The Annals of Applied Probability
2008, Vol. 18, No. 2, 379–426
DOI: 10.1214/105051607000000285

RECURSIVE COMPUTATION OF THE INVARIANT MEASURE OF A STOCHASTIC DIFFERENTIAL EQUATION DRIVEN BY A LÉVY PROCESS

BY FABIEN PANLOUP

Université Paris 6

We study some recursive procedures based on exact or approximate Euler schemes with decreasing step to compute the invariant measure of Lévy driven SDEs. We prove the convergence of these procedures toward the invariant measure under weak conditions on the moment of the Lévy process and on the mean-reverting of the dynamical system. We also show that an a.s. CLT for stable processes can be derived from our main results. Finally, we illustrate our results by several simulations.

1. Introduction.

1.1. *Objectives and motivations.* This paper is devoted to the computation of the invariant measure (denoted by ν) of ergodic stochastic processes which obey a stochastic differential equation (SDE) driven by a Lévy process. Practically, we want to construct a sequence of empirical measures $(\bar{\nu}_n(\omega, dx))_{n\geq 1}$ which can be recursively simulated and such that $\bar{\nu}_n(\omega, f) \to \nu(f)$ a.s. for a range of functions f containing bounded continuous functions.

In the case of Brownian diffusions, some methods have already been developed by several authors to approximate the invariant measure (see Section 1.3), but this paper seems to be the first one that deals with this problem in the case of general Lévy driven SDEs. The motivation for this generalization is the study of dynamical systems that are widely used in modeling. Indeed, there are many situations where the noise of the dynamical system is discontinuous or too intensive to be modeled by a Brownian motion. Let us consider an example that comes from the fragmentation-coalescence theory. In situations such as polymerization phenomenons, when temperature is near to its critical value, molecules constantly break-up and recombine. This situation has been modeled by Berestycki [2] through what he terms EFC (Exchangeable Fragmentation-Coalescence) process. The mass of the dust generated by this process (see [2] for more details) is a solution to a mean-reverting SDE for which the noise component is driven by a subordinator (an increasing Lévy process). We come back to this example in Section 7.

Received October 2005; revised April 2007.

AMS 2000 subject classifications. Primary 60H35, 60H10, 60J75; secondary 60F05.

Key words and phrases. Stochastic differential equation, Lévy process, invariant distribution, Euler scheme, almost sure central limit theorem.

For other examples of situations where models that use a Lévy driven SDE are adapted, we refer to Barndorff-Nielsen et al. [1] for examples in financial modeling (where ergodic Lévy driven SDEs are usually used to model the volatility of a financial market), Protter–Talay [23] for examples in finance, electrical engineering, ... or Deng [8], who models the spot prices of electricity by a mean-reverting Brownian diffusion perturbed by a compound Poisson noise.

1.2. *The stochastic differential equation.* According to the Lévy–Khintchine decomposition (for this result and for introduction to Lévy processes, see, e.g., Bertoin [4], Protter [22] or Sato [25]), an $\mathbb{R}^l$-valued Lévy process (L_t) with Lévy measure π admits the following decomposition: $L_t = \alpha t + \sqrt{Q} W_t + Y_t + N_t$, where $\alpha \in \mathbb{R}^l$, Q is a symmetric positive $l \times l$ real matrix, (W_t) is a l-dimensional standard Brownian motion, (Y_t) is a centered l-dimensional Lévy process with jumps bounded by 1 and characteristic function given for every $t \geq 0$ by

$$\mathbb{E}\{e^{i\langle u, Y_t\rangle}\} = \exp\left[t\left(\int_{\{|y|\leq 1\}} e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle \pi(dy)\right)\right]$$

and (N_t) is a compound Poisson process with parameters $\lambda = \pi(|y| > 1)$ and $\mu(dy) = 1_{\{|y|>1\}}\pi(dy)/\pi(|y| > 1)$ (λ denotes the parameter for the waiting time between the jumps of N and μ, the distribution of the jumps). Moreover, (W_t), (Y_t) and (N_t) are independent Lévy processes.

Following this decomposition, we consider an $\mathbb{R}^d$-valued càdlàg process (X_t) solution to the SDE

$$dX_t = b(X_{t^-})\,dt + \sigma(X_{t^-})\,dW_t + \kappa(X_{t^-})\,dZ_t, \tag{1}$$

where $b : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : \mathbb{R}^d \mapsto \mathbb{M}_{d,l}$ (set of $d \times l$ real matrices) and $\kappa : \mathbb{R}^d \mapsto \mathbb{M}_{d,l}$ are continuous with sublinear growth and (Z_t) is the sum of the jump components of the Lévy process: $Z_t = Y_t + N_t$.

In most papers dealing with Lévy driven SDEs, the SDE reads $dX_t = f(X_{t^-})\,dL_t$, where $(L_t)_{t\geq 0}$ is a Lévy process. Here, we separate each part of the Lévy process because they act differently on the dynamical system. We isolate the drift term because it usually produces the mean-reverting effect (which in turn induces the ergodicity of the SDE). The two other terms are both noises, but we distinguish them because they do not have the same behavior.

REMARK 1. In (1) we chose to write the jump component by compensating the jumps smaller than 1, but it is obvious that, for every $h > 0$, (X_t) is also solution to

$$dX_t = b^h(X_{t^-})\,dt + \sigma(X_{t^-})\,dW_t + \kappa(X_{t^-})\,dZ_t^h \tag{2}$$

with $b^h = b + \int_{\{|y|\in(1,h]\}} y\pi(dy)$ if $h > 1$, $b^h = b - \int_{\{|y|\in(h,1]\}} y\pi(dy)$ if $h < 1$, and $Z_t^h = Y_t^h + N_t^h$, where the characteristic function of Y_t^h is given for every $t \geq 0$ by

$$\mathbb{E}\{e^{i\langle u, Y_t^h\rangle}\} = \exp\left[t\left(\int_{\{|y|\leq h\}} e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle \pi(dy)\right)\right]$$

and (N_t^h) is a compound Poisson process with parameters $\lambda^h = \pi(|y| > h)$ and $\mu^h(dy) = 1_{\{|y|>h\}}\pi(dy)/\pi(|y| > h)$. By this remark, we want to emphasize that the formulation (1) is conventional and that the coefficient b in (1) is dependent on this choice. We will come back to this remark when we introduce the assumptions of the main results where we want, on the contrary, that they be intrinsic (see Remark 4).

Let us recall a result about existence, uniqueness and Markovian structure of the solutions of (1) (see [22]).

THEOREM 1. *Assume that b, σ and κ are locally Lipschitz functions with sublinear growth. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space satisfying the usual conditions and let X_0 be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$. Then, for any $(\mathcal{F}_t)$-Brownian motion $(W_t)_{t\geq 0}$, for any $(\mathcal{F}_t)$-measurable $(Z_t)_{t\geq 0}$ as previously defined, the SDE (1) admits a unique càdlàg solution $(X_t)_{t\geq 0}$ with initial condition X_0. Moreover, $(X_t)_{t\geq 0}$ is a Feller and Markov process.*

REMARK 2. Lévy driven SDEs are the largest subclass of SDEs driven by semimartingales such that the solutions have a Markovian structure. Indeed, a result due to Jacod and Protter (see [13]) shows that, under appropriate conditions on the coefficients, a stochastic process solution to a homogeneous SDE driven by a semimartingale is a strong Markov process if, and only if, the driving process is a Lévy process.

1.3. *Background on approximation of invariant measures for Brownian diffusions.* This problem has already been studied by several authors when (X_t) is a Brownian diffusion, that is, when $\kappa = 0$. In [28] Talay approximates $\nu(f)$ by $\bar{\nu}_n^\gamma(f) = 1/n \sum_{k=1}^n f(\bar{X}_{k-1}^\gamma)$, where $(\bar{X}_n^\gamma)_n$ denotes the Euler scheme with constant step γ. Denoting by ν^γ the invariant distribution of the homogeneous Markov chain $(\bar{X}_n^\gamma)_n$, he shows that $\bar{\nu}_n^\gamma \overset{n\to+\infty}{\Longrightarrow} \nu^\gamma$ and that $\nu^\gamma \overset{\gamma\to 0}{\Longrightarrow} \nu$, under some uniform ellipticity and Lyapunov-type stability assumptions. (A Markov process (X_t) with infinitesimal generator A satisfies a Lyapunov assumption if there exists a positive function $\mathcal{V}$ such that $\mathcal{V}(x) \to +\infty$ and $\limsup AV(x) = -\infty$ when $|x| \to +\infty$. Then, V is called a Lyapunov function for (X_t). Under this assumption, (X_t) admits a stationary, often ergodic when unique, distribution. The existence of such a Lyapunov function depends on the mean-reversion of the drift and on the intensity of the diffusions term (see, e.g., [5, 7, 12] and [19] for literature on Lyapunov stability).) In this procedure, γ and n correspond to the two types of errors that the discretization of this long time problem generates. Practically, one cannot efficiently manage them together. Indeed, when one implements this algorithm, one sets a positive real γ and then, one approximates the biased target ν^γ. In order to get rid of this problem, Lamberton and Pagès (see [14, 15]) replace the standard Euler scheme with constant step γ with an Euler scheme with decreasing step γ_n. Denoting by $(\bar{X}_n)_{n\geq 1}$ this Euler scheme and by $(\eta_k)_{k\geq 1}$ a sequence of weights such that $H_n = \sum_{k=1}^n \eta_k \overset{n\to+\infty}{\longrightarrow} +\infty$, they define a sequence of *weighted* empirical measures $(\bar{\nu}_n)$ and show under some Lyapunov assumptions (but without ellipticity assumptions) that if (η_n/γ_n) is nonincreasing,

$$\bar{\nu}_n(f) = \frac{1}{H_n}\sum_{k=1}^n \eta_k f(\bar{X}_{k-1}) \overset{n\to+\infty}{\longrightarrow} \nu(f) \qquad \text{p.s.},$$

for every continuous function f with polynomial growth (see [14, 15] for more details and [17] for extensions).

REMARK 3. These two approaches are significantly different. Talay's method strongly relies on the homogeneous Markovian structure of the constant step Euler scheme and on its classical "toolbox" (irreducibility, positive recurrence, ..., see, e.g., [18]). Since the Euler scheme with decreasing step is no longer homogeneous, Lamberton and Pagès develop another method based on stability of Markov chains and on martingale methods which can be extended to a nonhomogeneous setting (see [9]). This is why they do not need any ellipticity assumptions on the coefficients.

1.4. *Difficulties induced by the jumps of the Lévy process.* In this paper we adapt the Lamberton and Pagès approach. In order to obtain some similar results in the case of Lévy driven SDEs, one mainly has two kinds of obstacles to overcome.

From a dynamical point of view, the main difficulty comes from the moments of the jump component. Indeed, by contrast with the case of Brownian motion, the jump component can have only few moments (stable processes, e.g.), and it then generates some instability for the SDE.

The second obstacle appears in the simulation of the Euler scheme. Actually, only in some very particular cases can the jump component of a Lévy process be simulated (compound Poisson processes, stable processes, ...). In those cases, the Euler scheme [that we call *exact* Euler scheme and denote by (A)] can be built by using the true increments of (Z_t). Otherwise, one has to study some *approximate* Euler schemes where we replace the increments of Z_t with some approximations that can be simulated. The canonical way for approximating the jump component is to truncate its small jumps. Let $(u_n)_{n\geq 1}$ be a sequence of positive numbers such that $u_n < 1$ and (u_n) decreases to 0 and $(Y^n)_{n\geq 1}$ be the sequence of càdlàg processes defined by

$$Y_t^n = \sum_{0<s\leq t} \Delta Y_s 1_{\{\Delta Y_s \in D_n\}} - t\int_{D_n} y\pi(dy) \qquad \forall t \geq 0$$

with $D_n = \{y \in \mathbb{R}^l, u_n < |y| \leq 1\}$ and $\Delta Y_s = Y_s - Y_{s-}$. The process Y^n is a compensated compound Poisson process with parameters $\lambda_n = \pi(D_n)$ and $\mu_n(dy) =$

$1_{D_n}(y)\pi(dy)/\pi(D_n)$. It converges locally uniformly in L^2 to Y, that is, for every $T > 0$,

$$\mathbb{E}\left\{\sup_{0<t\leq T} |Y_t - Y_t^n|^2\right\} \overset{n\to+\infty}{\longrightarrow} 0 \qquad \text{(see [4]).} \tag{3}$$

We will denote by Z^n the process defined by $Z^n = Y^n + N$ and by (B) the Euler Scheme built with its increments. The increments of Z^n can be simulated if λ_n and the coefficient of the drift term can be calculated, and if μ_n can be simulated for all $n \in \mathbb{N}$. This is the case for a broad class of Lévy processes, thanks to classical techniques (rejection method, integral approximation, ...). If there exists $(u_n)_{n\geq 1}$ such that the increments of Z^n can be simulated for all $n \in \mathbb{N}$, we say that *the Lévy measure can be simulated*. However, the simulation time of Z_t^n depends on the average number of its jumps, that is, on $\pi(|y| > u_n)t$. When the truncation threshold tends to 0 (this is necessary to approach the true increment of the jump component), $\pi(|y| > u_n)$ explodes as soon as the Lévy measure is not finite (i.e., as soon as the true jump component is not a compound Poisson process). It implies that the simulation time of Z_t^n explodes for a fixed t. However, thanks to the decreasing step, it is possible to adapt the time step γ_n and the truncation threshold u_n so that the expectation of the number of jumps at each time step remains uniformly bounded. Following the same idea, it is also possible to choose some steps and some truncation thresholds so that the average number of jumps at each time step tends to 0. In this case, approximating the true component by the preceding compound Poisson process stopped at its first jump time (the first time when it jumps) can also be efficient [see Scheme (C)].

1.5. *Construction of the procedures.* Let $(\gamma_n)_{n\geq 1}$ be a decreasing sequence of positive real numbers such that $\lim \gamma_n = 0$ and $\Gamma_n = \sum_{k=1}^n \gamma_k \to +\infty$ when $n \to +\infty$. Let $(U_n)_{n\geq 1}$ be a sequence of i.i.d. square integrable centered $\mathbb{R}^l$-valued random variables with $\Sigma_{U_1} = I_d$. Finally, let $(\bar{Z}_n)_{n\geq 1}$, $(\bar{Z}_n^B)_{n\geq 1}$ and $(\bar{Z}_n^C)_{n\geq 1}$ be sequences of independent $\mathbb{R}^l$-valued random variables independent of $(U_n)_{n\geq 1}$, such that,

$$\bar{Z}_n \overset{(\mathbb{R}^l)}{=} Z_{\gamma_n}, \qquad \bar{Z}_n^B \overset{(\mathbb{R}^l)}{=} Z_{\gamma_n}^n \quad \text{and} \quad \bar{Z}_n^C \overset{(\mathbb{R}^l)}{=} Z_{\gamma_n \wedge T^n}^n \qquad \forall n \geq 1,$$

with $T^n = \inf\{s > 0, |\Delta Z_s^n| > 0\}$. Let $x \in \mathbb{R}^d$. The Euler Schemes (A), (B) and (C) are recursively defined by $\bar{X}_0 = \bar{X}_0^B = \bar{X}_0^C = x$ and for every $n \geq 1$,

$$\bar{X}_{n+1} = \bar{X}_n + \gamma_{n+1} b(\bar{X}_n) + \sqrt{\gamma_{n+1}}\sigma(\bar{X}_n)U_{n+1} + \kappa(\bar{X}_n)\bar{Z}_{n+1}, \tag{A}$$

$$\bar{X}_{n+1}^B = \bar{X}_n^B + \gamma_{n+1} b(\bar{X}_n^B) + \sqrt{\gamma_{n+1}}\sigma(\bar{X}_n^B)U_{n+1} + \kappa(\bar{X}_n^B)\bar{Z}_{n+1}^B, \tag{B}$$

$$\bar{X}_{n+1}^C = \bar{X}_n^C + \gamma_{n+1} b(\bar{X}_n^C) + \sqrt{\gamma_{n+1}}\sigma(\bar{X}_n^C)U_{n+1} + \kappa(\bar{X}_n^C)\bar{Z}_{n+1}^C. \tag{C}$$

We set $\mathcal{F}_n = \sigma(\bar{X}_k, k \leq n)$, $\mathcal{F}_n^B = \sigma(\bar{X}_k^B, k \leq n)$ and $\mathcal{F}_n^C = \sigma(\bar{X}_k^C, k \leq n)$. Let $(\eta_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $H_n = \sum_{k=1}^n \eta_k \to +\infty$. For each scheme, we define a sequence of weighted empirical measures by

$$\bar{\nu}_n = \frac{1}{H_n}\sum_{k=1}^{n}\eta_k\delta_{\bar{X}_{k-1}},$$
$$\bar{\nu}_n^B = \frac{1}{H_n}\sum_{k=1}^{n}\eta_k\delta_{\bar{X}^B_{k-1}} \quad\text{and}\quad \bar{\nu}_n^C = \frac{1}{H_n}\sum_{k=1}^{n}\eta_k\delta_{\bar{X}^C_{k-1}}. \tag{4}$$

For a function $f:\mathbb{R}^d \mapsto \mathbb{R}$, $(\bar{\nu}_n(f))$ can be recursively computed (so is the case for the two other schemes). Indeed, we have $\bar{\nu}_1(f) = f(x)$ and for every $n \geq 1$,

$$\bar{\nu}_{n+1}(f) = \bar{\nu}_n(f) + \frac{\eta_{n+1}}{H_{n+1}}\big(f(\bar{X}_{n+1}) - \bar{\nu}_n(f)\big).$$

Some comments about the approximate Euler schemes. In Scheme (B), since (u_n) decreases to 0, we discard fewer and fewer jumps of the true component when n grows. We will see in Theorem 2 below that this is the only condition on (u_n) for the convergence of $(\bar{\nu}_n^B)$. This means that we only need the law of $Z^n_{\gamma_n}$ to be an "asymptotically good approximation" of the law of Z_{γ_n}. Yet, as previously mentioned, there is a hidden constraint induced by the simulation time which is proportional to the average number $\pi(|y| > u_n)\gamma_n$ of jumps of Z^n on $[0, \gamma_n]$. In practice, we require $(\pi(|y| > u_n)\gamma_n)$ to be bounded.

Furthermore, if $\pi(|y| > u_n)\gamma_n \to 0$ (i.e., the average number of jumps at each step tends to 0), we will see that the first jump of Z^n is all that matters for the convergence of the empirical measure. This means that Scheme (C) becomes efficient.

