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DISTRIBUTION OF LEVELS IN HIGH-DIMENSIONAL RANDOM LANDSCAPES

BY ZAKHAR KABLUCHKO

Ulm University

We prove empirical central limit theorems for the distribution of levels of various random fields defined on high-dimensional discrete structures as the dimension of the structure goes to ∞. The random fields considered include costs of assignments, weights of Hamiltonian cycles and spanning trees, energies of directed polymers, locations of particles in the branching random walk, as well as energies in the Sherrington–Kirkpatrick and Edwards–Anderson models. The distribution of levels in all models listed above is shown to be essentially the same as in a stationary Gaussian process with regularly varying nonsummable covariance function. This type of behavior is different from the Brownian bridge-type limit known for independent or stationary weakly dependent sequences of random variables.

1. Statement of results.

1.1. *Introduction*. Strongly correlated random fields defined on high-dimensional discrete structures arise naturally in stochastic combinatorial optimization and in the physics of disordered systems. We will be interested in the properties of the empirical process formed by the levels of such random fields. The general setting is as follows. For every $n \in \mathbb{N}$, let $\{\mathbb{X}_n(t); t \in T_n\}$ be a zero-mean, unit-variance random field with a finite index set T_n. The empirical distribution function of the field $\mathbb{X}_n$ counts the proportion of values of $\mathbb{X}_n$ which are not greater than a given number $z \in \mathbb{R}$. It is defined as

$$F_n(z) = \frac{1}{|T_n|} \sum_{t \in T_n} 1_{\mathbb{X}_n(t) \le z}, \qquad z \in \mathbb{R}. \tag{1}$$

Here, $|T_n|$ denotes the cardinality of the finite set T_n. For a number of models of stochastic combinatorial optimization we will prove an empirical central limit theorem of the following form:

$$\bigl\{c_n\bigl(F_n(z) - \mathbb{E}F_n(z)\bigr); z \in \mathbb{R}\bigr\} \xrightarrow[n\to\infty]{\text{f.d.d.}} \{p(z)W; z \in \mathbb{R}\}. \tag{2}$$

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Here, c_n is a normalizing sequence, $\xrightarrow{\text{f.d.d.}}$ denotes the weak convergence of the finite-dimensional distributions, $p(z) = (2\pi)^{-1/2}e^{-z^2/2}$ is the standard Gaussian density and W is a random variable. Both c_n and W depend on the model under consideration, W being usually normal.

1.2. *Distribution of weights of subgraphs.* Our first result deals with the stochastic assignment problem. In this model, n jobs have to be assigned in a bijective way to n machines. The set of all assignments is denoted by T_n and is identified with the set of all permutations of n elements, so that $|T_n| = n!$. Let the cost of assigning a job i to the machine j be $\xi_{i,j}$, where $\{\xi_{i,j}; i, j \in \{1, \dots, n\}\}$ are independent copies of a random variable ξ satisfying $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 = 1$. The (normalized) cost of an assignment $t = (t(i))_{i=1}^n \in T_n$ is then defined by $\mathbb{X}_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_{i,t(i)}$.

THEOREM 1. *Let $\{\mathbb{X}_n(t); t \in T_n\}$ be the random landscape of the stochastic assignment problem. Then, $\mathbb{X}_n$ satisfies the empirical central limit theorem* (2) *with $c_n = \sqrt{n}$ and $W \sim N(0, 1)$.*

The next model we will consider is the mean-field stochastic traveling salesman problem. Denote by $G_n = (V_n, E_n)$ the undirected complete graph on a set V_n of $n \geq 3$ vertices with the set of edges E_n. A Hamiltonian path in G_n is a nonoriented closed path which contains every vertex of G_n exactly once. Let T_n be the set of Hamiltonian paths in G_n. Let the weight of an edge $e \in E_n$ be ξ_e, where $\{\xi_e; e \in E_n\}$ are independent copies of a random variable ξ satisfying $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 = 1$. The (normalized) weight of a Hamiltonian path $t \in T_n$ is then defined by $\mathbb{X}_n(t) = \frac{1}{\sqrt{n}} \sum_{e \in t} \xi_e$.

THEOREM 2. *Let $\{\mathbb{X}_n(t); t \in T_n\}$ be the random landscape of the mean-field stochastic traveling salesman problem. Then, $\mathbb{X}_n$ satisfies the empirical central limit theorem* (2) *with $c_n = \sqrt{n/2}$ and $W \sim N(0, 1)$.*

In the next theorem we will deal with the distribution of the weights of spanning trees in the complete graph. As above, let G_n be the undirected complete graph on n vertices with the set of edges denoted by E_n. A spanning tree is a connected subgraph of G_n which contains all the vertices of G_n and has no cycles. Note that the number of edges in any spanning tree of G_n is $n - 1$. Let T_n be the set of all spanning trees of the complete graph G_n, the cardinality of T_n being n^{n-2} by the Cayley formula. Let the weight of an edge $e \in E_n$ be ξ_e, where $\{\xi_e; e \in E_n\}$ are independent copies of a random variable ξ satisfying $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 = 1$. The (normalized) weight of a spanning tree $t \in T_n$ is then defined by $\mathbb{X}_n(t) = \frac{1}{\sqrt{n-1}} \sum_{e \in t} \xi_e$.

THEOREM 3. *Let the random landscape* $\{\mathbb{X}_n(t); t \in T_n\}$ *representing the weights of spanning trees be defined as above. Then,* $\mathbb{X}_n$ *satisfies the empirical central limit theorem* (2) *with* $c_n = \sqrt{n/2}$ *and* $W \sim N(0, 1)$.

REMARK 1. We believe that in all our results, the weak convergence of the finite-dimensional distributions can be replaced by the weak convergence in the Skorokhod space, but we will not deal with tightness questions here.

REMARK 2. In the setting of Theorems 1–3, $\lim_{n\to\infty} \mathbb{E}F_n(z) = \Phi(z)$ by the central limit theorem, where $\Phi(z)$ is the standard Gaussian distribution function. However, we cannot replace $\mathbb{E}F_n(z)$ by $\Phi(z)$ in (2). In order to justify such a replacement, a relation of the form $\mathbb{E}F_n(z) - \Phi(z) = o(1/c_n)$ as $n \to \infty$ would be needed. This relation is not true in general. If the distribution of ξ is nonlattice and $\mathbb{E}|\xi|^3 < +\infty$, then we have, by [13], page 210,

$$\mathbb{E}F_n(z) - \Phi(z) = n^{-1/2} Q(z) p(z) + o(n^{-1/2}), \qquad n \to \infty, \tag{3}$$

where $Q(z) = \frac{1}{6}\mathbb{E}[\xi^3](1 - z^2)$. In this case, Theorem 1 can be written in the form

$$\{\sqrt{n}(F_n(z) - \Phi(z)); z \in \mathbb{R}\} \xrightarrow[n\to\infty]{\text{f.d.d.}} \{p(z)(W + Q(z)); z \in \mathbb{R}\}, \tag{4}$$

where $W \sim N(0, 1)$. Similar considerations apply to Theorems 2, 3, 5, as well as to the case $d \geq 3$ of Theorem 4.

1.3. *Distribution of energies of directed polymers.* A d-dimensional directed polymer of length n is a sequence $t = (t(k))_{k=0}^n$ of sites in $\mathbb{Z}^d$ such that $t(0) = 0$, and $t(k)$ and $t(k + 1)$ are neighboring sites for all $k = 0, \dots, n - 1$. The set of all polymers of length n is denoted by T_n and contains $(2d)^n$ elements. Let $\{\xi_k(x); k \in \mathbb{N}, x \in \mathbb{Z}^d\}$ be independent copies of a random variable ξ satisfying $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 = 1$. For $d = 1, 2$, we additionally assume that $\mathbb{E}|\xi|^{2+\delta} < \infty$ for some $\delta > 0$. The (normalized) energy of a polymer $t \in T_n$ is defined by $\mathbb{X}_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^n \xi_k(t(k))$.

THEOREM 4. *Let* $\{\mathbb{X}_n(t); t \in T_n\}$ *be the random energy landscape of the directed polymer model. Then,* $\mathbb{X}_n$ *satisfies the empirical central limit theorem* (2) *with*

$$c_n = \begin{cases} \sqrt[4]{\pi n/4}, & d = 1, \\ \sqrt{\pi n/\log n}, & d = 2, \\ \sqrt{n}, & d \geq 3. \end{cases} \tag{5}$$

For $d = 1, 2$, *we have* $W \sim N(0, 1)$. *For* $d \geq 3$, *the random variable* W *has the same distribution as* $-\sum_{k=1}^\infty \sum_{x\in\mathbb{Z}^d} p_k(x)\xi_k(x)$, *where* $p_k(x)$ *is the probability that a simple* (*nearest-neighbor*) *random walk on* $\mathbb{Z}^d$ *starting at the origin is at* $x \in \mathbb{Z}^d$ *at time* $k \in \mathbb{N}$.

REMARK 3. If $\mathbb{E}|\xi|^3 < \infty$, then $|\mathbb{E}F_n(z) - \Phi(z)| \le C/\sqrt{n}$ by the Berry–Esseen inequality (see, e.g., [19], page 111). This implies that we can replace $\mathbb{E}F_n(z)$ by $\Phi(z)$ in (2) for $d = 1, 2$. Note that this does not apply to the case $d \ge 3$. In this case, we may use expansion (3) as in Remark 2.

1.4. *Distribution of particles in the branching random walk.* Branching random walk is a model combining a Galton–Watson branching process with a random spatial motion of particles. At time 0 there is a single particle on the real line located at 0. At time 1, this particle is replaced by a random number of offsprings whose displacements relative to the position of the parent particle are i.i.d. random variables. Then, every offspring generates new particles according to the same rules, and so on. All the random mechanisms involved are independent.

