  $g$ . Especially, we emphasize that no uniform ellipticity condition is necessary on  $\sigma$ .

1.1. *Known results on the approximation of quadratic BSDEs.* The design of efficient algorithms to solve BSDEs in any reasonable dimension has been intensively studied since the first work of Chevance [15]; see, for example, [5, 10, 11, 23, 43] and the references therein. In all these articles, the driver  $f$  of the BSDE is a Lipschitz function with respect to  $z$  and this assumption plays a key role in the proofs.

Up to now, there have been few results on the time-discretization and numerical simulation of quadratic BSDEs. We review now all the techniques that allow to compute the solution of quadratic BSDEs, to the best of our knowledge. None of them provide a suitable complete answer to the approximation of the BSDE (1.2).

First of all, when the generator has a specific form (roughly speaking the generator is a sum of a purely quadratic term  $z \mapsto C|z|^2$  and a Lipschitz function) it is possible to solve almost explicitly the quadratic BSDE by using an exponential transformation method, also called Cole–Hopf transform (see, e.g., [29]).

It is also possible to solve some specific quadratic Markovian BSDEs by solving a fully coupled forward backward system, that is, when  $Y$  and  $Z$  appear also in the

coefficients of (1.1). This is the method used by Delarue and Menozzi in [17, 18] where they solved in particular the deterministic KPZ equation. But approximation results for fully coupled forward–backward systems need strong assumptions on the regularity of coefficients and a uniform ellipticity assumption for  $\sigma$ . Moreover, their implementation is not straightforward (due to the coupling).

In some cases, one can also rely on “classical” schemes for Lipschitz BSDEs in order to numerically solve quadratic BSDEs. Indeed, when the terminal condition  $g$  is a bounded Lipschitz-continuous function and  $\sigma$  is bounded then it is known that  $Z$  is bounded by a constant  $M$  (see, e.g., Theorem 3.6 in [40]). Since the generator  $f$  is assumed to be locally Lipschitz with respect to  $z$ , one only needs to replace the generator  $f$  by a new generator  $\tilde{f}_M(\cdot, \cdot, \cdot) = f(\cdot, \cdot, \varphi_M(\cdot))$  where  $\varphi_M$  is the projection on the centered Euclidean ball of radius  $M$ . Then one can easily show that these two BSDEs with generators  $f$  and  $\tilde{f}_M$  have the same solution. It is then possible to solve the second BSDE with Lipschitz driver  $\tilde{f}_M$  to retrieve the solution to the quadratic BSDE. Let us remark that some exponential terms appear in the constant  $M$  which lead to a new generator with possibly huge Lipschitz constant with respect to  $z$  and may cause numerical difficulties; see [4].

In the general case,  $Z$  may be unbounded. Nevertheless, when  $g$  is a bounded Lipschitz function and  $\sigma$  is Lipschitz but not necessarily bounded the following nonuniform bound holds true

$$(1.3) \quad |Z_t| \leq C(1 + |X_t|) \quad \text{for all } t \leq T;$$

see, for example, Theorem 3.6 in [40].

Now, replacing the generator  $f$  with the Lipschitz generator  $\tilde{f}_M$  we obtain a solution  $(Y^M, Z^M)$  which is different from  $(Y, Z)$ . But it is possible to estimate the error between the two using the estimate on  $Z$ . The error is bounded by  $\frac{C_p}{M^p}$  for every  $p > 1$ ; see [28, 40]. Once again, since the new generator  $\tilde{f}_M$  is Lipschitz, we can easily apply classical numerical approximation schemes for Lipschitz BSDEs. Problems occur when one tries to obtain a rate of convergence for this technique. The classical (squared) error estimate for the discrete-time approximation of Lipschitz BSDEs is  $\frac{C}{n}$  with  $n$  the number of time steps, but the constant  $C$  depends strongly on the Lipschitz constant of  $\tilde{f}_M$  with respect to  $z$  and so it depends on  $M$ ; see, for example, [5, 43]. In fact, one obtains an upper bound for the time-discretization error (squared) of order  $Ce^{CM^2}n^{-1}$ , the exponential term resulting from the use of Gronwall’s lemma. Finally, an upper bound of the global error (squared) equals to

$$\frac{C_p}{M^p} + \frac{Ce^{CM^2}}{n}.$$

When  $M$  increases,  $n^{-1}$  will have to be small exponentially fast. The resulting rate of convergence turns out to be bad: setting  $M = (\log n)^{1/2}$  the global error bound becomes  $C_p(\log n)^{-p}$  which is not satisfactory.

To circumvent the above difficulties, one can impose a specific growth assumption on  $\sigma$ , leading to exponential moment control on  $X$ , in order to retrieve a better bound for the error between  $(Y, Z)$  and  $(Y^M, Z^M)$ . In this case, the global error becomes satisfactory; see Theorem 5.9 in [40]. Reasonable convergence rate can also be retrieved for unbounded locally Lipschitz-continuous terminal conditions, using estimates in the spirit of (1.3), but in the very restrictive case of constant  $\sigma$ ; see Theorem 5.7 in [40]. Note that dealing with an unbounded terminal condition is already a challenge for the theoretical study of (1.2).

In this paper, we focus on *Lipschitz-continuous bounded terminal condition* and *unbounded Lipschitz-continuous  $\sigma$* . This covers the case of models with great practical interest as geometric Brownian motion (Black–Scholes model). Using a similar truncation procedure as the one described above, we are able to obtain a bound on the time discretization error which does not depend on  $M$ . The global (squared) error bound is shown to be almost the classical one, that is to say  $\frac{C_\varepsilon}{n^{1-\varepsilon}}$ , for all  $\varepsilon > 0$ .

Let us conclude this review with the case of non-Lipschitz bounded terminal condition. In this case—even in the Lipschitz setting for the generator—new difficulties arise in the simulation of BSDEs; see, for example, [24]. In the quadratic case, when  $\sigma$  is bounded, it is possible to use estimates of the form

$$|Z_t| \leq \frac{C}{\sqrt{T-t}} \quad \text{or} \quad |Z_t| \leq \frac{C}{(T-t)^{(1-\alpha)/2}}$$

if the terminal condition is  $\alpha$ -Hölder; see [16, 39]. Thanks to these estimates one can replace the generator  $f$  by a Lipschitz generator such that the Lipschitz constant with respect to  $z$  depends on time and blows up near the time  $T$ . The time discretization problem is addressed in [39] and the approximation of discretized BSDEs thanks to least-squares regression is tackled in the paper [25]. In these two papers, the time-discretization grid is not uniform taking into account the estimates on  $Z$ . In particular, there are more points near the terminal time  $T$  than near the initial time. We think that it would be very interesting to try to extend our results and techniques in the case of irregular terminal conditions.

*1.2. Main results of the paper.* We now present in more depth our main results. As already mentioned, to tackle the problem of the numerical approximation of (1.2), we introduce a Lipschitz-continuous approximation of the driver  $f$ :  $f_N(\cdot, \cdot, \cdot) = f(\cdot, \cdot, \varphi_N(\cdot))$  and  $\varphi_N$  is the projection on the centered Euclidean ball of radius  $\rho N$  with  $\rho > 0$  chosen such that  $f_N$  is  $N$ -Lipschitz-continuous with respect to  $z$ .

Given a grid  $\pi = \{0 = t_0 < t_1 < \dots < t_n = T\}$  of the time interval  $[0, T]$ , we define  $h_i = t_{i+1} - t_i$  the time-step between times  $t_i$  and  $t_{i+1}$ , and  $h := \max_i h_i$  assuming that

(1.4)  $hn \leq C$  and there exists  $\theta \geq 1$  such that  $h_i n^\theta \geq C > 0, 0 \leq i < n$ .

Here and in the sequel,  $C$  is a positive constant, which may change from line to line but which does not depend on  $n$ . We denote it  $C_p$  if it depends on an extra parameter  $p$ .

DEFINITION 1.1. We denote  $(Y_i^\pi, Z_i^\pi)_{0 \leq i \leq n}$  the solution of the BTZ<sup>2</sup>-scheme satisfying:

- (i) the terminal condition is  $(Y_n^\pi, Z_n^\pi) = (g(X_n^\pi), 0)$ ,
- (ii) for  $i < n$ , the transition from step  $i + 1$  to step  $i$  is given by

$$(1.5) \quad \begin{cases} Y_i^\pi = \mathbb{E}_t[Y_{i+1}^\pi + h_i f_N(X_i^\pi, Y_i^\pi, Z_i^\pi)], \\ Z_i^\pi = \mathbb{E}_t[Y_{i+1}^\pi H_i^R], \end{cases}$$

where  $\mathbb{E}_t[\cdot]$  stands for  $\mathbb{E}[\cdot | \mathcal{F}_t]$ ,  $0 \leq t \leq T$ .

The discrete-time process  $(X_i^\pi)_{0 \leq i \leq n}$  is an approximation of  $(X_t)_{t \in [0, T]}$  and we choose to work here with the Euler scheme

$$\begin{cases} X_0^\pi = x, \\ X_{i+1}^\pi = X_i^\pi + b(X_i^\pi)h_i + \sigma(X_i^\pi)(W_{t_{i+1}} - W_{t_i}), \quad 0 \leq i < n. \end{cases}$$

The coefficients  $(H_i^R)_{0 \leq i < n}$  are some  $\mathbb{R}^{1 \times d}$  independent random vectors defined, given  $R > 0$ , by

$$(1.6) \quad (H_i^R)^\ell = \frac{-R}{\sqrt{h_i}} \vee \frac{W_{t_{i+1}}^\ell - W_{t_i}^\ell}{h_i} \wedge \frac{R}{\sqrt{h_i}}, \quad 1 \leq \ell \leq d.$$

We observe that  $(H_i^R)_{0 \leq i < n}$  satisfies

$$(1.7) \quad \begin{aligned} \mathbb{E}_t[H_i^R] &= 0, \\ h_i \mathbb{E}_t[(H_i^R)^\top H_i^R] &= h_i \mathbb{E}[(H_i^R)^\top H_i^R] = c_i I_{d \times d} \quad \text{and} \quad \frac{\lambda}{d} \leq c_i \leq \frac{\Lambda}{d}, \end{aligned}$$

where  $\lambda, \Lambda$  are positive constants that do not depend on  $R$ , for  $R$  big enough. Moreover, it is well known (see, e.g., [33]) that, under the Lipschitz continuity assumption on  $b$  and  $\sigma$ ,

$$(1.8) \quad \begin{aligned} \mathbb{E} \left[ \sup_{0 \leq i \leq n} |X_i^\pi|^{2p} \right] &\leq C_p \quad \text{and} \\ \max_{0 \leq i \leq n} \mathbb{E} \left[ \sup_{t \in [t_i, t_{i+1}]} |X_t - X_i^\pi|^{2p} \right] &\leq C_p h^p, \quad p \geq 1. \end{aligned}$$

Combining (1.7), (1.8) and the Lipschitz continuity property of  $f_N$ , an easy induction argument proves that  $(Y^\pi, Z^\pi)$  are square integrable, and thus conditional

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<sup>2</sup>Bouchard–Touzi–Zhang, the first authors to consider this scheme; see [5, 43].

expectations involved at each step of the algorithm are well defined. Moreover, assuming  $Kh < 1$  allows for the implicit definition of  $Y_i^\pi, i < n$ .

The first main result of the paper is the following theorem.

**THEOREM 1.1.** *Setting, for some  $\alpha \in (0, 1/2)$ ,*

$$(1.9) \quad N = n^\alpha \quad \text{and} \quad R = \log(n),$$

*we have, for all  $\eta > 0$ ,*

$$\mathbb{E} \left[ \sup_{0 \leq i \leq n} |Y_{t_i} - Y_i^\pi|^2 \right] + \mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z_s - Z_i^\pi|^2 ds \right] \leq C_{\alpha, \eta} h^{1-\eta}.$$

The choice of  $N$  and  $R$  as specific functions of  $n$  will be made clear in the following. The truncation procedure guarantees the stability of the scheme. Letting these constants grow with  $n$  guarantees the convergence of the scheme. Obviously, a good balance between the two has to be found.

To obtain this theorem, we first prove stability results for the scheme given in Definition 1.1. This is a priori not straightforward because the Lipschitz constant explodes. In order to do this, we use a linearization argument leading to a comparison theorem and relying on BMO martingales tools. We then study carefully the truncation error induced by the time-discretization. In particular, we have to revisit Zhang’s path regularity result.

One has to observe that the above scheme is still a theoretical one since it assumes a perfect computation of the conditional expectations. In practice, these conditional expectations have to be estimated. Many methods can be used and Theorem 1.1 is a key step toward a complete convergence analysis.

In this paper, we chose to compute the conditional expectation using a Markovian quantization procedure which is now quite well known. We refer to [26, 37] for general results about quantization and [2] for application to American options pricing and to [17] for application to coupled forward–backward SDEs. We present in Section 4 a fully implementable numerical scheme and prove the following upper bound for the convergence error:

$$|Y_0 - \widehat{Y}_0^\pi| \leq C_{\alpha, \eta} h^{(1/2)-\eta} \quad \text{for all } \eta > 0,$$

with  $(\widehat{Y}^\pi, \widehat{Z}^\pi)$  the solution of the scheme (1.1) where conditional expectations are replaced by implementable approximations. See Corollary 4.1 for a suitable choice of parameters.

The rest of this paper is organized as follows. In Section 2, we introduce the linearization tool for discrete schemes and we obtain some very useful estimates on  $(Y^\pi, Z^\pi)$  together with some stability results. Section 3 is devoted to the convergence analysis of the time discretization for quadratic BSDEs. In the last section, we give a fully implementable scheme, we study its convergence error and we provide some numerical illustrations.

**2. Preliminary results.** First of all, let us recall that under the assumptions on the generator  $f$  and the terminal condition  $g$  given in the previous section, existence and uniqueness result holds for (1.1) and (1.2). Moreover, the solution is known to have the following properties; see, for example, [1, 6, 34].

**PROPOSITION 2.1.** *The FBSDE (1.1) and (1.2) has a unique solution  $(X, Y, Z) \in \mathcal{S}^2 \times \mathcal{S}^\infty \times \mathcal{M}^2$ . Moreover, the martingale  $(\int_0^t Z_s dW_s)_{t \in [0, T]}$  belongs to the space of BMO martingales. The  $\mathcal{S}^\infty$  norm of  $Y$  and the BMO norm of  $(\int_0^t Z_s dW_s)_{t \in [0, T]}$  are bounded by a constant that depends only on  $T, |g|_\infty$ , and the constant that appears in the growth assumption on the generator  $f$ .*

BMO martingales theory plays a key role for a priori estimates needed in our study. For details about the theory, we refer the reader to [32]. We now recall the definition of a BMO martingale and introduce some notation. Let  $(M_t)_{0 \leq t \leq T}$  be a martingale for the filtration  $(\mathcal{G}_t)_{0 \leq t \leq T}$ . We say that  $M$  is a BMO martingale if it is a square integrable martingale such that

$$\|M\|_{\text{BMO}(\mathcal{G})}^2 := \sup_{\tau} \mathbb{E}[|M_T - M_{\tau-}|^2 | \mathcal{G}_\tau] < +\infty,$$

where the supremum is taken over all stopping times  $\tau \in [0, T]$ .

**2.1. Lipschitz approximation.** We first recall a key result concerning the Lipschitz approximation of quadratic BSDEs. We introduce  $(Y_t^N, Z_t^N)_{t \in [0, T]}$  the solution of the following BSDE:

$$(2.1) \quad Y_t^N = g(X_T) + \int_t^T f_N(X_s, Y_s^N, Z_s^N) ds - \int_t^T Z_s^N dW_s$$

recalling that  $f_N(\cdot, \cdot, \cdot) = f(\cdot, \cdot, \varphi_N(\cdot))$  and  $\varphi_N$  is the projection on the centered Euclidean ball of radius  $\rho N$  with  $\rho > 0$  chosen such that  $f_N$  is  $N$ -Lipschitz with respect to  $z$ .

**REMARK 2.1.** The results of Proposition 2.1 hold true for processes  $(X, Y^N, Z^N)$ . Importantly the  $\mathcal{S}^\infty$  norm of  $Y^N$  and the BMO norm of  $(\int_0^t Z_s^N dW_s)_{t \in [0, T]}$  are bounded by a constant that does not depend on  $N$ .