1.6. *Notations.* Throughout this paper, every positive real constant is denoted by C (it may vary from line to line). We denote the usual scalar product on $\mathbb{R}^d$ by $\langle\cdot,\cdot\rangle$ and the Euclidean norm by $|\cdot|$. For any $d \times l$ real matrix M, we define $\|M\| = \sup_{\{|x|\leq 1\}}|Mx|/|x|$. For a symmetric $d \times d$ real matrix M, we set $\lambda_M = \max(0, \lambda_1, \ldots, \lambda_d)$, where $\lambda_1, \ldots, \lambda_d$ denote the eigenvalues of M. For every $x \in \mathbb{R}^d$,

$$Mx^{\otimes 2} = x^*Mx \leq \lambda_M|x|^2. \tag{5}$$

We denote by $\mathcal{C}_b(\mathbb{R}^d)$ [resp. $\mathcal{C}_0(\mathbb{R}^d)$] the set of bounded continuous functions on $\mathbb{R}^d$ with values in $\mathbb{R}$ (resp. continuous functions that go to 0 at infinity) and $\mathcal{C}^2_K(\mathbb{R}^d)$, the set of $\mathcal{C}^2$-functions on $\mathbb{R}^d$ with values in $\mathbb{R}$ and compact support. One says that f is a p-Hölder function on E with values in F (where E and F are normed vector spaces) if

$$[f]_p = \sup_{x,y\in E}\frac{\|f(x)-f(y)\|_F}{\|x-y\|_E^p} < +\infty.$$

Finally, we say that $V:\mathbb{R}^d \mapsto \mathbb{R}^*_+$ is an EQ-function (*Essentially Quadratic* function) if V is $\mathcal{C}^2$, $\lim V(x) = +\infty$ when $|x| \to +\infty$, $|\nabla V| \leq C\sqrt{V}$ and D^2V is bounded. [In particular, V given by $V(x) = \rho + Sx^{\otimes 2}$, where ρ is a positive number and S is a definite and positive symmetric $d \times d$ real matrix, is an EQ-function.] For $p > 0$, one checks that $\|D^2(V^p)\| \le CV^{p-1}$, that V^p is a $2p$-Hölder function if $p \le 1/2$, and that $V^{p-1}\nabla V$ is a $(2p-1)$-Hölder function if $p \in (1/2, 1)$ (see Lemma 3). Hence, λ_p and c_p given by

$$\lambda_p := \frac{1}{2p} \sup_{x\in\mathbb{R}^d} \lambda_{V^{1-p}D^2(V^p)(x)} \quad \text{and}$$
$$c_p := \begin{cases} \left[\dfrac{V^p}{p}\right]_{2p}, & \text{if } p \in (0, 1/2], \\ [V^{p-1}\nabla V]_{2p-1}, & \text{if } p \in (1/2, 1] \end{cases} \tag{6}$$

are finite positive numbers.

1.7. *Organization of the paper.* The main results (Theorems 2 and 3) are stated in Section 2 and are proved in Sections 3, 4 and 5. First, we focus on the proof of these theorems for the exact Euler Scheme (A): in Section 3 we prove the almost sure tightness of $(\bar{\nu}_n)$ and in Section 4 we establish that every weak limiting distribution of $(\bar{\nu}_n)$ is invariant for the SDE (1). Second, in Section 5 we point out the main differences which arise in the proofs when considering the approximate Euler Schemes (B) and (C). In Section 6 we show that the almost sure central limit theorem for symmetric stable processes (see [3]) can be obtained as a consequence of our main theorems. Finally, in Section 7 we simulate the procedure on some concrete examples.

2. Main results. In Theorem 2 we obtain a result under simple conditions on the steps and on the weights. In Theorem 3 we show that, under more stringent conditions on the steps and on the weights, some assumptions on the coefficients of the SDE can be relaxed. Let us introduce the joint assumptions. First, we state some assumptions on the moments of the Lévy measure at $+\infty$ and 0:

$$(\mathrm{H}^1_\mathrm{p}): \int_{|y|>1} \pi(dy)|y|^{2p} < +\infty, \qquad (\mathrm{H}^2_\mathrm{q}): \int_{|y|\le 1} \pi(dy)|y|^{2q} < +\infty,$$

where p is a positive real number and $q \in [0, 1]$.

Assumption $(\mathrm{H}^1_\mathrm{p})$ is satisfied if, and only if, $\mathbb{E}|Z_t|^{2p} < +\infty$ for every $t \ge 0$ (see [1], Theorem 6.1). By the compensation formula (see [4]), $(\mathrm{H}^2_\mathrm{q})$ is satisfied if and only if $\mathbb{E}\{\sum_{0<s\le t} |\Delta Y_t|^{2q}\} < +\infty$, that is, if and only if (Y_t) has locally $2q$*-integrable variation*. We recall that $(\mathrm{H}^2_\mathrm{q})$ is always satisfied for $q = 1$ since $\int_{\{|y|\le 1\}} |y|^2\pi(dy) < \infty$ for any Lévy measure π.

Now, we introduce the Lyapunov assumption on the coefficients of the SDE and on π denoted by $(\mathrm{S_{a,p,q}})$. The parameter a specifies the intensity of the mean-reversion. We denote by $\tilde{b}$ the function defined by

$$\tilde{b} = \begin{cases} b, & \text{if } p \le 1/2 \le q, \\ b - \kappa \int_{\{|y|\le 1\}} y\pi(dy), & \text{if } p, q \le 1/2, \\ b + \kappa \int_{\{|y|>1\}} y\pi(dy), & \text{if } p > 1/2. \end{cases}$$

The function $\tilde{b}$ plays the role of the global drift of the dynamical system resulting from b and from the jump component (see Remark 4 for more precisions). Let $a \in (0, 1]$, $p > 0$ and $q \in [0, 1]$.

ASSUMPTION ($\mathrm{S_{a,p,q}}$). There exists an EQ-function V such that:

1. *Growth control*: $|b|^2 \le CV^a$,

$$\begin{cases} \mathrm{Tr}(\sigma\sigma^*) + \|\kappa\|^{2(p\vee q)} \le CV^{a+p-1}, & \text{if } p < 1, \\ \mathrm{Tr}(\sigma\sigma^*) + \|\kappa\|^2 \le CV^a, & \text{if } p \ge 1. \end{cases}$$

2. *Mean-reversion*: there exist $\beta \in \mathbb{R}$, $\alpha > 0$ such that, $\langle \nabla V, \tilde{b}\rangle + \phi_{p,q}(\sigma, \kappa, \pi, V) \le \beta - \alpha V^a$, where $\phi_{p,q}$ is given by

$$\phi_{p,q}(\sigma, \kappa, \pi, V) = \begin{cases} c_p m_{2p,\pi} \|\kappa\|^{2p} V^{1-p} 1_{q\le p}, & \text{if } p < 1, \\ \lambda_1\big(\mathrm{Tr}(\sigma\sigma^*) + m_{2,\pi}\|\kappa\|^2\big), & \text{if } p = 1, \\ d_p \lambda_p \Big(\mathrm{Tr}(\sigma\sigma^*) + m_{2,\pi}\|\kappa\|^2 + e_p m_{2p,\pi} \dfrac{\|\kappa\|^{2p}}{V^{p-1}}\Big), & \text{if } p > 1, \end{cases}$$

with $m_{r,\pi} = \int |y|^r \pi(dy)$, $d_p = 2^{(2(p-1)-1)_+}$, c_p and λ_p given by (6), and $e_p = [\sqrt{V}]_1^{2(p-1)}$.

Assumption ($\mathrm{S_{a,p,q}}$).2 can be viewed as a discretized version of "$AV^p \le \beta - \alpha V^{a+p-1}$," where A is the infinitesimal generator of (X_t) defined on a subset $\mathcal{D}(A)$ of $\mathcal{C}^2(\mathbb{R}^d)$ by

$$\begin{aligned} Af(x) &= \langle \nabla f, b\rangle(x) + \tfrac{1}{2}\mathrm{Tr}(\sigma^* D^2 f \sigma)(x) \\ &\quad + \int \big(f\big(x + \kappa(x)y\big) - f(x) - \langle \nabla f(x), \kappa(x)y\rangle 1_{\{|y|\le 1\}}\big)\pi(dy). \end{aligned} \tag{7}$$

Furthermore, one can check that if Assumption ($\mathrm{S_{a,p,q}}$) is fulfilled, then there exist $\bar{\beta} \in \mathbb{R}$ and $\bar{\alpha} > 0$ such that "$AV^p \le \bar{\beta} - \bar{\alpha}V^{a+p-1}$." This means that if V is the function whose existence is required in Assumption ($\mathrm{S_{a,p,q}}$), then V^p is a Lyapunov function for the stochastic process (X_t) and for the Euler scheme $(\bar{X}_n)$.

The left-hand side of ($\mathrm{S_{a,p,q}}$).2 is the sum of two antagonistic components: $\langle \nabla V, \tilde{b}\rangle$ produces the mean-reverting effect (see Example 1 for concrete cases), whereas the positive function $\phi_{p,q}$ expresses the noise induced by the Brownian and jump components. In particular, if the following tighter growth control condition holds,

$$|b|^2 \le C V^a, \qquad \begin{cases} \mathrm{Tr}(\sigma\sigma^*(x)) \overset{|x|\to+\infty}{=} o\big(V^{a-(1-p)_+}(x)\big), \\ \|\kappa(x)\|^2 \overset{|x|\to+\infty}{=} o(V^{\eta_{a,p,q}}(x)), \end{cases} \tag{8}$$

with $\eta_{a,p,q} = (p \vee q)^{-1}(a + p - 1)$ if $p \le 1$ and $\eta_{a,p,q} = a$ if $p > 1$, then the term $\phi_{p,q}$ becomes negligible and the mean-reversion assumption becomes

$$\langle \nabla V, \tilde{b} \rangle \le \beta - \alpha V^a.$$

REMARK 4. If we had chosen to compensate the jumps smaller than $h > 0$ rather than $h = 1$, the corresponding assumption would have been $(\mathrm{S}^{\mathrm{h}}_{\mathrm{a,p,q}})$, where $(\mathrm{S}^{\mathrm{h}}_{\mathrm{a,p,q}})$ is obtained from $(\mathrm{S}_{\mathrm{a,p,q}})$ by replacing b with b^h and $\tilde{b}$ with $\tilde{b^h}$ defined by

$$\tilde{b}^h = \begin{cases} b^h, & \text{if } p \le 1/2 \le q, \\ b^h - \kappa \displaystyle\int_{\{|y| \le h\}} y\pi(dy), & \text{if } p, q \le 1/2, \\ b^h + \kappa \displaystyle\int_{\{|y| > h\}} y\pi(dy), & \text{if } p > 1/2. \end{cases}$$

One can check that, for every $h > 0$, $(\mathrm{S}^{\mathrm{h}}_{\mathrm{a,p,q}}) \Longleftrightarrow (\mathrm{S}_{\mathrm{a,p,q}})$. This means that these assumptions do not depend on the choice of the truncation parameter h. Indeed, first, it is clear that $(\mathrm{S}^{\mathrm{h}}_{\mathrm{a,p,q}}).1 \Longleftrightarrow (\mathrm{S}_{\mathrm{a,p,q}}).1$. Second, when $p > 1/2$ or $p, q \le 1/2$, $(\mathrm{S}^{\mathrm{h}}_{\mathrm{a,p,q}}).2 \Longleftrightarrow (\mathrm{S}_{\mathrm{a,p,q}}).2$ because in these cases, $\tilde{b}^h = \tilde{b}$ for every $h > 0$. This can be explained by the existence of a formulation of the SDE that does not depend on h. Actually, when $p > 1/2$, we can rewrite the SDE (2) by replacing b^h with $\tilde{b}^h$ and, Z^h_t with $\hat{Z}^h_t = Z^h_t - t\int_{\{|y|>h\}} y\pi(dy)$, that is, we can compensate the big jumps. Since $(\hat{Z}^h_t) = (Z^\infty_t)$ for every $h > 0$, it follows that $\tilde{b}^h = \tilde{b}\,(= b^\infty)$ for every $h > 0$. There also exists an intrinsic formulation when $p, q \le 1/2$ because in this case, we can replace b^h with $\tilde{b}^h$ and Z^h_t with $\check{Z}^h_t = Z^h_t + t\int_{\{|y|\le h\}} y\pi(dy)$ (now, we do not compensate any jumps). Since $(\check{Z}^h_t) = (Z^0_t)$ for every $h > 0$, $\tilde{b}^h = \tilde{b}\,(= b^0)$. These formulations can be considered as the natural formulations of the dynamical system in these settings.

When $p \le 1/2 < q$, there is no intrinsic formulation of the SDE (even if π is symmetrical). Since b^h depends on h, it appears that the left-hand side of $(\mathrm{S}^{\mathrm{h}}_{\mathrm{a,p,q}}).2$ also depends on h. However, under the growth assumption on κ, one can check that $\langle \nabla V, b^h \rangle = \langle \nabla V, b \rangle + o(V^a)$ and it follows that the same conclusion still holds in this case.

We now state our first main result.

THEOREM 2. *Let $a \in (0,1]$, $p > 0$ and $q \in [0,1]$ such that* $(\mathrm{H}_{\mathrm{p}}^1)$, $(\mathrm{H}_{\mathrm{q}}^2)$ *and* $(\mathrm{S}_{\mathrm{a,p,q}})$ *are satisfied. Suppose that $\mathbb{E}\{|U_1|^{2(p\vee 1)}\} < +\infty$ and that the sequence $(\eta_n/\gamma_n)_{n\geq 1}$ is nonincreasing. Then*:

(1) *If $p/2+a-1>0$, the sequence $(\bar{\nu}_n)_{n\geq 1}$ is almost surely tight. Moreover, if $\kappa(x) \overset{|x|\to+\infty}{=} o(|x|)$ and $\mathrm{Tr}(\sigma\sigma^*) + \|\kappa\|^{2q} \leq CV^{p/2+a-1}$, then every weak limit of this sequence is an invariant probability for the SDE* (1). *In particular, if $(X_t)_{t\geq 0}$ admits a unique invariant probability ν, for every continuous function f such that $f = o(V^{p/2+a-1})$, $\lim_{n\to\infty} \bar{\nu}_n(f) = \nu(f)$ a.s.*

(2) *The same result holds for $(\bar{\nu}_n^B)_{n\geq 1}$.*

(3) *The same result holds for $(\bar{\nu}_n^C)_{n\geq 1}$ under the additional condition*

$$\pi(|y| > u_n)\gamma_n \overset{n\to+\infty}{\longrightarrow} 0. \tag{9}$$

We present below some examples which fulfill the conditions of Theorem 2. In the first we suppose that the dynamical system has a radial drift term and a noise generated by a centered jump Lévy process with a Lévy measure close to that of a symmetric stable process. In the second we suppose that the SDE is only driven by a jump Lévy process, but we suppose that it is not centered. This implies that even if the SDE has seemingly no drift term, a mean-reverting assumption can be still satisfied.

EXAMPLE 1. Let ϕ and ψ be positive, bounded and continuous functions on $\mathbb{R}^d$ such that

$$\phi(x) = \phi(-x) \qquad \forall x \in \mathbb{R}^d,$$
$$\underline{\phi} = \min_{\mathbb{R}^d} \phi(x) > 0 \quad \text{and} \quad \underline{\psi} = \inf_{\{|x|>1\}} \psi(x) > 0.$$

Consider (Z_t) defined as in the SDE (1) with Lévy measure π given by $\pi(dy) = \phi(y)/|y|^{d+r}\lambda_d(dy)$, where $r \in (0,2)$. When $\phi = C > 0$, the increments of (Z_t) can be exactly simulated because (Z_t) is a symmetric $\mathbb{R}^d$-stable process with order r. In the other cases, $\bar{Z}_n^B$ and $\bar{Z}_n^C$ can be simulated by the rejection method since the density of π is dominated by the density of a Pareto's law.

Let $\rho \in [0,2)$ and b be a continuous function defined by $b(x) = -\psi(x)x/|x|^\rho$. We consider (X_t) solution to

$$dX_t = b(X_{t^-})\,dt + \kappa(X_{t^-})\,dZ_t, \tag{10}$$

where κ is a continuous function such that $\|\kappa(x)\|^2 \leq C(1+|x|^2)^\epsilon$ with $\epsilon \leq 1$. A natural candidate for the function V is $V(x) = 1+|x|^2$. Indeed, since $\tilde{b} = b$ [because $\phi(y) = \phi(-y)$], one checks that there exists $\beta \in \mathbb{R}$ such that

$$\langle \nabla V(x), \tilde{b}(x)\rangle = -2\psi(x)|x|^{2-\rho} \leq \beta - \underline{\psi}V(x)^{1-\rho/2}.$$

We set $a := 1 - \rho/2$ and

$$\Delta(r) := \{(p,q) \in (0,+\infty) \times [0,1], \text{(H}^1_\text{p}\text{) and (H}^2_\text{q}\text{) hold}\}.$$

We have $\Delta(r) = (0, r/2) \times (r/2, 1)$. By (8), for every $(p,q) \in \Delta(r)$, $(\mathrm{S}_{\mathrm{a,p,q}})$ is satisfied if $\epsilon < (p+a-1)/q = (p-\rho/2)/q$. Hence, $(\bar{\nu}_n)$ is tight if there exists $(p,q) \in \Delta(r)$ such that $p/2 + a - 1 > 0$, that is, if $p > \rho$, and $(2p-\rho)/(2q) > \epsilon$. If, moreover, $\|\kappa\|^{2q} \le C(1+|x|^2)^{p/2+a-1}$, that is, if $(p-\rho)/(2q) \le \epsilon$, every weak limit ν is invariant for the SDE (10).

It follows that if the invariant distribution ν is unique, $\bar{\nu}_n \overset{\mathcal{L}}{\Longrightarrow} \nu$ a.s. as soon as $2\rho < r$ and $\epsilon < \sup\{(p-\rho)/(2q), (p,q) \in \Delta(r)\} = 1/2 - \rho/r$. Furthermore, $\bar{\nu}_n(f) \to \nu(f)$ a.s. for every continuous function f satisfying $f(x) \le C(1+|x|)^\theta$ with $\theta \in [0, (r/2-\rho)/2)$.

EXAMPLE 2. Let π be a Lévy measure on $\mathbb{R}$ such that $\int_{|y|\le 1} |y|\pi(dy) < +\infty$, $\int_{|y|>1} |y|^{2p}\pi(dy) < +\infty$ with $p \ge 2$ and $\int y\pi(dy) > 0$. Let (Z_t^0) be a real Lévy process with characteristic function given for every $t \ge 0$ by

$$\mathbb{E}\{e^{i\langle u, Z_t^0\rangle}\} = \exp\left[t\left(\int (e^{i\langle u,y\rangle} - 1)\pi(dy)\right)\right].$$

For instance, (Z_t^0) can be a subordinator with no drift term. We assume that $\kappa(x) = -\psi(x)x/|x|^\rho$ with $\rho \in [0,2)$ and ψ defined as in the preceding example. We then consider the SDE:

$$dX_t = \kappa(X_{t^-})\,dZ_t^0 = b(X_{t^-})\,dt + \kappa(X_{t^-})\,dZ_t,$$

with $b(x) = \kappa(x)\int_{\{|y|\le 1\}} y\pi(dy)$ and $Z_t = Z_t^0 - t\int_{\{|y|\le 1\}} y\pi(dy)$. Since $p > 1/2$, $\tilde{b}(x) = b(x) + \kappa(x)\int_{\{|y|>1\}} y\pi(dy) = \kappa(x)\int y\pi(dy)$. Setting $V(x) = 1 + x^2$, one checks that there exists $\beta \in \mathbb{R}$ such that

$$V'(x)\tilde{b}(x) = -2\psi(x)\int y\pi(dy)|x|^{2-\rho} \le \beta - \underline{\psi}\int y\pi(dy)V(x)^{1-\rho/2}.$$

We set $a = 1 - \rho/2$. Let $p \ge 2$ and $q \le 1/2$ such that $(\mathrm{H}^1_\mathrm{p})$ and $(\mathrm{H}^2_\mathrm{q})$ hold. First, checking that as soon as $\rho > 0$, $\|\kappa(x)\|^2 = o(1+|x|^2)^a$ when $|x| \to +\infty$, we derive from (8) that $(\mathrm{S}_{\mathrm{a,p,q}})$ is satisfied as soon as $\rho \in (0,2)$. Second, for every $p \ge 2$, $q \le 1$ and $a \in (0,1)$, one can check that $p/2 + a - 1 > 0$ and $\|\kappa(x)\|^{2q} \le C(1+|x|^2)^{p/2+a-1}$. Hence, Theorem 2 applies for every $\rho \in (0,2)$.