The formal definition is as follows. Let $\mathbb{T} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$ be an infinite tree with root $\varnothing$ (we agree that $\mathbb{N}^0 = \{\varnothing\}$), vertices of the form $t = (v_1, \dots, v_n)$, where $v_i \in \mathbb{N}$ and $n = 0$ corresponds to the root $t = \varnothing$ and edges connecting each such t with its successors $(v_1, \dots, v_n, k)$, where $k \in \mathbb{N}$. The number $l(t) = n$ is called the length of $t = (v_1, \dots, v_n)$. Let $\{Z_t; t \in \mathbb{T}\}$ be independent copies of a random variable Z which takes values in $\mathbb{N}$ and satisfies $m := \mathbb{E}Z > 1$ and $\mathbb{E}Z^2 < \infty$. The random variable Z_t should be thought of as the number of children of the particle coded by the vertex t. The nth generation of the branching random walk is the random set T_n consisting of all vertices $t = (v_1, \dots, v_n)$ of length $n \in \mathbb{N}$ such that $v_k \le Z_{(v_1,\dots,v_{k-1})}$ for every $k = 1, \dots, n$. Independently of the Z_t's, let $\{\xi_t; t \in \mathbb{T} \setminus \{\varnothing\}\}$ be independent copies of a random variable ξ such that $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = 1$. The random variable ξ_t should be thought of as the displacement of the particle coded by the vertex t relative to its parent. For $t = (v_1, \dots, v_n) \in \mathbb{T} \setminus \{\varnothing\}$ define $\mathbb{X}_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \xi_{(v_1,\dots,v_k)}$. Then, $\{\mathbb{X}_n(t); t \in T_n\}$ are the normalized positions of the particles in the nth generation of the branching random walk.

THEOREM 5. *The random field $\{\mathbb{X}_n(t), t \in T_n\}$ defined as above satisfies the empirical central limit theorem* (2) *with $c_n = \sqrt{n}$. The limiting random variable W has the same distribution as $-\lim_{n\to\infty} \sqrt{n}|T_n|^{-1} \sum_{t \in T_n} \mathbb{X}_n(t)$.*

In the case of Bernoulli-distributed displacements this theorem is due to [8]. The method of [8] relies strongly on the Markov property of the branching random walk. We will recover Theorem 5 as a particular case of our general approach.

1.5. *Distribution of energy levels in spin glasses.* Our last result concerns the distribution of energy levels in spin glasses. The general setting is as follows. For every $n \in \mathbb{N}$, let $G_n = (V_n, E_n)$ be an undirected graph without loops and multiple edges on a finite set of vertices V_n with the set of edges E_n. A spin configuration is a map $t : V_n \to \{-1, 1\}$. Let $T_n = \{-1, 1\}^{V_n}$ be the set of all spin configurations. Spins located at vertices v_1 and v_2 interact if there is an edge $e = \{v_1, v_2\} \in E_n$, the energy of the interaction being $t(v_1)t(v_2)J(e)$, where $\{J(e); e \in E_n\}$ are independent standard Gaussian random variables. The energy of a spin configuration $t \in T_n$ is defined by

$$\mathbb{X}_n(t) = |E_n|^{-1/2} \sum_{e=\{v_1,v_2\}\in E_n} t(v_1)t(v_2)J(e). \tag{6}$$

Examples are provided by the Sherrington–Kirkpatrick model in which G_n is the complete graph on n vertices, and the d-dimensional Edwards–Anderson model, in which G_n is the d-dimensional discrete box with side length n and nearest-neighbor interactions.

THEOREM 6. *Let $\{\mathbb{X}_n(t); t \in T_n\}$ be the energy landscape defined as in (6). If $\lim_{n\to\infty} |E_n| = \infty$, then the following empirical central limit theorem holds*:

$$\left\{|E_n|^{1/2} 2^{-|V_n|} \sum_{t\in T_n} \left(1_{\mathbb{X}_n(t)\le z} - \Phi(z)\right); z \in \mathbb{R}\right\} \xrightarrow[n\to\infty]{f.d.d.} \left\{\frac{zp(z)}{\sqrt{2}} W; z \in \mathbb{R}\right\}, \tag{7}$$

where Φ is the standard Gaussian distribution function and $W \sim N(0, 1)$.

1.6. *Discussion.* Empirical central limit theorems have been extensively studied for stationary sequences of random variables under various short-range dependence conditions. For example, it has been shown in [3, 21] that if $\{\mathbb{X}(n); n \in \mathbb{Z}\}$ is a stationary zero-mean, unit-variance Gaussian process whose covariance function $r(n) = \mathbb{E}[\mathbb{X}(0)\mathbb{X}(n)]$ satisfies $\sum_{n\in\mathbb{Z}} |r(n)| < \infty$, then

$$\left\{\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(1_{\mathbb{X}(k)\le z} - \Phi(z)\right); z \in \mathbb{R}\right\} \xrightarrow[n\to\infty]{\text{f.d.d.}} \{B(z); z \in \mathbb{R}\}, \tag{8}$$

where $\{B(z); z \in \mathbb{R}\}$ is a zero-mean Gaussian process with covariance function

$$\operatorname{Cov}(B(z_1), B(z_2)) = \sum_{k\in\mathbb{Z}} \operatorname{Cov}\left(1_{\mathbb{X}(0)\le z_1}, 1_{\mathbb{X}(k)\le z_2}\right), \qquad z_1, z_2 \in \mathbb{R}. \tag{9}$$

Similar results are available for stationary processes under mixing conditions [4], Chapter 22, [11, 24], stationary associated sequences [25], to cite only a few references.

There has been also much interest in proving empirical central limit theorems for stationary long-range dependent processes (see [10, 12, 22, 23], as well as the monographs [9, 15, 17] for further references). It has been shown that if $\{\mathbb{X}(n); n \in \mathbb{Z}\}$ is a stationary zero-mean, unit-variance Gaussian process whose covariance function r satisfies $r(n) = L(n)n^{-D}$ for some function L that varies slowly at $+\infty$ and some $D \in (0, 1)$, then

$$\left\{\frac{C_D}{L^{1/2}(n)n^{1-(1/2)D}} \sum_{k=1}^{n} \left(1_{\mathbb{X}(k)\le z} - \Phi(z)\right); z \in \mathbb{R}\right\} \xrightarrow[n\to\infty]{\text{f.d.d.}} \{p(z)W; z \in \mathbb{R}\}, \tag{10}$$

where $C_D > 0$ is some explicit constant and $W \sim N(0,1)$ (see [10, 22] for stronger results).

The models considered in the present paper look, at a first sight, rather different from stationary Gaussian processes with regularly varying covariance function. Nevertheless, as far as the empirical process is concerned, they behave in essentially the same way as in (10). A nonrigorous explanation of this phenomenon will be given in Section 1.7.

Let us also mention that several authors proved Poisson limit theorems for the local distribution of values of highly-correlated random fields in small windows [1, 2, 5–7]. As opposed to these results, we consider the distribution of values of random fields on a global scale.

1.7. *Idea of the proofs*. Let us describe a nonrigorous argument justifying our results. As an approximation to the models considered in Theorems 1–5, we take $\{\mathbb{X}_n(t); t \in T_n\}$ to be a Gaussian process with zero-mean, unit-variance marginals and a covariance structure given by $\mathbb{E}[\mathbb{X}_n(t_1)\mathbb{X}_n(t_2)] = \varepsilon_n$ for all $t_1 \neq t_2$, where $\varepsilon_n \in (0,1)$ is some sequence tending to 0 as $n \to \infty$. Intuitively, the sequence ε_n represents the order of the overlap of two generic assignments, Hamiltonian paths, etc.

Let F_n be the empirical distribution function of $\mathbb{X}_n$ defined as in (1). The process $\mathbb{X}_n$ can be represented (in distribution) as $\mathbb{X}_n(t) = \sqrt{1-\varepsilon_n}\mathbb{X}'_n(t) + \sqrt{\varepsilon_n}N$, where $\{\mathbb{X}'_n(t); t \in T_n\}$ and N are independent standard Gaussian random variables. Thus, we have a representation

$$F_n(z) = F'_n\left(\frac{z - \sqrt{\varepsilon_n}N}{\sqrt{1-\varepsilon_n}}\right), \qquad z \in \mathbb{R},$$

where $F'_n(z) = \frac{1}{|T_n|}\sum_{t \in T_n} 1_{\mathbb{X}'_n(t) \le z}$ is the empirical distribution function of $\mathbb{X}'_n$. By the central limit theorem, we have $F'_n \approx \Phi$ as $n \to \infty$ with a Brownian bridge error term of order $1/\sqrt{|T_n|}$, where Φ is the standard Gaussian distribution function. Now, the common feature of the models considered in Theorems 1–5 is that ε_n, the order of the correlation of two generic elements in T_n, is much larger than $1/|T_n|$. So, the order of the Brownian bridge fluctuations is much smaller than the order of the shift $\sqrt{\varepsilon_n}N$. Thus, we may write

$$F_n(z) = F'_n\left(\frac{z - \sqrt{\varepsilon_n}N}{\sqrt{1-\varepsilon_n}}\right) \approx \Phi\left(\frac{z - \sqrt{\varepsilon_n}N}{\sqrt{1-\varepsilon_n}}\right) \approx \Phi(z) - \sqrt{\varepsilon_n}p(z)N, \tag{11}$$

where $\approx$ means that we are ignoring terms of order $o_P(\sqrt{\varepsilon_n})$ as $n \to \infty$. This leads to a result of the form

$$\left\{\frac{1}{\sqrt{\varepsilon_n}}(F_n(z) - \Phi(z)); z \in \mathbb{R}\right\} \xrightarrow[n\to\infty]{\text{f.d.d.}} \{-p(z)N; z \in \mathbb{R}\}. \tag{12}$$

For example, let $\{\mathbb{X}(n); n \in \mathbb{Z}\}$ be a stationary zero-mean, unit-variance Gaussian process whose covariance function r satisfies $r(n) = L(n)n^{-D}$, where L is a slowly varying function and $D > 0$. Then, for generic k_1, k_2 in $T_n = \{1, \dots, n\}$, $\mathrm{Cov}(\mathbb{X}(k_1), \mathbb{X}(k_2))$ is of order $\varepsilon_n \approx L(n)n^{-D}$. If $D \in (0, 1)$, then ε_n is asymptotically larger than $1/|T_n|$ and the heuristic applies; cf. (10). In the models of Section 1.2 and in the branching random walk, we have $\varepsilon_n \approx 1/n$, whereas $|T_n|$ grows exponentially, so that again ε_n is larger than $1/|T_n|$. For directed polymers, ε_n depends on the dimension d and is again larger than $1/|T_n|$.