**THEOREM 2.2.** *For all  $q > 0$  and  $p \geq 1$ , there exists a constant  $C_{q,p} > 0$  such that*

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |Y_t - Y_t^N|^{2p}\right] + \mathbb{E}\left[\left(\int_0^T |Z_s - Z_s^N|^2 ds\right)^p\right] \leq \frac{C_{q,p}}{N^q}.$$

The proof of this theorem is given by Theorem 6.2 in [28] (see also Remark 5.5 in [40]).

REMARK 2.2. The control of the above error in terms of any power of  $N^{-1}$  legitimizes the choice to set  $N := n^\alpha$  for some  $\alpha > 0$ .

The above result is strongly linked to the following estimate on  $Z$ , and on  $Z^N$ , proved, for example, in [40], stated here for later use.

PROPOSITION 2.3. *Under our standing assumptions, for all  $t \in [0, T]$  and all  $N > 0$ ,*

$$|Z_t^N| + |Z_t| \leq C(1 + |X_t|).$$

Importantly,  $C$  does not depend on  $N$ .

We conclude this section by two technical lemmas.

LEMMA 2.1. *Setting, for all  $i < n$ ,*

$$(2.2) \quad \bar{Z}_i^N := \frac{1}{h_i} \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} Z_s^N ds \right],$$

then

$$\mathbb{E}_{t_i} \left[ \sum_{j=i}^{n-1} h_j |\bar{Z}_j^N|^2 \right] \leq C \quad \text{and} \quad |\bar{Z}_i^N| \leq C \left( 1 + \mathbb{E}_{t_i} \left[ \sup_{t_i \leq s \leq t_{i+1}} |X_s| \right] \right).$$

PROOF. 1. For the first claim, we observe that, for  $i \leq j < n$ ,

$$\mathbb{E}_{t_i} [ |\bar{Z}_j^N|^2 ] \leq \frac{1}{h_j} \mathbb{E}_{t_i} \left[ \int_{t_j}^{t_{j+1}} |Z_s^N|^2 ds \right].$$

Summing over  $j$  the previous inequality and using Remark 2.1, we obtain

$$\mathbb{E}_{t_i} \left[ \sum_{j=i}^{n-1} h_j |\bar{Z}_j^N|^2 \right] \leq \mathbb{E}_{t_i} \left[ \int_{t_i}^T |Z_s^N|^2 ds \right] \leq \left\| \int_0^\cdot Z_s^N dW_s \right\|_{\text{BMO}(\mathcal{F})} \leq C.$$

2. For the second claim, we compute

$$|\bar{Z}_i^N| = \frac{1}{h_i} \left| \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} Z_s^N ds \right] \right| \leq \mathbb{E}_{t_i} \left[ \sup_{t_i \leq s \leq t_{i+1}} |Z_s^N| \right] \leq C \left( 1 + \mathbb{E}_{t_i} \left[ \sup_{t_i \leq s \leq t_{i+1}} |X_s| \right] \right),$$

where we used Proposition 2.3.  $\square$

LEMMA 2.2. *We assume that  $\alpha \leq 1/2$ . Setting, for all  $i < n$ ,*

$$(2.3) \quad \tilde{Z}_i^N := \mathbb{E}_{t_i} \left[ Y_{t_{i+1}}^N \frac{(W_{t_{i+1}} - W_{t_i})^\top}{h_i} \right],$$

then

$$\mathbb{E}_{t_i} \left[ \sum_{j=i}^{n-1} h_j |\tilde{Z}_j^N|^2 \right] \leq C \quad \text{and} \quad |\tilde{Z}_i^N| \leq C \left( 1 + \mathbb{E}_{t_i} \left[ \sup_{t_i \leq s \leq t_{i+1}} |X_s|^4 \right]^{1/2} \right).$$

PROOF. 1. For the first claim, we observe that

$$\mathbb{E}_{t_i} \left[ \sum_{j=i}^{n-1} h_j |\tilde{Z}_j^N|^2 \right] \leq 2\mathbb{E}_{t_i} \left[ \sum_{j=i}^{n-1} h_j |\bar{Z}_j^N|^2 \right] + 2\mathbb{E}_{t_i} \left[ \sum_{j=i}^{n-1} h_j |\bar{Z}_j^N - \tilde{Z}_j^N|^2 \right].$$

The first term was already studied in Lemma 2.1. For the second term we compute, thanks to assumptions on  $f_N$ , Remark 2.1 and Cauchy–Schwarz inequality, for  $i \leq j < n$ ,

$$\begin{aligned} h_j \mathbb{E}_{t_i} [|\bar{Z}_j^N - \tilde{Z}_j^N|^2] &= h_j \mathbb{E}_{t_i} \left[ \left| \mathbb{E}_{t_j} \left[ \int_{t_j}^{t_{j+1}} f_N(X_s, Y_s^N, Z_s^N) ds \frac{W_{t_{j+1}} - W_{t_j}}{h_j} \right] \right|^2 \right] \\ &\leq h_j \mathbb{E}_{t_i} \left[ \int_{t_j}^{t_{j+1}} |f_N(X_s, Y_s^N, Z_s^N)|^2 ds \right] \\ &\leq C \left( h^2 + (1 + N^2 h) \mathbb{E}_{t_i} \left[ \int_{t_j}^{t_{j+1}} |Z_s^N|^2 ds \right] \right). \end{aligned}$$

Summing over  $j$ , we obtain

$$\mathbb{E}_{t_i} \left[ \sum_{j=i}^{n-1} h_j |\bar{Z}_j^N - \tilde{Z}_j^N|^2 \right] \leq C \left( 1 + \left\| \int_0^\cdot Z_s^N dW_s \right\|_{\text{BMO}(\mathcal{F})}^2 \right) \leq C.$$

2. For the second claim, once again we have

$$|\tilde{Z}_i^N| \leq |\bar{Z}_i^N| + |\bar{Z}_i^N - \tilde{Z}_i^N|.$$

The first term is dealt with combining Lemma 2.1 and Cauchy–Schwarz inequality. For the second term, we compute, thanks to the growth assumption on  $f_N$ , Remark 2.1, Proposition 2.3 and the Cauchy–Schwarz inequality,

$$\begin{aligned} (2.4) \quad |\bar{Z}_i^N - \tilde{Z}_i^N| &\leq C \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} |f_N(X_s, Y_s^N, Z_s^N)| ds \frac{|W_{t_{i+1}} - W_{t_i}|}{h_j} \right] \\ &\leq C \mathbb{E}_{t_i} \left[ \left( 1 + \sup_{t_i \leq s \leq t_{i+1}} |X_s|^2 \right) |W_{t_{i+1}} - W_{t_i}| \right] \end{aligned}$$

$$(2.5) \quad \leq Ch^{1/2} \left( 1 + \mathbb{E}_{t_i} \left[ \sup_{t_i \leq s \leq t_{i+1}} |X_s|^4 \right]^{1/2} \right). \quad \square$$

2.2. Linearization of the BTZ scheme.

DEFINITION 2.1. We consider the solution  $(Y_i, Z_i)_{0 \leq i \leq n}$  of the following BTZ scheme:

- (i) the terminal condition is given by  $Y_n = \xi$  for some  $\xi \in L^2(\mathcal{F}_T)$  and  $Z_n = 0$ ;

(ii) for  $0 \leq i < n$ , the transition from step  $i + 1$  to step  $i$  is given by

$$\begin{cases} Y_i = \mathbb{E}_{t_i}[Y_{i+1} + h_i F_i(Y_i, Z_i)], \\ Z_i = \mathbb{E}_{t_i}[Y_{i+1} H_i], \end{cases}$$

with  $(H_i)_{0 \leq i < n}$  some  $\mathbb{R}^{1 \times d}$  independent random vectors such that, for all  $0 \leq i < n$ ,  $H_i$  is  $\mathcal{F}_{t_{i+1}}$  measurable,  $\mathbb{E}_{t_i}[H_i] = 0$ ,

$$(2.6) \quad c_i I_{d \times d} = h_i \mathbb{E}[H_i^\top H_i] = h_i \mathbb{E}_{t_i}[H_i^\top H_i],$$

and

$$(2.7) \quad \frac{\lambda}{d} \leq c_i \leq \frac{\Lambda}{d},$$

where  $\lambda, \Lambda$  are positive constants. Let us remark that (2.6) and (2.7) imply that

$$(2.8) \quad \lambda \leq h_i \mathbb{E}[|H_i|^2] = h_i \mathbb{E}_{t_i}[|H_i|^2] \leq \Lambda.$$

For the reader’s convenience, we denote the above scheme by  $\mathcal{E}[(F_i), \xi]$ .

In the sequel, we use the following assumption on the coefficients of the scheme given in Definition 2.1.

ASSUMPTION (H1). (i) Functions  $F_i : \Omega \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$  are  $\mathcal{F}_{t_i} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable. They satisfy for some positive constants  $K_y$  and  $K_z^n$  which do not depend on  $i$  but  $K_z^n$  may depend on  $n$ ,

- $F_i(0, 0) \in L^2(\mathcal{F}_{t_i})$ ,
- $|F_i(y, z) - F_i(y', z')| \leq K_y |y - y'| + K_z^n |z - z'|$ .

(ii) For a given  $\varepsilon \in ]0, 1[$  which does not depend on  $n$ , we have that

$$h K_y < 1 - \varepsilon.$$

(iii) The following holds:

$$\left( \sup_{0 \leq i \leq n-1} h_i |H_i| \right) K_z^n < 1.$$

Observe that (H1)(ii) guarantees the well-posedness of the scheme.

We now give a representation result for the difference of two BTZ scheme solutions. Let  $(Y_i^1, Z_i^1)_{0 \leq i \leq n}$  be the solution of  $\mathcal{E}[(F_i^1), \xi^1]$  and  $(Y_i^2, Z_i^2)_{0 \leq i \leq n}$  be the solution of  $\mathcal{E}[(F_i^2), \xi^2]$ .

We denote  $\delta Y_i = Y_i^1 - Y_i^2$ ,  $\delta Z_i = Z_i^1 - Z_i^2$  and  $\delta F_i = F_i^1(Y_i^2, Z_i^2) - F_i^2(Y_i^2, Z_i^2)$ . Then, we have the following representation result.

PROPOSITION 2.4 (Euler scheme linearization). *Assume that  $F^1$  satisfies (H1)(i)–(ii). Setting, for  $0 \leq i \leq n$ ,*

$$E_i^\pi = \prod_{j=i}^{n-1} (1 + h_j H_j \gamma_j) \quad \text{and} \quad B_i^\pi = \prod_{j=i}^{n-1} (1 - h_j \beta_j),$$

with

$$\beta_j = \frac{F_j^1(Y_j^1, Z_j^1) - F_j^1(Y_j^2, Z_j^1)}{Y_j^1 - Y_j^2} \mathbb{1}_{\{Y_j^1 - Y_j^2 \neq 0\}}$$

and

$$\gamma_j = \frac{F_j^1(Y_j^2, Z_j^1) - F_j^1(Y_j^2, Z_j^2)}{|Z_j^1 - Z_j^2|^2} (Z_j^1 - Z_j^2)^\top \mathbb{1}_{\{Z_j^1 - Z_j^2 \neq 0\}},$$

then the following holds:

$$(2.9) \quad \delta Y_i = \mathbb{E}_{t_i} \left[ E_i^\pi (B_i^\pi)^{-1} \left( \delta Y_n + \sum_{k=i}^{n-1} h_k B_{k+1}^\pi \delta F_k \right) \right].$$

We used the convention  $\prod_{j=n}^{n-1} \cdot = 1$ .

PROOF. For  $0 \leq i \leq n - 1$ , we compute that

$$(2.10) \quad \delta Y_i = \mathbb{E}_{t_i} [\delta Y_{i+1} + h_i \beta_i \delta Y_i + h_i \delta Z_i \gamma_i + h_i \delta F_i].$$

Observing that  $\delta Z_i = \mathbb{E}_{t_i} [H_i \delta Y_{i+1}]$ , we obtain

$$\begin{aligned} \delta Y_i &= \frac{1}{1 - h_i \beta_i} \mathbb{E}_{t_i} [(1 + h_i H_i \gamma_i) \delta Y_{i+1} + h_i \delta F_i] \\ &= \frac{1}{1 - h_i \beta_i} \mathbb{E}_{t_i} [(1 + h_i H_i \gamma_i) (\delta Y_{i+1} + h_i \delta F_i)]. \end{aligned}$$

Under (H1)(ii), we observe that  $1 - h_i \beta_i \neq 0$  and the previous equality is well defined. Using an easy induction argument, we obtain

$$\delta Y_i = \mathbb{E}_{t_i} \left[ E_i^\pi (B_i^\pi)^{-1} \left( \delta Y_n + \sum_{k=i}^{n-1} h_k (E_{k+1}^\pi)^{-1} B_{k+1}^\pi \delta F_k \right) \right].$$

The proof is complete using the tower property of conditional expectation and the fact that  $\mathbb{E}_{t_{k+1}} [E_{k+1}^\pi] = 1$ .  $\square$

The previous representation leads to the following comparison result for the BTZ scheme.

COROLLARY 2.5 (Comparison theorem). Assume that  $F^1$  satisfies (H1). If

$$Y_n^1 \geq Y_n^2 \quad \text{and} \quad F_i^1(Y_i^2, Z_i^2) \geq F_i^2(Y_i^2, Z_i^2), \quad 0 \leq i \leq n - 1,$$

then we have that

$$Y_i^1 \geq Y_i^2, \quad 0 \leq i \leq n.$$

PROOF. We will use the BTZ scheme linearization given in Proposition 2.4. Since  $|\beta_i| \leq K_y$  and  $|\gamma_i| \leq K_z^n$ , the condition  $(\sup_{0 \leq i < n} h_i |H_i|) K_z^n < 1$  combined with  $hK_y < 1$ , implies that the coefficients  $E_i^\pi, B_i^\pi$  are positive, for  $i < n$ . Moreover, we assume that

$$Y_n^1 \geq Y_n^2 \quad \text{and} \quad F_i^1(Y_i^2, Z_i^2) \geq F_i^2(Y_i^2, Z_i^2), \quad 0 \leq i \leq n - 1,$$

so we have

$$\delta Y_n \geq 0 \quad \text{and} \quad \delta F_i \geq 0, \quad 0 \leq i \leq n - 1.$$

Thus, (2.9) gives us for all  $0 \leq i \leq n$

$$\delta Y_i = \mathbb{E}_{t_i} \left[ E_i^\pi (B_i^\pi)^{-1} \left( \delta Y_n + \sum_{k=i}^{n-1} h_k B_{k+1}^\pi \delta F_k \right) \right] \geq 0. \quad \square$$

REMARK 2.3. (i) As for the classical comparison theorem, the previous result stays true if we replace the condition

$$F^1 \text{ satisfies (H1) and } F_i^1(Y_i^2, Z_i^2) \geq F_i^2(Y_i^2, Z_i^2), \quad 0 \leq i \leq n - 1,$$

with

$$F^2 \text{ satisfies (H1) and } F_i^1(Y_i^1, Z_i^1) \geq F_i^2(Y_i^1, Z_i^1), \quad 0 \leq i \leq n - 1.$$

(ii) The comparison result for BSΔEs is already proved in [13] but without using the scheme linearization.

(iii) The truncation of the generator is essential to make the comparison theorem hold: Example 4.1 in [14] shows that comparison fails for quadratic BSΔEs with bounded terminal condition.

2.3. *A priori estimates (in the quadratic case).* In this part, we establish some a priori estimates for the solution of the BTZ scheme given by Definition 2.1 with quadratic generator. More precisely, we show that classical a priori estimates for quadratic BSDEs stay true for the corresponding BTZ scheme under suitable conditions. We consider schemes with essentially bounded terminal condition  $\xi$  and coefficients  $F$  satisfying more restrictive assumptions.

ASSUMPTION (H2). (i)  $\xi \in L^\infty(\mathcal{F}_T)$  and  $(F_i)_{0 \leq i \leq n-1}$  satisfy (H1),

(ii)  $F_i(0, 0) \in L^\infty(\mathcal{F}_i)$  for all  $0 \leq i \leq n - 1$  and there exists a constant  $\tilde{C}$  that does not depend on  $n$  and such that

$$\sup_{0 \leq i \leq n} |F_i(0, 0)| \leq \tilde{C},$$

(iii) there exist three positive constants  $K_y, \tilde{L}$  and  $\tilde{\Lambda}$  that do not depend on  $n$  and such that

$$(2.11) \quad |F_i(y, z)| \leq K_y|y| + \tilde{L}|z|^2 + \zeta_i \quad \text{with } \mathbb{E}_{t_i} \left[ \sum_{k=i}^n h_k |\zeta_k| \right] \leq \tilde{\Lambda}.$$

The first key estimate is related to the uniform boundedness in  $n$  of  $(Y_i)_{0 \leq i \leq n}$ .