The interest of Theorem 2 lies in the facility with which it can be put to use in concrete situations. For instance, in Scheme (A), we only have to take a sequence $(\gamma_n)_{n\ge 1}$ decreasing to 0, with infinite sum and $\eta_n = \gamma_n$. The next theorem (Theorem 3) requires tougher conditions on the sequences (γ_n) and (η_n), but it can be applied to SDEs where the coefficients do not necessarily verify all conditions of Theorem 2. It broadens the class of SDEs for which we can find an efficient procedure for the approximation of the invariant measure.

THEOREM 3. *Let* $a \in (0,1]$, $p > 0$ *and* $q \in [0,1]$ *such that* $(\mathrm{H}^1_{\mathrm{p}})$, $(\mathrm{H}^2_{\mathrm{q}})$ *and* $(\mathrm{S}_{\mathrm{a,p,q}})$ *are satisfied. Suppose that* $\mathbb{E}\{|U_1|^{2(p\vee 1)}\} < +\infty$. *Then*:

(1) *Let* $s \in (1,2]$ *satisfying the following additional conditions when* $p > 1/2$:

$$\begin{cases} s > \dfrac{2p}{2p + (a-1)(2p-1)/p}, & \text{if } \dfrac{1}{2} < p \le 1, \\ s > \dfrac{2p}{2p + a - 1}, & \text{if } p \ge 1. \end{cases} \tag{11}$$

If $p/s + a - 1 > 0$, *there exist some sequences* $(\gamma_n)_{n\ge 1}$ *and* $(\eta_n)_{n\ge 1}$ *such that* $(\bar{\nu}_n)_{n\ge 1}$ *is almost surely tight. Moreover, if* $\kappa(x) \overset{|x|\to+\infty}{=} o(|x|)$ *and* $\mathrm{Tr}(\sigma\sigma^*) + \|\kappa\|^{2q} \le CV^{p/s+a-1}$, *then every weak limit of this sequence is an invariant probability for the SDE* (1). *In particular, if* $(X_t)_{t\ge 0}$ *admits a unique invariant probability* ν, *for every continuous function* f *such that* $f = o(V^{p/s+a-1})$, $\lim_{n\to\infty} \bar{\nu}_n(f) = \nu(f)$ *a.s.*

(2) *The same result holds for* $(\bar{\nu}^B_n)_{n\ge 1}$.

(3) *The same result holds for* $(\bar{\nu}^C_n)_{n\ge 1}$ *under the additional condition* (9).

REMARK 5. The sequences $(\eta_n)_{n\ge 1}$ and $(\gamma_n)_{n\ge 1}$ must verify an explicit condition given in Proposition 1 below (see Remark 6 for a version adapted to polynomial steps and weights).

In the following example, we consider the same class of SDEs as in Example 1 in the nonintegrable case (i.e., $r \le 1$). One can observe that the mean-reversion condition and the growth condition on κ and on the functions f whose the procedure converges can be relaxed. We also give some explicit polynomial weights and steps for which Theorem 3 applies in this case.

EXAMPLE 3. Let $\rho \in [0,2)$ and $r \in (0,1]$, let b and κ be continuous functions defined as in Example 1. Consider (X_t) solution to the SDE (10) and assume that the invariant measure ν is unique. For $s \in (1,2]$, denote by $(\gamma_{n,s})$ and $(\eta_{n,s})$ some sequences of steps and weights satisfying $\gamma_{n,s} = Cn^{-r_1}$, $\eta_{n,s} = Cn^{-r_2}$ with $r_1 \le r_2$ and

$$0 < r_1 < 2\left(1 - \frac{1}{s}\right) \quad \text{and} \quad r_2 < 1 \quad \text{or} \quad 0 < r_1 \le 2\left(1 - \frac{1}{s}\right) \quad \text{and} \quad r_2 = 1.$$

Then, for these choices of steps and weights, $\bar{\nu}_n \overset{\mathcal{L}}{\Longrightarrow} \nu$ a.s. as soon as $s\rho < r$ and $\epsilon \in [0, 1/s - \rho/r)$ (this improves the condition: $2\rho < r$ and $\epsilon \in [0, 1/2 - \rho/r)$ of Example 1). Furthermore, $\bar{\nu}_n(f) \to \nu(f)$ a.s. for every continuous function f satisfying $|f(x)| \le C(1 + |x|)^\theta$ with $\theta \in [0, (r/s - \rho)/2)$ (this improves the condition: $\theta \in [0, (r/2 - \rho)/2)$ of Example 1).

3. Almost sure tightness of $(\bar{\nu}_n(w,dx))_{n\in\mathbb{N}}$. The main result of this section is Proposition 1. We need to introduce the function $f_{a,p}$ defined for all $s\in(1,2]$ by

$$f_{a,p}(s)=\begin{cases} s, & \text{if } s\geq 2p,\\ \dfrac{p+a-1}{p/s+(a-1)/(2(p\wedge 1))}\wedge s, & \text{if } s<2p.\end{cases}\tag{12}$$

Assume that $p+a-1>0$. Then, $s\mapsto f_{a,p}(s)$ is a nondecreasing function which satisfies $f_{1,p}(s)=s$ for all $p>0$ and $f_{a,p}(2)=2$. Note that $f_{a,p}(s)>1$ if and only if s satisfies assumption (11).

PROPOSITION 1. *Let $a\in(0,1]$, $p>0$ and $q\in(0,1]$ such that $(\mathrm{H}^1_{\mathrm{p}})$, $(\mathrm{H}^2_{\mathrm{q}})$ and $(\mathrm{S}_{\mathrm{a,p,q}})$ are satisfied. Assume that $\mathbb{E}\{|U_1|^{2(p\vee 1)}\}<+\infty$ and $(\eta_n/\gamma_n)_{n\geq 1}$ is nonincreasing.*

(1) *Then,*

$$\sup_{n\geq 1}\bar{\nu}_n(V^{p/2+a-1})<+\infty \qquad a.s.$$

Consequently, if $\frac{p}{2}+a-1>0$, the sequence $(\bar{\nu}_n)_{n\in\mathbb{N}}$ is a.s. tight.

(2) *Let $s\in(1,2)$ such that assumption (11) is satisfied. Assume that $(\eta_n)_{n\geq 1}$ and $(\gamma_n)_{n\geq 1}$ are such that*

$$\left(\frac{1}{\gamma_n}\left(\frac{\eta_n}{H_n\sqrt{\gamma_n}}\right)^{f_{a,p}(s)}\right) \text{ is nonincreasing and } \sum_{n\geq 1}\left(\frac{\eta_n}{H_n\sqrt{\gamma_n}}\right)^{f_{a,p}(s)}<+\infty.\tag{13}$$

Then, $\sup_{n\geq 1}\bar{\nu}_n(V^{p/s+a-1})<+\infty$ a.s. and the sequence $(\bar{\nu}_n)_{n\in\mathbb{N}}$ is a.s. tight as soon as $p/s+a-1>0$.

REMARK 6. If $\gamma_n=Cn^{-r_1}$ and $\eta_n=Cn^{-r_2}$ with $r_1\leq r_2$, then assumption (13) reads

$$\begin{aligned} &r_2<1 \quad\text{and}\quad 0<r_1<\bar{r}_1:=2\left(1-\frac{1}{f_{a,p}(s)}\right)\quad\text{or}\\ &r_2=1 \quad\text{and}\quad 0<r_1\leq\bar{r}_1.\end{aligned}\tag{14}$$

The proof of Proposition 1 is organized as follows: first, in Section 3.1 (see Proposition 2) we establish a fundamental recursive control of the sequence $(V^p(\bar{X}_n))$: we show that $(\mathrm{R}_{\mathrm{a,p}})$: There exist $n_0\in\mathbb{N}$, $\alpha'>0$, $\beta'>0$ such that $\forall n\geq n_0$,

$$E\{V^p(\bar{X}_{n+1})|\mathcal{F}_n\}\leq V^p(\bar{X}_n)+\gamma_{n+1}V^{p-1}(\bar{X}_n)\big(\beta'-\alpha' V^a(\bar{X}_n)\big).\tag{15}$$

For this step, we rely on Lemma 2 that provides a control of the moments of the increments of the jump component in terms of p and q.

Second, in Section 3.3 we make use of martingale techniques in order to derive some consequences from $(\mathrm{R}_{\mathrm{a,p}})$. In Lemma 5 we establish a L^p-control of the Euler scheme with arguments close to [15]. This control is fundamental for the proof of Corollary 1 where we show the following property:

$(\mathrm{C}_{\mathrm{p,s}})$: There exist $\rho \in (1,2]$ and a sequence (π_n) of $\mathcal{F}_n$-measurable random variables such that

$$\sum_{n\geq 1}\left(\frac{\eta_n}{H_n\gamma_n}\right)^{\rho}\mathbb{E}\{|V^{p/s}(\bar{X}_n)-\pi_{n-1}|^{\rho}\}<+\infty. \tag{16}$$

This step is used to obtain a L^ρ-martingale control (see proof of Lemma 1). We will see in the proof of Corollary 1 that the choice of the sequence (π_n) depends on p and q. In particular, even if q does not appear in $(\mathrm{C}_{\mathrm{p,s}})$, this assumption indirectly depends on this parameter. The same remark holds for $(\mathrm{R}_{\mathrm{a,p}})$. In the following lemma, we show that these two steps are all what we have to show for the proof of Proposition 1.

LEMMA 1. *Let $p>0$, $a\in(0,1]$ and $s\in(1,2]$ such that $(\mathrm{H}^1_{\mathrm{p}})$, $(\mathrm{R}_{\mathrm{a,p}})$ and $(\mathrm{C}_{\mathrm{p,s}})$ are fulfilled. Assume that $\mathbb{E}\{|U_1|^{2(p\vee 1)}\}<+\infty$ and that (η_n/γ_n) is nonincreasing. Then,*

$$\sup_{n\geq 1}\bar{\nu}_n(V^{p/s+a-1})<+\infty \qquad a.s. \tag{17}$$

and the sequence $(\bar{\nu}_n)_{n\geq 1}$ is a.s. tight as soon as $p/s+a-1>0$.

PROOF. By a convexity argument (see Lemma 3 of [15]), one shows that $(\mathrm{R}_{\mathrm{a,p}}) \Longrightarrow (\mathrm{R}_{\mathrm{a},\bar{\mathrm{p}}})$ for all $\bar{p}\in(0,p]$. Hence, for all $s\in(1,2]$, there exists $n_0\in\mathbb{N}$, $\hat{\alpha}>0$ and $\hat{\beta}>0$ such that $\forall k\geq n_0$,

$$\mathbb{E}\{V^{p/s}(\bar{X}_k)|\mathcal{F}_{k-1}\}\leq V^{p/s}(\bar{X}_{k-1})+\gamma_k V^{p/s-1}(\bar{X}_{k-1})\big(\hat{\beta}-\hat{\alpha}V^a(\bar{X}_{k-1})\big). \tag{18}$$

For $R>0$, set $\varepsilon(R)=\sup_{\{|x|>R\}}V^{-a}(x)$ and $M(R)=\sup_{\{|x|\leq R\}}V^{p/s-1}(x)$. We have

$$V^{p/s-1}(x)\leq\varepsilon(R)V^{p/s+a-1}(x)+M(R). \tag{19}$$

Since $V(x)\to+\infty$ when $|x|\to+\infty$ (resp. since V is bounded on every compact set), $\varepsilon(R)\to 0$ when $R\to+\infty$ [resp. $M(R)$ is finite for every $R>0$]. Hence, for every $\varepsilon>0$, there exists $M_\varepsilon>0$ such that $V^{p/s-1}\leq\varepsilon V^{p/s+a-1}+M_\varepsilon$. By setting $\varepsilon=\hat{\alpha}/(2\hat{\beta})$, $\tilde{\alpha}=\hat{\alpha}/2$ and $\tilde{\beta}=\hat{\beta}M_\varepsilon$, we deduce that $V^{p/s-1}(\hat{\beta}-\hat{\alpha}V^a)\leq\tilde{\beta}-\tilde{\alpha}V^{p/s+a-1}$. Hence, we derive from (18) that

$$V^{p/s+a-1}(\bar{X}_{k-1})\leq\frac{V^{p/s}(\bar{X}_{k-1})-\mathbb{E}\{V^{p/s}(\bar{X}_k)|\mathcal{F}_{k-1}\}}{\tilde{\alpha}\gamma_k}+\frac{\tilde{\beta}}{\tilde{\alpha}}\qquad\forall k\geq n_0.$$

It follows that (17) holds if

$$\sup_{n\geq n_0+1}\left(\frac{1}{H_n}\sum_{k=n_0+1}^{n}\frac{\eta_k}{\gamma_k}\big(V^{p/s}(\bar{X}_{k-1})-\mathbb{E}\{V^{p/s}(\bar{X}_k)|\mathcal{F}_{k-1}\}\big)\right) < +\infty \qquad \text{a.s.} \tag{20}$$

We then prove (20). We decompose the above sum as follows:

$$\begin{aligned}
&\frac{1}{H_n}\sum_{k=n_0+1}^{n}\frac{\eta_k}{\gamma_k}\big(V^{p/s}(\bar{X}_{k-1})-\mathbb{E}\{V^{p/s}(\bar{X}_k)|\mathcal{F}_{k-1}\}\big)\\
&\quad=-\frac{1}{H_n}\sum_{k=n_0+1}^{n}\frac{\eta_k}{\gamma_k}\Delta V^{p/s}(\bar{X}_k)\\
&\qquad+\frac{1}{H_n}\sum_{k=n_0+1}^{n}\frac{\eta_k}{\gamma_k}\big(V^{p/s}(\bar{X}_k)-\mathbb{E}\{V^{p/s}(\bar{X}_k)|\mathcal{F}_{k-1}\}\big),
\end{aligned}$$

where $\Delta V^{p/s}(\bar{X}_k)=V^{p/s}(\bar{X}_k)-V^{p/s}(\bar{X}_{k-1})$. First, an Abel's transform yields

$$\begin{aligned}
-\frac{1}{H_n}\sum_{k=n_0+1}^{n}\frac{\eta_k}{\gamma_k}\Delta V^{p/s}(\bar{X}_k)&=\frac{1}{H_n}\left(\frac{\eta_{n_0}}{\gamma_{n_0}}V^{p/s}(\bar{X}_{n_0})-\frac{\eta_n}{\gamma_n}V^{p/s}(\bar{X}_n)\right)\\
&\quad+\frac{1}{H_n}\left(\sum_{k=n_0+1}^{n}\left(\frac{\eta_k}{\gamma_k}-\frac{\eta_{k-1}}{\gamma_{k-1}}\right)V^{p/s}(\bar{X}_{k-1})\right)\\
&\leq\frac{\eta_{n_0}}{H_n\gamma_{n_0}}V^{p/s}(\bar{X}_{n_0}),
\end{aligned}$$

where we used in the last inequality that (η_n/γ_n) is nonincreasing. Hence, since $H_n \overset{n\to+\infty}{\longrightarrow} +\infty$ and $\frac{\eta_{n_0}}{H_n\gamma_{n_0}}V^{p/s}(\bar{X}_{n_0}) \overset{n\to+\infty}{\longrightarrow} 0$ a.s.,

$$\sup_{n\geq n_0}\left(-\frac{1}{H_n}\sum_{k=n_0+1}^{n}\frac{\eta_k}{\gamma_k}\Delta V^{p/s}(\bar{X}_k)\right) < +\infty \qquad \text{a.s.} \tag{21}$$

Second, one denotes by $(M_n)_{n\in\mathbb{N}}$ the martingale defined by

$$M_n=\sum_{k=1}^{n}\frac{\eta_k}{H_k\gamma_k}\big(V^{p/s}(\bar{X}_k)-\mathbb{E}\{V^{p/s}(\bar{X}_k)|\mathcal{F}_{k-1}\}\big). \tag{22}$$

Let $\rho\in(1,2]$ and (π_k) be a sequence of $\mathcal{F}_k$-measurable random variables such that (16) holds. We derive from the elementary inequality $|u+v|^\rho\leq 2^{\rho-1}(|u|^\rho+$

$|v|^\rho$) that

$$\begin{aligned}\mathbb{E}\{\big|V^{p/s}(\bar{X}_k)-\mathbb{E}\{V^{p/s}(\bar{X}_k)|\mathcal{F}_{k-1}\}\big|^\rho\}&\\ \le C\mathbb{E}\{|V^{p/s}(\bar{X}_k)-\pi_{k-1}|^\rho\}+C\mathbb{E}\{\big|\mathbb{E}\{(\pi_{k-1}-V^{p/s}(\bar{X}_k))|\mathcal{F}_{k-1}\}\big|^\rho\}&\\ \le C\mathbb{E}\{|V^{p/s}(\bar{X}_k)-\pi_{k-1}|^\rho\},&\end{aligned}$$

thanks to the Jensen inequality. Hence, $(\mathrm{C_{p,s}})$ yields $\sum_{k\ge1}\mathbb{E}\{|\Delta M_k|^\rho\}<+\infty$ a.s. Since $\rho>1$, it follows from Chow's theorem (see [11]) that $M_n \overset{n\to\infty}{\longrightarrow} M_\infty$ a.s. where M_∞ is finite a.s. Then, Kronecker's lemma yields

$$\frac{1}{H_n}\sum_{k=n_0+1}^{n}\frac{\eta_k}{\gamma_k}\big(V^{p/s}(\bar{X}_k)-\mathbb{E}\{V^{p/s}(\bar{X}_k)|\mathcal{F}_{k-1}\}\big)\overset{n\to\infty}{\longrightarrow}0 \qquad \text{a.s.} \tag{23}$$

Hence, (20) follows from (21) and (23). Finally, since $\lim_{|x|\to+\infty}V^{p/s+a-1}(x)=+\infty$ when $p/s+a-1>0$, we derive from a classical tightness criteria (see, e.g., [9], page 41) that $(\bar{\nu}_n)_{n\ge1}$ is a.s. tight as soon as $p/s+a-1>0$. □

3.1. *A recursive stability relation.*

PROPOSITION 2. *Let $p>0$, $q\in[0,1]$ and $a\in(0,1]$. Assume* $(\mathrm{H^1_p})$, $(\mathrm{H^2_q})$ *and* $(\mathrm{S_{a,p,q}})$. *If, moreover,* $\mathbb{E}\{|U_1|^{2(p\vee1)}\}<+\infty$, *then* $(\mathrm{R_{a,p}})$ *holds.*

The idea of the proof of Proposition 2 is to obtain an inequality of the following type:

$$\mathbb{E}\{V^p(\bar{X}_{n+1})-V^p(\bar{X}_n)|\mathcal{F}_n\}\le\gamma_{n+1}pV^{p-1}(\bar{X}_n)\Phi(\bar{X}_n)+R_n,$$

where $\Phi=\langle\nabla V,\tilde{b}\rangle+\phi_{p,q}(\sigma,\kappa,\pi,V)$ [see $(\mathrm{S_{a,p,q}})$.2] and R_n is asymptotically negligible in a sense made clear in the proof. To this end, we begin by three lemmas. In Lemma 2 we study the behavior of the moments of (Z_t) near 0. Then, in Lemma 3, we state some properties of the derivatives of V^p in terms of p and in the last one (Lemma 4) we control the contribution of the jump component on the conditional expectation (conditioned by $\mathcal{F}_n$) of the increment $V^p(\bar{X}_{n+1})-V^p(\bar{X}_n)$.