On a more rigorous level, our proofs will be based on an adaptation of the reduction method of [22]. This method was introduced in the setting of stationary Gaussian processes with regularly varying covariance. The idea is to approximate the empirical distribution function by a certain expansion involving Hermite polynomials. Recall that the Hermite polynomials form an orthogonal system with respect to the weight p, the standard Gaussian density; see Section 3.1 for precise definitions. Every function which is square integrable with respect to the weight p can be expanded into a Hermite–Fourier series. For the function $f(x) = 1_{x \le z} - \Phi(z)$ (here, $z \in \mathbb{R}$ is fixed), the first two terms in the Hermite–Fourier expansion are

$$f(x) = 1_{x \le z} - \Phi(z) = -p(z)x - \tfrac{1}{2} z p(z)(x^2 - 1) + \cdots. \tag{13}$$

To prove Theorems 1–5, we will show that the random variable $\sum_{t \in T_n} (1_{\mathbb{X}_n(t) \le z} - \mathbb{P}[\mathbb{X}_n(t) \le z])$ can be approximated in the L^2-sense by the random variable $-p(z) \sum_{t \in T_n} \mathbb{X}_n(t)$ corresponding to the first term of the expansion (13). The statements justifying this approximation are Lemma 1 and Proposition 1 below. For the proof of Theorem 6, we need a more accurate approximation involving the second Hermite polynomial since there, we have $\sum_{t \in T_n} \mathbb{X}_n(t) = 0$ by symmetry reasons. In the setting of Theorem 6, we will prove that $\sum_{t \in T_n} (1_{\mathbb{X}_n(t) \le z} - \Phi(z))$ can be approximated by $-\frac{1}{2} z p(z) \sum_{t \in T_n} (\mathbb{X}_n^2(t) - 1)$.

1.8. *Notation*. Let us collect the notation which will be used throughout the paper. The standard Gaussian density and distribution function are denoted by $p(z) = (2\pi)^{-1/2} e^{-z^2/2}$ and $\Phi(z) = \int_{-\infty}^{z} p(t)\,dt$, respectively. We denote by ξ a random variable satisfying $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 = 1$. Let Φ_n be the distribution function of $(\xi_1 + \cdots + \xi_n)/\sqrt{n}$, where $\{\xi_i; i \in \mathbb{N}\}$ are independent copies of ξ. By the central limit theorem, $\lim_{n \to \infty} \Phi_n(z) = \Phi(z)$ for every $z \in \mathbb{R}$. Throughout, C is a large positive constant whose value may change from line to line.

2. Proofs for combinatorial models.

2.1. *Local limit theorems*. We start by recalling two classical local limit theorems which will be needed in our proofs. The first of them deals with lattice random variables. Recall that a random variable is called lattice if its values are of the form $b + h\mathbb{Z}$ for some $b \in \mathbb{R}$ and $h \ge 0$.

THEOREM 7 ([13], page 233, or [19], page 187). *Let* $\{\xi_i; i \in \mathbb{N}\}$ *be independent copies of a random variable* ξ *satisfying* $\mathbb{E}\xi = 0$ *and* $\mathbb{E}\xi^2 = 1$. *Assume that the values of* ξ *are of the form* $b + h\mathbb{Z}$, *where* $h > 0$ *is maximal with this property. Then, the following asymptotic relation holds uniformly in* $z \in nb + h\mathbb{Z}$:

$$\mathbb{P}[\xi_1 + \cdots + \xi_n = z] = \frac{h}{\sqrt{n}} p\left(\frac{z}{\sqrt{n}}\right) + o\left(\frac{1}{\sqrt{n}}\right), \qquad n \to \infty. \tag{14}$$

The next theorem is an analogue of Theorem 7 for nonlattice distributions. Recall the notation introduced at the end of Section 1.

THEOREM 8 ([20]). *Let* ξ *be a nonlattice random variable satisfying* $\mathbb{E}\xi = 0$ *and* $\mathbb{E}\xi^2 = 1$. *Then, the following asymptotic relation holds uniformly in* $z_1, z_2 \in \mathbb{R}$:

$$\Phi_n(z_2) - \Phi_n(z_1) = \Phi(z_2) - \Phi(z_1) + o(1)(|z_2 - z_1| + n^{-1/2}), \qquad n \to \infty. \tag{15}$$

COROLLARY 1. *Regardless of whether* ξ *is lattice or nonlattice, there is a constant* $C > 0$ *depending on* ξ *such that for all* $n \in \mathbb{N}$ *and* $z_1, z_2 \in \mathbb{R}$,

$$|\Phi_n(z_2) - \Phi_n(z_1)| \le C|z_2 - z_1| + Cn^{-1/2}. \tag{16}$$

PROOF. If the distribution of ξ is nonlattice, then the corollary follows immediately from (15) and the fact that the function Φ is Lipschitz. Suppose that ξ is lattice as in Theorem 7. Without restriction of generality, let $z_1 < z_2$ and define $I_n = (nb + h\mathbb{Z}) \cap (\sqrt{n}z_1, \sqrt{n}z_2]$. Then by Theorem 7,

$$\Phi_n(z_2) - \Phi_n(z_1) = \sum_{z \in I_n} \left(\frac{h}{\sqrt{n}} p\left(\frac{z}{\sqrt{n}}\right) + o\left(\frac{1}{\sqrt{n}}\right)\right) \le \sum_{z \in I_n} \frac{C}{\sqrt{n}},$$

where the o-term is uniform in $z \in \mathbb{R}$. Since the cardinality of I_n differs from $h^{-1}\sqrt{n}(z_2 - z_1)$ by at most 1, we obtain the statement of the corollary. □

REMARK 4. With an additional assumption $\mathbb{E}|\xi|^3 < \infty$, Corollary 1 follows from the Berry–Esseen inequality (see [19], page 111).

2.2. *The main lemma*. The next lemma will play a crucial role in the sequel. Essentially, it provides an estimate for the dependence between the random variables $1_{X_1 \le z}$ and $1_{X_2 \le z}$, where X_1 and X_2 are two normalized sums of i.i.d. random variables having a nontrivial overlap. In our applications, X_1 and X_2 will be the normalized weights of two Hamiltonian paths, spanning trees, etc. We will regularize $1_{X_1\le z}$ and $1_{X_2\le z}$ by subtracting certain terms motivated by the Hermite expansion of the function $f(x)=1_{x\le z}$.

LEMMA 1. *Let $\{\xi_i; i\in\mathbb{N}\}$ be independent copies of a random variable ξ satisfying $\mathbb{E}\xi=0$ and $\mathbb{E}\xi^2=1$. Let $z\in\mathbb{R}$ be fixed. Given $r\in\mathbb{N}\cup\{0\}$, $n\in\mathbb{N}$ with $r\le n$, define two random variables Y_1, Y_2 by*

$$Y_i=1_{X_i\le z}-\Phi_n(z)+p(z)X_i,\qquad i=1,2, \tag{17}$$

where $X_1=\frac{1}{\sqrt{n}}\sum_{i=1}^n\xi_i$ and $X_2=\frac{1}{\sqrt{n}}\sum_{i=n-r+1}^{2n-r}\xi_i$. Then, there is a constant C depending only on the distribution of ξ such that for all $r\in\mathbb{N}\cup\{0\}$, $n\in\mathbb{N}$ with $r\le n$, we have

$$0\le\mathbb{E}[Y_1Y_2]\le C\frac{r}{n}. \tag{18}$$

Further, if $\varepsilon_n>0$ is any sequence with $\lim_{n\to\infty}\varepsilon_n=0$, *then there is a sequence δ_n such that* $\lim_{n\to\infty}\delta_n=0$ *and for every $r\in\mathbb{N}\cup\{0\}$, $n\in\mathbb{N}$ with $r\le\varepsilon_n n$, we have*

$$0\le\mathbb{E}[Y_1Y_2]\le\delta_n\frac{r}{n}. \tag{19}$$

PROOF. Since the statement is trivially fulfilled for $r=0$ and $r=n$, we assume $0<r<n$ henceforth. It will be convenient to introduce the following notation: for $u\in\mathbb{R}$, we write

$$\rho=\frac{r}{n}\in(0,1),\qquad z(u)=\frac{z-u\sqrt{\rho}}{\sqrt{1-\rho}}. \tag{20}$$

It follows from (17) that we have

$$\mathbb{E}[Y_1Y_2]=\operatorname{Cov}(1_{X_1\le z},1_{X_2\le z})+2p(z)\mathbb{E}[1_{X_1\le z}X_2]+p^2(z)\rho. \tag{21}$$

We start by considering the first term on the right-hand side of (21). We are going to show that

$$\begin{aligned}&\operatorname{Cov}(1_{X_1\le z},1_{X_2\le z})\\&\quad=\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\big(\Phi_{n-r}(z(u))-\Phi_{n-r}(z(v))\big)^2\Phi_r(du)\Phi_r(dv).\end{aligned} \tag{22}$$

Define three independent random variables $\tilde{X}_1$, $\tilde{X}$, $\tilde{X}_2$ by

$$\tilde{X}_1=\frac{1}{\sqrt{n}}\sum_{i=1}^{n-r}\xi_i,\qquad \tilde{X}=\frac{1}{\sqrt{n}}\sum_{i=n-r+1}^{n}\xi_i,\qquad \tilde{X}_2=\frac{1}{\sqrt{n}}\sum_{i=n+1}^{2n-r}\xi_i. \tag{23}$$

Note that $X_1=\tilde{X}_1+\tilde{X}$ and $X_2=\tilde{X}_2+\tilde{X}$. The distribution function of $\tilde{X}/\sqrt{\rho}$ is Φ_r. Conditioning on the event $\tilde{X}/\sqrt{\rho}\in du$ and using the independence of

$\tilde{X}_1, \tilde{X}, \tilde{X}_2$, we obtain

$$
\begin{aligned}
\mathbb{E}[1_{X_1 \le z} 1_{X_2 \le z}] &= \mathbb{P}[\tilde{X}_1 + \tilde{X} \le z, \tilde{X}_2 + \tilde{X} \le z] \\
&= \int_{\mathbb{R}} \big(\mathbb{P}[\tilde{X}_1 \le z - u\sqrt{\rho}]\big)^2 \Phi_r(du) \\
&= \int_{\mathbb{R}} \Phi_{n-r}^2(z(u))\ \Phi_r(du) \\
&= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \big(\Phi_{n-r}^2(z(u)) + \Phi_{n-r}^2(z(v))\big) \Phi_r(du) \Phi_r(dv).
\end{aligned} \tag{24}
$$