PROPOSITION 2.6. *Assume (H2)(i)–(ii) holds true. Then*

$$|Y_i| \leq \left( |\xi|_\infty + T \sup_{0 \leq i \leq n-1} |F_i(0, 0)|_\infty \right) e^{CK_y/\varepsilon} \leq (|\xi|_\infty + T\tilde{C}) e^{CK_y/\varepsilon}.$$

PROOF. We introduce  $(Y_i^2, Z_i^2)_{0 \leq i \leq n}$  the solution of the BTZ scheme  $\mathcal{E}[(F_i^2), |\xi|_\infty]$  with  $F_i^2(y, z) = |F_i(0, 0)|_\infty + K_y|y|$ . We observe that the terminal condition and the generator of this scheme are deterministic functions which implies that  $Z_i^2 = 0$  for all  $0 \leq i \leq n$ . We are able to compare  $F_i$  and  $F_i^2$  under (H2)(i)–(ii):

$$F_i(Y_i^2, Z_i^2) = F_i(Y_i^2, 0) \leq |F_i(0, 0)|_\infty + K_y|Y_i^2| = F_i^2(Y_i^2, Z_i^2).$$

Since  $\xi \leq |\xi|_\infty$  we can apply the comparison theorem given in Corollary 2.5:

$$\begin{aligned} Y_i \leq Y_i^2 &= \frac{|\xi|_\infty}{\prod_{k=i}^{n-1} (1 - h_k K_y)} + \sum_{j=i}^{n-1} \frac{h_j |F_j(0, 0)|_\infty}{\prod_{k=i}^j (1 - h_k K_y)} \\ &\leq |\xi|_\infty \left( 1 + \frac{h K_y}{\varepsilon} \right)^{n-i} + \sum_{j=i}^{n-1} h_j |F_j(0, 0)|_\infty \left( 1 + \frac{h K_y}{\varepsilon} \right)^{j-i+1} \\ &\leq \left( |\xi|_\infty + T \sup_{0 \leq j \leq n-1} |F_j(0, 0)|_\infty \right) e^{CK_y/\varepsilon}. \end{aligned}$$

Using similar arguments, we obtain that

$$Y_i \geq \left( -|\xi|_\infty - T \sup_{0 \leq j \leq n-1} |F_j(0, 0)|_\infty \right) e^{CK_y/\varepsilon}$$

which completes the proof.  $\square$

The second estimate is related to  $(Z_i)_{0 \leq i \leq n}$ .

PROPOSITION 2.7. *Under (H2), we have that*

$$\mathbb{E}_{t_i} \left[ \sum_{k=i}^{n-1} h_k |Z_k|^2 \right] \leq C, \quad 0 \leq i \leq n - 1.$$

PROOF. Since (H2) holds, we can apply Proposition 2.6 and get

$$\sup_{0 \leq i \leq n} |Y_i| \leq (|\xi|_\infty + T\tilde{C})e^{CK_y/\varepsilon} := m.$$

We split the proof in two steps, depending on the value of  $m$ .

1. In this first step, we assume that

$$(2.12) \quad 2m\tilde{L} \leq \frac{d}{2\Lambda}.$$

We observe that the BTZ scheme can be rewritten

$$Y_i = Y_{i+1} + h_i F_i(Y_i, Z_i) - h_i c_i^{-1} Z_i H_i^\top - \Delta M_i,$$

where  $c_i$  is given by (2.6) and  $\Delta M_i$  is an  $\mathcal{F}_{i+1}$ -measurable random variable satisfying  $\mathbb{E}_{t_i}[\Delta M_i] = 0$ ,  $\mathbb{E}_{t_i}[|\Delta M_i|^2] < \infty$  and  $\mathbb{E}_{t_i}[\Delta M_i H_i] = 0$ . Using the identity  $|y|^2 = |x|^2 + 2x(y - x) + |y - x|^2$ , we obtain, setting  $x = Y_i$  and  $y = Y_{i+1}$ ,

$$\begin{aligned} |Y_{i+1}|^2 &= |Y_i|^2 + 2Y_i(-h_i F_i(Y_i, Z_i) + h_i c_i^{-1} Z_i H_i^\top + \Delta M_i) \\ &\quad + |-h_i F_i(Y_i, Z_i) + h_i c_i^{-1} Z_i H_i^\top + \Delta M_i|^2. \end{aligned}$$

Taking the conditional expectation w.r.t.  $\mathcal{F}_{t_i}$  in the previous equality, we obtain using (H2)(iii) and (2.6),

$$\begin{aligned} \mathbb{E}_{t_i}[|Y_{i+1}|^2] &\geq |Y_i|^2 - 2Y_i h_i F_i(Y_i, Z_i) + \mathbb{E}_{t_i}[|h_i c_i^{-1} Z_i H_i^\top|^2] \\ &\geq |Y_i|^2 - 2mh_i(K_y m + \tilde{L}|Z_i|^2 + |\zeta_i|) + h_i(c_i)^{-2} Z_i h_i \mathbb{E}_{t_i}[H_i^\top H_i] Z_i^\top \\ &\geq |Y_i|^2 - 2mh_i(K_y m + \tilde{L}|Z_i|^2 + |\zeta_i|) + h_i(c_i)^{-1} |Z_i|^2 \\ &\geq |Y_i|^2 - 2m^2 K_y h_i + \left(\frac{d}{\Lambda} - 2m\tilde{L}\right) h_i |Z_i|^2 - 2mh_i |\zeta_i|. \end{aligned}$$

Finally, an easy induction over  $i$  allows to obtain

$$\begin{aligned} \mathbb{E}_{t_i} \left[ \sum_{k=i}^{n-1} h_k |Z_k|^2 \right] &\leq \frac{1}{d/\Lambda - 2m\tilde{L}} (\mathbb{E}_{t_i}[|Y_n|^2] - |Y_i|^2 + 2m^2 K_y T + 2m\tilde{\Lambda}) \\ &\leq \frac{2m^2 + 2m^2 K_y T + 2m\tilde{\Lambda}}{d/\Lambda - 2m\tilde{L}}. \end{aligned}$$

Since the previous bound does not depend on  $n$ , the result is proved in this special case.

2a. To prove the result in the general case, we use similar arguments as in [42]: we cut  $\xi$  and  $(F_i(0, 0))$  in pieces small enough such that we are able to use step 1. Let us set an integer  $\kappa \in \mathbb{N}^*$  that does not depend on  $n$  and such that

$$(2.13) \quad \frac{4m\tilde{L}}{\kappa} \leq \frac{d}{2\Lambda}.$$

For each  $a \in \{1, \dots, \kappa\}$ , we denote  $(Y_i^a, Z_i^a)_{0 \leq i \leq n}$  the solution of  $\mathcal{E}[(\Phi_i^a), \xi^a]$  with  $\xi^a = \frac{\xi}{\kappa}$  and

$$\Phi_i^a(y, z) = F_i\left(y + \sum_{q=1}^{a-1} Y_i^q, z + \sum_{q=1}^{a-1} Z_i^q\right) - F_i\left(\sum_{q=1}^{a-1} Y_i^q, \sum_{q=1}^{a-1} Z_i^q\right) + \frac{F_i(0, 0)}{\kappa}.$$

We observe that

$$(2.14) \quad Y_i = \sum_{a=1}^{\kappa} Y_i^a \quad \text{and} \quad Z_i = \sum_{a=1}^{\kappa} Z_i^a.$$

Since (H2)(i)–(ii) holds true for  $(\Phi_i^a)$  and  $\xi^a$ , we can apply Proposition 2.6 and remark that

$$(2.15) \quad \begin{aligned} \sup_{0 \leq i \leq n} |Y_i^a| &\leq \left( |\xi^a|_{\infty} + \sup_{0 \leq i \leq n-1} |\Phi_i^a(0, 0)|_{\infty} T \right) e^{CK_y/\varepsilon} \\ &\leq \left( \frac{|\xi|_{\infty}}{\kappa} + \frac{\sup_{0 \leq i \leq n-1} |F_i(0, 0)|_{\infty} T}{\kappa} \right) e^{CK_y/\varepsilon} \\ &\leq \frac{m}{\kappa}. \end{aligned}$$

2b. In this last step, we use an induction argument to show

$$(2.16) \quad \mathbb{E}_{t_i} \left[ \sum_{k=i}^{n-1} h_k |Z_k^a|^2 \right] \leq C, \quad 0 \leq i < n,$$

for all  $a \in \{1, \dots, \kappa\}$ . Combined with (2.14), this proves the proposition in the general case. We have proved in the first step that (2.16) is true for  $a = 1$ . Now let us assume that it is true up to  $a < \kappa$ . Then we compute that

$$\begin{aligned} |\Phi_i^{a+1}(y, z)| &\leq \left| F_i\left(y + \sum_{q=1}^a Y_i^q, z + \sum_{q=1}^a Z_i^q\right) \right| \\ &\quad + \left| F_i\left(\sum_{q=1}^a Y_i^q, \sum_{q=1}^a Z_i^q\right) \right| + \frac{|F_i(0, 0)|}{\kappa} \\ &\leq K_y|y| + 2\tilde{L}|z|^2 + \varsigma_i^a, \end{aligned}$$

where  $\varsigma_i^a = 2K_y|\sum_{q=1}^a Y_i^q| + 3\tilde{L}|\sum_{q=1}^a Z_i^q|^2 + 2|\varsigma_i| + |F_i(0, 0)|_{\infty}/\kappa$ . Assumption (H2)(iii), bound (2.15) and the induction hypothesis yield that

$\mathbb{E}_{t_i}[\sum_{k=i}^n h_k |\zeta_k^a|] \leq C$  for all  $0 \leq i < n$ . Then we have that  $\Phi^{a+1}$  satisfies Assumption (H2) with  $2\tilde{L}$  instead of  $\tilde{L}$  and  $\zeta^a$  instead of  $\zeta$ . Since we have assumed that (2.13) holds true, then we can apply step 1 to obtain

$$\mathbb{E}_{t_i} \left[ \sum_{k=i}^{n-1} h_k |Z_k^{a+1}|^2 \right] \leq C, \quad 0 \leq i < n,$$

which completes the proof.  $\square$

We conclude this section by applying previous results to the scheme given in Definition 1.1.

**COROLLARY 2.8.** *Under assumptions of Theorem 1.1 the following holds true, for  $n$  large enough,*

$$\sup_{0 \leq i \leq n} \left( |Y_i^\pi| + \mathbb{E}_{t_i} \left[ \sum_{k=i}^{n-1} |Z_k^\pi|^2 h_k \right] \right) \leq C.$$

**PROOF.** We simply observe that with our special choice of parameters  $R$  and  $N$ , we have for  $n$  large enough

$$\left( \sup_{0 \leq i \leq n-1} h_i |H_i^R| \right) n^\alpha \leq \sqrt{h} \sqrt{d} R n^\alpha \leq \frac{C \sqrt{d} \log n}{n^{1/2-\alpha}} < 1,$$

and that the generator of the scheme given in Definition 1.1 satisfies (H2) (with  $K_z^n = N := n^\alpha$ ). The result follows then from a direct application of Proposition 2.6 and Proposition 2.7.  $\square$

**REMARK 2.4.** In a slightly different framework, Gobet and Turkedjiev have already obtained the Corollary 2.8 in [25] by direct calculations without using the linearization technique.

**2.4. Scheme stability.** In this part, we will establish some bounds on the difference between two schemes. Firstly, we introduce a perturbed version of the scheme given in Definition 2.1.

**DEFINITION 2.2.** (i) The terminal condition is given by  $\tilde{Y}_n = \tilde{\xi}$  for some  $\tilde{\xi} \in L^\infty(\mathcal{F}_T)$  and  $\tilde{Z}_n = 0$ ;  
 (ii) for  $0 \leq i < n$

$$\begin{cases} \tilde{Y}_i = \mathbb{E}_{t_i} [\tilde{Y}_{i+1} + h_i F_i(\tilde{Y}_i, \tilde{Z}_i)] + \zeta_i^Y, \\ \tilde{Z}_i = \mathbb{E}_{t_i} [\tilde{Y}_{i+1} H_i]. \end{cases}$$

Perturbations  $\zeta_i^Y$  are  $\mathcal{F}_{t_i}$ -measurable and square integrable random variables. Moreover, we assume that

$$(2.17) \quad \sup_{0 \leq i < n} \mathbb{E}_{t_i} \left[ \sum_{j=i}^{n-1} |\tilde{Z}_j|^2 h_j \right] < C.$$

2.4.1. *Stability results for the Y component.* Setting  $\delta Y_i := Y_i - \tilde{Y}_i$  and  $\delta Z_i := Z_i - \tilde{Z}_i$ , we obtain a key stability result for the Y component.

PROPOSITION 2.9. *Assume that Assumption (H1) holds true. Then, for all  $0 \leq i \leq n$ ,*

$$|\delta Y_i| \leq C \mathbb{E}_{t_i}^{\mathbb{Q}^\pi} \left[ |\delta Y_n| + \sum_{j=i}^{n-1} |\zeta_j^Y| \right],$$

where

$$\frac{d\mathbb{Q}^\pi}{d\mathbb{Q}} = E_0^\pi = \prod_{j=0}^{n-1} (1 + h_j H_j \gamma_j)$$

and

$$(2.18) \quad \gamma_j = \frac{F_j(\tilde{Y}_j, Z_j) - F_j(\tilde{Y}_j, \tilde{Z}_j)}{|Z_j - \tilde{Z}_j|^2} (Z_j - \tilde{Z}_j)^\top \mathbb{1}_{\{Z_j - \tilde{Z}_j \neq 0\}}.$$

PROOF. Using the Euler scheme linearization given in Proposition 2.4 and observing  $\delta F_k = \frac{-\zeta_k^Y}{h_k}$ , it follows from (2.9) that

$$|\delta Y_i| \leq \mathbb{E}_{t_i} \left[ |E_i^\pi| |B_i^\pi|^{-1} \left( |\delta Y_n| + \sum_{k=i}^{n-1} |B_{k+1}^\pi| |\zeta_k^Y| \right) \right].$$

Moreover,

$$|B_i^\pi|^{-1} |B_{k+1}^\pi| \leq \left( \frac{1}{1 - hK_y} \right)^{k+1-i} \leq \left( 1 + \frac{hK_y}{\varepsilon} \right)^{k+1-i} \leq e^{(CK_y)/\varepsilon},$$

leading to

$$|\delta Y_i| \leq C \mathbb{E}_{t_i} \left[ |E_i^\pi| \left( |\delta Y_n| + \sum_{k=i}^{n-1} |\zeta_k^Y| \right) \right].$$

Under (H1)(iii), we get that  $E_i^\pi > 0$  for all  $0 \leq i \leq n$  and then

$$\left( \prod_{j=0}^k (1 + h_j H_j \gamma_j) \right)_{0 \leq k \leq n}$$

is a positive martingale with expectation equal to 1. The measure  $\mathbb{Q}^\pi$  is thus a probability measure.  $\square$

2.4.2. *Estimates on  $\mathbb{Q}^\pi$ .* In order to retrieve nice estimates on the probability measure  $\mathbb{Q}^\pi$ , we need to introduce a new assumption.

ASSUMPTION (H3). (i) (H2) holds true and  $(\sup_{0 \leq i \leq n-1} h_i |H_i|) K_z^n < 1 - \varepsilon$  with  $\varepsilon$  a positive constant that does not depend on  $n$ ,

(ii)  $F_i$  are  $\tilde{L}$ -locally Lipschitz functions with respect to  $z: \forall y \in \mathbb{R}, \forall z, z' \in \mathbb{R}^{1 \times d}, \forall 0 \leq i \leq n - 1$ ,

$$|F_i(y, z) - F_i(y, z')| \leq \tilde{L}(1 + |z| + |z'|)|z - z'|,$$

with  $\tilde{L}$  a constant that does not depend on  $n$ .