LEMMA 2. (i) *Let $p>0$ such that* $(\mathrm{H^1_p})$ *holds. Then, for every $h>0$, there exists a locally bounded function ψ_h such that*

$$\forall t\ge0 \qquad \mathbb{E}\{|N_t^h|^{2p}\}=\int_{|y|>h}|y|^{2p}\pi(dy)\big(t+\psi_h(t)t^2\big). \tag{24}$$

(ii) *Let $q\in[0,1]$ such that* $(\mathrm{H^2_q})$ *holds. Then, for every $h>0$,*

$$\begin{cases}\mathbb{E}\left\{\left|Y_t^h+t\displaystyle\int_{|y|\le h}y\pi(dy)\right|^{2q}\right\}\le t\displaystyle\int_{|y|\le h}|y|^{2q}\pi(dy), & \textit{if } q\le1/2,\\ \mathbb{E}\{|Y_t^h|^{2q}\}\le C_q t\displaystyle\int_{|y|\le h}|y|^{2q}\pi(dy), & \textit{if } q\in(1/2,1].\end{cases}$$

(iii) *Let* $p \in [1, +\infty)$ *such that* $(\mathrm{H}^1_{\mathrm{p}})$ *holds. Then, there exists* $\eta > 1$ *such that, for every* $T > 0$, *for every* $\varepsilon > 0$, *there exists* $C_{\varepsilon,T,p} > 0$ *such that,*

$$\forall t \in [0, T] \qquad \mathbb{E}\{|\hat{Z}_t|^{2p}\} \le t\left(\int |y|^{2p}\pi(dy) + \varepsilon\right) + C_{\varepsilon,T,p}t^{\eta},$$

where $(\hat{Z}_t)$ *is the compensated jumps process defined by* $\hat{Z}_t = Z_t - t\int_{|y|>1} y\pi(dy)$. *In particular,* $\mathbb{E}|\hat{Z}_t|^2 = t\int |y|^2\pi(dy)$.

REMARK 7. In this lemma we obtain, in particular, a control of the expansion of $t \mapsto \mathbb{E}\{|D_t|^r\}$ in the neighborhood of 0 (where D denotes one of the above jump components and r, a positive number). We have the following type of inequality: $\mathbb{E}\{|D_t|^r\} \le c_r t + O(t^{\eta})$, where c_r is a nonnegative real constant and $\eta > 1$. In the first and in the last inequality, we minimize this value because it has a direct impact on the coefficients of the function $\phi_{p,q}$ and then, on the mean-reverting assumption (see Lemma 4 for details). Note that we cannot have $c_r = 0$ in the inequalities of Lemma 2. Indeed, according to the Kolmogorov criterion, a Lévy process D that satisfies $\mathbb{E}\{|D_t|^r\} \le Ct^{\eta}$ in the neighborhood of 0 is pathwise continuous [for the Brownian motion, $c_r = 0$ as soon as $r > 2$ since $\mathbb{E}\{|W_t|^r\} = o(t^{r/2})$]. When $p > 1$, this feature generates a specific contribution of the jump component on the mean-reverting assumption $((\mathrm{S}_{\mathrm{a,p,q}}).2)$. This contribution appears in $\phi_{p,q}$ where there is an additional term of order $2p$ coming only from the jump component.

PROOF OF LEMMA 2. (i) $(N^h_t)_{t\ge 0}$ is a compound Poisson process with parameters $\lambda_h = \pi(|y| > h)$ and $\mu^h(dy) = 1_{\{|y|>h\}}\pi(dy)/\pi(|y| > h)$. Hence, (N^h_t) can be written as follows: $N^h_t = \sum_{n\ge 1} R_n 1_{T_n \le t}$, where $(R_n)_{n\ge 1}$ is a sequence of i.i.d. r.v. with law μ^h and $(T_n)_{n\in\mathbb{N}}$ is the sequence of the jump times of a Poisson process with parameter λ_h independent of $(R_n)_{n\ge 1}$. We have

$$\mathbb{E}\{|N^h_t|^{2p}\} = \sum_{n\ge 1}\mathbb{E}\left\{\left|\sum_{i=1}^n R_i\right|^{2p}\right\}e^{-\lambda_h t}\frac{(\lambda_h t)^n}{n!} = \lambda_h t\mathbb{E}\{|R_1|^{2p}\}F_{\lambda_h}(t)$$

$$\text{where } F_{\lambda}(t) = e^{-\lambda t}\sum_{n\ge 0}\frac{\mathbb{E}\{|\sum_{i=1}^{n+1} R_i|^{2p}\}}{\mathbb{E}\{|R_1|^{2p}\}}\frac{(\lambda t)^n}{(n+1)!}.$$

By the elementary inequality (this inequality will be usually needed in the sequel for the control of the moments of some sums of jumps)

$$\forall a_1, \dots, a_n \in \mathbb{R}^l, \forall \alpha > 0 \qquad \left|\sum_{i=1}^n a_i\right|^{\alpha} \le n^{(\alpha-1)_+}\sum_{i=1}^n |a_i|^{\alpha}, \tag{25}$$

used with $\alpha = 2p$, we obtain

$$\frac{\mathbb{E}\{|\sum_{i=1}^{n+1} R_i|^{2p}\}}{(n+1)!\mathbb{E}\{|R_1|^{2p}\}} \le \frac{\mathbb{E}\{(n+1)^{(2p-1)_+}\sum_{i=1}^{n+1}|R_i|^{2p}\}}{(n+1)!\mathbb{E}\{|R_1|^{2p}\}} = \frac{(n+1)^{(2p-1)_+}}{n!}.$$

It follows that F_{λ_h} is an analytic function on $\mathbb{R}$ such that $F_{\lambda_h}(0)=1$. Therefore,

$$F_{\lambda_h}(t)=1+t\psi_h(t)\quad \text{with } |\psi_h(t)|\le C(p,h,\lambda_h)\quad \forall t\in[0,T].$$

Since $\mathbb{E}\{|R_1|^{2p}\}=\frac{1}{\lambda_h}\int_{\{|y|>h\}}|y|^{2p}\pi(dy)$, the first equality is obvious.

(ii) If $\int_{|y|\le h}|y|^{2q}\pi(dy)<+\infty$ with $q\le 1/2$, then Y^h has locally bounded variations and $Y_t^h+t\int_{|y|\le h}y\pi(dy)=\sum_{0<s\le t}\Delta Y_t^h$. Inequality (25) with $\alpha=2q$ and the compensation formula yield

$$\begin{aligned}\mathbb{E}\left\{\left|Y_t^h+t\int_{|y|\le h}y\pi(dy)\right|^{2q}\right\}&\le \mathbb{E}\left\{\sum_{0<s\le t}|\Delta Y_t^h|^{2q}\right\}\\ &=t\int_{y\le h}|y|^{2q}\pi(dy).\end{aligned}$$

Now, let $q\in(1/2,1]$. As Y^h is a martingale, we derive from the Burkholder–Davis–Gundy (BDG) inequality (see [4]) that

$$\mathbb{E}\{|Y_t^h|^{2q}\}\le C_q\mathbb{E}\left\{\left(\sum_{0<s\le t}|\Delta Y_s^h|^2\right)^q\right\}.$$

The second inequality follows from inequality (25) with $\alpha=q$ and from the compensation formula.

(iii) One first considers case $p=1$. The process (M_t) defined by $M_t=|\hat{Z}_t|^2-t\int|y|^2\pi(dy)$ is a martingale. Then, in particular, $\mathbb{E}\{|\hat{Z}_t|^2\}=t\int|y|^2\pi(dy)$. Suppose now that $p>1$. In order to simplify the notation, we assume that $T<1$. The BDG inequality yields

$$\mathbb{E}\{|Y_t^h|^{2p}\}\le C_p\mathbb{E}\left\{\left(\sum_{0<s\le t}|\Delta Y_s^h|^2\right)^p\right\}. \tag{26}$$

For every integer $k\ge 1$, $M_{t,k}:=\sum_{0<s\le t}|\Delta Y_s^h)|^{2^k}-t\int_{\{|y|\le h\}}|y|^{2^k}\pi(dy)$ is a martingale. By inequality (25) and the BDG inequality applied to $(M_{t,k})$, we obtain

$$\begin{aligned}&\mathbb{E}\left\{\left(\sum_{0<s\le t}|\Delta Y_s^h|^{2^k}\right)^{p/2^{k-1}}\right\}\\ &\quad\le C\left(\mathbb{E}\{|M_{t,k}|^{p/2^{k-1}}\}+\left(t\int_{\{|y|\le h\}}|y|^{2^k}\pi(dy)\right)^{p/2^{k-1}}\right)\\ &\quad\le C\left(\mathbb{E}\left\{\left(\sum_{0<s\le t}|\Delta Y_s^h|^{2^{k+1}}\right)^{p/2^k}\right\}+\left(t\int_{\{|y|\le h\}}|y|^{2^k}\pi(dy)\right)^{p/2^{k-1}}\right).\end{aligned}$$

Set $k_0 = \inf\{k \geq 1, 2^k \geq p\}$. Iterating the preceding relation yields

$$\mathbb{E}\left\{\left(\sum_{0<s\leq t} |\Delta Y_s^h|^2\right)^p\right\} \leq C\mathbb{E}\left\{\left(\sum_{0<s\leq t} |\Delta Y_s^h|^{2^{k_0+1}}\right)^{p/2^{k_0}}\right\} + C\sum_{k=1}^{k_0}\left(t\int_{\{|y|\leq h\}} |y|^{2^k}\pi(dy)\right)^{p/2^{k-1}}.$$

By construction, $p/2^{k_0} \leq 1$. We then derive from inequality (25) with $\alpha = p/2^{k_0}$, from the compensation formula and from (26) that

$$\begin{aligned}\mathbb{E}\{|Y_t^h|^{2p}\} &\leq C_p\mathbb{E}\left\{\left(\sum_{0<s\leq t} |\Delta Y_s^h|^2\right)^p\right\}\\ &\leq C_p t\int_{\{|y|\leq h\}} |y|^{2p}\pi(dy) + C_{p,h}t^{\eta_1}\end{aligned} \tag{27}$$

with $\eta_1 = p/2^{k_0-1} > 1$. We now consider $(\hat{Z}_t)$. For every $h \in (0, +\infty)$, we have $\hat{Z}_t = Y_t^h + \hat{N}_t^h$, where $\hat{N}_t^h = N_t^h - t\int_{|y|>h} y\pi(dy)$. Using the elementary inequality,

$$\forall u, v \in \mathbb{R}_+, \forall \alpha \geq 1 \qquad (u+v)^\alpha \leq u^\alpha + \alpha 2^{\alpha-1}(u^{\alpha-1}v + v^\alpha), \tag{28}$$

we derive from (24) that

$$\forall \alpha > 1 \qquad \mathbb{E}\{|\hat{N}_t^h|^\alpha\} \leq t\int_{|y|>h} |y|^\alpha\pi(dy) + C_{\alpha,h}t^{\alpha\wedge 2}. \tag{29}$$

Using (28) and the independence between $(\hat{N}_t^h)$ and (Y_t^h) also yields

$$\mathbb{E}\{|\hat{Z}_t|^{2p}\} \leq \mathbb{E}\{|\hat{N}_t^h|^{2p}\} + C(\mathbb{E}\{|\hat{N}_t^h|\}^{2p-1}\mathbb{E}\{|Y_t^h|\} + \mathbb{E}\{|Y_t^h|^{2p}\}).$$

Since $\mathbb{E}\{|Y_t^h|^2\} = t\int_{\{|y|\leq h\}} |y|^2\pi(dy)$, we derive from the Jensen inequality that $\mathbb{E}\{|Y_t^h|\} \leq C_h\sqrt{t}$. Hence, by (27) and (29), it follows that, for every $h > 0$ and $t \leq T$,

$$\mathbb{E}\{|\hat{Z}_t|^{2p}\} \leq t\int_{\{|y|>h\}} |y|^{2p}\pi(dy) + C^1_{p,h}t^{3/2\wedge\eta_1} + C^2_p t\int_{\{|y|\leq h\}} |y|^{2p}\pi(dy)$$

with $\eta_1 > 1$, $C^1_{p,h} > 0$ and $C^2_p > 0$. Let ε be a positive number. As C^2_p does not depend on h, and $\int_{|y|\leq h} |y|^{2p}\pi(dy) \to 0$ when $h \to 0$, we can choose $h_\varepsilon > 0$ such that $C^2_p\int_{|y|\leq h_\varepsilon} |y|^{2p}\pi(dy) \leq \varepsilon$. That yields the announced inequality. □

LEMMA 3. *Let V be an EQ-function defined on $\mathbb{R}^d$. Then*:

(a) *If $p \in [0, 1/2]$, V^p is α-Hölder for any $\alpha \in [2p, 1]$ and if $p \in (0, 1]$, $\nabla(V^p)$ is α-Hölder for any $\alpha \in [2p-1, 1] \cap (0, 1]$.*

(b) *Let* $x, y \in \mathbb{R}^d$ *and* $\xi \in [x, x+y]$ *and set* $\underline{v} = \min\{V(x), x \in \mathbb{R}^d\}$. *If* $p \le 1$,

$$\tfrac{1}{2} D^2(V^p)(\xi) y^{\otimes 2} \le p \underline{v}^{p-1} \lambda_p |y|^2. \tag{30}$$

If, moreover, $|y| \le (1-\varepsilon) \frac{\sqrt{V}(x)}{[\sqrt{V}]_1}$ *with* $\varepsilon \in (0,1]$, *then,*

$$\tfrac{1}{2} D^2(V^p)(\xi) y^{\otimes 2} \le p \lambda_p \varepsilon^{2(p-1)} V^{p-1}(x) |y|^2. \tag{31}$$

If $p > 1$,

$$\tfrac{1}{2} D^2(V^p)(\xi) y^{\otimes 2} \le p \lambda_p 2^{(2(p-1)-1)_+} \big(V^{p-1}(x) + [\sqrt{V}]_1 |y|^{2(p-1)}\big) |y|^2. \tag{32}$$

PROOF. Consider a continuous function $f : \mathbb{R}^d \mapsto \mathbb{R}$. Let $\alpha \in (0,1]$ such that $|f|^{1/\alpha}$ is Lipschitz. Then, f is an α-Hölder function. This argument yields (a) (see [21] for details). Now, let us pass to (b). We have

$$D^2(V^p) = p V^{p-1} \left(D^2 V + (p-1) \frac{\nabla V \otimes \nabla V}{V} \right), \tag{33}$$

where $(\nabla V \otimes \nabla V)_{i,j} = (\nabla V)_i (\nabla V)_j$. Since $V^{p-1} \le \underline{v}^{p-1}$ if $p \le 1$, we derive (30) from relations (5) and (6). For (31), we consider $\xi = x + \theta y$ with $\theta \in [0,1]$ and $|y| \le (1-\varepsilon) \frac{\sqrt{V}(x)}{[\sqrt{V}]_1}$. As $\sqrt{V}$ is a Lipschitz function,

$$\sqrt{V}(\xi) \ge \sqrt{V}(x) - [\sqrt{V}]_1 |y| \ge \varepsilon \sqrt{V}(x) \quad \Longrightarrow \quad V^{p-1}(\xi) \le \varepsilon^{2(p-1)} V^{p-1}(x).$$

Hence, inequality (31) follows from (6). If $p > 1$,

$$\sqrt{V}(\xi) \le \sqrt{V}(x) + [\sqrt{V}]_1 |y| \quad \Longrightarrow \quad V^{p-1}(\xi) \le \big(\sqrt{V}(x) + [\sqrt{V}]_1 |y|\big)^{2(p-1)}.$$

We then derive (32) from (25) [with $\alpha = 2(p-1)$ and $n = 2$] and from (6). □

LEMMA 4. *Let* $p \in (0,1)$, $q \in [0,1]$ *and* $a \in (0,1]$. *Assume* $(\mathrm{H}_\mathrm{p}^1)$, $(\mathrm{H}_\mathrm{q}^2)$ *and* $(\mathrm{S}_{\mathrm{a,p,q}})$.1. *Then, for every* $\varepsilon > 0$, *there exists* $h_\varepsilon \in [0, +\infty]$, $T_\varepsilon > 0$ *and* $C_\varepsilon > 0$ *such that for every* $x, z \in \mathbb{R}^d$, *for every* $t \le T_\varepsilon$,

$$\begin{aligned} &\mathbb{E}\{V^p(z + \kappa(x) Z_t^{h_\varepsilon}) - V^p(z)\} \\ &\quad \le t \left(p c_p \int |y|^{2p} \pi(dy) 1_{\{q \le p\}} \|\kappa(x)\|^{2p} + \varepsilon V^{p+a-1}(x) + C_\varepsilon \right), \end{aligned} \tag{34}$$

with c_p *given by* (6), $h_\varepsilon \in (0,1]$ *if* $p \le 1/2 < q$, $h_\varepsilon = 0$ *if* $p, q \le 1/2$ *and* $h_\varepsilon = +\infty$ *if* $p \in (1/2, 1)$.