In a similar way, we obtain

$$
\begin{aligned}
\mathbb{E}[1_{X_1 \le z}] \mathbb{E}[1_{X_2 \le z}] &= (\mathbb{P}[\tilde{X}_1 + \tilde{X} \le z])^2 \\
&= \left(\int_{\mathbb{R}} \Phi_{n-r}(z(u)) \Phi_r(du) \right)^2 \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi_{n-r}(z(u)) \Phi_{n-r}(z(v)) \Phi_r(du) \Phi_r(dv).
\end{aligned} \tag{25}
$$

Bringing (24) and (25) together, we obtain (22). Let us consider the second term on the right-hand side of (21). Conditioning on $\tilde{X}/\sqrt{\rho} \in du$, we obtain

$$
\begin{aligned}
&2p(z)\mathbb{E}[1_{X_1 \le z} X_2] \\
&\quad = 2p(z)\mathbb{E}[1_{\tilde{X}_1 + \tilde{X} \le z} \tilde{X}] \\
&\quad = 2p(z)\sqrt{\rho} \int_{\mathbb{R}} u \mathbb{P}[\tilde{X}_1 \le z - u\sqrt{\rho}] \Phi_r(du) \\
&\quad = 2p(z)\sqrt{\rho} \int_{\mathbb{R}} u \Phi_{n-r}(z(u)) \Phi_r(du) \\
&\quad = p(z)\sqrt{\rho} \int_{\mathbb{R}} \int_{\mathbb{R}} (u - v)\big(\Phi_{n-r}(z(u)) - \Phi_{n-r}(z(v))\big) \Phi_r(du) \Phi_r(dv).
\end{aligned} \tag{26}
$$

Also, we have

$$
\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} (u - v)^2 \Phi_r(du) \Phi_r(dv) = 1. \tag{27}
$$

Bringing (21), (22), (26), (27) together, we obtain

$$
\mathbb{E}[Y_1 Y_2] = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \Delta^2(u, v) \Phi_r(du) \Phi_r(dv), \tag{28}
$$

where $\Delta(u, v)$ is given by

$$
\Delta(u, v) = \Phi_{n-r}(z(u)) - \Phi_{n-r}(z(v)) + p(z)(u - v)\sqrt{\rho}. \tag{29}
$$

Let us now prove the first statement of the lemma. Note that (28) implies that $\mathbb{E}[Y_1 Y_2] \ge 0$. It follows from (17) that $\mathbb{E}Y_1^2 = \mathbb{E}Y_2^2 \le 9$. By the Cauchy–Schwarz inequality, equation (18) is fulfilled for $r \in [n/2, n]$ and every $n \in \mathbb{N}$ with $C = 18$. Let us henceforth assume that $r \le n/2$ (and so, $\rho \le 1/2$). Applying Corollary 1 and recalling (20), we obtain, both in the lattice and in the nonlattice case,

$$
\begin{aligned}
|\Phi_{n-r}(z(u)) - \Phi_{n-r}(z(v))| &\le C\Big(\frac{|u-v|\sqrt{\rho}}{\sqrt{1-\rho}} + \frac{1}{\sqrt{n-r}}\Big) \\
&\le C(|u-v|+1)\sqrt{\rho}.
\end{aligned}
\tag{30}
$$

It follows from (29) and (30) that $|\Delta(u,v)| \le C(|u-v|+1)\sqrt{\rho}$. Hence,

$$\Delta^2(u,v) \le C\big((u-v)^2+1\big)\rho. \tag{31}$$

Inserting this into (28) yields

$$\mathbb{E}[Y_1Y_2] \le C\rho \int_{\mathbb{R}}\int_{\mathbb{R}} \big((u-v)^2+1\big)\Phi_r(du)\Phi_r(dv) = 3C\rho. \tag{32}$$

This completes the proof of (18).

Let us prove the second statement of the lemma. It suffices to show that for every $\delta > 0$ there is $N = N(\delta)$ such that for every $n > N$ and $r \le \varepsilon_n n$, we have $\mathbb{E}[Y_1Y_2] \le \delta\rho$. It follows from (31), (27) and the weak convergence of Φ_r to Φ as $r \to \infty$ that we can choose $B = B(\delta)$ such that for all $n, r \in \mathbb{N}$ with $r \le n/2$,

$$\int_{\mathbb{R}^2\setminus[-B,B]^2} \Delta^2(u,v)\Phi_r(du)\Phi_r(dv) < \delta\rho. \tag{33}$$

Assume first that the distribution of ξ is nonlattice. We always assume that $r \le \varepsilon_n n$. By Theorem 8, the following holds uniformly in $u, v \in [-B, B]$ as $n \to \infty$:

$$
\begin{aligned}
\Phi_{n-r}(z(u)) - \Phi_{n-r}(z(v)) &= p(z)\big(z(u)-z(v)\big) + o\big(\sqrt{\rho}\big) \\
&= -p(z)\big((u-v)+o(1)\big)\sqrt{\rho}.
\end{aligned}
$$

Together with (29), this implies that $\Delta(u,v) = o(\sqrt{\rho})$ uniformly in $u, v \in [-B, B]$ as $n \to \infty$. It follows that for n large enough,

$$\int_{[-B,B]^2} \Delta^2(u,v)\Phi_r(du)\Phi_r(dv) < \delta\rho. \tag{34}$$

This, together with (33) and (28), completes the proof in the nonlattice case.

Assume now that the random variable ξ is lattice with values in the set $b + h\mathbb{Z}$, with h being maximal with this property. Let $u, v \in [-B, B] \cap r^{-1/2}(rb + h\mathbb{Z})$ with $u < v$. Note that by (20), $z(u) - z(v) \in (n-r)^{-1/2}h\mathbb{Z}$. Hence, the number of points in the set

$$I_{n,r}(u,v) := (z(v), z(u)] \cap (n-r)^{-1/2}\big((n-r)b + h\mathbb{Z}\big)$$

is equal to $h^{-1}(n-r)^{1/2}(z(u)-z(v))$. By Theorem 7,

$$\begin{aligned}\Phi_{n-r}(z(u)) - \Phi_{n-r}(z(v)) &= \sum_{x \in I_{n,r}(u,v)} \mathbb{P}\left[\frac{\xi_1 + \cdots + \xi_{n-r}}{\sqrt{n-r}} = x\right] \\ &= \sum_{x \in I_{n,r}(u,v)} \left(\frac{h}{\sqrt{n-r}} p(x) + o\left(\frac{1}{\sqrt{n-r}}\right)\right) \\ &= (v-u)p(z)\sqrt{\rho} + o(\sqrt{\rho}).\end{aligned}$$

It follows that $\Delta(u,v) = o(\sqrt{\rho})$ as $n \to \infty$ uniformly in $u, v \in [-B, B] \cap r^{-1/2}(rb + h\mathbb{Z})$. Hence, equation (34) holds for n large enough and the proof is complete. $\square$

2.3. *An empirical central limit theorem for overlapping sums.* In this section, we state and prove a result from which we will deduce Theorems 1–5. It is an empirical central limit theorem for overlapping sums of independent random variables. Let $\{\xi_e; e \in E\}$ be independent copies of a random variable ξ satisfying $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 = 1$, where E is some countable index set. For every $n \in \mathbb{N}$, let $T_n \subset 2^E$ be a finite collection of (typically, overlapping) subsets of E, each subset having cardinality n. Define a random field $\{\mathbb{X}_n(t); t \in T_n\}$ by

$$\mathbb{X}_n(t) = \frac{1}{\sqrt{n}} \sum_{e \in t} \xi_e. \tag{35}$$

Let $\Phi_n(z) = \mathbb{P}[\mathbb{X}_n(t) \le z]$, where $z \in \mathbb{R}$, be the distribution function of $\mathbb{X}_n(t)$. The covariance function of the random field $\mathbb{X}_n$ is given by $\rho_n(t_1, t_2) = \frac{1}{n}|t_1 \cap t_2|$. Define also $s_n \ge 0$ by

$$s_n^2 = \operatorname{Var}\left[\sum_{t \in T_n} \mathbb{X}_n(t)\right] = \sum_{t_1, t_2 \in T_n} \rho_n(t_1, t_2). \tag{36}$$

PROPOSITION 1. *Let the random field $\{\mathbb{X}_n(t); t \in T_n\}$ be defined as above. Assume that for some random variable V and some sequence $\varepsilon_n > 0$ with $\lim_{n\to\infty} \varepsilon_n = 0$, the following two conditions are satisfied*:

$$\frac{1}{s_n} \sum_{t \in T_n} \mathbb{X}_n(t) \xrightarrow[n\to\infty]{d} V, \tag{37}$$

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{t_1, t_2 \in T_n} \rho_n(t_1, t_2) 1_{\rho_n(t_1,t_2) > \varepsilon_n} = 0. \tag{38}$$

Then, the following convergence of stochastic processes holds true:

$$\left\{\frac{1}{s_n} \sum_{t \in T_n} \left(1_{\mathbb{X}_n(t) \le z} - \Phi_n(z)\right); z \in \mathbb{R}\right\} \xrightarrow[n\to\infty]{f.d.d.} \{-p(z)V; z \in \mathbb{R}\}. \tag{39}$$

PROOF. For $z \in \mathbb{R}$, define a zero-mean random field $\{\mathbb{Y}_n(t; z); t \in T_n\}$ by

$$\mathbb{Y}_n(t; z) = 1_{\mathbb{X}_n(t) \le z} - \Phi_n(z) + p(z)\mathbb{X}_n(t). \tag{40}$$

We will show that

$$\lim_{n\to\infty} \operatorname{Var}\left[\frac{1}{s_n}\sum_{t\in T_n} \mathbb{Y}_n(t; z)\right] = \lim_{n\to\infty} \frac{1}{s_n^2} \sum_{t_1,t_2\in T_n} \mathbb{E}[\mathbb{Y}_n(t_1; z)\mathbb{Y}_n(t_2; z)] = 0. \tag{41}$$

By the first part of Lemma 1, we have for every $t_1, t_2 \in T_n$,

$$0 \le \mathbb{E}[\mathbb{Y}_n(t_1; z)\mathbb{Y}_n(t_2; z)] \le C\rho_n(t_1, t_2). \tag{42}$$