PROPOSITION 2.10. *Assume that (H3) holds true. Then  $M_t := \sum_{t_i \leq t} h_i \gamma_i H_i$ , with  $(\gamma_i)_{0 \leq i \leq n-1}$  given by (2.18), is a BMO martingale for the discontinuous filtration  $\mathcal{F}^n$  defined by  $\mathcal{F}_t^n := \mathcal{F}_{t_i}$  when  $t_i \leq t < t_{i+1}$ . Moreover, there exists a constant  $C$  that does not depend on  $n$  such that*

$$\|M\|_{\text{BMO}(\mathcal{F}^n)} \leq C.$$

PROOF. We have to show that there exists a constant  $C$  that does not depend on  $n$  such that, for all stopping time  $S \leq T$ ,

$$\mathbb{E}[|M_T - M_{S-}|^2 | \mathcal{F}_S] \leq C.$$

Thanks to remark (76.4) in Chapter VII of [21], we know that it is sufficient to show that for all  $0 \leq i < n$ ,

$$\mathbb{E}_{t_i} \left[ \sum_{j=i}^{n-1} |h_j H_j \gamma_j|^2 \right] \leq C.$$

To prove this point, we use the fact that  $F_i$  is a  $\tilde{L}$ -locally Lipschitz function with respect to  $z$  and (2.8):

$$\begin{aligned} & \mathbb{E}_{t_i} \left[ \sum_{j=i}^{n-1} |h_j H_j \gamma_j|^2 \right] \\ & \leq 3\tilde{L}^2 + 3\tilde{L}^2 \mathbb{E}_{t_i} \left[ \sum_{j=i}^{n-1} |h_j H_j|^2 |\tilde{Z}_j|^2 \right] + 3\tilde{L}^2 \mathbb{E}_{t_i} \left[ \sum_{j=i}^{n-1} |h_j H_j|^2 |Z_j|^2 \right] \\ & \leq 3\tilde{L}^2 + 3\tilde{L}^2 \Lambda \mathbb{E}_{t_i} \left[ \sum_{j=i}^{n-1} |\tilde{Z}_j|^2 h_j \right] + 3\tilde{L}^2 \Lambda \mathbb{E}_{t_i} \left[ \sum_{j=i}^{n-1} |Z_j|^2 h_j \right]. \end{aligned}$$

The proof is complete combining (2.17) with Proposition 2.7.  $\square$

Since  $M$  is a BMO martingale, we retrieve some strong properties for this process.

PROPOSITION 2.11. Assume that (H3) holds true. Then the Doléans–Dade exponential  $E_t := \prod_{j \leq t} (1 + h_j H_j \gamma_j)$  is a uniformly integrable martingale for the filtration  $\mathcal{F}^n$  satisfying the “reverse Hölder inequality”

$$\mathbb{E}_t \left[ \frac{E_T^{p^*}}{E_t^{p^*}} \right] \leq C, \quad 0 \leq t \leq T,$$

for some  $p^* > 1$  and  $C > 0$  that depend only on  $\|M\|_{\text{BMO}(\mathcal{F}^n)}$  and  $\varepsilon$ . In particular, we can choose them independently of  $n$ . As a direct corollary, we have that  $M$  is a  $L^{p^*}$  bounded martingale.

PROOF. The first theorem in [31] states that  $(E_t)_{0 \leq t \leq 1}$  is a uniformly integrable martingale satisfying the “reverse Hölder inequality” for some  $p^* > 1$ . We just have to check that we can choose  $C$  and  $p^*$  that only depend on  $\|M\|_{\text{BMO}(\mathcal{F}^n)}$  and  $\varepsilon$ . First, thanks to Theorem 2 in [30] we know that there exist positive constants  $a$  and  $K$  such that

$$(2.19) \quad \mathbb{E}_\tau \left[ \left( \frac{E_T}{E_\tau} \right)^a \right] \leq K,$$

for any stopping time  $\tau$ . By checking carefully the proof of this theorem, we remark that  $a$  is chosen such that

$$k_a := \frac{4a^2 + a}{\varepsilon^2} < \frac{1}{\|M\|_{\text{BMO}(\mathcal{F}^n)}}$$

and then  $K$  is set

$$K := \frac{1}{1 - k_a \|M\|_{\text{BMO}(\mathcal{F}^n)}^2}.$$

To conclude, we use Lemma 3 in [31] that says that if  $M$  satisfies (2.19), then it satisfies a “reverse Hölder inequality.” By checking carefully the proof of this lemma, we can see that constants  $C$  and  $p^*$  in the “reverse Hölder inequality” are only obtained thanks to  $a$ ,  $K$  and  $\varepsilon$ .  $\square$

Combining the previous proposition with Proposition 2.9, we obtain, using Hölder’s inequality, the following result.

COROLLARY 2.1. Assume that (H3) holds true. Then there exist constants  $C > 0$  and  $q^* > 1$  that do not depend on  $n$  and such that, for all  $0 \leq i \leq n$ ,

$$|\delta Y_i| \leq C \left( \mathbb{E}_{t_i} [|\delta Y_n|^{q^*}]^{1/q^*} + \mathbb{E}_{t_i} \left[ \left( \sum_{j=i}^{n-1} |\zeta_j^Y| \right)^{q^*} \right]^{1/q^*} \right).$$

$q^*$  is the conjugate exponent of  $p^*$  given in Proposition 2.11.

REMARK 2.5. If  $\zeta_i^Y = \zeta_i^{Y,1} + \zeta_i^{Y,2}$ , it is easy to see that one may just apply Corollary 2.1 on the first part of the perturbation:

$$|\delta Y_i| \leq C \left( \mathbb{E}_{t_i} [|\delta Y_n|^{q^*}]^{1/q^*} + \mathbb{E}_{t_i} \left[ \left( \sum_{j=i}^{n-1} |\zeta_j^{Y,1}| \right)^{q^*} \right]^{1/q^*} + \mathbb{E}_{t_i}^{\mathbb{Q}^\pi} \left[ \sum_{j=i}^{n-1} |\zeta_j^{Y,2}| \right] \right),$$

$0 \leq i \leq n.$

2.4.3. *Stability result for the Z component.*

PROPOSITION 2.12. *Assume that (H3) holds true. Then*

$$\mathbb{E} \left[ \sum_{i=0}^{n-1} h_i |\delta Z_i|^2 \right] \leq C \left( \mathbb{E} [|\delta Y_n|^2] + \mathbb{E} \left[ \sum_{i=0}^{n-1} \frac{|\zeta_i^Y|^2}{h_i} \right] + \mathbb{E} \left[ \sup_{0 \leq i \leq n-1} |\delta Y_i|^4 \right]^{1/2} \right).$$

PROOF. As in the proof of Proposition 2.7, we first observe that equation (2.10) can be rewritten

$$\delta Y_i = \delta Y_{i+1} + h_i \beta_i \delta Y_i + h_i \delta Z_i \gamma_i + \zeta_i^Y - h_i c_i^{-1} \delta Z_i H_i^\top - \delta \Delta M_i,$$

where  $\delta \Delta M_i$  is an  $\mathcal{F}_{t_{i+1}}$  random variable satisfying  $\mathbb{E}_{t_i} [\delta \Delta M_i] = 0$ ,  $\mathbb{E}_{t_i} [|\delta \Delta M_i|^2] < \infty$  and  $\mathbb{E}_{t_i} [\delta \Delta M_i H_i] = 0$ . Using the identity  $|y|^2 = |x|^2 + 2x(y - x) + |y - x|^2$  and taking the conditional expectation, we compute, setting  $x = \delta Y_i$  and  $y = \delta Y_{i+1}$ ,

$$\begin{aligned} \mathbb{E}_{t_i} [|\delta Y_{i+1}|^2] &\geq |\delta Y_i|^2 - 2|\delta Y_i|^2 h_i \beta_i - 2h_i \delta Y_i \delta Z_i \gamma_i \\ &\quad - 2\delta Y_i \zeta_i^Y + c_i^{-1} h_i \delta Z_i c_i^{-1} h_i \mathbb{E}_{t_i} [H_i^\top H_i] \delta Z_i^\top. \end{aligned}$$

It follows from (2.6) and (2.7) applied to the previous inequality that

$$|\delta Y_i|^2 + \frac{d}{\Lambda} h_i |\delta Z_i|^2 \leq \mathbb{E}_{t_i} [|\delta Y_{i+1}|^2] + 2\delta Y_i \zeta_i^Y + 2h_i \delta Y_i \delta Z_i \gamma_i + 2|\delta Y_i|^2 h_i \beta_i$$

and Young’s inequality leads to

$$|\delta Y_i|^2 + \frac{d}{2\Lambda} h_i |\delta Z_i|^2 \leq \mathbb{E}_{t_i} [|\delta Y_{i+1}|^2] + h_i \left( 1 + 2K_y + \frac{2\Lambda |\gamma_i|^2}{d} \right) |\delta Y_i|^2 + \frac{|\zeta_i^Y|^2}{h_i}.$$

Summing over  $i$  the previous inequality, we obtain

$$\mathbb{E} \left[ \sum_{i=0}^{n-1} h_i |\delta Z_i|^2 \right] \leq C \mathbb{E} [|\delta Y_n|^2] + C \mathbb{E} \left[ \sum_{i=0}^{n-1} h_i (1 + |\gamma_i|^2) |\delta Y_i|^2 \right] + C \mathbb{E} \left[ \sum_{i=0}^{n-1} \frac{|\zeta_i^Y|^2}{h_i} \right].$$

Applying Hölder’s inequality, we get

$$\begin{aligned} \mathbb{E} \left[ \sum_{i=0}^{n-1} h_i |\delta Z_i|^2 \right] &\leq C \mathbb{E} [|\delta Y_n|^2] + C \mathbb{E} \left[ \sum_{i=0}^{n-1} \frac{|\zeta_i^Y|^2}{h_i} \right] \\ &\quad + C \mathbb{E} \left[ \sup_{0 \leq i \leq n-1} |\delta Y_i|^4 \right]^{1/2} \mathbb{E} \left[ \left( 1 + \sum_{i=0}^{n-1} |\gamma_i|^2 h_i \right)^2 \right]^{1/2}. \end{aligned}$$

To complete the proof, we just have to show that

$$\mathbb{E} \left[ \left( \sum_{i=0}^{n-1} h_i |\gamma_i|^2 \right)^2 \right] \leq C.$$

Using the Burkholder–Davis–Gundy inequality for the discrete martingale  $(\sum_{i=0}^j h_i H_i \gamma_i)_{0 \leq j \leq n}$ , the previous inequality holds true if we have

$$\mathbb{E} \left[ \left( \sup_{0 \leq j \leq n-1} \sum_{i=0}^j h_i H_i \gamma_i \right)^4 \right] \leq C.$$

Thanks to Proposition 2.10 we know that  $M_t = \sum_{i \leq t} h_i H_i \gamma_i$  is a BMO martingale with a BMO norm that does not depend on  $n$ . To complete the proof, we use an energy inequality or the John–Nirenberg inequality; see, for example, Theorem 109 and inequality (109.5) in Chapter VI of [21], and obtain

$$\mathbb{E} \left[ \left( \sup_{0 \leq j \leq n-1} \sum_{i=0}^j h_i H_i \gamma_i \right)^4 \right] \leq C$$

with  $C$  that depends only on  $\|M\|_{\text{BMO}(\mathcal{F}^n)}$ .  $\square$

**3. Convergence analysis of the discrete-time approximation.** The aim of this part is to study the error between the solution  $(Y, Z)$  of the BSDE (1.2) and  $(Y^\pi, Z^\pi)$  the solution of the BTZ scheme given in Definition 1.1, recalling (1.9). Thanks to Theorem 2.2 we know that we just have to estimate the error between  $(Y^N, Z^N)$  and  $(Y^\pi, Z^\pi)$ .

Let us first observe that we can apply results of the previous section to  $(Y^\pi, Z^\pi)$ .

LEMMA 3.1. *Under same assumptions as Theorem 1.1, the scheme given in Definition 1.1 satisfies (H3).*

PROOF. With our special choice of parameters  $R$  and  $N$ , there exists  $\varepsilon > 0$  such that for  $n$  big enough we have  $K_{f,y} h \leq \frac{CK_{f,y}}{n} < 1 - \varepsilon$ . Moreover, we have also for  $n$  large enough

$$\left( \sup_{0 \leq i \leq n-1} h_i |H_i^R| \right) n^\alpha \leq \sqrt{h} R n^\alpha \leq \frac{\sqrt{C} \log n}{n^{1/2-\alpha}} \leq 1 - \varepsilon. \quad \square$$

3.1. *Expression of the perturbing error.* We first observe that  $(Y^N, Z^N)$  can be rewritten as a perturbed BTZ scheme. Namely, setting  $\tilde{Y}_i := Y_{t_i}^N$ , for all  $i \leq n$ , we have

$$(3.1) \quad \begin{cases} \tilde{Y}_i = \mathbb{E}_{t_i} [\tilde{Y}_{i+1} + h_i f_N(X_i^\pi, \tilde{Y}_i, \tilde{Z}_i)] + \zeta_i^Y, \\ \tilde{Z}_i = \mathbb{E}_{t_i} [\tilde{Y}_{i+1} H_i^R], \end{cases}$$

with

$$(3.2) \quad \zeta_i^Y = \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} f_N(X_s, Y_s^N, Z_s^N) - f_N(X_i^\pi, Y_{t_i}^N, \tilde{Z}_i) ds \right].$$

The following lemma will allow us to use the results of the last section.

LEMMA 3.2. *The perturbed scheme  $(\tilde{Y}_i, \tilde{Z}_i)_{i \leq n}$  satisfies, for all  $0 \leq k \leq n - 1$ ,*

$$\mathbb{E}_{t_k} \left[ \sum_{i=k}^{n-1} |\tilde{Z}_i|^2 h_i \right] \leq C.$$

PROOF. Observe that

$$(3.3) \quad \mathbb{E}_{t_k} \left[ \sum_{i=k}^{n-1} h_i |\tilde{Z}_i|^2 \right] \leq C \left( \mathbb{E}_{t_k} \left[ \sum_{i \geq k} |\tilde{Z}_i - \tilde{Z}_i^N|^2 h_i \right] + \mathbb{E}_{t_k} \left[ \sum_{i \geq k} |\tilde{Z}_i^N|^2 h_i \right] \right),$$

where

$$\tilde{Z}_i^N := \mathbb{E}_{t_i} \left[ Y_{t_{i+1}}^N \frac{\Delta W_i}{h_i} \right].$$

Applying Lemma 2.2, we obtain

$$(3.4) \quad \mathbb{E}_{t_k} \left[ \sum_{i=k}^{n-1} h_i |\tilde{Z}_i|^2 \right] \leq C \left( 1 + \mathbb{E}_{t_k} \left[ \sum_{i \geq k} |\tilde{Z}_i - \tilde{Z}_i^N|^2 h_i \right] \right).$$

Moreover, we compute

$$\begin{aligned} \mathbb{E}_{t_k} \left[ \sum_{i \geq k} |\tilde{Z}_i - \tilde{Z}_i^N|^2 h_i \right] &= \mathbb{E}_{t_k} \left[ \sum_{i \geq k} \left| \mathbb{E}_{t_i} \left[ (Y_{t_{i+1}}^N - Y_{t_i}^N) \left( H_i^R - \frac{\Delta W_i}{h_i} \right) \right] \right|^2 h_i \right] \\ &\leq C \sum_{i \geq k} \mathbb{E}_{t_k} [ |Y_{t_{i+1}}^N - Y_{t_i}^N|^2 ], \end{aligned}$$

where we used Cauchy–Schwarz inequality, recalling (2.8).

We then compute, thanks to assumptions on  $f_N$  and Remark 2.1,

$$\begin{aligned} &\mathbb{E}_{t_k} [ |Y_{t_{i+1}}^N - Y_{t_i}^N|^2 ] \\ &\leq C \left( h_i \mathbb{E}_{t_k} \left[ \int_{t_i}^{t_{i+1}} |f_N(X_s, Y_s^N, Z_s^N)|^2 ds \right] + \mathbb{E}_{t_k} \left[ \int_{t_i}^{t_{i+1}} |Z_s^N|^2 ds \right] \right) \\ &\leq C \left( h^2 + (1 + N^2 h) \mathbb{E}_{t_k} \left[ \int_{t_i}^{t_{i+1}} |Z_s^N|^2 ds \right] \right). \end{aligned}$$

Summing over  $i$ , recalling Remark 2.1, we obtain

$$(3.5) \quad \mathbb{E}_{t_k} \left[ \sum_{i \geq k} |\tilde{Z}_i - \tilde{Z}_i^N|^2 h_i \right] \leq C \left( 1 + \left\| \int_0^\cdot Z_s^N dW_s \right\|_{\text{BMO}(\mathcal{F})}^2 \right) \leq C.$$

The proof is complete combining the above inequality with (3.4).  $\square$

3.2. *Regularity.* In the following, we need regularity results on  $(X, Y^N, Z^N)$ . The specificity here is that we need the estimates under the probability measure  $\mathbb{P}$  and  $\mathbb{Q}^\pi$ . The first result deals with the path regularity of  $Y$  under the probability measure  $\mathbb{P}$ . It is a mere generalization of Theorem 5.5 in [28].