PROOF. Set $\Delta(z, x, U) = V^p(z + \kappa(x) U) - V^p(z)$. We first consider the case $p \le 1/2$ and $q > 1/2$. Let $h \in (0, \infty)$. Since $Z_t^h = Y_t^h + N_t^h$, we can decompose $\Delta(z, x, Z_t^h)$ as follows:

$$\Delta(z, x, Z_t^h) = \Delta(z + \kappa(x) N_t^h, x, Y_t^h) + \Delta(z, x, N_t^h).$$

One controls each term of the right-hand side. On the one hand, as V^p is $2p$-Hölder with constant $[V^p]_{2p} = pc_p$ [see (6)], we deduce from Lemma 2(i) that

$$\begin{aligned}\mathbb{E}\{\Delta(z, x, N_t^h)\} &\le pc_p \|\kappa(x)\|^{2p}\mathbb{E}\{|N_t^h|^{2p}\}\\ &\le pc_p \int_{|y|>h} |y|^{2p}\pi(dy)\|\kappa(x)\|^{2p}\big(t + \psi_h(t)t^2\big),\end{aligned} \tag{35}$$

where ψ_h is a locally bounded function. On the other hand, we set $\tilde{z} = z + \kappa(x)N_t^h$. By the Taylor formula,

$$\Delta(\tilde{z}, x, Y_t^h) = \langle \nabla(V^p)(\tilde{z}), \kappa(x)Y_t^h\rangle + \langle \nabla(V^p)(\xi) - \nabla(V^p)(\tilde{z}), \kappa(x)Y_t^h\rangle$$

with $\xi \in [\tilde{z}, \tilde{z} + \kappa(x)Y_t^h]$. As (N_t^h) and (Y_t^h) are independent and Y_t^h is centered, $\mathbb{E}\{\langle \nabla(V^p)(\tilde{z}), \kappa(x)Y_t^h\rangle\} = 0$. By Lemma 3, $V^{p-1}\nabla V = \nabla(V^p)/p$ is $(2q-1)$-Hölder (because $2q - 1 \in [2p-1, 1] \cap (0, 1]$ in this case). Then, it follows from Lemma 2(ii).2 that

$$\begin{aligned}\mathbb{E}\big\{\Delta\big(z + \kappa(x)N_t^h, x, Y_t^h\big)\big\} &\le p[V^{p-1}\nabla V]_{2q-1}\|\kappa(x)\|^{2q}\mathbb{E}\{|Y_t^h|^{2q}\}\\ &\le C\|\kappa(x)\|^{2q} t \int_{|y|\le h} |y|^{2q}\pi(dy).\end{aligned} \tag{36}$$

Let $\varepsilon > 0$. First, by $(\mathrm{S_{a,p,q}})$.1, $\|\kappa(x)\|^{2q} \le CV^{p+a-1}$. Then, using that $\int_{|y|\le h} |y|^{2q}\pi(dy) \to 0$ when $h \to 0$, we can fix $h_\varepsilon \in (0, 1]$ such that

$$\mathbb{E}\big\{\Delta\big(z + \kappa(x)N_t^{h_\varepsilon}, x, Y_t^{h_\varepsilon}\big)\big\} \le \frac{\varepsilon}{2} t V^{p+a-1}(x). \tag{37}$$

Second, since ψ_{h_ε} is locally bounded, it follows from (35) that there exists C_ε^1 such that, for every $t \le 1$, $\mathbb{E}\{\Delta(z, x, N_t^{h_\varepsilon})\} \le C_\varepsilon^1 t\|\kappa(x)\|^{2p}$. Now, as $p < q$, for every $\delta > 0$, there exists $C_\delta^2 > 0$ such that $\|\kappa(x)\|^{2p} \le \delta V^{p+a-1} + C_\delta^2$ [see (19) for similar arguments]. Hence, setting $\delta_\varepsilon = \varepsilon/(2C_\varepsilon^1)$ yields

$$\mathbb{E}\{\Delta(z, x, N_t^{h_\varepsilon})\} \le t\left(\frac{\varepsilon}{2} V^{p+a-1}(x) + C_\varepsilon\right) \tag{38}$$

with $C_\varepsilon = C_\varepsilon^1 C_{\delta_\varepsilon}^2$. Then, adding up (37) and (38) yields the result when $p \le 1/2 < q$.

When $p, q \le 1/2$, we deal with $(\check{Z}_t) = (Z_t^0)$. For every $h > 0$, $\check{Z}_t = \check{Y}_t^h + N_t^h$, where $\check{Y}_t^h = Y_t^h + t\int_{\{|y|\le h\}} y\pi(dy)$. Hence, for every $h > 0$,

$$\Delta(z, x, \check{Z}_t) = \Delta\big(z + \kappa(x)N_t^h, x, \check{Y}_t^h\big) + \Delta(z, x, N_t^h).$$

If $q \le p$, π satisfies $(\mathrm{H_p^2})$. Since $p \le 1/2$, V^p is $2p$-Hölder. Therefore, by Lemma 2(ii).1,

$$\mathbb{E}\big\{\Delta\big(z + \kappa(x)N_t^h, x, \check{Y}_t^h\big)\big\} \le pc_p t\|\kappa(x)\|^{2p} \int_{|y|\le h} |y|^{2p}\pi(dy).$$

By summing up this inequality and (35), we deduce (34). When $p < q \leq 1/2$, we use that V^p is $2q$-Hölder (see Lemma 3) and a proof analogous to the case $p \leq 1/2 < q$ yields the result.

Finally, we consider the case $p > 1/2$ where we deal with $\hat{Z}_t = Z_t^{\infty}$. For every $h > 0$, we have $\hat{Z}_t = Y_t^h + \hat{N}_t^h$, where $\hat{N}_t^h = N_t^h - \int_{\{|y>h\}} y\pi(dy)$. For every $h > 0$, $\Delta(z, x, \hat{Z}_t)$ can be written as follows:

$$\Delta(z, x, \hat{Z}_t) = \Delta\big(z + \kappa(x)\hat{N}_t^h, x, Y_t^h\big) + \Delta(z, x, \hat{N}_t^h). \tag{39}$$

One the one hand, by the same process as that used for (36) and by inequality (29), we have

$$\begin{aligned}\mathbb{E}\{\Delta(z, x, \hat{N}_t^h)\} &\leq p[V^{p-1}\nabla V]_{2p-1}\|\kappa(x)\|^{2p}\mathbb{E}\{|\hat{N}_t^h|^{2p}\}\\ &\leq t\|\kappa(x)\|^{2p}\bigg(\int_{|y|>h}|y|^{2p}\pi(dy) + C_h t^{2p-1}\bigg).\end{aligned}$$

On the other hand, by Lemma 3, $V^{p-1}\nabla V$ is $(2(p \vee q) - 1)$-Hölder. Hence, (36) is still valid in this case if we replace q with $p \vee q$. By using this control for the first term of the right-hand side of (39), we obtain if $q \leq p$

$$\begin{aligned}&\mathbb{E}\{\Delta(z, x, \hat{Z}_t)\}\\ &\quad\leq t\|\kappa(x)\|^{2p}\bigg(\int_{\{|y|>h\}}|y|^{2p}\pi(dy) + \int_{|y|\leq h}|y|^{2p}\pi(dy) + C_h t^{2p-1}\bigg).\end{aligned}$$

The result follows in this case. When $q > p$, the sequel of the proof is similar to the case $p \leq 1/2 < q$. $\square$

3.2. *Proof of Proposition* 2. For this proof, one needs to study separately the $p < 1$ and $p \geq 1$ cases. We detail the first case. When $p \geq 1$, we briefly indicate the process of the proof which is close to that of Lemma 3 of [15].

Case $p < 1$. For $h > 1$, we set $\bar{Z}_n^h = \bar{Z}_n - \gamma_n \int_{\{1<|y|\leq h\}} y\pi(dy)$ and for $h \in (0, 1)$, $\bar{Z}_n^h = \bar{Z}_n + \gamma_n \int_{\{h<|y|\leq 1\}} y\pi(dy)$. If $q \leq 1/2$ (resp. $p > 1/2$), we can take $h = 0$ (resp. $h = +\infty$). Thus, we can write

$$\begin{aligned}&\Delta\bar{X}_{n+1} := \bar{X}_{n+1} - \bar{X}_n = \sum_{k=1}^{3}\Delta\bar{X}_{n+1,k}^h \qquad \text{with } \Delta\bar{X}_{n+1,1}^h = \gamma_{n+1}b^h(\bar{X}_n),\\ &\Delta\bar{X}_{n+1,2}^h = \sqrt{\gamma_{n+1}}\sigma(\bar{X}_n)U_{n+1} \quad \text{and} \quad \Delta\bar{X}_{n+1,3}^h = \kappa(\bar{X}_n)\bar{Z}_{n+1}^h.\end{aligned} \tag{40}$$

The idea is to study the difference $V^p(\bar{X}_{n+1}) - V^p(\bar{X}_n)$ as the sum of three terms that correspond to the above decomposition. For $k = 1, 2, 3$, set $\bar{X}_{n+1,k}^h = \bar{X}_n^h + \sum_{i=1}^{k}\Delta\bar{X}_{n+1,i}^h$.

(i) First term: There exists $n_1 \in \mathbb{N}$ such that, for every $n \geq n_1$

$$\begin{aligned} &\mathbb{E}\{V^p(\bar{X}^h_{n+1,1}) - V^p(\bar{X}_n)|\mathcal{F}_n\} \\ &\quad \leq p\gamma_{n+1}\frac{\langle \nabla V, b^h\rangle}{V^{1-p}}(\bar{X}_n) + C\gamma_{n+1}^2 V^{a+p-1}(\bar{X}_n). \end{aligned} \tag{41}$$

Indeed, from Taylor's formula,

$$V^p(\bar{X}^h_{n+1,1}) - V^p(\bar{X}_n) = p\gamma_{n+1}\frac{\langle \nabla V, b^h\rangle}{V^{1-p}}(\bar{X}_n) + \frac{1}{2}D^2(V^p)(\xi^1_{n+1})(\Delta\bar{X}^h_{n+1,1})^{\otimes 2},$$

where $\xi^1_{n+1} \in [\bar{X}_n, \bar{X}_n + \gamma_{n+1}b^h(\bar{X}_n)]$. Set $x = \bar{X}_n$ and $y = \gamma_{n+1}b^h(\bar{X}_n)$. Since $\gamma_n \overset{n\to+\infty}{\longrightarrow} 0$ and $|b^h| \leq C\sqrt{V}$ by $(\mathrm{S}_{\mathrm{a,p,q}})$.1, there exists $n_1 \in \mathbb{N}$ such that, for $n \geq n_1$, $|y| \leq \frac{\sqrt{V}(x)}{2[\sqrt{V}]_1}$ a.s. Thus, we can apply the second inequality of Lemma 3(b) with $\varepsilon = 1/2$ and deduce (41) from $(\mathrm{S}_{\mathrm{a,p,q}})$.1.

(ii) Second term: For every $\varepsilon > 0$, there exists $n_{2,\varepsilon} \in \mathbb{N}$ such that, for every $n \geq n_{2,\varepsilon}$,

$$\mathbb{E}\{V^p(\bar{X}^h_{n+1,2}) - V^p(\bar{X}^h_{n+1,1})|\mathcal{F}_n\} \leq \varepsilon\gamma_{n+1}V^{a+p-1}(\bar{X}_n) + C^1_\varepsilon\gamma_{n+1}. \tag{42}$$

Let us prove this inequality. Since $\mathbb{E}\{U_{n+1}|\mathcal{F}_n\} = 0$, we deduce from Taylor's formula that

$$\mathbb{E}\{V^p(\bar{X}^h_{n+1,2}) - V^p(\bar{X}^h_{n+1,1})|\mathcal{F}_n\} = \tfrac{1}{2}\mathbb{E}\{D^2(V^p)(\xi^2_{n+1})(\Delta\bar{X}_{n+1,2})^{\otimes 2}|\mathcal{F}_n\}$$

with $\xi^2_{n+1} \in [\bar{X}^h_{n+1,1}; \bar{X}^h_{n+1,2}]$. Set $x = \bar{X}^h_{n+1,1}$ and $y = \sqrt{\gamma_{n+1}}\sigma(x)U_{n+1}$. By $(\mathrm{S}_{\mathrm{a,p,q}})$.1, $\|\sigma(x)\| \leq C_\sigma\sqrt{V}(x)$ because $p+a-1 \leq 1$. Then, the conditions of (31) are satisfied with $\varepsilon = 1/2$ if $|U_{n+1}| \leq \rho_{n+1} = 1/(2C_\sigma[\sqrt{V}]_1\sqrt{\gamma_{n+1}})$. Therefore,

$$\begin{aligned} &\mathbb{E}\{D^2(V^p)(\xi^2_{n+1})(\Delta\bar{X}_{n+1,2})^{\otimes 2}1_{\{|U_{n+1}|\leq\rho_{n+1}\}}|\mathcal{F}_n\} \\ &\quad \leq C\gamma_{n+1}V^{p-1}(\bar{X}_n)\operatorname{Tr}(\sigma\sigma^*)(\bar{X}_n) \\ &\quad \leq C\gamma_{n+1}V^{a+2(p-1)}(\bar{X}_n) \end{aligned}$$

since $\operatorname{Tr}(\sigma\sigma^*) \leq CV^{p+a-1}$ when $p < 1$. By (30) and $(\mathrm{S}_{\mathrm{a,p,q}})$.1, we also have

$$\mathbb{E}\{D^2(V^p)(\xi^2_{n+1})(\Delta\bar{X}_{n+1,2})^{\otimes 2}1_{\{|U_{n+1}|>\rho_{n+1}\}}|\mathcal{F}_n\} \leq C\delta_{n+1}\gamma_{n+1}V^{p+a-1}(\bar{X}_n),$$

where $\delta_n = \mathbb{E}\{|U_n|^2 1_{\{|U_n|>\rho_n\}}\}$. Now, let $\varepsilon > 0$. First, since $a + 2(p-1) < a + p - 1$ when $p < 1$, there exists $C_\varepsilon > 0$ such that $V^{p-1}\operatorname{Tr}(\sigma\sigma^*) \leq \varepsilon V^{a+p-1} + C_\varepsilon$ [see (19) for similar arguments]. Second, since $\rho_n \to +\infty$, $\delta_n \to 0$. Thus, there exists $n_{2,\varepsilon} \in \mathbb{N}$ such that, for every $n \geq n_{2,\varepsilon}$, $C\delta_n V^{p+a-1} \leq \varepsilon V^{p+a-1}$. The combination of these two arguments yields (42).

(iii) Third term: For every $\varepsilon > 0$, there exists $h_\varepsilon \in [0, \infty]$, $C_\varepsilon^2 > 0$ and $n_{3,\varepsilon}$ such that, for all $n \geq n_{3,\varepsilon}$,

$$\begin{aligned}(43)\quad &\mathbb{E}\{V^p(\bar{X}^{h_\varepsilon}_{n+1,3}) - V^p(\bar{X}^{h_\varepsilon}_{n+1,2})|\mathcal{F}_n\}\\ &\quad\leq \gamma_{n+1}\biggl(pc_p\int|y|^{2p}\pi(dy)1_{\{q\leq p\}}\|\kappa(\bar{X}_n)\|^{2p} + \varepsilon V^{p+a-1}(\bar{X}_n) + C_\varepsilon^2\biggr)\end{aligned}$$

with $h_\varepsilon \in (0, 1]$ if $p \leq 1/2 < q$, $h_\varepsilon = 0$ if $p, q \leq 1/2$ and $h_\varepsilon = +\infty$ if $p \in (1/2, 1)$. This step is a consequence of Lemma 4: since U_{n+1} and $\bar{Z}^h_{n+1}$ are independent, we have

$$\mathbb{E}\{V^p(\bar{X}^{h_\varepsilon}_{n+1,3}) - V^p(\bar{X}^{h_\varepsilon}_{n+1,2})|\mathcal{F}_n\} = \mathbb{E}\{G_{h_\varepsilon}(\bar{X}^{h_\varepsilon}_{n+1,2}, \bar{X}_n)|\mathcal{F}_n\},$$

where $G_h(z, x) = \mathbb{E}\{V^p(z + \kappa(x)Z^h_t) - V^p(z)\}$. Then, Lemma 4 yields (43).

We can now prove the proposition. Let $\varepsilon > 0$. By adding (41), (42) and (43) and using that $\gamma_n^2 \leq \varepsilon\gamma_n$ for sufficiently large n (since $\gamma_n \to 0$), we obtain that there exists $n_\varepsilon \in \mathbb{N}$, $h_\varepsilon > 0$ and $C_\varepsilon > 0$ such that, for every $n \geq n_\varepsilon$,

$$\begin{aligned}(44)\quad &\mathbb{E}\{V^p(\bar{X}_{n+1})|\mathcal{F}_n\}\\ &\quad\leq V^p(\bar{X}_n) + \gamma_{n+1}(\varepsilon V^{p+a-1}(\bar{X}_n) + C_\varepsilon)\\ &\qquad + \gamma_{n+1}pV^{p-1}(\bar{X}_n)\\ &\qquad\quad\times\biggl(\langle\nabla V, b^{h_\varepsilon}\rangle + 1_{q\leq p}c_p\int|y|^{2p}\pi(dy)\|\kappa\|^{2p}V^{1-p}\biggr)(\bar{X}_n).\end{aligned}$$

When $p, q \leq 1/2$ (resp. $p > 1/2$), $h_\varepsilon = 0$ (resp. $h_\varepsilon = \infty$). We deduce that $\langle\nabla V, b^{h_\varepsilon}\rangle = \langle\nabla V, \tilde{b}\rangle$ because $\tilde{b} = b^0$ (resp. $\tilde{b} = b^\infty$) when $p, q \leq 1/2$ (resp. $p > 1/2$). We then recognize the left-hand side of $(\mathrm{S_{a,p,q}})$.2 in (44). When $p \leq 1/2 < q$, $h_\varepsilon \in (0, \infty)$ and $\langle\nabla V, b^{h_\varepsilon}\rangle = \langle\nabla V, \tilde{b}\rangle + \Phi_{h_\varepsilon}$, where $\Phi_h(x) = \langle\nabla V(x), \kappa(x)\int_{\{h_\varepsilon<|y|\leq 1\}} y\pi(dy)\rangle$. Therefore, by $(\mathrm{S_{a,p,q}})$.2, we obtain

$$\begin{aligned}&\mathbb{E}\{V^p(\bar{X}_{n+1})|\mathcal{F}_n\}\\ &\quad\leq V^p(\bar{X}_n) + \gamma_{n+1}pV^{p-1}(\bar{X}_n)(\beta - \alpha V^a(\bar{X}_n))\\ &\qquad + \gamma_{n+1}(\varepsilon V^{p+a-1}(\bar{X}_n) + C_\varepsilon + 1_{\{p\leq 1/2<q\}}pV^{p-1}(\bar{X}_n)\Phi_{h_\varepsilon}(\bar{X}_n)).\end{aligned}$$

When $p, q \leq 1/2$ or $p > 1/2$, we set $\varepsilon = p\alpha/2$ and obtain $(\mathrm{R_{a,p}})$ with $\beta' = p\beta + C_\varepsilon/\underline{v}^{p-1}$ and $\alpha' = p\alpha/2$. When $p \leq 1/2 < q$, by $(\mathrm{S_{a,p,q}})$.1, one checks that, for every $\varepsilon > 0$, there exists $\tilde{C}_\varepsilon > 0$ such that $V^{p-1}|\Phi_{h_\varepsilon}| \leq \varepsilon V^{p+a-1} + \tilde{C}_\varepsilon$ and the result follows.

Case $p \geq 1$. Thanks to Taylor's formula,

$$\begin{aligned}V^p(\bar{X}_{n+1}) &= V^p(\bar{X}_n) + \gamma_{n+1}\langle\nabla(V^p)(\bar{X}_n), \Delta X_{n+1}\rangle\\ &\quad + \tfrac{1}{2}D^2(V^p)(\xi_{n+1})(\Delta\bar{X}_{n+1})^{\otimes 2},\end{aligned}$$

where $\xi_{n+1} \in [\bar{X}_n, \bar{X}_{n+1}]$ and

$$\Delta \bar{X}_{n+1} = \bar{X}_{n+1} - \bar{X}_n = \gamma_{n+1} b^{\infty}(\bar{X}_n) + \sqrt{\gamma_{n+1}}\sigma(\bar{X}_n)U_{n+1} + \kappa(\bar{X}_n)\bar{Z}^{\infty}_{n+1}$$

with $\bar{Z}^{\infty}_n = \bar{Z}_n - \gamma_n \int_{\{|y|>1\}} y\pi(dy)$. Using that $\tilde{b} = b^{\infty}$ in this case and that $\mathbb{E}\{U_{n+1}|\mathcal{F}_n\} = \mathbb{E}\{\bar{Z}^{\infty}_{n+1}|\mathcal{F}_n\} = 0$ yields

$$\begin{aligned}\mathbb{E}\{V^p(\bar{X}_{n+1})|\mathcal{F}_n\} &= V^p(\bar{X}_n) + p\gamma_{n+1}V^{p-1}(\bar{X}_n)\langle \nabla V, \tilde{b}\rangle(\bar{X}_n)\\ &\quad + \tfrac{1}{2}\mathbb{E}\{D^2(V^p)(\xi_{n+1})(\Delta\bar{X}_{n+1})^{\otimes 2}|\mathcal{F}_n\}.\end{aligned}$$

The sequel of the proof consists in studying the last term of this equality. The main tools for this are the last inequality of Lemma 3 which provides a control of $D^2(V^p)(\xi_{n+1})(\Delta\bar{X}_{n+1})^{\otimes 2}$ and Lemma 2(iii), which gives a control of the moments of the jump component (see [21] for details or [15] for a similar proof).