This allows us to estimate the contribution of those terms in (41) which satisfy $\rho_n(t_1, t_2) > \varepsilon_n$. It follows from (42) and (38) that as $n \to \infty$,

$$\sum_{\substack{t_1,t_2\in T_n\\ \rho_n(t_1,t_2)>\varepsilon_n}} \mathbb{E}[\mathbb{Y}_n(t_1; z)\mathbb{Y}_n(t_2; z)] \le C \sum_{\substack{t_1,t_2\in T_n\\ \rho_n(t_1,t_2)>\varepsilon_n}} \rho_n(t_1, t_2) = o(s_n^2). \tag{43}$$

Let us consider the terms with $\rho_n(t_1, t_2) \le \varepsilon_n$. It follows from the second part of Lemma 1 that there is a sequence $\delta_n > 0$ such that $\lim_{n\to\infty} \delta_n = 0$ and for every t_1, t_2 such that $\rho_n(t_1, t_2) \le \varepsilon_n$, we have

$$0 \le \mathbb{E}[\mathbb{Y}_n(t_1; z)\mathbb{Y}_n(t_2; z)] \le \delta_n\rho_n(t_1, t_2). \tag{44}$$

It follows from (44) and (36) that as $n \to \infty$,

$$\sum_{\substack{t_1,t_2\in T_n\\ \rho_n(t_1,t_2)\le\varepsilon_n}} \mathbb{E}[\mathbb{Y}_n(t_1; z)\mathbb{Y}_n(t_2; z)] \le \delta_n \sum_{\substack{t_1,t_2\in T_n\\ \rho_n(t_1,t_2)\le\varepsilon_n}} \rho_n(t_1, t_2) = o(s_n^2). \tag{45}$$

Combining (43) and (45), we obtain (41).

Take some $z_1, \dots, z_d \in \mathbb{R}$. Recalling (40), we may write for every $i = 1, \dots, d$,

$$\frac{1}{s_n}\sum_{t\in T_n}\bigl(1_{\mathbb{X}_n(t)\le z_i} - \Phi_n(z_i)\bigr) = -\frac{p(z_i)}{s_n}\sum_{t\in T_n}\mathbb{X}_n(t) + \frac{1}{s_n}\sum_{t\in T_n}\mathbb{Y}_n(t; z_i).$$

The first term on the right-hand side converges to $-p(z_i)V$ in distribution by (37), whereas the second term converges to 0 in probability by (41). This completes the proof. $\square$

2.4. *Proofs of Theorems 1–3*. In this section we derive Theorems 1–3 as consequences of Proposition 1. We will replace condition (38) by the following one:

$$\sum_{t_1,t_2\in T_n} r_n^2(t_1, t_2) = O(ns_n^2), \qquad n \to \infty. \tag{46}$$

Here, $r_n(t_1,t_2)=|t_1\cap t_2|$ is the overlap of the sets $t_1,t_2\in T_n$. Condition (46) implies that (38) holds with $\varepsilon_n=1/\sqrt{n}$. Indeed, we have, as $n\to\infty$,

$$\sum_{t_1,t_2\in T_n}\rho_n(t_1,t_2)1_{\rho_n(t_1,t_2)>1/\sqrt{n}}\le\sqrt{n}\sum_{t_1,t_2\in T_n}\rho_n^2(t_1,t_2)=o(s_n^2).$$

PROOF OF THEOREM 1. The number of assignments on the set of n elements is given by $|T_n|=n!$. To apply Proposition 1, we take $E=\mathbb{N}\times\mathbb{N}$ and identify an assignment $t\in T_n$ with the subset $\{(i,t(i)); i=1,\dots,n\}$ of E. To verify condition (37) of Proposition 1, note that

$$\sum_{t\in T_n}\mathbb{X}_n(t)=\frac{1}{\sqrt{n}}\sum_{t\in T_n}\sum_{i=1}^{n}\xi_{i,t(i)}=\frac{(n-1)!}{\sqrt{n}}\sum_{i,j=1}^{n}\xi_{i,j}. \tag{47}$$

It follows that s_n^2 defined in (36) is given by

$$s_n^2=\operatorname{Var}\Biggl[\sum_{t\in T_n}\mathbb{X}_n(t)\Biggr]=n!(n-1)!. \tag{48}$$

The central limit theorem together with (47) and (48) implies that the random variable $s_n^{-1}\sum_{t\in T_n}\mathbb{X}_n(t)$ converges weakly to the standard Gaussian distribution as $n\to\infty$. This verifies condition (37) with $V\sim N(0,1)$.

Let us verify condition (46). Let $\tilde{t}\in T_n$ be the identical assignment, that is, $\tilde{t}(i)=i$, $i=1,\dots,n$. We have

$$\sum_{t_1,t_2\in T_n}r_n^2(t_1,t_2)=n!\sum_{t\in T_n}r_n^2(t,\tilde{t})=n!\sum_{t\in T_n}\Biggl(\sum_{i=1}^{n}1_{t(i)=i}\Biggr)^2=2(n!)^2,$$

where the last equality follows from the well-known fact that the expectation of the squared number of fixed points in a random permutation is 2. Together with (48), this verifies condition (46). The proof is completed by applying Proposition 1. □

PROOF OF THEOREM 2. To apply Proposition 1, we take E to be the set of all two-element subsets of $\mathbb{N}$ and identify the set V_n of vertices of the complete graph G_n with $\{1,\dots,n\}$. Then, any (nonoriented) Hamiltonian path $t\in T_n$ can be viewed as a subset of E. Let us verify condition (37) of Proposition 1. The number of Hamiltonian paths in the complete graph G_n, $n\ge 3$, is given by $|T_n|=\frac{1}{2}(n-1)!$. The number of Hamiltonian paths containing a given edge is easily seen to be $(n-2)!$. Hence,

$$\sum_{t\in T_n}\mathbb{X}_n(t)=\frac{1}{\sqrt{n}}\sum_{t\in T_n}\sum_{e\in t}\xi_e=\frac{1}{\sqrt{n}}\sum_{e\in E_n}\xi_e\sum_{t\in T_n}1_{e\in t}=\frac{(n-2)!}{\sqrt{n}}\sum_{e\in E_n}\xi_e. \tag{49}$$

Note that the number of edges in G_n is $|E_n|=\frac{1}{2}n(n-1)$. It follows that s_n^2 defined in (36) is given by

$$s_n^2=\operatorname{Var}\Biggl[\sum_{t\in T_n}\mathbb{X}_n(t)\Biggr]=\frac{1}{2}(n-1)!(n-2)!. \tag{50}$$

By the central limit theorem, combined with (49) and (50), the random variable $s_n^{-1}\sum_{t\in T_n}\mathbb{X}_n(t)$ converges weakly to the standard Gaussian distribution as $n\to\infty$. This verifies condition (37) of Proposition 1.

We prove that (46) holds. We have

$$\begin{aligned}\sum_{t_1,t_2\in T_n} r_n^2(t_1,t_2) &= \sum_{t_1,t_2\in T_n}\Big(\sum_{e\in E_n} 1_{e\in t_1}1_{e\in t_2}\Big)^2\\ &= \sum_{t_1,t_2\in T_n}\sum_{e,f\in E_n} 1_{e\in t_1}1_{e\in t_2}1_{f\in t_1}1_{f\in t_2} \qquad (51)\\ &= \sum_{e,f\in E_n}\Big(\sum_{t\in T_n} 1_{e\in t}1_{f\in t}\Big)^2.\end{aligned}$$

The sum $\sum_{t\in T_n} 1_{e\in t}1_{f\in t}$ represents the number of Hamiltonian paths containing the edges e and f. If $e=f$, then there are $(n-2)!$ such paths. If the edges e and f have exactly one common vertex, then the number of Hamiltonian paths containing e and f is easily seen to be $(n-3)!$. Finally, if the edges e and f do not have a common vertex, then the number of paths containing both e and f is $2(n-3)!$. The number of pairs $(e,f)\in E_n^2$ having exactly one common vertex is $6\binom{n}{3}$, and the number of pairs $(e,f)\in E_n^2$ without a common vertex is $6\binom{n}{4}$. It follows from (51) that $\sum_{t_1,t_2\in T_n} r_n^2(t_1,t_2)$ is equal to

$$\binom{n}{2}((n-2)!)^2+6\binom{n}{3}((n-3)!)^2+24\binom{n}{4}((n-3)!)^2.$$

This expression is of order $O(ns_n^2)$ as $n\to\infty$. It follows that (46) is fulfilled. The proof is completed by applying Proposition 1. □

PROOF OF THEOREM 3. By Cayley's theorem, the number of spanning trees on n vertices is given by $|T_n|=n^{n-2}$. Since each spanning tree has $n-1$ edges, and since there are $n(n-1)/2$ edges, any edge is contained in $2n^{n-3}$ trees. Hence,

$$\sum_{t\in T_n}\mathbb{X}_n(t)=\frac{1}{\sqrt{n-1}}\sum_{t\in T_n}\sum_{e\in t}\xi_e=\frac{2n^{n-3}}{\sqrt{n-1}}\sum_{e\in E_n}\xi_e. \qquad (52)$$

It follows that

$$s_n^2=\operatorname{Var}\Big[\sum_{t\in T_n}\mathbb{X}_n(t)\Big]=2n^{2n-5}. \qquad (53)$$

By the central limit theorem together with (52) and (53), the random variable $s_n^{-1}\sum_{t\in T_n}\mathbb{X}_n(t)$ converges weakly to the standard Gaussian distribution as $n\to\infty$.