PROPOSITION 3.1 (*Y-part*). *For all  $p \leq 1$ , we have*

$$(3.6) \quad \sup_{0 \leq j \leq n-1} \mathbb{E} \left[ \sup_{t_j \leq s \leq t_{j+1}} |Y_s^N - Y_{t_j}^N|^{2p} \right] \leq C_p h^p.$$

The second result is a slight modification of the well-known Zhang path regularity theorem, whose proof is postponed to the ArXiv version of his paper.

PROPOSITION 3.2 (*Z-part*). *For all  $p \geq 1$  and  $\eta > 0$ , we have*

$$\mathbb{E} \left[ \sup_{0 \leq i \leq n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi} \left[ \sum_{j=i}^{n-1} \left( \int_{t_j}^{t_{j+1}} |Z_s^N - \bar{Z}_j^N|^2 ds \right)^{1+\eta} \right]^p \right] \leq C_{\eta,p} h^{p(1+\eta)}.$$

Let us remark that the previous proposition stays true when we replace  $\mathbb{Q}^\pi$  by  $\mathbb{P}$ : it is a mere generalization of Theorem 5.6 in [28].

3.3. *Discretization error for the Y-component.*

PROPOSITION 3.3. *There exists  $q^* > 1$  and, for all  $\eta > 0$  and  $p \geq 1$ , there exist constants  $C_p$  and  $C_{\alpha,\eta,p}$  such that*

$$\begin{aligned} & \mathbb{E} \left[ \sup_{0 \leq i \leq n} |Y_{t_i} - Y_i^\pi|^{2p} \right] \\ & \leq C_{\alpha,\eta,p} h^{p(1-\eta)} + C_p \mathbb{E} \left[ \sup_{0 \leq j \leq n} |X_{t_j} - X_j^\pi|^{2pq^*} \right]^{1/q^*} \\ & \quad + C_p \max_{0 \leq j \leq n-1} \left( \mathbb{E} \left[ \left| H_j^R - \frac{\Delta W_j}{h_j} \right| \right]^{4p} + \mathbb{E} \left[ \left| H_j^R - \frac{\Delta W_j}{h_j} \right| \right]^{2p} \right). \end{aligned}$$

Before giving the proof, let us emphasize that  $q^*$  is the exponent given by Corollary 2.1 and so it is the conjugate exponent of  $p^*$  given by Proposition 2.11.

PROOF OF PROPOSITION 3.3. The proof is divided in several steps.

1. We first observe that

$$(3.7) \quad \begin{aligned} & \mathbb{E} \left[ \sup_{0 \leq i \leq n} |Y_{t_i} - Y_i^\pi|^{2p} \right] \\ & \leq C_p \left( \mathbb{E} \left[ \sup_{0 \leq i \leq n} |Y_{t_i} - Y_{t_i}^N|^{2p} \right] + \mathbb{E} \left[ \sup_{0 \leq i \leq n} |Y_{t_i}^N - Y_i^\pi|^{2p} \right] \right). \end{aligned}$$

To bound the first term in the right-hand side of the above equation, we apply Theorem 2.2 and get

$$\mathbb{E}\left[\sup_{0 \leq i \leq n} |Y_{t_i} - Y_{t_i}^N|^{2p}\right] \leq C_{\alpha,p} h^p,$$

recalling (1.9).

2. To control the error between the solution  $Y^N$  and the scheme  $Y^\pi$ , we will combine the stability results proved in the previous section with a careful analysis of the perturbation error  $(\zeta_i^Y)_{0 \leq i < n}$  given by (3.2). We first observe that

$$\begin{aligned} \zeta_i^Y &= \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} f_N(X_s, Y_s^N, Z_s^N) - f_N(X_i^\pi, Y_s^N, Z_s^N) ds \right] \\ &\quad + \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} f_N(X_i^\pi, Y_s^N, Z_s^N) - f_N(X_i^\pi, Y_{t_i}^N, Z_s^N) ds \right] \\ &\quad + \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} f_N(X_i^\pi, Y_{t_i}^N, Z_s^N) - f_N(X_i^\pi, Y_{t_i}^N, \bar{Z}_i^N) ds \right] \\ &\quad + \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} f_N(X_i^\pi, Y_{t_i}^N, \bar{Z}_i^N) - f_N(X_i^\pi, Y_{t_i}^N, \tilde{Z}_i^N) ds \right] \\ &\quad + \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} f_N(X_i^\pi, Y_{t_i}^N, \tilde{Z}_i^N) - f_N(X_i^\pi, Y_{t_i}^N, \tilde{Z}_i) ds \right] \\ &:= \zeta_i^{Y,x} + \zeta_i^{Y,y} + \zeta_i^{Y,\bar{z}} + \zeta_i^{Y,\tilde{z}} + \zeta_i^{Y,w}, \end{aligned}$$

recalling (2.2) and (2.3).

Using Lemma 3.1 and Lemma 3.2, we apply Proposition 2.9 and Corollary 2.1 (see also Remark 2.5) to obtain

$$\begin{aligned} &|Y_{t_i}^N - Y_i^\pi| \\ &\leq C \mathbb{E}_{t_i} \left[ \left( \sum_{j=0}^{n-1} |\zeta_j^{Y,x}| \right)^{q^*} \right]^{1/q^*} + C \mathbb{E}_{t_i} \left[ \left( \sum_{j=0}^{n-1} |\zeta_j^{Y,y}| \right)^{q^*} \right]^{1/q^*} \\ &\quad + C \mathbb{E}_{t_i} \left[ \left( \sum_{j=0}^{n-1} |\zeta_j^{Y,w}| \right)^{q^*} \right]^{1/q^*} + C \mathbb{E}_{t_i} \left[ \left( \sum_{j=0}^{n-1} |\zeta_j^{Y,\bar{z}}| \right)^{q^*} \right]^{1/q^*} \\ &\quad + C \mathbb{E}_{t_i} [ |Y_{t_n}^N - Y_n^\pi|^{q^*} ]^{1/q^*} + C \mathbb{E}_{t_i} \left[ \left\{ \prod_{j=i}^{n-1} (1 + h_j H_j^R \gamma_j^{N,n}) \right\} \left[ \sum_{j=i}^{n-1} |\zeta_j^{Y,\tilde{z}}| \right] \right]. \end{aligned}$$

A convexity inequality and Doob maximal inequality allow us to write, for all  $p \geq 1$ ,

$$(3.8) \quad \mathbb{E}\left[\sup_{0 \leq i \leq n} |Y_{t_i}^N - Y_i^\pi|^{2p}\right] \leq C(\mathcal{E}_p^x + \mathcal{E}_p^y + \mathcal{E}_p^w + \mathcal{E}_p^{\bar{z}} + \mathcal{E}_p^{\tilde{z}}),$$

with

$$\mathcal{E}_p^x := \mathbb{E}[|Y_{t_n}^N - Y_n^\pi|^{2pq^*}]^{1/q^*} + C\mathbb{E}\left[\left(\sum_{j=0}^{n-1} |\zeta_j^{Y,x}| \right)^{2pq^*}\right]^{1/q^*}$$

coming from the approximation of  $X$  by  $X^\pi$  in the terminal condition and the generator,

$$\mathcal{E}_p^y := \mathbb{E}\left[\left(\sum_{j=0}^{n-1} |\zeta_j^{Y,y}| \right)^{2pq^*}\right]^{1/q^*}$$

coming from the approximation of  $Y^N$  by  $\sum_{i=0}^{n-1} Y_{t_i}^N \mathbb{1}_{t_i \leq t < t_{i+1}}$  in the generator,

$$\mathcal{E}_p^w := \mathbb{E}\left[\left(\sum_{j=0}^{n-1} |\zeta_j^{Y,w}| \right)^{2pq^*}\right]^{1/q^*}$$

coming from the approximation of  $\Delta W_i$  by  $h_i H_i$ ,

$$\mathcal{E}_p^{\tilde{z}} := \mathbb{E}\left[\left(\sum_{j=0}^{n-1} |\zeta_j^{Y,\tilde{z}}| \right)^{2pq^*}\right]^{1/q^*}$$

coming from the approximation of  $\sum_{i=0}^{n-1} \bar{Z}_i^N \mathbb{1}_{t_i \leq t < t_{i+1}}$  by  $\sum_{i=0}^{n-1} \tilde{Z}_i^N \mathbb{1}_{t_i \leq t < t_{i+1}}$  in the generator, and finally

$$\mathcal{E}_p^{\tilde{z}} := n^p \mathbb{E}\left[ \sup_{0 \leq i \leq n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi} \left[ \sum_{j=i}^{n-1} |\zeta_j^{Y,\tilde{z}}|^2 \right]^p \right],$$

due to the approximation of  $Z^N$  by  $\sum_{i=0}^{n-1} \bar{Z}_i^N \mathbb{1}_{t_i \leq t < t_{i+1}}$  in the generator.

We will now bound these five terms.

2a. Since  $g$  is Lipschitz continuous, we have

$$(3.9) \quad \mathbb{E}[|Y_{t_n}^N - Y_n^\pi|^{2pq^*}]^{1/q^*} \leq C_p \mathbb{E}[|X_{t_n}^\pi - X_T|^{2pq^*}]^{1/q^*}.$$

Similarly, since  $f_N$  is Lipschitz-continuous in its  $x$ -variable,

$$(3.10) \quad \begin{aligned} \mathbb{E}\left[\left(\sum_{j=0}^{n-1} |\zeta_j^{Y,x}| \right)^{2pq^*}\right]^{1/q^*} &\leq C_p \sup_{0 \leq j \leq n-1} \mathbb{E}\left[\left(\sup_{t_j \leq s \leq t_{j+1}} |X_s - X_j^\pi| \right)^{2pq^*}\right]^{1/q^*} \\ &\leq C_p \sup_{0 \leq j \leq n-1} \mathbb{E}\left[\sup_{t_j \leq s \leq t_{j+1}} |X_s - X_{t_j}|^{2pq^*}\right]^{1/q^*} \\ &\quad + C_p \sup_{0 \leq j \leq n-1} \mathbb{E}[|X_{t_j} - X_j^\pi|^{2pq^*}]^{1/q^*}. \end{aligned}$$

Classical result on the path regularity of SDE's solutions yields

$$(3.11) \quad \sup_{0 \leq j \leq n-1} \mathbb{E} \left[ \sup_{t_j \leq s \leq t_{j+1}} |X_s - X_{t_j}|^{2pq^*} \right]^{1/q^*} \leq C_p h^p.$$

Combining (3.9)–(3.10)–(3.11), we obtain

$$(3.12) \quad \mathcal{E}_p^x \leq C_p h^p + C_p \mathbb{E} \left[ \sup_{0 \leq j \leq n} |X_{t_j} - X_j^\pi|^{2pq^*} \right]^{1/q^*}.$$

2b. We easily compute that

$$\mathbb{E} \left[ \left( \sum_{j=0}^{n-1} |\zeta_j^{Y,y}| \right)^{2pq^*} \right]^{1/q^*} \leq C_p n^{-1} \sum_{j=0}^{n-1} \mathbb{E} \left[ \sup_{t_j \leq s \leq t_{j+1}} |Y_s^N - Y_{t_j}^N|^{2pq^*} \right]^{1/q^*}.$$

Applying inequality (3.6), this leads to

$$(3.13) \quad \mathcal{E}_p^y \leq C_p h^p.$$

2c. Using (H3)(ii) and Remark 2.1, we have

$$\begin{aligned} |\zeta_j^{Y,w}| &\leq Ch_j(1 + |\tilde{Z}_j^N| + |\tilde{Z}_j|) |\tilde{Z}_j^N - \tilde{Z}_j| \\ &\leq Ch_j(1 + |\tilde{Z}_j^N|) (|\tilde{Z}_j^N - \tilde{Z}_j|^2 + |\tilde{Z}_j^N - \tilde{Z}_j|) \\ &\leq Ch_j(1 + |\tilde{Z}_j^N|) \\ &\quad \times \left( \mathbb{E}_{t_j} \left[ |Y_{t_{j+1}}^N| \left| H_j^R - \frac{\Delta W_j}{h_j} \right| \right]^2 + \mathbb{E}_{t_j} \left[ |Y_{t_{j+1}}^N| \left| H_j^R - \frac{\Delta W_j}{h_j} \right| \right] \right) \\ &\leq Ch_j(1 + |\tilde{Z}_j^N|) \left( \mathbb{E} \left[ \left| H_j^R - \frac{\Delta W_j}{h_j} \right| \right]^2 + \mathbb{E} \left[ \left| H_j^R - \frac{\Delta W_j}{h_j} \right| \right] \right), \end{aligned}$$

and thus, we obtain

$$\begin{aligned} &\mathbb{E} \left[ \left( \sum_{j=0}^{n-1} |\zeta_j^{Y,w}| \right)^{2pq^*} \right]^{1/q^*} \\ &\leq C_p \max_{0 \leq j \leq n-1} \left( \mathbb{E} \left[ \left| H_j^R - \frac{\Delta W_j}{h_j} \right| \right]^2 + \mathbb{E} \left[ \left| H_j^R - \frac{\Delta W_j}{h_j} \right| \right] \right)^{2p} \\ &\quad \times \left( 1 + \mathbb{E} \left[ \max_{0 \leq i \leq n-1} |\tilde{Z}_i^N|^{2pq^*} \right]^{1/q^*} \right). \end{aligned}$$

Using Lemma 2.2, we compute

$$\begin{aligned} \mathbb{E} \left[ \max_{0 \leq i \leq n-1} |\tilde{Z}_i^N|^{2pq^*} \right]^{1/q^*} &\leq C_p \left( 1 + \mathbb{E} \left[ \max_{0 \leq i \leq n-1} \mathbb{E}_{t_i} \left[ \sup_{0 \leq s \leq T} |X_s|^4 \right]^{pq^*} \right]^{1/q^*} \right) \\ &\leq C_p \left( 1 + \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s|^{4pq^*} \right]^{1/q^*} \right) \\ &\leq C_p, \end{aligned}$$

where we used the Doob maximal inequality. Finally, we obtain

$$(3.14) \quad \mathcal{E}_p^w \leq C_p \max_{0 \leq j \leq n-1} \left( \mathbb{E} \left[ \left| H_j^R - \frac{\Delta W_j}{h_j} \right| \right]^{4p} + \mathbb{E} \left[ \left| H_j^R - \frac{\Delta W_j}{h_j} \right| \right]^{2p} \right).$$

2d. Using (H3)(ii), (2.5), Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} |\xi_j^{Y, \tilde{z}}| &\leq Ch_j(1 + |\tilde{Z}_j^N| + |\bar{Z}_j^N|) |\tilde{Z}_j^N - \bar{Z}_j^N| \\ &\leq Ch^{1/2} h_j (1 + |\tilde{Z}_j^N| + |\bar{Z}_j^N|) \left( 1 + \mathbb{E}_{t_j} \left[ \sup_{t_j \leq s \leq t_{j+1}} |X_s|^4 \right]^{1/2} \right) \\ &\leq Ch^{1/2} h_j \left( 1 + \mathbb{E}_{t_j} \left[ \sup_{t_j \leq s \leq t_{j+1}} |X_s|^4 \right] \right). \end{aligned}$$

Then by same arguments than in part 2c we obtain

$$(3.15) \quad \mathcal{E}_p^{\tilde{z}} \leq C_p h^p.$$

2e. The last term is the more involved. Since the functions  $f$  and  $f_N$  are locally Lipschitz with respect to  $z$ , compute  $|\xi_j^{Y, \tilde{z}}|$ :

$$|\xi_j^{Y, \tilde{z}}| \leq C \mathbb{E}_{t_j} \left[ \left( 1 + \sup_{t_j \leq s \leq t_{j+1}} |Z_s^N| + |\bar{Z}_j^N| \right) \int_{t_j}^{t_{j+1}} |Z_s^N - \bar{Z}_j^N| ds \right],$$

and so,

$$(3.16) \quad |\xi_j^{Y, \tilde{z}}|^2 \leq Ch_j \mathbb{E}_{t_j} \left[ \left( 1 + \sup_{t_j \leq s \leq t_{j+1}} |Z_s^N|^2 + |\bar{Z}_j^N|^2 \right) \int_{t_j}^{t_{j+1}} |Z_s^N - \bar{Z}_j^N|^2 ds \right].$$