3.3. *Consequences of Proposition* 2. In Proposition 2, we established $(\mathrm{R}_{\mathrm{a,p}})$. According to Lemma 1, it suffices now to prove $(\mathrm{C}_{\mathrm{p,s}})$. This property is established in Corollary 1 and is a consequence of Proposition 2 [under additional assumptions on (γ_n) and (η_n) when $s < 2$]. More precisely, we first show in Lemma 5 that a supermartingale property can be derived from $(\mathrm{R}_{\mathrm{a,p}})$ and that this property provides an L^{p+a-1}-control of the sequence $(V(\bar{X}_n))$ [see (45)]. Second, we show in Corollary 1 that we can derive $(\mathrm{C}_{\mathrm{p,s}})$ from this lemma.

LEMMA 5. *Let $a \in (0, 1]$ and $p > 0$. Assume $(\mathrm{H}^1_{\mathrm{p}})$ and $(\mathrm{R}_{\mathrm{a,p}})$. Let $(\theta_n)_{n\in\mathbb{N}}$ be a nonincreasing sequence of nonnegative numbers such that $\sum_{n\geq 1}\theta_n\gamma_n < \infty$. Then, there exists $n_0 \geq 0$, $\hat{\alpha} > 0$ and $\hat{\beta} > 0$ such that $(S_n)_{n\geq n_0}$ defined by*

$$S_n = \theta_n V^p(\bar{X}_n) + \hat{\alpha}\sum_{k=1}^{n}\theta_k\gamma_k V^{p+a-1}(\bar{X}_{k-1}) + \hat{\beta}\sum_{k>n}\theta_k\gamma_k$$

is a nonnegative L^1-supermartingale. In particular,

$$\sum_{n\geq 1}\theta_n\gamma_n\mathbb{E}\{V^{p+a-1}(\bar{X}_{n-1})\} < +\infty \quad \textit{and} \quad \mathbb{E}\{V^p(\bar{X}_n)\} \overset{n\to+\infty}{=} O\left(\frac{1}{\theta_n}\right). \tag{45}$$

PROOF. Since b, σ and κ have sublinear growth and $\bar{Z}_n \in L^{2p}$ for every $n \geq 1$, we can check by induction that, for every $n \geq 0$, $V^p(\bar{X}_n)$ is integrable. Denote by $(\Delta_n)_{n\geq 1}$ the sequence of martingale increments defined by $\Delta_n = V^p(\bar{X}_n) - \mathbb{E}\{V^p(\bar{X}_n)|\mathcal{F}_{n-1}\}$. By $(\mathrm{R}_{\mathrm{a,p}})$, there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$,

$$\begin{aligned}&\theta_{n+1}V^p(\bar{X}_{n+1})\\ &\quad\leq \theta_{n+1}\Delta_{n+1} + \theta_{n+1}\mathbb{E}\{V^p(\bar{X}_{n+1})|\mathcal{F}_n\}\\ &\quad\leq \theta_{n+1}\Delta_{n+1} + \theta_{n+1}\big(V^p(\bar{X}_n) + \gamma_{n+1}V^{p-1}(\bar{X}_n)\big(\beta' - \alpha' V^a(\bar{X}_n)\big)\big).\end{aligned}$$

By the same argument as in (19), one can find $\hat{\alpha} > 0$ and $\hat{\beta} > 0$ such that $V^{p-1}(\beta' - \alpha' V^a) \le \hat{\beta} - \hat{\alpha} V^{p+a-1}$. Since (θ_n) is nonincreasing, we deduce that

$$\theta_{n+1}\big(V^p(\bar{X}_{n+1}) + \hat{\alpha}\gamma_{n+1}V^{p+a-1}(\bar{X}_n)\big) \\ \le \theta_n V^p(\bar{X}_n) + \theta_{n+1}\Delta_{n+1} + \theta_{n+1}\gamma_{n+1}\hat{\beta}.$$

Adding "$\hat{\alpha}\sum_{k=1}^n \theta_k\gamma_k V^{p+a-1}(\bar{X}_{k-1}) + \hat{\beta}\sum_{k>n+1}\theta_k\gamma_k$" to both sides of the inequality yields

$$S_{n+1} \le S_n + \theta_{n+1}\Delta_{n+1} \quad \Longrightarrow \quad \mathbb{E}\{S_{n+1}|\mathcal{F}_n\} \le S_n \qquad \forall n \ge n_0.$$

Since $S_{n_0} \in L^1$, it follows that $(S_n)_{n\ge n_0}$ is a nonnegative supermartingale and then, that $\sup\mathbb{E}\{S_n\} < +\infty$. The result is obvious. $\square$

COROLLARY 1. *Let $a \in (0,1]$, $p > 0$ and $q \in [0,1]$. Assume $(\mathrm{H}^1_{\mathrm{p}})$, $(\mathrm{H}^2_{\mathrm{q}})$ and $(\mathrm{S}_{\mathrm{a,p,q}})$. If $\mathbb{E}\{|U_1|^{2(p\vee 1)}\} < +\infty$ and $(\eta_n/\gamma_n)_{n\in\mathbb{N}}$ is nonincreasing,*

$$\sum_{n\ge1}\left(\frac{\eta_n}{H_n\gamma_n}\right)^2 \mathbb{E}\big\{\big|V^{p/2}(\bar{X}_n) - V^{p/2}\big(\bar{X}_{n-1} + \gamma_n\tilde{b}(\bar{X}_{n-1})\big)\big|^2\big\} < +\infty. \tag{46}$$

In particular, $(\mathrm{C}_{\mathrm{p},2})$ holds with $\rho = 2$ and $\pi_n = V^{p/2}(\bar{X}_n + \gamma_n\tilde{b}(\bar{X}_n))$.

Furthermore, if conditions (11) and (13) are satisfied for $s \in (1,2)$,

$$\sum_{n\ge1}\left(\frac{\eta_n}{H_n\gamma_n}\right)^{f_{a,p}(s)} \mathbb{E}\big\{\big|V^{p/s}(\bar{X}_n) - V^{p/s}\big(\bar{X}_{n-1} + \gamma_n\tilde{b}(\bar{X}_{n-1})\big)\big|^{f_{a,p}(s)}\big\} \\ < +\infty. \tag{47}$$

In particular, $(\mathrm{C}_{\mathrm{p,s}})$ holds with $\rho = f_{a,p}(s)$ and $\pi_n = V^{p/s}(\bar{X}_n + \gamma_n\tilde{b}(\bar{X}_n))$.

PROOF. Let us begin the proof by two useful remarks. First, (46) is a particular case of (47) since $f_{a,p}(2) = 2$ and (13) is always satisfied in this case. Indeed, as $(\eta_n/\gamma_n)_{n\in\mathbb{N}}$ is nonincreasing, so is $(\frac{1}{\gamma_n}(\frac{\eta_n}{H_n\sqrt{\gamma_n}})^2)$, and

$$\sum_{n\ge1}\left(\frac{\eta_n}{H_n\sqrt{\gamma_n}}\right)^2 \le \frac{\eta_1}{\gamma_1}\sum_{n\ge1}\frac{\eta_n}{H_n^2} \le \frac{\eta_1}{\gamma_1}\sum_{n\ge1}\frac{\Delta H_n}{H_n^2} \le C\left(1 + \int_{\eta_1}^{\infty}\frac{dt}{t^2}\right) < \infty, \tag{48}$$

with $\Delta H_n = H_n - H_{n-1}$. Then, it suffices to prove (47). Second, by Lemma 5 applied with $\theta_n = \frac{1}{\gamma_n}(\frac{\eta_n}{H_n\sqrt{\gamma_n}})^{f_{a,p}(s)}$, we have

$$\sum_{n\ge1}\left(\frac{\eta_n}{H_n\sqrt{\gamma_n}}\right)^{f_{a,p}(s)}\mathbb{E}\{V^{p+a-1}(\bar{X}_{n-1})\} < +\infty. \tag{49}$$

Hence, one checks that (47) holds as soon as

$$\mathbb{E}\big\{\big|V^{p/s}(\bar{X}_n) - V^{p/s}\big(\bar{X}_{n-1} + \gamma_n\tilde{b}(\bar{X}_{n-1})\big)\big|^{f_{a,p}(s)}\big\} \\ \le C\gamma_n^{f_{a,p}(s)/2}\mathbb{E}\{V^{p+a-1}(\bar{X}_{n-1})\}. \tag{50}$$

Thus, we only need to prove (50). We inspect the $p/s \le 1/2$ and $p/s > 1/2$ cases successively.

Case $p/s \le 1/2$. In this case, $f_{a,p}(s) = s$. We keep the notation introduced in (40), with $h = 1$ if $p \le 1/2 < q$, $h = 0$ if $p, q \le 1/2$ and $h = +\infty$ if $p > 1/2$, and derive from (25) that

$$\begin{aligned} &\bigl|V^{p/s}(\bar{X}_n) - V^{p/s}\bigl(\bar{X}_{n-1} + \gamma_n \tilde{b}(\bar{X}_{n-1})\bigr)\bigr|^s \\ &\qquad \le C|V^{p/s}(\bar{X}^h_{n,2}) - V^{p/s}(\bar{X}^h_{n,1})|^s + C|V^{p/s}(\bar{X}_n) - V^{p/s}(\bar{X}^h_{n,2})|^s. \end{aligned} \tag{51}$$

We study successively the two right-hand side members. First, by the Taylor formula,

$$|V^{p/s}(\bar{X}^h_{n,2}) - V^{p/s}(\bar{X}^h_{n,1})|^s \le C\gamma_n^{s/2}|\langle V^{p/s-1}\nabla V(\xi^1_n), \sigma(\bar{X}_{n-1})U_n\rangle|^s$$

with $\xi^1_n \in [\bar{X}^h_{n,1}; \bar{X}^h_{n,2}]$. The function $V^{p/s-1}\nabla V$ is bounded. Hence, since $\|\sigma\|^s \le C\,\mathrm{Tr}(\sigma\sigma^*)$ (because $s \le 2$), we derive from $(\mathrm{S}_{\mathrm{a,p,q}})$.1 that

$$\mathbb{E}\{|V^{p/s}(\bar{X}^h_{n,2}) - V^{p/s}(\bar{X}^h_{n,1})|^s|\mathcal{F}_{n-1}\} \le C\gamma_n^{s/2}V^{a+p-1}(\bar{X}_{n-1}). \tag{52}$$

Second, since U_n and $\bar{Z}^h_n$ are independent,

$$\begin{aligned} \mathbb{E}\{|V^{p/s}(\bar{X}_n) - V^{p/s}(\bar{X}^h_{n,2})|^s\} &= \mathbb{E}\{\Upsilon_h(\bar{X}^h_{n,2}, \bar{X}_{n-1}, \gamma_n)|\mathcal{F}_{n-1}\} \\ &\qquad \text{where } \Upsilon_h(z,x,\gamma) = \mathbb{E}\bigl\{\bigl|V^{p/s}\bigl(z + \kappa(x)Z^h_\gamma\bigr) - V^{p/s}(z)\bigr|^s\bigr\}. \end{aligned}$$

By (51) and (52), one checks that (50) holds if there exists $C > 0$ such that, for every $z, x \in \mathbb{R}^d$ and $\gamma \le \gamma_1$,

$$\Upsilon_h(z,x,\gamma) \le C\gamma^{s/2}V^{a+p-1}(x), \tag{53}$$

where $h = 1$ (resp. $h = 0$, resp. $h = +\infty$) if $p \le 1/2 < q$ (resp. if $p, q \le 1/2$, resp. if $p > 1/2$). Then, it suffices to prove (53). First, when $p \le 1/2 < q$, we have $Z^1_\gamma = Z_\gamma = Y_\gamma + N_\gamma$. On the one hand, since $V^{p/s}$ is a $2p/s$-Hölder function [see Lemma 3(a)], it follows from $(\mathrm{S}_{\mathrm{a,p,q}})$.1 and Lemma 2(i) that

$$\begin{aligned} \mathbb{E}\bigl\{\bigl|V^{p/s}\bigl(z + \kappa(x)N_\gamma\bigr) - V^{p/s}(z)\bigr|^s\bigr\} &\le C\|\kappa(x)\|^{2p}\mathbb{E}\{|N_\gamma|^{2p}\} \\ &\le C\gamma V^{p+a-1}(x). \end{aligned} \tag{54}$$

On the other hand, $V^{p/s}$ is a $2q/s$-Hölder function when $q/s \le 1/2$ (because $2p/s \le 2q/s \le 1$ in this case). Hence, using this property if $q/s \le 1/2$ and the Taylor formula if $q/s > 1/2$ yields

$$\begin{aligned} &\mathbb{E}\bigl\{\bigl|V^{p/s}\bigl(z + \kappa(x)(N_\gamma + Y_\gamma)\bigr) - V^{p/s}\bigl(z + \kappa(x)N_\gamma\bigr)\bigr|^s\bigr\} \\ &\qquad \le C\begin{cases} \|\kappa(x)\|^{2q}\mathbb{E}\{|Y_\gamma|^{2q}\}, & \text{if } q/s \le 1/2, \\ \mathbb{E}\{|\langle V^{p/s-1}\nabla V(\xi_2), \kappa(x)Y_\gamma\rangle|^s\}, & \text{if } q/s > 1/2, \end{cases} \end{aligned} \tag{55}$$

with $\xi_2 \in [z+\kappa(x)N_\gamma, z+\kappa(x)(N_\gamma+Y_\gamma)]$. By Lemma 2(ii).2, $\mathbb{E}\{|Y_\gamma|^{2q}\} \le C\gamma$. It follows from Jensen's inequality that

$$(56)\qquad \mathbb{E}\{|V^{p/s}(z+\kappa(x)Z_\gamma) - V^{p/s}(z+\kappa(x)N_\gamma)|^s\} \le C\gamma^{s/(2q)\wedge 1}\|\kappa(x)\|^{s\wedge 2q}.$$

One checks that $\|\kappa\|^{s\wedge 2q} \le CV^{p+a-1}$ under $(\mathrm{S}_{\mathrm{a,p,q}})$.1 and that $\gamma + \gamma^{s/(2q)\wedge 1} \le C\gamma^{s/2}$ for every $\gamma \le \gamma_1$. Hence, summing up (54) and (56) and using (25) yields (53) (with $h=1$) when $p \le 1/2 < q$.

When $p, q \le 1/2$ (resp. $p > 1/2$), we have to check that (53) holds with $h = 0$ (resp. with $h = +\infty$). Then, we need to use a decomposition of the jump component adapted to the value of h. We split up Z_γ^0 (resp. Z_γ^∞) as follows: $Z_\gamma^0 = \check{Y}_\gamma + N_\gamma$ with $\check{Y}_\gamma = Y_\gamma + \gamma\int_{\{|y|\le 1\}} y\pi(dy)$ [resp. $Z_\gamma^\infty = Y_\gamma + \hat{N}_\gamma$ with $\hat{N}_\gamma = N_\gamma - \gamma\int_{\{|y|>1\}} y\pi(dy)$]. Then, when $p, q \le 1/2$ (resp. $p > 1/2$), the idea is to replace Y_γ with $\check{Y}_\gamma$ (resp. N_γ with $\hat{N}_\gamma$) in the left-hand sides of (54) and (55) and to derive some adapted controls from Lemma 2 and inequality (29). Since the proof is close to that of the $p \le 1/2 < q$ case, we leave it to the reader.

Case $p/s > 1/2$. Since $p > 1/2$, we use the notation introduced in (40) with $h = +\infty$. We recall that $\bar{X}_{n-1} + \gamma_n \tilde{b}(\bar{X}_{n-1}) = \bar{X}_{n,1}^\infty$. Then, applying the following inequality,

$$(57)\qquad \forall u, v \ge 0, \forall \alpha \ge 1 \qquad |u^\alpha - v^\alpha| \le C_\alpha(|u-v|u^{\alpha-1} + |u-v|^\alpha)$$

with $u = \sqrt{V}(\bar{X}_n)$, $v = \sqrt{V}(\bar{X}_{n,1}^\infty)$ and $\alpha = (2p)/s$, we obtain

$$\begin{aligned}|V^{p/s}(\bar{X}_n) - V^{p/s}(\bar{X}_{n,1}^\infty)| &\le C|\sqrt{V}(\bar{X}_n) - \sqrt{V}(\bar{X}_{n,1}^\infty)|V^{p/s-1/2}(\bar{X}_{n-1})\\ &\quad + C|\sqrt{V}(\bar{X}_n) - \sqrt{V}(\bar{X}_{n,1}^\infty)|^{2p/s}.\end{aligned}$$

We deduce from $(\mathrm{S}_{\mathrm{a,p,q}})$.1 that

$$|\bar{X}_n - \bar{X}_{n,1}^\infty| \le \begin{cases} CV^{(a+p-1)/(2p)}(\bar{X}_{n-1})(\sqrt{\gamma_n}|U_n| + |\bar{Z}_n^\infty|), & \text{if } p<1,\\ CV^{a/2}(\bar{X}_{n-1})(\sqrt{\gamma_n}|U_n| + |\bar{Z}_n^\infty|), & \text{if } p \ge 1.\end{cases}$$

Since $\sqrt{V}$ is Lipschitz, one then checks that

$$\begin{aligned}&|V^{p/s}(\bar{X}_n) - V^{p/s}(\bar{X}_{n,1}^\infty)|\\ &\quad \le CV^r(\bar{X}_{n-1})(\sqrt{\gamma_n}|U_n| + |\bar{Z}_n^\infty| + \gamma_n^{p/s}|U_n|^{2p/s} + |\bar{Z}_n^\infty|^{2p/s}),\\ &\qquad \text{where } r = \begin{cases}\left(\dfrac{p}{s} + \dfrac{a-1}{2p}\right) \vee \left(\dfrac{a+p-1}{s}\right), & \text{if } p<1,\\ \left(\dfrac{p}{s} + \dfrac{a-1}{2}\right) \vee \dfrac{ap}{s}, & \text{if } p \ge 1.\end{cases}\end{aligned}$$

One derives from Lemma 2(iii) and from the Jensen inequality that, for $\alpha > 0$, $\mathbb{E}\{|\bar{Z}_n^\infty|^\alpha\} = O(\gamma_n^{(\alpha/2)\wedge 1})$. Therefore, since $2p/s \ge 1/2$ and $f_{a,p}(s) \le 2$, we have

$$\mathbb{E}\{(\sqrt{\gamma_n}|U_n| + |\bar{Z}_n^\infty| + \gamma_n^{p/s}|U_n|^{2p/s} + |\bar{Z}_n^\infty|^{2p/s})^{f_{a,p}(s)}\} = O(\gamma_n^{f_{a,p}(s)/2}).$$

Second, one deduces from the definition of $f_{a,p}$ that $rf_{a,p}(s) \leq a + p - 1$. Therefore, inequality (50) follows. $\square$

By Lemma 1, Corollary 1 concludes the proof of Proposition 1 and then, the part which is concerned with the tightness of $(\bar{\nu}_n(\omega, dx))_{n\geq 1}$. The only thing left to prove the theorem for Scheme (A) is thus to identify the limit. This is the aim of the next section.