Let us verify condition (46). As in (51), we have

$$\sum_{t_1,t_2\in T_n} r_n^2(t_1,t_2)=\sum_{e,f\in E_n}\left(\sum_{t\in T_n} 1_{e\in t}1_{f\in t}\right)^2.$$

Given two edges e and f, we will compute the number of spanning trees $N_n(e,f)=\sum_{t\in T_n} 1_{e\in t}1_{f\in t}$ in the complete graph G_n containing these two edges. For $e=f$, we have shown that this number is equal to $2n^{n-3}$. We claim that if the edges e and f have exactly one common vertex, then $N_n(e,f)=3n^{n-4}$, whereas if e and f do not have common vertices, then $N_n(e,f)=4n^{n-4}$. For completeness, we will prove this by using the transfer current theorem giving an interpretation of random spanning trees in terms of electric networks (see [18], Section 8.2). It says that the probability that a uniformly chosen spanning tree (in any finite graph) contains two given edges e and f is given by the determinant

$$\det\begin{pmatrix} Y(e,e) & Y(e,f)\\ Y(f,e) & Y(f,f)\end{pmatrix},\tag{54}$$

where $Y(g,h)$ denotes the (signed) current which flows through the (somehow oriented) edge h if a battery is hooked up between the ends of the (somehow oriented) edge $g=(v_1,v_2)$ with such voltage that the total current flowing through the graph is 1. By Kirchhoff's laws and symmetry reasons, we have $Y(g,g)=2/n$, $Y(g,h)=1/n$ if h is of the form (v_1,v) for some vertex $v\neq v_2$, and $Y(g,h)=1/n$ if $h=(v,v_2)$ for some vertex $v\neq v_1$. If g and h have no vertices in common, then $Y(g,h)=0$. Inserting this into (54) and recalling that the total number of spanning trees in T_n is n^{n-2}, we obtain the above mentioned formulae for $N_n(e,f)$.

Recall from the proof of Theorem 2 that the number of pairs $(e,f)\in E_n^2$ having exactly one common vertex is $6\binom{n}{3}$, whereas the number of pairs $(e,f)\in E_n^2$ having no vertices in common is $6\binom{n}{4}$. Thus,

$$\sum_{t_1,t_2\in T_n} r_n^2(t_1,t_2)=4\binom{n}{2}n^{2(n-3)}+54\binom{n}{3}n^{2(n-4)}+96\binom{n}{4}n^{2(n-4)}.$$

The right-hand side is of order $O(ns_n^2)$ as $n\to\infty$. This completes the proof of (46). $\square$

2.5. *Proof of Theorem* 4. We will verify conditions (37) and (38) of Proposition 1. Recall that $p_k(x)$ is the probability that a simple (nearest-neighbor) d-dimensional random walk which starts at the origin, visits the site $x\in\mathbb{Z}^d$ at time $k\in\mathbb{N}\cup\{0\}$. Note that with s_n defined by (36), we have

$$\sum_{t\in T_n}\mathbb{X}_n(t)=\frac{(2d)^n}{\sqrt{n}}\sum_{k=1}^n\sum_{x\in\mathbb{Z}^d} p_k(x)\xi_k(x),\tag{55}$$

$$s_n^2=\frac{(2d)^{2n}}{n}\sum_{k=1}^n\sum_{x\in\mathbb{Z}^d} p_k^2(x).\tag{56}$$

First, we find an asymptotic formula for $\sum_{k=1}^{n}\sum_{x\in\mathbb{Z}^d} p_k^2(x)$ as $n\to\infty$. A symmetry argument shows that $\sum_{x\in\mathbb{Z}^d} p_k^2(x)=p_{2k}(0)$. Also, by the multidimensional local limit theorem (e.g., [16], Section 1.2), $p_{2k}(0)\sim 2^{1-d}(\pi k/d)^{-d/2}$ as $k\to\infty$. Thus, in the case $d\ge 3$ we have

$$S^2:=\sum_{k=1}^{\infty}\sum_{x\in\mathbb{Z}^d} p_k^2(x)<\infty. \tag{57}$$

For $d=1,2$, we obtain the following asymptotics as $n\to\infty$:

$$\sum_{k=1}^{n}\sum_{x\in\mathbb{Z}^d} p_k^2(x)\sim 2^{1-d}\sum_{k=1}^{n}\left(\frac{d}{\pi k}\right)^{d/2}\sim\begin{cases}2\sqrt{\dfrac{n}{\pi}}, & d=1,\\ \dfrac{1}{\pi}\log n, & d=2.\end{cases} \tag{58}$$

In the case $d\ge 3$, combining (55)–(57), we obtain the following relation verifying condition (37):

$$\frac{1}{s_n}\sum_{t\in T_n}\mathbb{X}_n(t)\xrightarrow[n\to\infty]{d}\frac{1}{S}\sum_{k=1}^{\infty}\sum_{x\in\mathbb{Z}^d} p_k(x)\xi_k(x),$$

where the series on the right-hand side converges in the L^2-sense.

In the case $d=1,2$, we will verify condition (37) by proving that the random variable $\frac{1}{s_n}\sum_{t\in T_n}\mathbb{X}_n(t)$ converges as $n\to\infty$ to the standard Gaussian distribution. To this end, we will show that a triangular array in which the nth row consists of the random variables $\{p_k(x)\xi_k(x); k=1,\dots,n, x\in\mathbb{Z}^d\}$ (with only finitely of them being nonzero) satisfies the Lyapunov condition: for some $\delta>0$ and as $n\to\infty$,

$$\sum_{k=1}^{n}\sum_{x\in\mathbb{Z}^d}\mathbb{E}[|p_k(x)\xi_k(x)|^{2+\delta}]=o\left(\left(\sum_{k=1}^{n}\sum_{x\in\mathbb{Z}^d} p_k^2(x)\right)^{(2+\delta)/2}\right). \tag{59}$$

Note that $\sup_{x\in\mathbb{Z}^d} p_k(x)=O(k^{-d/2})$ as $k\to\infty$ by the multidimensional local limit theorem (see [16], Section 1.2). Recalling the assumption $\mathbb{E}|\xi|^{2+\delta}<\infty$, we have

$$\begin{aligned}\sum_{k=1}^{n}\sum_{x\in\mathbb{Z}^d}\mathbb{E}[|p_k(x)\xi_k(x)|^{2+\delta}]&=C\sum_{k=1}^{n}\sum_{x\in\mathbb{Z}^d} p_k^{2+\delta}(x)\\ &\le C\sum_{k=1}^{n}\left(k^{-(1+\delta)d/2}\sum_{x\in\mathbb{Z}^d} p_k(x)\right)\\ &=C\sum_{k=1}^{n}k^{-(1+\delta)d/2}\\ &\le Cn^{-(1+\delta)d/2+1}.\end{aligned}$$

It follows from (58) that for $d = 1, 2$, the Lyapunov condition (59) holds. To complete the verification of condition (37) of Proposition 1, recall (55), (56) and apply the Lyapunov central limit theorem.

Let us verify condition (38) for every $d \in \mathbb{N}$. Arguing as in (51), we obtain

$$\sum_{t_1,t_2\in T_n} r_n^2(t_1,t_2) = \sum_{\substack{k_1,k_2=1,\dots,n\\ x_1,x_2\in\mathbb{Z}^d}} \left(\sum_{t\in T_n} 1_{t(k_1)=x_1} 1_{t(k_2)=x_2}\right)^2. \tag{60}$$

The sum $\sum_{t\in T_n} 1_{t(k_1)=x_1} 1_{t(k_2)=x_2}$ counts the polymers $t \in T_n$ with the property $t(k_1) = x_1$, $t(k_2) = x_2$. For $1 \le k_1 \le k_2 \le n$, the number of such paths is $(2d)^n p_{k_1}(x_1) p_{k_2-k_1}(x_2 - x_1)$. It follows that

$$\begin{aligned}\sum_{t_1,t_2\in T_n} r_n^2(t_1,t_2) &\le 2\cdot(2d)^{2n} \sum_{1\le k_1\le k_2\le n} \sum_{x_1,x_2\in\mathbb{Z}^d} p_{k_1}^2(x_1) p_{k_2-k_1}^2(x_2-x_1)\\ &\le 2\cdot(2d)^{2n}\left(\sum_{k=0}^n \sum_{x\in\mathbb{Z}^d} p_k^2(x)\right)^2.\end{aligned}$$

With $\varepsilon_n = n^{-1/4}$, it follows that for any dimension $d \in \mathbb{N}$,

$$\sum_{\substack{t_1,t_2\in T_n\\ r_n(t_1,t_2)>\varepsilon_n n}} r_n(t_1,t_2) \le \frac{1}{n^{3/4}} \sum_{t_1,t_2\in T_n} r_n^2(t_1,t_2) = o(n s_n^2), \qquad n\to\infty,$$

where the last step follows from (56) combined with (57) (in the case $d \ge 3$) or (58) (in the case $d = 1, 2$). This verifies condition (38). The proof of Theorem 4 can be now completed by applying Proposition 1.

2.6. *Proof of Theorem* 5. Given two vertices $t_1 = (v_1, \dots, v_n) \in \mathbb{T}$ and $t_2 = (w_1, \dots, w_n) \in \mathbb{T}$ of length $n \in \mathbb{N}$ denote by $r_n(t_1, t_2) = \min\{i \in \mathbb{N} : v_i \ne w_i\} - 1$ the number of common ancestors, excluding $\varnothing$, of t_1 and t_2. The next lemma will be needed in the proof of Theorem 5.