Let us remark that in the previous bound, the term inside the conditional expectation is a  $\mathcal{F}_{t_{j+1}}$ -measurable random variable, so we have

$$\begin{aligned} &\mathbb{E}_{t_j} \left[ \left( 1 + \sup_{t_j \leq s \leq t_{j+1}} |Z_s^N|^2 + |\bar{Z}_j^N|^2 \right) \int_{t_j}^{t_{j+1}} |Z_s^N - \bar{Z}_j^N|^2 ds \right] \\ &= \mathbb{E}_{t_j}^{\mathbb{Q}^\pi} \left[ \frac{1}{1 + h_j H_j^R \gamma_j^{N,n}} \left( 1 + \sup_{t_j \leq s \leq t_{j+1}} |Z_s^N|^2 + |\bar{Z}_j^N|^2 \right) \right. \\ &\quad \left. \times \int_{t_j}^{t_{j+1}} |Z_s^N - \bar{Z}_j^N|^2 ds \right] \\ &\leq \frac{1}{\varepsilon} \mathbb{E}_{t_j}^{\mathbb{Q}^\pi} \left[ \left( 1 + \sup_{t_j \leq s \leq t_{j+1}} |Z_s^N|^2 + |\bar{Z}_j^N|^2 \right) \int_{t_j}^{t_{j+1}} |Z_s^N - \bar{Z}_j^N|^2 ds \right] \\ &\leq \frac{1}{\varepsilon} \mathbb{E}_{t_j}^{\mathbb{Q}^\pi} \left[ \left( 1 + \sup_{0 \leq s \leq T} |Z_s^N|^2 + \max_{0 \leq i \leq n-1} |\bar{Z}_i^N|^2 \right) \int_{t_j}^{t_{j+1}} |Z_s^N - \bar{Z}_j^N|^2 ds \right] \end{aligned}$$

since  $1/(1 + h_j H_j^R \gamma_j^{N,n}) \leq 1/\varepsilon$  under (H3). Then (3.16) becomes

$$(3.17) \quad |\zeta_j^{Y, \bar{z}}|^2 \leq Ch_j \mathbb{E}_{t_j}^{\mathbb{Q}^\pi} \left[ \left( 1 + \sup_{0 \leq s \leq T} |Z_s^N|^2 + \max_{0 \leq i \leq n-1} |\bar{Z}_i^N|^2 \right) \times \int_{t_j}^{t_{j+1}} |Z_s^N - \bar{Z}_j^N|^2 ds \right].$$

Thanks to Proposition 2.3 and Lemma 2.1 we can simplify the first part of our estimate:

$$\sup_{0 \leq s \leq T} |Z_s^N| \leq C \left( 1 + \sup_{0 \leq s \leq T} |X_s| \right)$$

and

$$\begin{aligned} \max_{0 \leq i \leq n-1} |\bar{Z}_i^N| &\leq C \left( 1 + \max_{0 \leq i \leq n-1} \mathbb{E}_{t_i} \left[ \sup_{t_i \leq s \leq t_{i+1}} |X_s| \right] \right) \\ &\leq C \left( 1 + \max_{0 \leq i \leq n-1} \mathbb{E}_{t_i} \left[ \sup_{0 \leq s \leq T} |X_s| \right] \right). \end{aligned}$$

Inserting these two bounds into (3.17), we obtain

$$\begin{aligned} \mathcal{E}_p^{\bar{z}} &\leq C \mathbb{E} \left[ \sup_{0 \leq i \leq n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi} \left[ \left( 1 + \max_{0 \leq j \leq n} \mathbb{E}_{t_j} \left[ \sup_{0 \leq s \leq T} |X_s|^2 \right] \right) \right. \right. \\ &\quad \left. \left. \times \sum_{j=i}^{n-1} \int_{t_j}^{t_{j+1}} |Z_s^N - \bar{Z}_j^N|^2 ds \right]^p \right], \end{aligned}$$

and, using Hölder’s inequality and a convexity inequality, we get for any  $\eta > 0$

$$(3.18) \quad \begin{aligned} \mathcal{E}_p^{\bar{z}} &\leq C_{\eta,p} \left( 1 + \mathbb{E} \left[ \sup_{0 \leq i \leq n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi} \left[ \max_{0 \leq j \leq n} \mathbb{E}_{t_j} \left[ \sup_{0 \leq s \leq T} |X_s|^2 \right]^{(1+\eta)/\eta} \right]^p \right]^{\eta/(1+\eta)} \right) \\ &\quad \times \mathbb{E} \left[ \sup_{0 \leq i \leq n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi} \left[ \left( \sum_{j=i}^{n-1} \int_{t_j}^{t_{j+1}} |Z_s^N - \bar{Z}_j^N|^2 ds \right)^{1+\eta} \right]^p \right]^{1/(1+\eta)} \\ &\leq C_{\eta,p} h^{-(p\eta)/(1+\eta)} \\ &\quad \times \left( 1 + \mathbb{E} \left[ \sup_{0 \leq i \leq n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi} \left[ \max_{0 \leq j \leq n} \mathbb{E}_{t_j} \left[ \sup_{0 \leq s \leq T} |X_s|^2 \right]^{(1+\eta)/\eta} \right]^p \right]^{\eta/(1+\eta)} \right) \\ &\quad \times \mathbb{E} \left[ \sup_{0 \leq i \leq n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi} \left[ \sum_{j=i}^{n-1} \left( \int_{t_j}^{t_{j+1}} |Z_s^N - \bar{Z}_j^N|^2 ds \right)^{1+\eta} \right]^p \right]^{1/(1+\eta)}. \end{aligned}$$

We can easily upper bound the first part of the last estimate. Indeed, thanks to Proposition 2.11 we are able to use once again Hölder’s inequality with  $p^*$  and  $q^*$ :

$$\begin{aligned} &\mathbb{E} \left[ \sup_{0 \leq i \leq n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi} \left[ \max_{0 \leq j \leq n} \mathbb{E}_{t_j} \left[ \sup_{0 \leq s \leq T} |X_s|^2 \right]^{(1+\eta)/\eta} \right]^p \right]^{\eta/(1+\eta)} \\ &\leq \mathbb{E} \left[ \sup_{0 \leq i \leq n-1} \mathbb{E}_{t_i} \left[ \prod_{j=i}^{n-1} (1 + h_j H_j^R \gamma_j^{N,n})^{p^*} \right]^{p/p^*} \right] \end{aligned}$$

$$\begin{aligned}
 & \times \mathbb{E}_{t_i} \left[ \max_{0 \leq j \leq n} \mathbb{E}_{t_j} \left[ \sup_{0 \leq s \leq T} |X_s|^2 \right]^{(q^*(1+\eta))/\eta} \right]^{p/q^*} ]^{\eta/(1+\eta)} \\
 & \leq C_{\eta,p} \mathbb{E} \left[ \sup_{0 \leq i \leq n-1} \mathbb{E}_{t_i} \left[ \max_{0 \leq j \leq n} \mathbb{E}_{t_j} \left[ \sup_{0 \leq s \leq T} |X_s|^2 \right]^{(q^*(1+\eta))/\eta} \right]^{p/q^*} \right]^{\eta/(1+\eta)} \\
 & \leq C_{\eta,p} \mathbb{E} \left[ \sup_{0 \leq i \leq n-1} \mathbb{E}_{t_i} \left[ \max_{0 \leq j \leq n} \mathbb{E}_{t_j} \left[ \sup_{0 \leq s \leq T} |X_s|^2 \right]^{(q^*(1+\eta))/\eta} \right]^{2p} \right]^{\eta/(2q^*(1+\eta))}.
 \end{aligned}$$

To conclude now, we just have to use Doob maximal inequality and classical estimates on  $X$  to obtain

$$\begin{aligned}
 & \mathbb{E} \left[ \sup_{0 \leq i \leq n-1} \mathbb{E}_{t_i} \left[ \max_{0 \leq j \leq n} \mathbb{E}_{t_j} \left[ \sup_{0 \leq s \leq T} |X_s|^2 \right]^{(q^*(1+\eta))/\eta} \right]^{2p} \right]^{\eta/(2q^*(1+\eta))} \\
 & \leq C_{\eta,p} \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s|^{(2pq^*(1+\eta))/\eta} \right]^{\eta/(2q^*(1+\eta))} \leq C_{\eta,p}.
 \end{aligned}$$

Finally, (3.18) becomes

$$\begin{aligned}
 (3.19) \quad \mathcal{E}_p^{\bar{z}} & \leq C_{\eta,p} h^{-(p\eta)/(1+\eta)} \\
 & \times \mathbb{E} \left[ \sup_{0 \leq i \leq n-1} \mathbb{E}_{t_i}^{\mathbb{Q}^\pi} \left[ \sum_{j=i}^{n-1} \left( \int_{t_j}^{t_{j+1}} |Z_s^N - \bar{Z}_j^N|^2 ds \right)^{1+\eta} \right]^p \right]^{1/(1+\eta)}.
 \end{aligned}$$

Applying Proposition 3.2, we deduce from the last inequality

$$(3.20) \quad \mathcal{E}_p^{\bar{z}} \leq C_{\eta,p} h^{p/(1+\eta)} = C_{\eta,p} h^{p(1-\tilde{\eta})},$$

with  $\tilde{\eta} = 1 - 1/(1 + \eta)$ . Since (3.20) is true for all  $\eta > 0$ , then it is true for all  $\tilde{\eta} > 0$  and then we can replace  $\tilde{\eta}$  by  $\eta$ .

3. Inserting estimates (3.12)–(3.13)–(3.14)–(3.20) in (3.8) completes the proof of the proposition.  $\square$

### 3.4. Discretization error for the $Z$ -component.

PROPOSITION 3.4. *There exists  $q^* > 1$  (the same as in Proposition 3.3) such that for all  $\eta > 0$ ,*

$$\begin{aligned}
 & \mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z_s - Z_i^\pi|^2 ds \right] \\
 & \leq C_{\alpha,\eta} h^{1-\eta} + C \mathbb{E} \left[ \sup_{0 \leq j \leq n} |X_{t_j} - X_j^\pi|^{4q^*} \right]^{1/(2q^*)} \\
 & \quad + C \max_{0 \leq j \leq n-1} \left( \mathbb{E} \left[ \left| H_j^R - \frac{\Delta W_j}{h_j} \right|^4 \right] + \mathbb{E} \left[ \left| H_j^R - \frac{\Delta W_j}{h_j} \right|^2 \right] \right).
 \end{aligned}$$

PROOF. The proof is divided in several steps.

1. First, thanks to Theorem 2.2 we know that we just have to estimate the error between  $Z^N$  and  $Z^\pi$ . We then observe

$$\begin{aligned} & \mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z_s^N - Z_i^\pi|^2 ds \right] \\ & \leq 4\mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z_s^N - \bar{Z}_i^N|^2 ds \right] + 4\mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\bar{Z}_i^N - \tilde{Z}_i^N|^2 ds \right] \\ & \quad + 4\mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\tilde{Z}_i^N - \tilde{Z}_i|^2 ds \right] + 4\mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\tilde{Z}_i - Z_i^\pi|^2 ds \right]. \end{aligned}$$

Applying Theorem 5.6 in [28], we obtain

$$\mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z_s^N - \bar{Z}_i^N|^2 ds \right] \leq Ch.$$

Moreover, by using (2.5) and classical estimates on  $X$ , we directly have that

$$\mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\bar{Z}_i^N - \tilde{Z}_i^N|^2 ds \right] \leq Ch.$$

Finally, by using the fact that  $Y^N$  is bounded uniformly in  $n$  (see Remark 2.1) we easily compute that

$$\begin{aligned} \mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\tilde{Z}_i^N - \tilde{Z}_i|^2 ds \right] & \leq \mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}_{t_i} \left[ |Y_{t_{i+1}}^N| \left| H_i^R - \frac{\Delta W_i}{h_i} \right|^2 ds \right] \right] \\ & \leq C \max_{0 \leq j \leq n-1} \mathbb{E} \left[ \left| H_j^R - \frac{\Delta W_j}{h_j} \right|^2 \right]. \end{aligned}$$

Thus, we conclude that

$$\begin{aligned} \mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z_s^N - Z_i^\pi|^2 ds \right] & \leq Ch + C \max_{0 \leq j \leq n-1} \mathbb{E} \left[ \left| H_j^R - \frac{\Delta W_j}{h_j} \right|^2 \right] \\ & \quad + \mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\tilde{Z}_i - Z_i^\pi|^2 ds \right]. \end{aligned}$$

2. Applying the stability results of Proposition 2.12, we obtain

$$\begin{aligned} (3.21) \quad \mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\tilde{Z}_i - Z_i^\pi|^2 ds \right] & \leq C\mathbb{E}[|Y_{t_n}^N - Y_n^\pi|^2] + C\mathbb{E} \left[ \sum_{i=0}^{n-1} \frac{|\zeta_i^Y|^2}{h_i} \right] \\ & \quad + C\mathbb{E} \left[ \sup_{0 \leq i \leq n-1} |Y_{t_i}^N - Y_i^\pi|^4 \right]^{1/2}. \end{aligned}$$

Using the same arguments as in proof of Proposition 3.3 with the simpler setting  $p = 1$  and  $\mathbb{Q}^\pi = \mathbb{P}$  (these arguments also require to show Proposition 3.2 with  $\mathbb{Q}^\pi = \mathbb{P}$ ), one retrieves that

$$\begin{aligned} & \mathbb{E}[|Y_{t_n}^N - Y_n^\pi|^2] + \mathbb{E}\left[\sum_{i=0}^{n-1} \frac{|\zeta_i^Y|^2}{h_i}\right] \\ & \leq C_\eta h^{1-\eta} + C\mathbb{E}\left[\sup_{0 \leq j \leq n} |X_{t_j} - X_j^\pi|^2\right] \\ & \quad + C \max_{0 \leq j \leq n-1} \left( \mathbb{E}\left[\left|H_j^R - \frac{\Delta W_j}{h_j}\right|^4\right] + \mathbb{E}\left[\left|H_j^R - \frac{\Delta W_j}{h_j}\right|^2\right] \right). \end{aligned}$$

Plugging the last inequality in equation (3.21) and applying Proposition 3.3, with  $p = 2$ , we obtain

$$\begin{aligned} & \mathbb{E}\left[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\tilde{Z}_i - Z_i^\pi|^2 ds\right] \\ & \leq C_\eta h^{1-\eta} + C\mathbb{E}\left[\sup_{0 \leq j \leq n} |X_{t_j} - X_j^\pi|^{4q^*}\right]^{1/(2q^*)} \\ & \quad + C \max_{0 \leq j \leq n-1} \left( \mathbb{E}\left[\left|H_j^R - \frac{\Delta W_j}{h_j}\right|^4\right] + \mathbb{E}\left[\left|H_j^R - \frac{\Delta W_j}{h_j}\right|^2\right] \right). \end{aligned}$$

Combining this last inequality with step 1 completes the proof of the proposition.  $\square$

3.5. *Proof of Theorem 1.1.* We have to combine Proposition 3.3 with  $p = 1$ , Proposition 3.4 with classical estimates on the Euler scheme for SDE, recall (1.8), and classical results about Gaussian distribution tails. Indeed, we compute that

$$\begin{aligned} (3.22) \quad \mathbb{E}\left[\left|H_i^R - \frac{\Delta W_i}{h_i}\right|\right] & \leq \mathbb{E}\left[\left|H_i^R - \frac{\Delta W_i}{h_i}\right|^2\right]^{1/2} \leq \left(\frac{2d}{h_i} \int_R^{+\infty} x^2 \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx\right)^{1/2} \\ & \leq C \left(\frac{Re^{-R^2/2}}{h_i}\right)^{1/2} \leq C \left(\frac{\log n}{e^{1/2(\log n)^2 - \theta \log n}}\right)^{1/2} \leq \frac{C}{n}, \end{aligned}$$

recall (1.4).