4. Identification of the weak limits of $(\bar{\nu}_n(\omega, dx))_{n\geq 1}$. In this section we show that every weak limiting distribution of $(\bar{\nu}_n(\omega, dx))_{n\geq 1}$ is invariant for $(X_t)_{t\geq 0}$. For this purpose, we will rely on the Echeverria–Weiss theorem (see [7], page 238, [14] and [16]). This is a criterion for invariance based on the infinitesimal generator A of (X_t) defined by (7). By the Echeverria–Weiss theorem, we know that if $A(\mathcal{C}_K^2(\mathbb{R}^d)) \subset \mathcal{C}_0(\mathbb{R}^d)$, a probability ν is invariant for the SDE if for every $f \in \mathcal{C}_K^2(\mathbb{R}^d)$, $\nu(Af) = 0$. One can check that $A(\mathcal{C}_K^2(\mathbb{R}^d)) \subset \mathcal{C}_0(\mathbb{R}^d)$ if $\|\kappa(x)\| = o(|x|)$ when $|x| \to +\infty$ (and this condition cannot be improved in general). Hence, under this condition on κ, it follows that every weak limiting distribution of $(\bar{\nu}_n)$ is invariant if for every $f \in \mathcal{C}_K^2(\mathbb{R}^d)$, $\bar{\nu}_n(Af) \to 0$. The main result of this section is then the following proposition.

PROPOSITION 3. *Let $a \in (0, 1]$, $p > 0$ and $q \in [0, 1]$. Assume* $(\mathrm{H}_\mathrm{p}^1)$, $(\mathrm{H}_\mathrm{q}^2)$, $(\mathrm{S}_{\mathrm{a,p,q}})$.1. *Assume that $\|\kappa(x)\| \overset{|x|\to+\infty}{=} o(|x|)$ and that $(\eta_n/\gamma_n)_{n\geq 1}$ is nonincreasing. If, moreover,*

$$\sup_{n\geq 1} \bar{\nu}_n\big(\|\kappa\|^{2q} + \mathrm{Tr}(\sigma\sigma^*)\big) < \infty \quad \textit{and}$$
$$\sum_{k\geq 1} \frac{\eta_k^2}{H_k^2 \gamma_k} \mathbb{E}\{V^{a+p-1}(\bar{X}_{k-1})\} < +\infty, \tag{58}$$

then,

$$\forall f \in \mathcal{C}_K^2(\mathbb{R}^d),\ \textit{a.s.}, \qquad \int Af\, d\bar{\nu}_n \overset{n\to\infty}{\longrightarrow} 0. \tag{59}$$

Consequently, a.s., every weak limiting distribution of $(\bar{\nu}_n(\omega, dx))_{n\geq 1}$ is invariant for the SDE (1).

REMARK 8. This proposition is sufficient to conclude the proof because the two assumptions in (58) hold under the assumptions of Theorem 2 (resp. Theorem 3). Indeed, since $\|\kappa\|^{2q} + \mathrm{Tr}(\sigma\sigma^*) \leq CV^{p/s+a-1}$ with $s = 2$ in Theorem 2 [resp. with s satisfying (11) in Theorem 3], the first is a consequence of Proposition 1. Likewise, the second is a consequence of Lemma 5 applied with $\theta_n = (\eta_n/(H_n\gamma_n))^2$ [see (48)].

4.1. *Proof of Proposition* 3. The proof of Proposition 3 is built in two successive steps that are represented by Propositions 4 and 5. In Proposition 4 we claim that showing that $\bar{\nu}_n(Af) \to 0$ a.s. is equivalent to showing that $1/H_n \sum_{k=1}^n (\eta_k/\gamma_k)\mathbb{E}\{f(\bar{X}_k) - f(\bar{X}_{k-1})|\mathcal{F}_{k-1}\} \to 0$ a.s. Then, in Proposition 5 we show that this last term does tend to 0.

PROPOSITION 4. *Assume that the assumptions of Proposition* 3 *are fulfilled. Then, for every* $f \in \mathcal{C}_K^2(\mathbb{R}^d)$,

$$\lim_{n\to\infty} \frac{1}{H_n}\sum_{k=1}^n \eta_k\left(\frac{\mathbb{E}\{f(\bar{X}_k) - f(\bar{X}_{k-1})|\mathcal{F}_{k-1}\}}{\gamma_k} - Af(\bar{X}_{k-1})\right) = 0 \qquad a.s. \tag{60}$$

We begin the proof by a technical lemma.

LEMMA 6. *Let* $\Phi : \mathbb{R}^d \mapsto \mathbb{R}^l$ *be a continuous function with compact support,* $\Psi : \mathbb{R}^d \mapsto \mathbb{R}_+$, *a locally bounded function,* $(h_1^\theta)_{\theta\in[0,1]}$ *and* $(h_2^\theta)_{\theta\in[0,1]}$ *two families of Borel functions defined on* $\mathbb{R}^d \times \mathbb{R}_+$ *with values in* $\mathbb{R}^d$ *satisfying the following assumptions*:

- *There exists* $\delta_0 > 0$ *such that*

$$\inf_{\theta\in[0,1],\gamma\in[0,\delta_0]} \big(|h_1^\theta|(x,\gamma) + |h_2^\theta|(x,\gamma)\big) \overset{|x|\to+\infty}{\longrightarrow} +\infty. \tag{61}$$

- *For every compact set* K,

$$\sup_{x\in K,\theta\in[0,1]} |h_1^\theta(x,\gamma) - h_2^\theta(x,\gamma)| \overset{\gamma\to 0}{\longrightarrow} 0. \tag{62}$$

Then, for every sequence $(x_k)_{k\in\mathbb{N}}$ *of* $\mathbb{R}^d$,

$$\frac{1}{H_n}\sum_{k=1}^n \eta_k \sup_{\theta\in[0,1]} \|\Phi(h_1^\theta(x_{k-1},\gamma_k)) - \Phi(h_2^\theta(x_{k-1},\gamma_k))\|\Psi(x_{k-1}) \overset{n\to+\infty}{\longrightarrow} 0.$$

PROOF. Φ has a compact support, therefore, we derive from (61) that there exists $M_{\delta_0} > 0$ such that, for every $|x| > M_{\delta_0}$, $\gamma \le \delta_0$ and $\theta \in [0,1]$,

$$\Phi(h_1^\theta(x,\gamma)) = \Phi(h_2^\theta(x,\gamma)) = 0.$$

Consider $\rho \mapsto w(\rho,\Phi) = \sup\{\eta > 0, \sup_{|x-y|\le\eta} |\Phi(x) - \Phi(y)| \le \rho\}$. As Φ is uniformly continuous, $w(\rho,\Phi) > 0$ for every $\rho > 0$. Thanks to (62), for every $\rho > 0$, there exists $\delta_\rho \le \delta_0$ such that, for every $\gamma \le \delta_\rho$, $\theta \in [0,1]$,

$$\sup_{|x|\le M_{\delta_0}} |h_1^\theta(x,\gamma) - h_2^\theta(x,\gamma)| \le w(\rho,\Phi).$$

As $\gamma_k \overset{k\to+\infty}{\longrightarrow} 0$, there exists $k_\rho \in \mathbb{N}$ such that $\gamma_k \le \delta_\rho$ for $k \ge k_\rho$. By using that $H_n \overset{n\to+\infty}{\longrightarrow} +\infty$, we deduce that

$$\limsup_{n\to+\infty} \frac{1}{H_n}\sum_{k=1}^{n} \eta_k \sup_{\theta\in[0,1]} \|\Phi(h_1^\theta(x_{k-1},\gamma_k)) - \Phi(h_2^\theta(x_{k-1},\gamma_k))\|\Psi(x_{k-1}) \le \rho\overline{\Psi}_{\delta_0},$$

where $\overline{\Psi}_{\delta_0} := \sup\{|\Psi(x)|, |x| \le M_{\delta_0}\} < +\infty$ since Ψ is locally bounded. The result follows. $\square$

PROOF OF PROPOSITION 4. We have to inspect successively the $q \in (1/2, 1]$ and $q \in [0, 1/2]$ cases.

Case $q \in (1/2, 1]$. Let $f \in \mathcal{C}_K^2(\mathbb{R}^d)$. Decompose the infinitesimal generator as the sum of three terms defined by

$$A_1 f(x) = \langle \nabla f, b\rangle(x), \qquad A_2 f(x) = \mathrm{Tr}(\sigma^* D^2 f \sigma)(x),$$

$$A_3 f(x) = \int \bigl(f\bigl(x + \kappa(x)y\bigr) - f(x) - \langle \nabla f(x), \kappa(x)y\rangle 1_{\{|y|\le 1\}}\bigr)\pi(dy).$$

Set $\bar{X}_{k,1} = \bar{X}_{k-1} + \gamma_k b(\bar{X}_{k-1})$, $\bar{X}_{k,2} = \bar{X}_{k,1} + \sqrt{\gamma_k}\sigma(\bar{X}_{k-1})U_k$ and $\bar{X}_{k,3} = \bar{X}_{k,2} + \sqrt{\gamma_k}\kappa(\bar{X}_{k-1})\bar{Z}_k$. We then part the proof into three steps:

Step 1. $$\frac{1}{\gamma_k}\mathbb{E}\{f(\bar{X}_{k,1}) - f(\bar{X}_{k-1})/\mathcal{F}_{k-1}\} = A_1 f(\bar{X}_{k-1}) + R_1(\gamma_k, \bar{X}_{k-1})$$

$$\text{with } \frac{1}{H_n}\sum_{1}^{n} \eta_k R_1(\gamma_k, \bar{X}_{k-1}) \overset{n\to\infty}{\longrightarrow} 0.$$

Step 2. $$\frac{1}{\gamma_k}\mathbb{E}\{f(\bar{X}_{k,2}) - f(\bar{X}_{k,1})|\mathcal{F}_{k-1}\} = A_2 f(\bar{X}_{k-1}) + R_2(\gamma_k, \bar{X}_{k-1})$$

$$\text{with } \frac{1}{H_n}\sum_{1}^{n} \eta_k R_2(\gamma_k, \bar{X}_{k-1}) \overset{n\to\infty}{\longrightarrow} 0.$$

Step 3. $$\frac{1}{\gamma_k}\mathbb{E}\{f(\bar{X}_{k}) - f(\bar{X}_{k,2})|\mathcal{F}_{k-1}\} = A_3 f(\bar{X}_{k-1}) + R_3(\gamma_k, \bar{X}_{k-1})$$

$$\text{with } \frac{1}{H_n}\sum_{1}^{n} \eta_k R_3(\gamma_k, \bar{X}_{k-1}) \overset{n\to\infty}{\longrightarrow} 0.$$

The combination of the three steps yields Proposition 4. We refer to Proposition 4 of [15] for steps 1 and 2 and focus on the last step where the specificity of our jump Lévy setting appears. Since $\bar{X}_{k-1}$ is $\mathcal{F}_{k-1}$-measurable and $\bar{Z}_k$, U_k and $\mathcal{F}_{k-1}$ are independent, we have

$$\mathbb{E}\bigl\{f\bigl(\bar{X}_{k,2} + \kappa(\bar{X}_{k-1})\bar{Z}_k\bigr)|\mathcal{F}_{k-1}\bigr\} = Q_{\gamma_k} f(\bar{X}_{k-1}),$$

$$\text{where } Q_\gamma f(x) = \int_{\mathbb{R}^d} \mathbb{E}\bigl\{f\bigl(S_{x,\gamma,u} + \kappa(x)Z_\gamma\bigr)\bigr\}\mathbb{P}_{U_1}(du),$$

with $S_{x,\gamma,u}=x+\gamma b(x)+\sqrt{\gamma}\sigma(x)u$. Set $V_t=S_{x,\gamma,u}+\kappa(x)Z_t$. Applying Itô's formula to $(f(V_t))_{t\geq 0}$ yields

$$\begin{aligned} f(V_t)=f(S_{x,\gamma,u})+\int_0^t\langle\nabla f(V_{s^-}),\kappa(x)\,dY_s\rangle \\ +\sum_{0<s\leq t}\tilde{H}^f\big(S_{x,\gamma,u}+\kappa(x)Z_{s^-},x,\Delta Z_s\big)\end{aligned}\tag{63}$$

$$\text{where, } \tilde{H}^f(z,x,y)=f\big(z+\kappa(x)y\big)-f(z)-\langle\nabla f(z),\kappa(x)y\rangle 1_{\{|y|\leq 1\}}.\tag{64}$$

The process $(\int_0^t\langle\nabla f(V_{s^-}),\kappa(x)\,dY_s\rangle)$ is a true martingale since ∇f is bounded. The compensation formula and a change of variable yield

$$\begin{aligned}&\mathbb{E}\{f\big(S_{x,\gamma,u}+\kappa(x)Z_\gamma\big)\}\\ &\quad=\mathbb{E}\{f(V_\gamma)\}\\ &\quad=f(S_{x,\gamma,u})+\gamma\mathbb{E}\bigg\{\int_0^1 dv\int\pi(dy)\tilde{H}^f\big(S_{x,\gamma,u}+\kappa(x)Z_{v\gamma},x,y\big)\bigg\}.\end{aligned}$$

Since $A_3f(x)=\int\pi(dy)\tilde{H}^f(x,x,y)=\mathbb{E}\{\int_0^1 dv\int\pi(dy)\tilde{H}^f(x,x,y)\}$, it follows from the previous inequality that

$$\frac{1}{\gamma_k}\mathbb{E}\{f(\bar{X}_k)-f(\bar{X}_{k,2})|\mathcal{F}_{k-1}\}=A_3f(\bar{X}_{k-1})+R_3(\gamma_k,\bar{X}_{k-1}),$$

where,

$$R_3(\gamma,x)=\int\mathbb{E}\bigg\{\int_0^1 dv\int\pi(dy)\Delta\tilde{H}^f\big(S_{x,\gamma,u}+\kappa(x)Z_{v\gamma},x,x,y\big)\bigg\}\mathbb{P}_{U_1}(du)$$

with $\Delta\tilde{H}^f(z_1,z_2,x,y)=\tilde{H}^f(z_1,x,y)-\tilde{H}^f(z_2,x,y)$. We upper-bound R_3 by two terms: $R_{3,1}$ and $R_{3,2}$ that are associated to the small and big jumps components of (Z_t), namely,

$$\begin{aligned}&R_{3,1}(\gamma,x)\\ &\quad=\int\int_0^1 dv\int_{\{|y|\leq 1\}}\pi(dy)\mathbb{E}\big|\Delta\tilde{H}^f\big(S_{x,\gamma,u}+\kappa(x)Z_{v\gamma},x,x,y\big)\big|\mathbb{P}_{U_1}(du),\\ &R_{3,2}(\gamma,x)\\ &\quad=\int\int_0^1 dv\int_{\{|y|>1\}}\pi(dy)\mathbb{E}\big|\Delta\tilde{H}^f\big(S_{x,\gamma,u}+\kappa(x)Z_{v\gamma},x,x,y\big)\big|\mathbb{P}_{U_1}(du).\end{aligned}$$

We study successively $R_{3,1}$ and $R_{3,2}$. From Taylor's formula, we have for every y such that $|y|\leq 1$

$$\big|\Delta\tilde{H}^f\big(S_{x,\gamma,u}+\kappa(x)Z_{v\gamma},x,x,y\big)\big|\leq\tfrac{1}{2}R(Z,\gamma,x,u,v,y)|\kappa(x)y|^2,$$

where

$$R(Z,\gamma,x,u,v,y) = \sup_{\theta\in[0,1]} \|D^2 f\big(S_{x,\gamma,u}+\kappa(x)(Z_{v\gamma}+\theta y)\big) - D^2 f\big(x+\theta\kappa(x)y\big)\|.$$

By setting $\Phi = D^2 f$, $\Psi(x) = \|\kappa(x)\|^2|y|^2$,

$$h_1^\theta(x,\gamma) = S_{x,\gamma,u}+\kappa(x)(Z_{v\gamma}+\theta y) \quad \text{and} \quad h_2^\theta(x,\gamma) = x+\theta\kappa(x)y,$$

we want to show that the assumptions of Lemma 6 are a.s. fulfilled for every *fixed* u, v and y.

First, since $\kappa(x) \overset{|x|\to+\infty}{=} o(|x|)$, there exists a continuous function ε such that $\kappa(x) = |x|\varepsilon(x)$ and $\varepsilon(x) \overset{|x|\to\infty}{\longrightarrow} 0$. Therefore, as b and σ have sublinear growth, one checks that there exist some positive real constants C_1 and C_2 such that

$$\begin{cases} |S_{x,\gamma,u}+\kappa(x)(Z_{v\gamma}+\theta y)| \geq |x|\big(1-\gamma C_1 - (|Z_{v\gamma}|+|y|)|\varepsilon(x)|\big) - C_2, \\ |x+\theta\kappa(x)y| \geq |x|\big(1-|\varepsilon(x)||y|\big). \end{cases} \tag{65}$$

Let δ_0 be a positive number such that $1-\delta_0 C_1 > 0$. Since (Z_t) is locally bounded (as a càdlàg process) and $\varepsilon(x) \overset{|x|\to\infty}{\longrightarrow} 0$, there exists a.s. $M>0$,

$$\inf_{|x|>M,\gamma\in[0,\delta_0]} \big(1-\gamma C_1 - (|Z_{v\gamma}|+|y|)|\varepsilon(x)|\big) > 0.$$

It follows that a.s.,

$$\inf_{\theta\in[0,1],\gamma\in[0,\delta_0]} \big(|h_1^\theta|(x,\gamma)+|h_2^\theta|(x,\gamma)\big) \overset{|x|\to\infty}{\longrightarrow} +\infty.$$

Second, let K be a compact set of $\mathbb{R}^d$. We check that (62) holds. We have

$$\begin{aligned} &\sup_{x\in K,\theta\in[0,1]} |h_1^\theta(x,\gamma) - h_2^\theta(x)| \\ &\qquad \leq \sup_{x\in K}\big(\gamma|b(x)| + \sqrt{\gamma}\|\sigma(x)\||u| + \|\kappa(x)\|\,|Z_{v\gamma}|\big) \overset{\gamma\to 0}{\longrightarrow} 0 \qquad \text{a.s.} \end{aligned} \tag{66}$$

because b,σ,κ are locally bounded and $\lim_{t\to 0} Z_t = 0$ a.s. Thus, by Lemma 6, for any sequence $(x_k)_{k\in\mathbb{N}}$ of $\mathbb{R}^d$, for every $(u,v,y)\in\mathbb{R}^d\times[0,1]\times B_d(0,1)$,

$$\frac{1}{H_n}\sum_{k=1}^n \eta_k \Delta\tilde{H}^f\big(S_{x_{k-1},\gamma_k,u}+\kappa(x_{k-1})Z_{v\gamma_k}, x_{k-1}, x_{k-1}, y\big) \overset{n\to\infty}{\longrightarrow} 0 \qquad \text{a.s.} \tag{67}$$

Now, since ∇f and $D^2 f$ are bounded, we derive from Taylor's formula that, for every $z_1, z_2 \in \mathbb{R}^d$,

$$|\tilde{H}^f(z_2,x,y) - \tilde{H}^f(z_1,x,y)|1_{\{|y|\leq 1\}} \leq \begin{cases} 2\|\nabla f\|_\infty \|\kappa(x)\|\,|y|1_{\{|y|\leq 1\}}, \\ 2\|D^2 f\|_\infty \|\kappa(x)\|^2|y|^2 1_{\{|y|\leq 1\}}. \end{cases}$$

Then, for every $q \in (1/2, 1]$,

$$(68)\qquad \left|\Delta \tilde{H}^f\big(S_{x,\gamma,u} + \kappa(x) Z_{v\gamma}, x, x, y\big)\right| 1_{\{|y|\le 1\}} \le C\|\kappa(x)\|^{2q}|y|^{2q} 1_{\{|y|\le 1\}},$$

where $C = 2\max(\|\nabla f\|_\infty, \|D^2 f\|_\infty)$. Therefore, by assumption $(\mathrm{H}^2_{\mathrm{q}})$, we finally derive from (67), (68) and from the Lebesgue dominated convergence theorem that

$$(69)\qquad \begin{aligned} &\frac{1}{H_n}\sum_{k=1}^n \eta_k R_{3,1}(\gamma_k, x_{k-1}) \overset{n\to\infty}{\longrightarrow} 0 \\ &\qquad\qquad \text{if } \sup_{n\in\mathbb{N}} \frac{1}{H_n}\sum_{k=1}^n \eta_k \|\kappa(x_{k-1})\|^{2q} < \infty. \end{aligned}$$

We apply this result to $(x_k) = (\bar{X}_k)$. By (58), $\sup_{n\in\mathbb{N}} \bar{\nu}_n(\|\kappa\|^{2q}) < \infty$ a.s. Hence, it follows that $1/H_n \sum_{k=1}^n \eta_k R_{3,1}(\gamma_k, \bar{X}_{k-1}) \overset{n\to\infty}{\longrightarrow} 0$ a.s.