LEMMA 2. *Fix* $k \in \mathbb{N}$. *Define a stochastic process* $\{V_n^{(k)}; n \in \mathbb{N}\}$ *by*

$$V_n^{(k)} = \frac{1}{m^{2n}} \sum_{\substack{t_1,t_2\in T_n\\ t_1\ne t_2}} r_n^k(t_1,t_2). \tag{61}$$

Then, the limit $V_\infty^{(k)} := \lim_{n\to\infty} V_n^{(k)}$ *exists in* $(0,\infty)$ *a.s.*

PROOF. Let $\mathcal{A}_n = \sigma\{Z_t; l(t) < n\}$ be the σ-algebra generated by the genealogical structure of the first n generations of the branching random walk. By definition, the random variable $V_n^{(k)}$ is $\mathcal{A}_n$-measurable. We will show that the sequence $\{V_n^{(k)}; n \in \mathbb{N}\}$ is a submartingale with respect to the filtration $\{\mathcal{A}_n; n \in \mathbb{N}\}$. We have

$$V_{n+1}^{(k)} = \frac{1}{m^{2n+2}} \sum_{\substack{t_1,t_2 \in T_n \\ t_1 \neq t_2}} Z_{t_1} Z_{t_2} r_n^k(t_1,t_2) + \frac{1}{m^{2n+2}} \sum_{t \in T_n} Z_t(Z_t - 1) n^k.$$

By our assumptions, $m = \mathbb{E}Z_t > 1$ and $\gamma_2 := \mathbb{E}[Z_t(Z_t - 1)] \in (0, \infty)$. It follows that

$$\mathbb{E}[V_{n+1}^{(k)} | \mathcal{A}_n] = V_n^{(k)} + \frac{\gamma_2 n^k}{m^{2n+2}} |T_n| > V_n^{(k)}, \tag{62}$$

whence the submartingale property. The sequence $\{V_n^{(k)}; k \in \mathbb{N}\}$ is bounded in L^1, since applying (62) recursively, we obtain

$$\mathbb{E}[V_{n+1}^{(k)}] = \mathbb{E}[V_n^{(k)}] + \frac{\gamma_2 n^k}{m^{n+2}} = \cdots = \gamma_2 \sum_{i=1}^{n} \frac{i^k}{m^{i+2}}. \tag{63}$$

By the martingale convergence theorem, $V_\infty^{(k)} = \lim_{n\to\infty} V_n^{(k)}$ exists in $[0, \infty)$ a.s. To see that the limit is nonzero a.s., consider particles in generation n which are offsprings of some fixed particle in generation 1. It is a classical fact that the number of these offsprings divided by m^{n-1} converges to an a.s. nonzero random variable (see [14], page 13). Since for any of these two offsprings t_1, t_2, we have $r_n(t_1, t_2) \geq 1$, it follows that $V_\infty^{(k)} > 0$ a.s. □

PROOF OF THEOREM 5. Given vertices $t_1, t_2 \in \mathbb{T}$ of length $n \in \mathbb{N}$, note that $\rho_n(t_1, t_2) := \mathbb{E}[\mathbb{X}_n(t_1)\mathbb{X}_n(t_2)] = \frac{1}{n} r_n(t_1, t_2)$. For $n \in \mathbb{N}$, let $s_n > 0$ be a random variable defined by

$$s_n^2 = \sum_{t_1,t_2 \in T_n} \rho_n(t_1, t_2).$$

First we prove that we have a.s. finite random variables V, W defined by

$$V = \lim_{n\to\infty} \frac{1}{s_n} \sum_{t \in T_n} \mathbb{X}_n(t), \qquad W = -\lim_{n\to\infty} \frac{\sqrt{n}}{|T_n|} \sum_{t \in T_n} \mathbb{X}_n(t). \tag{64}$$

By Lemma 2, we have

$$\lim_{n\to\infty} \sqrt{n} m^{-n} s_n = \lim_{n\to\infty} \sqrt{V_n^{(1)} + m^{-2n} n |T_n|} = \sqrt{V_\infty^{(1)}} \in (0, \infty) \qquad \text{a.s.}, \tag{65}$$

where we have also used that $\lim_{n\to\infty} m^{-n}|T_n|$ exists in $(0, \infty)$ a.s. (see [14], page 13). It has been observed in [8] that $\{\sqrt{n} m^{-n} \sum_{t \in T_n} \mathbb{X}_n(t); n \in \mathbb{N}\}$ is an L^2-bounded martingale with respect to the filtration $\{\mathcal{B}_n; n \in \mathbb{N}\}$, where $\mathcal{B}_n$ is the

σ-algebra generated by the genealogical structure $\{Z_t; l(t) < n\}$ and the displacements $\{\xi_t; l(t) \le n\}$ of the first n generations of the branching random walk. By the martingale convergence theorem and (65), we obtain that the limits in (64) exist a.s. Also, it follows from Lemma 2 and (65) that

$$\lim_{n\to\infty} \frac{1}{ns_n^2} \sum_{t_1,t_2\in T_n} r_n^2(t_1,t_2) = \lim_{n\to\infty} \frac{m^{2n}}{ns_n^2}\left(V_n^{(2)} + \frac{n^2}{m^{2n}}|T_n|\right) = \frac{V_\infty^{(2)}}{V_\infty^{(1)}}, \tag{66}$$

which is finite a.s.

The proof of Theorem 5 can be completed as follows. Since the set T_n of particles in the nth generation is random, we cannot apply Proposition 1 directly. To overcome this difficulty, we will use a conditioning argument. We may assume that the random variables $\{Z_t; t \in \mathbb{T}\}$ representing the numbers of children are defined on a probability space $(\Omega_Z, \mathcal{A}_Z, \mu_Z)$ and the random variables $\{\xi_t; t \in \mathbb{T} \setminus \{\varnothing\}\}$ representing the displacements are defined on $(\Omega_\xi, \mathcal{A}_\xi, \mu_\xi)$. Then, we can define the branching random walk on the product $\Omega_Z \times \Omega_\xi$ of both the spaces. Fix some $\tau \in \Omega_Z$ and restrict all random variables to the set $\{\tau\} \times \Omega_\xi$ endowed with the probability measure $\delta_\tau \times \mu_\xi$, where δ_τ is the Dirac measure at τ. Essentially, this means that we fix the realization of the Galton–Watson tree but do not fix the displacements of the particles. Note that the set T_n becomes deterministic after such restriction. It follows from (64) and (66) that conditions (37) and (46) of Proposition 1 are fulfilled (in the restricted setting) for μ_Z-a.e. $\tau \in \Omega_Z$. Applying Proposition 1, we obtain that for μ_Z-a.e. $\tau \in \Omega_Z$,

$$\left\{\frac{\sqrt{n}}{|T_n|} \sum_{t\in T_n} \left(1_{\mathbb{X}_n(t)\le z} - \Phi_n(z)\right); z \in \mathbb{R}\right\} \xrightarrow[n\to\infty]{\text{f.d.d.}} \{p(z)W; z \in \mathbb{R}\},$$

where the random variables under consideration are restricted to the space $\{\tau\} \times \Omega_\xi$. To complete the proof, integrate over $\tau \in \Omega_Z$. $\square$

3. Proof of Theorem 6.

3.1. *Hermite polynomials.* We need to recall some facts about Hermite polynomials. Recall that $p(z) = (2\pi)^{-1/2} e^{-z^2/2}$ is the standard Gaussian density. Let $L^2(\mathbb{R}, p)$ be the set of all measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $\|f\|^2_{L^2(\mathbb{R},p)} := \int_{\mathbb{R}} f^2(z) p(z)\, dz$ is finite. The space $L^2(\mathbb{R}, p)$ is a separable Hilbert space endowed with the scalar product $\langle f, g\rangle_{L^2(\mathbb{R},p)} = \int_{\mathbb{R}} f(z) g(z) p(z)\, dz$. The (normalized) Hermite polynomials $h_0, h_1, \ldots$ are defined by $h_n(z) = (-1)^n (n!)^{-1/2} e^{z^2/2} \frac{d^n}{dz^n} e^{-z^2/2}$. The sequence $\{h_n\}_{n=0,1,\ldots}$ is an orthonormal basis in $L^2(\mathbb{R}, p)$. For the proof of the next lemma see [21], Lemma 1.1, or [15], page 55.

LEMMA 3. *Let (X, Y) be a zero-mean Gaussian vector with $\mathbb{E}X^2 = \mathbb{E}Y^2 = 1$ and $\mathbb{E}[XY] = \rho$. Then, for every $i, j \in \mathbb{N} \cup \{0\}$,*

$$\mathbb{E}[h_i(X)h_j(Y)] = \begin{cases} \rho^i, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Given $f \in L^2(\mathbb{R}, p)$ and $k \in \mathbb{N}$, we denote by $P_k f$ the orthogonal projection of f onto the orthogonal complement of the k-dimensional linear subspace spanned by the first k Hermite polynomials $h_0, \dots, h_{k-1}$. That is,

$$(P_k f)(z) = \sum_{i=k}^{\infty} \langle f, h_i \rangle_{L^2(\mathbb{R},p)} h_i(z) = f(z) - \sum_{i=0}^{k-1} \langle f, h_i \rangle_{L^2(\mathbb{R},p)} h_i(z). \tag{67}$$

LEMMA 4. *Let (X, Y) be a zero-mean Gaussian vector with $\mathbb{E}X^2 = \mathbb{E}Y^2 = 1$ and $\mathbb{E}[XY] = \rho$. Then, for any $f, g \in L^2(\mathbb{R}, p)$ and $k \in \mathbb{N}$,*

$$|\mathbb{E}[P_k f(X) P_k g(Y)]| \le |\rho|^k \|f\|_{L^2(\mathbb{R},p)} \|g\|_{L^2(\mathbb{R},p)}. \tag{68}$$

PROOF. Write $f_i = \langle f, h_i \rangle_{L^2(\mathbb{R},p)}$ and $g_i = \langle g, h_i \rangle_{L^2(\mathbb{R},p)}$ for $i \in \mathbb{N} \cup \{0\}$. We have $P_k f(X) = \sum_{i=k}^{\infty} f_i h_i(X)$ and $P_k g(Y) = \sum_{i=k}^{\infty} g_i h_i(Y)$. Using Lemma 3 and the inequality $|\rho| \le 1$, we obtain

$$|\mathbb{E}[P_k f(X) P_k g(Y)]| = \left| \sum_{i=k}^{\infty} \rho^i f_i g_i \right| \le |\rho|^k \sum_{i=0}^{\infty} |f_i||g_i|.$$

To complete the proof, apply the Cauchy–Schwarz inequality. □

3.2. *Reduction method.* The following proposition is an empirical central limit theorem for Gaussian processes.