### 4. Numerical scheme.

4.1. *Definition and convergence.* In this part, we propose a fully implementable numerical scheme based on a Markovian quantization method; see, for

example, [26, 37] for general results about quantization and [2, 17] for a setting related to ours. To this end, given  $\delta > 0$  and  $\kappa \in \mathbb{N}^*$ , we consider the bounded lattice grid:

$$\Gamma = \{x \in \delta\mathbb{Z}^d \mid |x^j| \leq \kappa\delta, 1 \leq j \leq d\}.$$

Observe that there are  $(2\kappa)^d + 1$  points in  $\Gamma$ . We then introduce a projection operator  $\Pi$  on the grid  $\Gamma$  centered in  $X_0$  given by, for  $x \in R^d$ ,

$$(\Pi[x])^j = \begin{cases} \delta \lfloor \delta^{-1}(x^j - X_0^j) + \frac{1}{2} \rfloor + X_0^j, & \text{if } |x^j - X_0^j| \leq \kappa\delta, \\ \kappa\delta, & \text{if } x^j - X_0^j > \kappa\delta, \\ -\kappa\delta, & \text{if } x^j - X_0^j < -\kappa\delta. \end{cases}$$

To compute the conditional expectation appearing in the scheme given in Definition 1.1, we use an optimal quantization of Gaussian random variables  $(\Delta W_i)$ . These random variables are approximated by a sequence of centered random variables  $(\Delta \widehat{W}_i = \sqrt{h_i} G_M(\frac{\Delta W_i}{\sqrt{h_i}}))$  with discrete support. Here,  $G_M$  denotes the projection operator on the optimal quantization grid for the standard Gaussian distribution with  $M$  points in the support; see [26, 37] for details.<sup>3</sup> Moreover, it is shown in [26] that

$$(4.1) \quad \mathbb{E}[|\Delta W_i - \Delta \widehat{W}_i|^p]^{1/p} \leq C_{p,d} \sqrt{h} M^{-1/d}.$$

In this context, we introduce the following discrete/truncated version of the Euler scheme:

$$(4.2) \quad \begin{cases} \widehat{X}_0^\pi = X_0, \\ \widehat{X}_{i+1}^\pi = \Pi[\widehat{X}_i^\pi + h_i b(\widehat{X}_i^\pi) + \sigma(\widehat{X}_i^\pi) \Delta \widehat{W}_i]. \end{cases}$$

We observe that  $\widehat{X}^\pi$  is a Markovian process living on  $\Gamma$  and satisfying  $|\widehat{X}_i^\pi| \leq C(|X_0| + \kappa\delta)$ , for all  $i \leq n$ .

We then adapt the scheme given in Definition 1.1 to this framework.

DEFINITION 4.1. We denote  $(\widehat{Y}^\pi, \widehat{Z}^\pi)_{0 \leq i \leq n}$  the solution of the BTZ-scheme satisfying:

- (i) the terminal condition is  $(\widehat{Y}_n^\pi, \widehat{Z}_n^\pi) = (g(\widehat{X}_n^\pi), 0)$ ;
- (ii) for  $i < n$ , the transition from step  $i + 1$  to step  $i$  is given by

$$(4.3) \quad \begin{cases} \widehat{Y}_i^\pi = \mathbb{E}_{t_i}[\widehat{Y}_{i+1}^\pi + h_i f_N(\widehat{X}_i^\pi, \widehat{Y}_i^\pi, \widehat{Z}_i^\pi)], \\ \widehat{Z}_i^\pi = \mathbb{E}_{t_i}[\widehat{Y}_{i+1}^\pi \widehat{H}_i^R]. \end{cases}$$

The coefficients  $(\widehat{H}_i^R)$  are defined, given  $R > 0$ , by

$$(4.4) \quad (\widehat{H}_i^R)^\ell = \frac{-R}{\sqrt{h_i}} \vee \frac{(\Delta \widehat{W}_i)^\ell}{h_i} \wedge \frac{R}{\sqrt{h_i}}, \quad 1 \leq \ell \leq d.$$

The parameters  $R$  and  $N$  are chosen as in (1.9).

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<sup>3</sup>The grids can be downloaded from the website: <http://www.quantize.maths-fi.com/>.

PROPOSITION 4.1.  $(\widehat{Y}^\pi, \widehat{Z}^\pi)$  is a Markovian process. More precisely, for all  $i \in \{0, \dots, n\}$ , there exist two functions  $u^\pi(t_i, \cdot) : \Gamma \rightarrow \mathbb{R}$  and  $v^\pi(t_i, \cdot) : \Gamma \rightarrow \mathbb{R}^{1 \times d}$  such that

$$\widehat{Y}^\pi = u^\pi(t_i, \widehat{X}_i^\pi) \quad \text{and} \quad \widehat{Z}_i^\pi = v^\pi(t_i, \widehat{X}_i^\pi).$$

These functions can be computed on the grid by the following backward induction: for all  $i \in \{0, \dots, n\}$  and  $x \in \Gamma$ ,

$$(4.5) \quad \begin{cases} v^\pi(t_i, x) = \mathbb{E} \left[ u^\pi(t_{i+1}, \Pi(x + h_i b(x) + \sqrt{h_i} \sigma(x) G_M(U))) \frac{G_M^R(U)}{\sqrt{h_i}} \right], \\ u^\pi(t_i, x) = \mathbb{E} [u^\pi(t_{i+1}, \Pi(x + h_i b(x) + \sqrt{h_i} \sigma(x) G_M(U)))] \\ \quad + h f_N(t_i, x, u^\pi(t_i, x), v^\pi(t_i, x)) \quad \text{for } i < n, \end{cases}$$

with  $U \sim \mathcal{N}(0, 1)$  and  $(G_M^R(\cdot))^\ell = (-R) \vee (G_M(\cdot))^\ell \wedge R$ , for  $\ell \in \{1, \dots, d\}$ .

The terminal condition is given by  $u^\pi(t_n, x) = g(x)$  and  $v^\pi(t_n, x) = 0$ .

REMARK 4.1. Observe that the above scheme is implicit in  $u^\pi(t_i, x)$ . We then use a Picard iteration to compute this term in practice, the error is very small because  $hK_y \ll 1$  and we do not study it here.

THEOREM 4.1. For all  $r > 0$  and  $\eta > 0$ , the following holds:

$$|Y_0 - \widehat{Y}_0^\pi| \leq C_{\alpha, \eta} h^{(1/2) - \eta} + C_r n (\kappa \delta)^{-r} + C(\delta n + n^{\alpha + (1/2)}) M^{-1/d}.$$

From the above theorem, we straightforwardly deduce the following corollary.

COROLLARY 4.1. Setting  $\delta = n^{-3/2}$ ,  $\kappa = n^{3/2 + \tilde{\eta}}$  and  $M = n^{(1+\alpha)d}$ , we obtain

$$|Y_0 - \widehat{Y}_0^\pi| \leq C_{\alpha, \eta, \tilde{\eta}} h^{(1/2) - \eta},$$

for all  $\eta > 0$ ,  $\tilde{\eta} > 0$  and  $0 < \alpha < \frac{1}{2}$ .

PROOF OF THEOREM 4.1. 1. Error on  $Y$ : We first observe that

$$|Y_0 - \widehat{Y}_0^\pi| \leq |Y_0 - Y_0^\pi| + |Y_0^\pi - \widehat{Y}_0^\pi|.$$

Applying Theorem 1.1, we obtain

$$|Y_0 - \widehat{Y}_0^\pi| \leq C_{\alpha, \eta} h^{(1/2) - \eta} + |Y_0^\pi - \widehat{Y}_0^\pi|.$$

For the second term, we simply rewrite  $(\widehat{Y}^\pi, \widehat{Z}^\pi)$  as a perturbation of the scheme given in Definition 1.1, namely

$$\widehat{Y}_i^\pi = \mathbb{E}_{t_i} [\widehat{Y}_{i+1}^\pi + h_i f_N(X_i^\pi, \widehat{Y}_i^\pi, \mathbb{E}_{t_i} [\widehat{Y}_{i+1}^\pi H_i^R])] + \zeta_i^Y$$

with

$$\zeta_i^Y := h_i (f_N(\widehat{X}_i^\pi, \widehat{Y}_i^\pi, \widehat{Z}_i^\pi) - f_N(X_i^\pi, \widehat{Y}_i^\pi, \mathbb{E}_{t_i} [\widehat{Y}_{i+1}^\pi H_i^R])).$$

Applying Proposition 2.7 for the two schemes and the Corollary 2.1, we obtain for some  $q > 1$ ,

$$(4.6) \quad |Y_0^\pi - \widehat{Y}_0^\pi| \leq C \left( \mathbb{E}[|X_n^\pi - \widehat{X}_n^\pi|^q]^{1/q} + \mathbb{E} \left[ \left( \sum_{i=0}^{n-1} |\zeta_i^{Y,x}| \right)^q \right]^{1/q} + \mathbb{E} \left[ \left( \sum_{i=0}^{n-1} |\zeta_i^{Y,z}| \right)^q \right]^{1/q} \right),$$

where

$$\begin{aligned} \zeta_i^{Y,x} &:= h_i(f_N(\widehat{X}_i^\pi, \widehat{Y}_i^\pi, \widehat{Z}_i^\pi) - f_N(X_i^\pi, \widehat{Y}_i^\pi, \widehat{Z}_i^\pi)), \\ \zeta_i^{Y,z} &:= h_i(f_N(X_i^\pi, \widehat{Y}_i^\pi, \widehat{Z}_i^\pi) - f_N(X_i^\pi, \widehat{Y}_i^\pi, \mathbb{E}_i[\widehat{Y}_{i+1}^\pi, H_i^R])). \end{aligned}$$

We easily compute that

$$(4.7) \quad \mathbb{E} \left[ \left( \sum_{i=0}^{n-1} |\zeta_i^{Y,x}| \right)^q \right]^{1/q} \leq C \mathbb{E} \left[ \sup_i |X_i^\pi - \widehat{X}_i^\pi|^q \right]^{1/q}$$

and

$$(4.8) \quad \mathbb{E} \left[ \left( \sum_{i=0}^{n-1} |\zeta_i^{Y,z}| \right)^q \right]^{1/q} \leq C n^\alpha \sup_i \mathbb{E}[|H_i^R - \widehat{H}_i^R|^q]^{1/q}.$$

From (4.1), it follows that

$$\mathbb{E}[|H_i^R - \widehat{H}_i^R|^q]^{1/q} \leq C n^{1/2} M^{-1/d}.$$

Combining the above estimations with (4.6), we obtain

$$(4.9) \quad |Y_0^\pi - \widehat{Y}_0^\pi| \leq C \left( \mathbb{E} \left[ \sup_i |X_i^\pi - \widehat{X}_i^\pi|^q \right]^{1/q} + n^{\alpha+(1/2)} M^{-1/d} \right).$$

2. We now study the first term in the right-hand side of the above equation, namely the error on the forward component.

Let  $\tilde{X}^\pi$  denote the Euler scheme for  $X$  where we replace  $\Delta W_i$  by  $\Delta \widehat{W}_i$ , that is,

$$\tilde{X}_{i+1}^\pi = \tilde{X}_i^\pi + h_i b(\tilde{X}_i^\pi) + \sigma(\tilde{X}_i^\pi) \Delta \widehat{W}_i.$$

We then split the error into two terms:

$$\begin{aligned} &\mathbb{E} \left[ \sup_i |X_i^\pi - \widehat{X}_i^\pi|^q \right]^{1/q} \\ &\leq C \left( \mathbb{E} \left[ \sup_i |X_i^\pi - \tilde{X}_i^\pi|^{2q} \right]^{1/(2q)} + \mathbb{E} \left[ \sup_i |\tilde{X}_i^\pi - \widehat{X}_i^\pi|^{2q} \right]^{1/(2q)} \right). \end{aligned}$$

2a. We now write  $\tilde{X}^\pi$  as a perturbation of  $X^\pi$ , namely

$$\tilde{X}_{i+1}^\pi = \tilde{X}_i^\pi + h_i b(\tilde{X}_i^\pi) + \sigma(\tilde{X}_i^\pi) \Delta W_i + \xi_i^{\tilde{X}}$$

with

$$\zeta_i^{\tilde{X}} = \sigma(\tilde{X}_i^\pi)(\Delta \widehat{W}_i - \Delta W_i).$$

Applying Lemma A.1, we obtain

$$\mathbb{E} \left[ \sup_{0 \leq j \leq n} |X_j^\pi - \tilde{X}_j^\pi|^{2q} \right]^{1/(2q)} \leq C \mathbb{E} \left[ \left( \sum_{j=0}^n |\zeta_j^{\tilde{X}}| \right)^{2q} \right]^{1/(2q)}.$$

Moreover, we compute

$$\mathbb{E} \left[ \left( \sum_{j=0}^n |\zeta_j^{\tilde{X}}| \right)^{2q} \right] \leq n^{2q-1} \sum_{j=0}^n \mathbb{E}[|\zeta_j^{\tilde{X}}|^{2q}] \leq C n^q M^{-2q/d}$$

since

$$\begin{aligned} \mathbb{E}[|\zeta_j^{\tilde{X}}|^{2q}] &\leq C \mathbb{E}[(1 + |\tilde{X}_j^\pi|)^{4q}]^{1/2} \mathbb{E}[|\Delta \widehat{W}_j - \Delta W_j|^{4q}]^{1/2} \\ &\leq C h^q M^{-(2q)/d}. \end{aligned}$$

Combining the above estimation, we obtain

$$\mathbb{E} \left[ \sup_{0 \leq j \leq n} |X_j^\pi - \tilde{X}_j^\pi|^{2q} \right]^{1/(2q)} \leq C \sqrt{n} M^{-1/d}.$$

2b. We now write  $\widehat{X}^\pi$  as a perturbation of  $\tilde{X}^\pi$ , namely

$$\widehat{X}_{i+1}^\pi = \widehat{X}_i^\pi + h_i b(\widehat{X}_i^\pi) + \sigma(\widehat{X}_i^\pi) \Delta \widehat{W}_i + \zeta_i^{\widehat{X}},$$

with

$$\zeta_i^{\widehat{X}} = \Pi[\check{X}_{i+1}] - \check{X}_{i+1} \quad \text{and} \quad \check{X}_{i+1} := \widehat{X}_i^\pi + h_i b(\widehat{X}_i^\pi) + \sigma(\widehat{X}_i^\pi) \Delta \widehat{W}_i.$$

Applying Lemma A.1, we get

$$\mathbb{E} \left[ \sup_{0 \leq j \leq n} |\tilde{X}_j^\pi - \widehat{X}_j^\pi|^{2q} \right]^{1/(2q)} \leq C \mathbb{E} \left[ \left( \sum_{j=0}^n |\zeta_j^{\widehat{X}}| \right)^{2q} \right]^{1/(2q)}.$$

From the definition of the projection operator, we have that, for all  $r > 1$ ,

$$|\zeta_j^{\widehat{X}}| \leq \delta + |\check{X}_{i+1}| \mathbf{1}_{\{|\check{X}_{i+1}| > \kappa \delta\}} \leq \delta + \frac{|\check{X}_{i+1}|^{r+1}}{(\kappa \delta)^r}$$

which leads to

$$\mathbb{E} \left[ \sup_{0 \leq j \leq n} |\tilde{X}_j^\pi - \widehat{X}_j^\pi|^{2q} \right]^{1/(2q)} \leq C n \left( \delta + \frac{1}{(\kappa \delta)^r} \mathbb{E} \left[ \sup_{0 \leq j \leq n} |\check{X}_j|^{2q(r+1)} \right]^{1/(2q)} \right).$$

The proof for this step is complete observing that  $\mathbb{E}[\sup_j |\check{X}_j|^{2q(r+1)}]^{1/(2q)} \leq C_r$ .

3. The proof is concluded by inserting the above estimate in (4.9).  $\square$

4.2. *A numerical example.* We illustrate in this part the convergence of the algorithm given in Definition 4.1 with  $d \in \{1, 2, 3\}$ . To this end, we consider the following quadratic Markovian BSDE:

$$\begin{cases} X_t^\ell = X_0^\ell + \int_0^t \nu X_s^\ell dW_s^\ell, & \ell \in \{1, 2, 3\}, \\ Y_t = g(X_1) + \int_t^1 \frac{a}{2} \|Z_s\|^2 ds - \int_t^1 Z_s dW_s, \end{cases} \quad 0 \leq t \leq 1,$$

where  $a, \nu$  and  $(X_0^\ell)_{\ell \in \{1,2,3\}}$  are given real positive parameters and  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  is a bounded Lipschitz function.

Applying Itô’s formula, one can show that the solution is given by

$$Y_t = \frac{1}{a} \log(\mathbb{E}_t[\exp(ag(X_1))]), \quad t \leq 1.$$

For any given  $g, \nu$  and  $a$ , it is possible to estimate the solution  $Y_0$  at time 0 using an approximation of the Gaussian distribution at time  $T = 1$ , since  $X_1^\ell = X_0^\ell e^{-(\nu^2/2) + \nu W_1^\ell}$ .