PROPOSITION 2. *For every $n \in \mathbb{N}$, let $\{\mathbb{X}_n(t); t \in T_n\}$ be a zero-mean, unit-variance Gaussian process. Let $\rho_n(t_1, t_2) = \mathbb{E}[\mathbb{X}_n(t_1)\mathbb{X}_n(t_2)]$ be the covariance function of $\mathbb{X}_n$. Define $\varsigma_n \ge 0$ by*

$$\varsigma_n^2 := \operatorname{Var}\left[\sum_{t \in T_n} (\mathbb{X}_n^2(t) - 1)\right] = 2 \sum_{t_1, t_2 \in T_n} \rho_n^2(t_1, t_2). \tag{69}$$

Suppose that for some random variable V and for some sequence $\varepsilon_n > 0$ satisfying $\lim_{n\to\infty} \varepsilon_n = 0$, the following three conditions hold:

$$\lim_{n\to\infty} \frac{1}{\varsigma_n^2} \sum_{t_1, t_2 \in T_n} \rho_n(t_1, t_2) = 0, \tag{70}$$

$$\frac{1}{\varsigma_n} \sum_{t \in T_n} (\mathbb{X}_n^2(t) - 1) \xrightarrow[n\to\infty]{d} V, \tag{71}$$

$$\lim_{n\to\infty} \frac{1}{\varsigma_n^2} \sum_{t_1, t_2 \in T_n} \rho_n^2(t_1, t_2) 1_{|\rho_n(t_1,t_2)| > \varepsilon_n} = 0. \tag{72}$$

Then, the following convergence of stochastic processes holds true:

$$\left\{\frac{1}{\varsigma_n}\sum_{t\in T_n}\left(1_{\mathbb{X}_n(t)\le z}-\Phi(z)\right); z\in\mathbb{R}\right\}\xrightarrow[n\to\infty]{f.d.d.}\left\{-\frac{1}{2}zp(z)V; z\in\mathbb{R}\right\}. \tag{73}$$

PROOF. The proof is based on the reduction method of [22]. For $x, z \in \mathbb{R}$, write $f(x; z) = 1_{x\le z}$. For $z \in \mathbb{R}$, define a zero-mean random field $\{\mathbb{Y}_n(t; z); t \in T_n\}$ by

$$\mathbb{Y}_n(t; z) := (P_3 f(\cdot; z))(\mathbb{X}_n(t)), \tag{74}$$

where P_3 is the projection operator given in (67). Since the first three Hermite polynomials are given by $h_0(x) = 1$, $h_1(x) = x$, $h_2(x) = \frac{1}{\sqrt{2}}(x^2 - 1)$, this means that

$$\mathbb{Y}_n(t; z) = 1_{\mathbb{X}_n(t)\le z} - \Phi(z) + p(z)\mathbb{X}_n(t) + \tfrac{1}{2}zp(z)\left(\mathbb{X}_n^2(t) - 1\right). \tag{75}$$

By Lemma 4 with $k = 3$ and $f = g$, we have $\mathbb{E}[\mathbb{Y}_n(t_1; z)\mathbb{Y}_n(t_2; z)] \le C|\rho_n(t_1, t_2)|^3$ for every $t_1, t_2 \in T_n$, where the constant C does not depend on $z \in \mathbb{R}$. It follows that

$$\begin{aligned}\operatorname{Var}\left[\sum_{t\in T_n}\mathbb{Y}_n(t; z)\right] &= \sum_{t_1,t_2\in T_n}\mathbb{E}[\mathbb{Y}_n(t_1; z)\mathbb{Y}_n(t_2; z)]\\ &\le C\sum_{t_1,t_2\in T_n}|\rho_n(t_1, t_2)|^3\\ &\le C\varepsilon_n\sum_{t_1,t_2\in T_n}\rho_n^2(t_1, t_2) + C\sum_{\substack{t_1,t_2\in T_n\\ |\rho_n(t_1,t_2)|>\varepsilon_n}}\rho_n^2(t_1, t_2)\\ &= o(\varsigma_n^2)\end{aligned} \tag{76}$$

as $n \to \infty$, where the last step follows from the assumption $\lim_{n\to\infty}\varepsilon_n = 0$ and condition (72). Take some $z_1, \ldots, z_d \in \mathbb{R}$. Then, for every $i = 1, \ldots, d$, it follows from (75) that we have the following decomposition:

$$\begin{aligned}&\frac{1}{\varsigma_n}\sum_{t\in T_n}\left(1_{\mathbb{X}_n(t)\le z_i} - \Phi(z_i)\right)\\ &\quad = -\frac{p(z_i)}{\varsigma_n}\sum_{t\in T_n}\mathbb{X}_n(t) - \frac{z_i p(z_i)}{2\varsigma_n}\sum_{t\in T_n}\left(\mathbb{X}_n^2(t) - 1\right)\\ &\qquad + \frac{1}{\varsigma_n}\sum_{t\in T_n}\mathbb{Y}_n(t; z_i).\end{aligned}$$

As $n \to \infty$, the first term converges to 0 in probability by condition (70). The second term converges in distribution to $-\frac{1}{2}z_i p(z_i)V$ by condition (71).

Finally, the third term converges to 0 in probability by (76). This completes the proof. □

3.3. *Completing the proof of Theorem 6.* We will verify the conditions of Proposition 2. Given a spin configuration $t \in T_n$ and an edge $e = \{v_1, v_2\} \in E_n$, we write $t \diamond e = t(v_1)t(v_2) \in \{+1, -1\}$. Recall that the energy of a spin configuration $t \in T_n$ is given by

$$\mathbb{X}_n(t) = |E_n|^{-1/2} \sum_{e \in E_n} (t \diamond e) J(e), \tag{77}$$

where $\{J(e); e \in E_n\}$ are independent standard Gaussian random variables.

We start by verifying condition (70) of Proposition 2. Since $\sum_{t \in T_n} (t \diamond e) = 0$ for every edge $e \in E_n$, we have

$$\begin{aligned} \sum_{t \in T_n} \mathbb{X}_n(t) &= |E_n|^{-1/2} \sum_{t \in T_n} \sum_{e \in E_n} (t \diamond e) J(e) \\ &= |E_n|^{-1/2} \sum_{e \in E_n} J(e) \sum_{t \in T_n} (t \diamond e) \\ &= 0. \end{aligned}$$

Hence, $\sum_{t_1, t_2 \in T_n} \rho_n(t_1, t_2) = 0$, which implies that condition (70) holds.

Let us verify condition (71) of Proposition 2. Note that for every different edges $e_1, e_2 \in E_n$, we have $\sum_{t \in T_n} (t \diamond e_1)(t \diamond e_2) = 0$. Hence,

$$\begin{aligned} &\sum_{t \in T_n} (\mathbb{X}_n^2(t) - 1) \\ &\quad = |E_n|^{-1} \sum_{t \in T_n} \sum_{e_1, e_2 \in E_n} \big((t \diamond e_1)(t \diamond e_2) J(e_1) J(e_2) - 1_{e_1 = e_2}\big) \qquad (78) \\ &\quad = \frac{|T_n|}{|E_n|} \sum_{e \in E_n} (J^2(e) - 1). \end{aligned}$$

By Lemma 3, $\mathbb{E}[(\mathbb{X}_n^2(t_1) - 1)(\mathbb{X}_n^2(t_2) - 1)] = 2\rho_n^2(t_1, t_2)$. It follows from this and (78) that

$$\varsigma_n^2 = \operatorname{Var}\Big[\sum_{t \in T_n} (\mathbb{X}_n^2(t) - 1)\Big] = \frac{|T_n|^2}{|E_n|^2} \operatorname{Var}\Big[\sum_{e \in E_n} (J^2(e) - 1)\Big] = \frac{2|T_n|^2}{|E_n|}. \tag{79}$$

The central limit theorem together with (78) and (79) implies that condition (71) is satisfied with $V \sim N(0, 1)$.

Let us verify condition (72) of Proposition 2. It follows from (77) that for every $t_1, t_2 \in T_n$, $\rho_n(t_1, t_2) = |E_n|^{-1} \sum_{e \in E_n} (t_1 \diamond e)(t_2 \diamond e)$. Define a spin configuration

$\tilde{t} \in T_n$ by requiring that $\tilde{t}(v) = 1$ for every vertex $v \in V_n$. It follows that

$$\begin{aligned}
\sum_{t_1,t_2 \in T_n} \rho_n^4(t_1,t_2) &= |T_n| \sum_{t \in T_n} \rho_n^4(\tilde{t}, t) \\
&= |T_n||E_n|^{-4} \sum_{t \in T_n} \left(\sum_{e \in E_n} (t \diamond e) \right)^4 \qquad (80) \\
&= |T_n||E_n|^{-4} \sum_{e_1,e_2,e_3,e_4 \in E_n} \sum_{t \in T_n} \prod_{k=1}^{4} (t \diamond e_k).
\end{aligned}$$

It will be convenient to write $\eta(e_1,\dots,e_4) = \sum_{t \in T_n} \prod_{k=1}^4 (t \diamond e_k)$. If some vertex $v \in V_n$ belongs to exactly one or exactly three of the edges $e_1,\dots,e_4$, then $\eta(e_1,\dots,e_4) = 0$ by spin flip symmetry. Consider some quadruple $e_1,\dots,e_4$ for which $\eta(e_1,\dots,e_4) \neq 0$. We will show that there are at most $C|E_n|^2$ such quadruples. The union of all vertices belonging to $e_1,\dots,e_4$ consists of 2 or 4 elements. In both cases, we can find $i, j \in 1,\dots,4$ such that the union of vertices belonging to $e_1,\dots,e_4$ coincides with the union of the vertices of e_i, e_j. There are at most $|E_n|^2$ possibilities to choose e_i and e_j and a bounded number of choices for the remaining two edges. To summarize, there are at most $C|E_n|^2$ terms of the form $\eta(e_1,\dots,e_4)$ which are nonzero, and any such term is bounded by $|T_n|$. It follows from these considerations and (80) that

$$\sum_{t_1,t_2 \in T_n} \rho_n^4(t_1,t_2) \le |T_n||E_n|^{-4} \cdot C|E_n|^2|T_n| \le C\frac{|T_n|^2}{|E_n|^2}. \qquad (81)$$

Now we are able to verify condition (72). Since $\lim_{n\to\infty} |E_n| = \infty$, we can choose $\varepsilon_n > 0$ in such a way that $\lim_{n\to\infty} \varepsilon_n = 0$ but $\lim_{n\to\infty} \varepsilon_n^2 |E_n| = \infty$. Recalling (79) and (81), we obtain

$$\frac{1}{\varsigma_n^2} \sum_{t_1,t_2 \in T_n} \rho_n^2(t_1,t_2) 1_{|\rho_n(t_1,t_2)| > \varepsilon_n} \le \frac{1}{\varsigma_n^2 \varepsilon_n^2} \sum_{t_1,t_2 \in T_n} \rho_n^4(t_1,t_2) \le \frac{C}{\varepsilon_n^2 |E_n|},$$

which converges to 0 as $n \to \infty$. This completes the verification of condition (72) of Proposition 2.

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INSTITUTE OF STOCHASTICS
ULM UNIVERSITY
HELMHOLTZSTR. 18
89069 ULM
GERMANY
E-MAIL: zakhar.kabluchko@uni-ulm.de