4.2.1. *Illustration when  $d = 2$ .* For our numerical illustration,  $g$  is given by

$$g : x \mapsto 3 \sum_{\ell=1}^2 \sin^2(x^\ell),$$

and we set  $\nu = 1, X_0^1 = X_0^2 = 1$ .

Given  $n$  the number of time steps in the approximation grid, we consider

$$N(n) = n^{1/4} \quad \text{and} \quad R(n) = \log(n),$$

recalling (1.9). We will refer to the scheme given in Definition 4.1 with this set of parameters  $(N, R)$  as the “adaptive truncation” scheme. We discuss in Section 4.2.3 below the choice of  $\alpha$ .

The graph on Figure 1 shows the convergence of the algorithm for time step varying from 5 to 40. In the simulation, we fixed  $M$  to be large enough ( $M = 100$ ), so that the error in the space discretization can be neglected in the analysis.

The expected convergence rate should be between 0.5, that is to say the minimal rate proved in this paper, and 1 the general optimal rate for the Euler scheme; see, for example, [11, 22]. We found a rate 0.6 which then seems reasonable. Note that all the convergence rate estimated below are also in the predicted range.

On Figure 2, we illustrate qualitatively the importance of the truncation procedure.

When  $a = 1$ , we already observed that the scheme given in Definition 4.1 is converging nicely. It appears that for this specific choice of parameters  $X_0, \nu, g$  and  $a$ ,

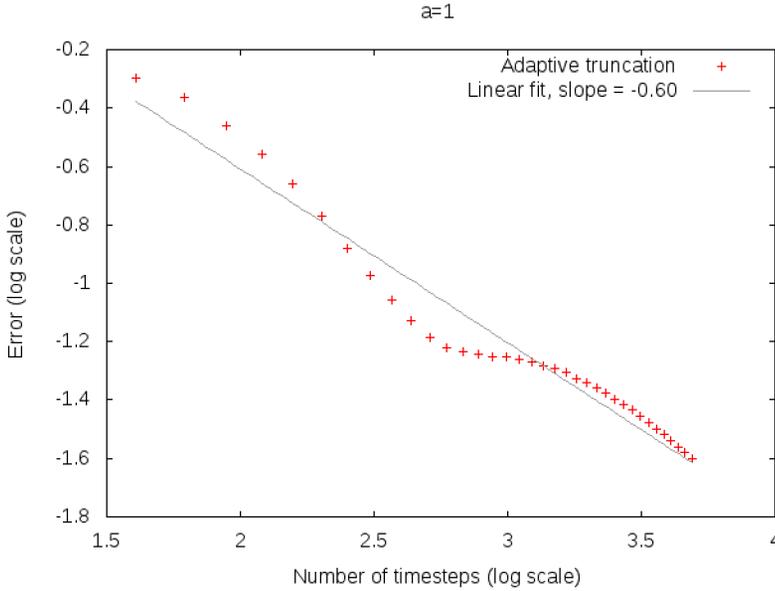


FIG. 1. Empirical convergence of the scheme given in Definition 4.1.

the usual BTZ-scheme, referred to as “no truncation” scheme, is also converging. But, when  $a$  becomes bigger, the usual BTZ-scheme becomes unstable.

On Figure 2, we consider  $a = 3.5$ . In this case, the behavior of the usual BTZ-scheme is interesting. First, let us mention that we plot a truncated error which explains the flat alignment of some points. This shows that the scheme is not stable. It manages though to be stabilised when the number of time step is big enough ( $h$  small enough). We are not able to explain yet this behavior. The detailed study of the numerical stability (or unstability) of the BTZ-scheme in the quadratic setting is outside the scope of this paper. These questions are left for further research. In the (more classical) Lipschitz case, we refer the reader to [12].

We also observe that the “adaptive truncation” scheme is converging nicely, even for this large value of  $a$ .

4.2.2. *Illustration when  $d = 3$ .* For our numerical illustration, we tested the usual BTZ-scheme and the adaptively truncated scheme given in Definition 4.1 ( $\alpha = 1/4$ ) for various models, that is, various terminal conditions  $g : \mathbb{R}^3 \rightarrow \mathbb{R}$  and values of  $a$ . In practice, we used the following parameters:

- (i) Model I:  $g(x) = 3 \sin^2(\sum_{\ell=1}^3 x^\ell)$  and  $a = 5$ .
- (ii) Model II:  $g(x) = 3 \sum_{\ell=1}^3 \sin^2(x^\ell)$  and  $a = 5$ .
- (iii) Model III:  $g(x) = 4 \operatorname{atan}(\sum_{\ell=1}^3 x^\ell)$  and  $a = 5$ .
- (iv) Model IV:  $g(x) = 3 \wedge [x^1 - x^2]_+ + [2 - x^3]_+$  and  $a = 4$ .

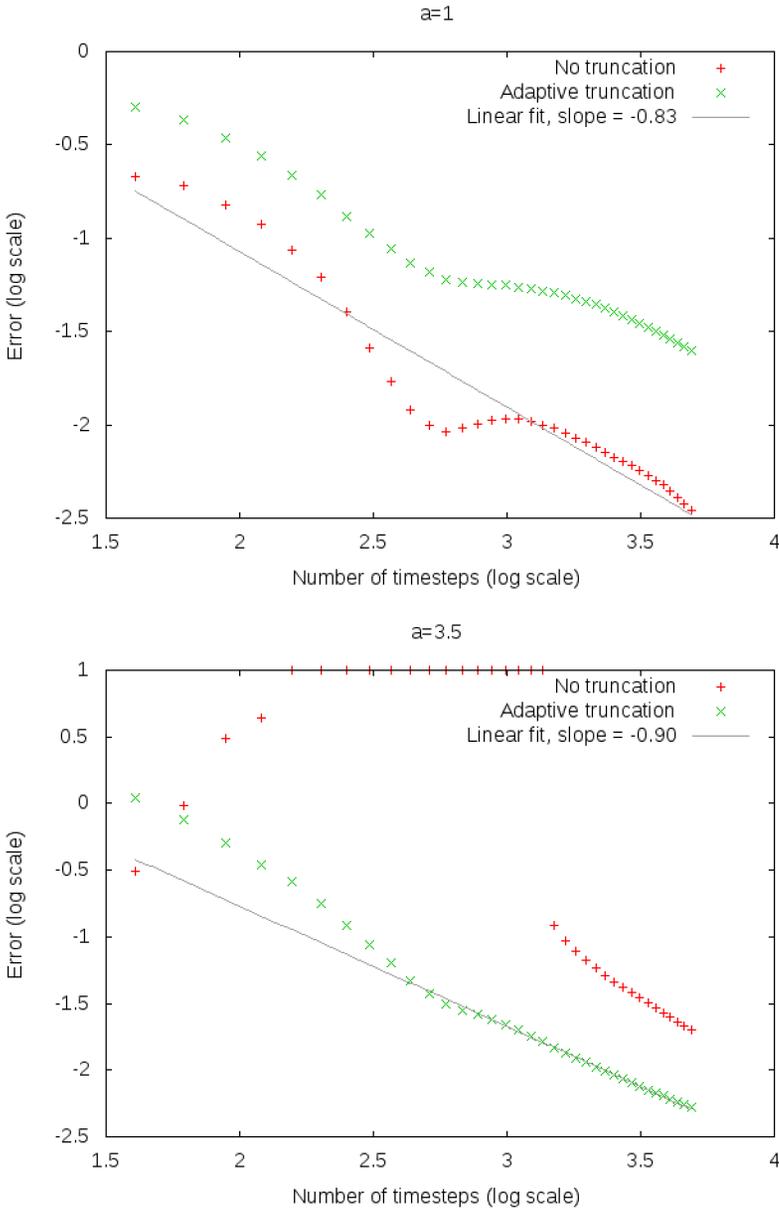


FIG. 2. Comparison of schemes' convergence.

We set the number of time steps  $N = 12$ .<sup>4</sup> We gather in the Table 1 the results we obtained. The true value is estimated using the Cole–Hopf transform and we indicate, when relevant, the relative error between parenthesis.

<sup>4</sup>It takes 1/2 hour to obtain one value on an ultrabook with Intel Core i7-3667U CPU @ 2.00 GHz (4 cores).

TABLE 1

Comparison between the truncated and the untruncated scheme for different models in dimension 3

Scheme/Model	I	II	III	IV
True value	2.67	7.53	5.38	3.96
No truncation	$7.06 \times 10^6$	$4.98 \times 10^{59}$	5.31 (<2%)	$1.13 \times 10^{29}$
Adaptive truncation	2.69 (<1%)	7.29 (~3%)	5.31 (<2%)	4.37 (~10%)

For this large value of  $a$ , the adaptively truncated scheme is always able to compute good estimates of the true value. This is only the case for Model III when using the BTZ-scheme. For the other models, the usual BTZ-scheme is unstable.

4.2.3. *Influence of the  $\alpha$  parameter.* To conclude this numerical illustration, we would like to comment on the choice of  $\alpha$ . To do this, we work with  $d = 1$  in order to be able to use quite a lot of time steps ( $n = 250$ ). Moreover, we set  $\nu = 0.4$ ,  $a = 5$  and  $g = 3 \sin^2$ . We plot on Figure 3 the convergence error of the scheme for  $\alpha = 0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{5}{8}$  thus varying the truncation parameter  $N = n^\alpha$ . The theoretical convergence result of Corollary 4.1 states no dependence upon  $\alpha$  for the convergence rate when  $\alpha \in (0, \frac{1}{2})$ . This is of course an asymptotic result. Nevertheless, we are able to observe this on Figure 3 for  $\alpha = \frac{1}{8}, \frac{1}{4}, \frac{3}{8}$  noticing small discrepancies for low  $n$  and some “unstable” for  $\alpha = 3/8$ . For  $\alpha = 0$ —meaning that the truncation is fixed to 1—we observe that the scheme comes close to the

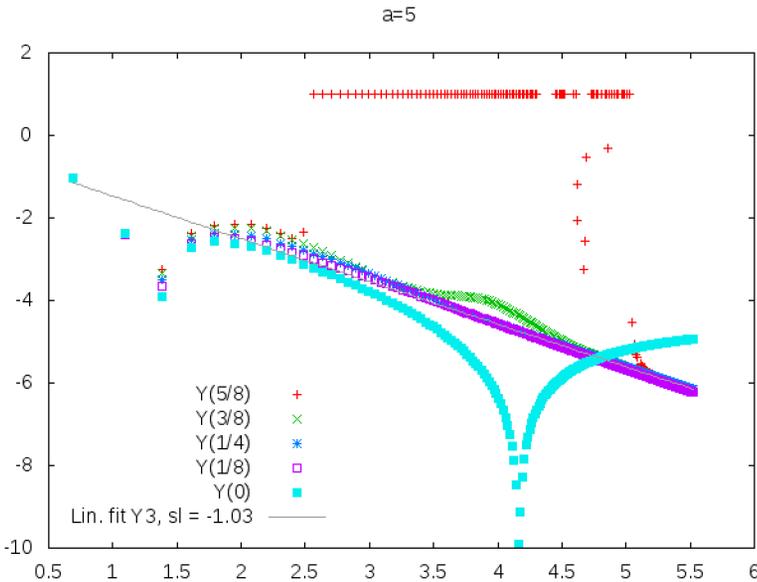


FIG. 3. Convergence profile for different  $\alpha$ - $Y(\alpha)$ .

correct value but then diverges, as expected. For  $\alpha = \frac{5}{8}$ , the scheme is unstable but manages to stabilize for large  $n$ . This numerical example is quite interesting as it illustrates the different behaviours of the scheme in terms of  $\alpha$ . In general, the choice of  $\alpha$  should depend on the various parameters of the problem  $X_0, \nu, a$  and  $\|g\|_\infty$  specially for small  $n$ . The optimal choice of  $\alpha$  (balancing convergence error and stability) is an interesting question that requires a deeper understanding of the qualitative behavior of the scheme in terms of the model parameters. These questions are left for further research.

APPENDIX

**A.1. Stability result for the Euler scheme of an SDE.**

LEMMA A.1. *Let us consider  $q \geq 1$  and two forward schemes  $(X_i)_{0 \leq i \leq n}$  and  $(\tilde{X}_i)_{0 \leq i \leq n}$  given by*

$$\begin{aligned} X_{i+1} &= X_i + h_i b(X_i) + \sigma(X_i) \sqrt{h_i} N_i, \\ \tilde{X}_{i+1} &= \tilde{X}_i + h_i b(\tilde{X}_i) + \sigma(\tilde{X}_i) \sqrt{h_i} N_i + \zeta_i, \end{aligned}$$

with  $(\zeta_i)_{0 \leq i < n}$  some random variables in  $L^{2q}$  and  $(N_i)_{0 \leq i < n}$  some independent and centered random variables in  $L^{2q}$  such that  $N_i$  is  $\mathcal{F}_i$  measurable for all  $0 \leq i < n$  and  $\mathbb{E}_i[N_i^2] = \mathbb{E}[N_i^2] \leq C$  with  $C$  that does not depend on  $n$ . Then we have the following stability result:

$$\mathbb{E} \left[ \sup_{0 \leq k \leq n} |X_k - \tilde{X}_k|^{2q} \right] \leq C_q |X_0 - \tilde{X}_0|^{2q} + C_q \mathbb{E} \left[ \left( \sum_{j=0}^{n-1} |\zeta_j| \right)^{2q} \right].$$

PROOF. By considering the difference between the two schemes, we have

$$\begin{aligned} X_i - \tilde{X}_i &= X_0 - \tilde{X}_0 + \sum_{j=0}^{i-1} h_j [b(X_j) - b(\tilde{X}_j)] \\ &\quad + \sum_{j=0}^{i-1} \sqrt{h_j} [\sigma(X_j) - \sigma(\tilde{X}_j)] N_j + \sum_{j=0}^{i-1} \zeta_j, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq k \leq i} |X_k - \tilde{X}_k|^{2q} \right] &\leq C_q |X_0 - \tilde{X}_0|^{2q} + C_q \mathbb{E} \left[ \left( \sum_{j=0}^{i-1} |\zeta_j| \right)^{2q} \right] \\ &\quad + C_q \mathbb{E} \left[ \sup_{0 \leq k \leq i} \left| \sum_{j=0}^{k-1} h_j [b(X_j) - b(\tilde{X}_j)] \right|^{2q} \right] \\ &\quad + C_q \mathbb{E} \left[ \sup_{0 \leq k \leq i} \left| \sum_{j=0}^{k-1} \sqrt{h_j} [\sigma(X_j) - \sigma(\tilde{X}_j)] N_j \right|^{2q} \right]. \end{aligned}$$

Recalling that  $b$  and  $\sigma$  are Lipschitz and by using a convexity inequality and the Burkholder–Davis–Gundy inequality, we obtain

$$\begin{aligned} \mathbb{E}\left[\sup_{0 \leq k \leq i} |X_k - \tilde{X}_k|^{2q}\right] &\leq C_q |X_0 - \tilde{X}_0|^{2q} + C_q \mathbb{E}\left[\left(\sum_{j=0}^{i-1} |\zeta_j|\right)^{2q}\right] \\ &\quad + C_q \sum_{j=0}^{i-1} h_j \mathbb{E}\left[\sup_{0 \leq k \leq j} |X_k - \tilde{X}_k|^{2q}\right] \\ &\quad + C_q \mathbb{E}\left[\left(\sum_{j=0}^{i-1} h_j |X_j - \tilde{X}_j|^2\right)^q\right] \\ &\leq C_q |X_0 - \tilde{X}_0|^{2q} + C_q \mathbb{E}\left[\left(\sum_{j=0}^{n-1} |\zeta_j|\right)^{2q}\right] \\ &\quad + C_q \sum_{j=0}^{i-1} h_j \mathbb{E}\left[\sup_{0 \leq k \leq j} |X_k - \tilde{X}_k|^{2q}\right]. \end{aligned}$$

The proof is concluded by a direct application of the discrete Gronwall's lemma.  $\square$

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DEPARTMENT OF MATHEMATICS  
IMPERIAL COLLEGE LONDON  
180 QUEEN'S GATE  
LONDON, SW7 2AZ  
UNITED KINGDOM  
E-MAIL: [j.chassagneux@imperial.ac.uk](mailto:j.chassagneux@imperial.ac.uk)

INSTITUT DE MATHÉMATIQUES DE BORDEAUX  
UNIVERSITÉ DE BORDEAUX  
IMB, UMR 5251  
F-33400 TALENCE  
FRANCE  
E-MAIL: [adrien.richou@math.univ-bordeaux1.fr](mailto:adrien.richou@math.univ-bordeaux1.fr)