Now, let us focus on $R_{3,2}$. Set $\Delta f(z_1, z_2) = f(z_1) - f(z_2)$. Then,

$$\begin{aligned} R_{3,2}(\gamma, x) &= \int \mathbb{E}\left\{\int_0^1 dv \int_{\{|y|>1\}} \pi(dy) \Delta f\big(x, S_{x,\gamma,u} + \kappa(x) Z_{v\gamma}\big)\right\} \mathbb{P}_{U_1}(du) \\ &\quad + \int \mathbb{E}\left\{\int_0^1 dv \int_{\{|y|>1\}} \pi(dy) \right. \\ &\qquad\qquad \left. \times \Delta f\big(S_{x,\gamma,u} + \kappa(x)(Z_{v\gamma} + y), x + \kappa(x) y\big)\right\} \mathbb{P}_{U_1}(du). \end{aligned}$$

One proceeds as before. By using Lemma 6, one begins by showing that, for any sequence $(x_k)_{k\in\mathbb{N}}$, for every $(u, v, y) \in [0,1] \times \mathbb{R}^d \times B_d(0,1)^c$, a.s.,

$$(70)\qquad \begin{aligned} &\frac{1}{H_n}\sum_{k=1}^n \eta_k \Delta f\big(x_{k-1}, S_{x_{k-1},\gamma_k,u} + \kappa(x_{k-1}) Z_{u\gamma_k}\big) \overset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \\ &\frac{1}{H_n}\sum_{k=1}^n \eta_k \Delta f\big(S_{x_{k-1},\gamma_k,u} + \kappa(x_{k-1})(Z_{v\gamma_k} + y), x_{k-1} + \kappa(x_{k-1}) y\big) \overset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

By the dominated convergence theorem [which can be applied because $\pi(|y| > 1) < \infty$ and f is bounded], we deduce that, for any sequence $(x_k)_{k\in\mathbb{N}}$,

$$\frac{1}{H_n}\sum_{k=1}^n \eta_k R_{3,2}(\gamma_k, x_{k-1}) \overset{n\to\infty}{\longrightarrow} 0 \qquad \text{a.s.}$$

This completes the proof of Step 3 when $q \in (1, 2]$.

Case $q \le 1/2$. The reader can check that the assumption $q \in (1/2, 1]$ is used only once: when we want to apply the dominated convergence theorem for $R_{3,1}$ [see (68)]. Since inequality (68) is not true when $q < 1/2$, we need to decompose the infinitesimal generator in a slightly different way:

$$A_1 f(x) = \langle \nabla f, b^0 \rangle(x),$$
$$A_2 f(x) = \operatorname{Tr}(\sigma^* D^2 f \sigma)(x),$$
$$A_3 f(x) = \int \big(f\big(x + \kappa(x)y\big) - f(x)\big)\pi(dy).$$

Note that this decomposition is only possible when $q \le 1/2$. That means that with the notation (40), we decompose $\Delta \bar{X}_k$ with $h = 0$ and inspect the three induced steps. We do not go into further details since the proof is similar to the case $q > 1/2$. □

PROPOSITION 5. *Assume that the assumptions of Proposition 3 are fulfilled. Then*,

$$\lim_{n\to\infty} \frac{1}{H_n} \sum_{k=1}^{n} \frac{\eta_k}{\gamma_k} \mathbb{E}\{f(\bar{X}_k) - f(\bar{X}_{k-1}) | \mathcal{F}_{k-1}\} = 0 \qquad a.s.$$

PROOF. We do not detail the proof of this proposition which is an adaptation of Proposition 3 in [15]. □

5. Proof of the main theorems for Schemes (B) and (C). The aim of this section is to give a general idea of the proof for Schemes (B) and (C) and to overcome the main difficulties induced by the approximation of the jump component. For Scheme (A), main theorems have been proven in two successive steps. First, we focused on tightness results (Proposition 1) and then proved that every weak limiting distribution is invariant for $(X_t)_{t\ge 0}$ (Proposition 3). We follow the same process for Schemes (B) and (C). We will successively explain for both schemes why Proposition 1 and Proposition 3 remain valid.

5.1. *Almost sure tightness of $\bar{\nu}_n^B(\omega, dx)$ and $\bar{\nu}_n^C(\omega, dx)$.* The tightness result for Schemes (B) and (C) is strictly identical to Proposition 1 [in particular, assumption (9) is not necessary for tightness]. Looking carefully into the proof of this theorem for Scheme (A) shows that the properties of the jumps that we use are the following: the control of the moments of the jump components (Lemma 2) which is fundamental for Proposition 2, and independence between $(\bar{Y}_n)_{n\in\mathbb{N}}$, $(\bar{N}_n)_{n\in\mathbb{N}}$ and $(U_n)_{n\in\mathbb{N}}$. We show in Lemma 7 below that the controls of Lemma 2 hold true for the moments of the jump components of Schemes (B) and (C). Then, since Scheme (B) satisfies the independence properties, Proposition 1 follows in this case. In Scheme (C), $(\bar{Y}_n^C)_{n\in\mathbb{N}}$ and $(\bar{N}_n^C)_{n\in\mathbb{N}}$ are no longer independent. It raises several technical difficulties in the proof of Proposition 2 in case $p < 1$, but the process of the proof is the same. So, we only state a variant of Lemma 2 (see [21] for details).

LEMMA 7. *Let T_0 be a positive number and $T^n = \inf\{s > 0, |\Delta Z_t^n| > 0\}$.*

(i) *Let $p > 0$ such that* $(\mathrm{H}_\mathrm{p}^1)$ *holds. Then, for every $t \le T_0$ and $h > 0$,*

$$\mathbb{E}\{|N_{t\wedge T^n}^h|^{2p}\} \le t \int_{|y|>h} |y|^{2p}\pi(dy) \qquad \text{if } p > 0.$$

(ii) *Let τ be an $(\mathcal{F}_t)$-stopping time and $q \in [0,1]$ such that* $(\mathrm{H}_\mathrm{q}^2)$ *holds. Set $D_n^h = \{y, |y| \in (u_n, h]\}$ and $Y_t^{h,n} = \sum_{0<s\le t} \Delta Y_s^h 1_{\{\Delta Y_s^h \in D_n^h\}} - t\int_{D_n^h} y\pi(dy)$. Then,*

$$\begin{cases} \mathbb{E}\left\{\left|Y_{t\wedge\tau}^{h,n} + (t\wedge\tau)\displaystyle\int_{D_n^h} y\pi(dy)\right|^{2q}\right\} \le t\displaystyle\int_{|y|\le h} |y|^{2q}\pi(dy), & \text{if } q \in [0,1/2], \\ \mathbb{E}\{|Y_{t\wedge\tau}^{h,n}|^{2q}\} \le C_q t \displaystyle\int_{|y|\le h} |y|^{2q}\pi(dy), & \text{if } q \in (1/2,1]. \end{cases}$$

(iii) *Let $p \ge 1$ such that* $(\mathrm{H}_\mathrm{p}^1)$ *holds. Set $\hat{Z}_t^n = Z_t^n - t\int_{\{|y|>1\}} y\pi(dy)$. Then, there exists $\eta > 1$ such that, for every $T_0 > 0$, for every $\varepsilon > 0$, there exists $C_{\varepsilon,T_0,p} > 0$, $n_0 \in \mathbb{N}$ such that, for every $t \ge T_0$ and $n \ge n_0$,*

$$\mathbb{E}\{|\hat{Z}_t^n|^{2p}\} \le t\left(\int |y|^{2p}\pi(dy) + \varepsilon\right) + C_{\varepsilon,T_0,p} t^\eta$$

and

$$\mathbb{E}\{|\hat{Z}_{t\wedge T^n}^n|^{2p}\} \le t\left(\int |y|^{2p}\pi(dy) + \varepsilon\right) + C_{\varepsilon,T_0,p} t^\eta.$$

PROOF. The proof is left to the reader. □

REMARK 9. In (iii), the control is only valid for n sufficiently large but that does not make any problem since $(\mathrm{R}_{\mathrm{a,p}})$ just needs to be valid for sufficiently large n.

5.2. *Identification of the limit of $(\bar{\nu}_n^B)_{n\in\mathbb{N}}$ and $(\bar{\nu}_n^C)_{n\in\mathbb{N}}$.* The theorem which is obtained for $(\bar{\nu}_n^B)_{n\in\mathbb{N}}$ and $(\bar{\nu}_n^C)_{n\in\mathbb{N}}$ is strictly identical to Proposition 3 under the additional condition (9) for Scheme (C). We recall that the proof of Proposition 3 is based on two steps: Propositions 4 and 5. Proposition 5 is still valid without additional difficulties. However, the proof of the analogous result to Proposition 4 raises some new difficulties. Denote by $A^{k,B}$ and $A^{k,C}$ the operators on $\mathcal{C}_2^K(\mathbb{R}^d)$ with values in $\mathcal{C}_b(\mathbb{R}^d,\mathbb{R})$ defined by

$$A^{k,B} f(x) = \langle \nabla f, b\rangle(x) + \tfrac{1}{2}\mathrm{Tr}(\sigma^* D^2 f\sigma)(x) + \int_{\{|y|\ge u_k\}} \tilde{H}^f(x,x,y)\pi(dy),$$

$$A^{k,C} f(x) = A^{k,B} f(x) - \bigl(1 - \alpha_k(\gamma_k)\bigr)\int_{\{|y|\ge u_k\}} \tilde{H}^f(x,x,y)\pi(dy),$$

where $\alpha_k(t) = \frac{1-e^{-\pi(|y|>u_k)t}}{\pi(|y|>u_k)t}$. "$Af - A^{k,B}f$" and "$Af - A^{k,C}f$" can be viewed as the principal part of the weak error induced by the approximation in Schemes (B) and (C) [$A^{k,B}$ is the infinitesimal generator of (X_t^k), where (X_t^k) is solution to the SDE (1) driven by (Z_t^k) instead of (Z_t)]. Thus, one may expect that this error be negligible in the sense of our problem. This is the aim of Lemma 8.

LEMMA 8. *Assume* $(\mathrm{H}_\mathrm{q}^2)$. *Let* $(x_k)_{k\in\mathbb{N}}$ *be a sequence such that*

$$\sup_{n\geq 1}\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\|\kappa(x_{k-1})\|^{2q} < \infty. \tag{71}$$

Then, for every function $f \in \mathcal{C}_2^K(\mathbb{R}^d,\mathbb{R})$,

$$\lim_{n\to+\infty}\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\big(Af(x_{k-1}) - A^{k,B}f(x_{k-1})\big) = 0$$

and if $\pi(D_n)\gamma_n \overset{n\to+\infty}{\longrightarrow} 0$,

$$\lim_{n\to+\infty}\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\big(Af(x_{k-1}) - A^{k,C}f(x_{k-1})\big) = 0.$$

PROOF. Note that $A^{k,B}f(x) - Af(x) = \int_{\{|y|<u_k\}}\tilde{H}^f(x,x,y)\pi(dy)$. When $q \geq 1/2$, we deduce from Taylor's formula and the boundedness of ∇f and $D^2 f$ that there exists $C_q > 0$ such that

$$|\tilde{H}^f(x,x,y)|1_{\{|y|\leq u_k\}} \leq C_q\|\kappa(x)\|^{2q}|y|^{2q}1_{\{|y|\leq u_k\}}.$$

When $q \leq 1/2$, since f is a $2q$-Hölder function,

$$\begin{aligned}|\tilde{H}^f(x,x,y)|1_{\{|y|\leq u_k\}} &\leq [f]_{2q}\|\kappa(x)\|^{2q}|y|^{2q}1_{\{|y|\leq u_k\}}\\ &\quad + \sup_{x\in\operatorname{supp} f}|\nabla f(x)|\cdot\|\kappa(x)\|\,|y|1_{\{|y|\leq u_k\}}.\end{aligned}$$

By setting $v_{k,q} = \int_{\{|y|<u_k\}}|y|^{2q}\pi(dy)$, we have

$$|Af(x_{k-1}) - A^{k,B}f(x_{k-1})| \leq \begin{cases} C\big(v_{k,q}\|\kappa(x_{k-1})\|^{2q} + v_{k,1}\big), & \text{if } q\leq 1/2,\\ Cv_{k,q}\|\kappa(x_{k-1})\|^{2q}, & \text{if } q\geq 1/2.\end{cases}$$

Since $v_{k,\alpha} \overset{k\to\infty}{\longrightarrow} 0$ for every $\alpha \geq q$ under assumption $(\mathrm{H}_\mathrm{q}^2)$, the first result follows from (71). One deduces the second inequality by checking that

$$|A^{k,B}f(x) - A^{k,C}f(x)| \leq C\pi(D_k)\gamma_k\big(1 + \|\kappa(x)\|^{2q}\big). \qquad \square$$

Set

$$R_3^{B,k}(\gamma_k,\bar{X}_{k-1}^B)=\frac{\mathbb{E}\{f(\bar{X}_k^B)-f(\bar{X}_{k-1}^B)|\mathcal{F}_{k-1}^B\}}{\gamma_k}-A^{k,B}f(\bar{X}_{k-1}^B),$$

$$R_3^{C,k}(\gamma_k,\bar{X}_{k-1}^C)=\frac{\mathbb{E}\{f(\bar{X}_k^C)-f(\bar{X}_{k-1}^C)|\mathcal{F}_{k-1}^C\}}{\gamma_k}-A^{k,C}f(\bar{X}_{k-1}^C).$$

The rest of the proof then amounts to proving that

$$\lim_{n\to\infty}\frac{1}{H_n}\sum_{k=1}^n\eta_k R_3^{B,k}(\gamma_k,\bar{X}_{k-1}^B)=0$$

and

$$\lim_{n\to\infty}\frac{1}{H_n}\sum_{k=1}^n\eta_k R_3^{C,k}(\gamma_k,\bar{X}_{k-1}^C)=0.$$

We do not detail this proof based on the same approach as the proof of Proposition 4 (see [21] for more details). However, we want to derive the main difficulties from the proof. For Scheme (B), one deduces from the Ito formula that

$$|R_3^{k,B}(\gamma,x)|\le\int\int_0^1 dv\int\pi(dy)\mathbb{E}\big|\Delta\tilde{H}^f\big(S_{x,\gamma,u}+\kappa(x)Z_{v\gamma}^k,x,x,y\big)\big|\mathbb{P}_{U_1}(du).$$

The right-hand term can be written $R_{3,1}^{B,k}(\gamma,x)+R_{3,2}^{B,k}(\gamma,x)$, where $R_{3,1}^{B,k}(\gamma,x)$ [resp. $R_{3,2}^{B,k}(\gamma,x)$] is simply derived from $R_{3,1}(\gamma,x)$ [resp. $R_{3,2}(\gamma,x)$], defined in the proof of Proposition 3, by replacing Z with Z^k. We focus on $R_{3,1}^{B,k}$. One observes that the controls (65) and (66) used for $R_{3,1}$ no longer work since the jump component depends on n. An idea is to use the Skorokhod representation theorem (see, e.g., [24]) in order to replace (Z^k) by a uniformly controllable sequence.

LEMMA 9. *There exist a sequence of càdlàg processes $(\tilde{Z}^n)$ and a càdlàg process $\tilde{Z}$ such that $\tilde{Z}^n\stackrel{\mathcal{L}}{=}Z^n$ for every $n\ge1$, $\tilde{Z}\stackrel{\mathcal{L}}{=}Z$ and $\tilde{Z}^n\to\tilde{Z}$ a.s. for the Skorokhod topology. In particular,*

$$\sup_{n\in\mathbb{N}}\sup_{0\le s\le T}|\tilde{Z}_s^n|<+\infty\qquad\forall T>0\quad\textit{and}$$
$$\limsup_{n\to+\infty,\gamma\to0}\sup_{0\le s\le\gamma}|\tilde{Z}_s^n|=0\qquad a.s.\tag{72}$$

PROOF. Z^n converges locally uniformly in L^2 toward Z, hence, in distribution for the Skorokhod (Polish) topology. Thanks to the Skorokhod representation theorem, there exists $(\tilde{Z}^n)_{n\in\mathbb{N}}$ and $\tilde{Z}$ with $\tilde{Z}^n\stackrel{\mathcal{L}}{=}Z^n$ and $\tilde{Z}\stackrel{\mathcal{L}}{=}Z$ such that $\tilde{Z}^n$ tends a.s. toward $\tilde{Z}$ for Skorokhod topology. The assertion (72) easily follows from the continuity of $\alpha\mapsto\|\alpha\|_{\sup}$ and $\alpha\mapsto\alpha(0)$ for the Skorokhod topology. $\square$

Since $R^{B,k}_{3,1}$ only depends on the law of Z^n, one can replace Z^n with $\tilde{Z}^n$. Then, we use (72) as an alternative to the local boundedness and the continuity at $t=0$ of (Z_t) needed in (65) and (66) respectively. A result analogous to (67) follows. The idea is the same for $R^{B,k}_{3,2}$.

Finally, for Scheme (C), the result essentially follows from the following remark:

$$\sup_{0<s\le t} |Z^n_{s\wedge T^n}| \le \sup_{0<s\le t} |Z^n_s|.$$

This means that the remainders in Scheme (C) are easier to control than those of Scheme (B). For more details, we refer to [21].

6. A theoretical application. The "classical" a.s. CLT due to Brosamler [6] and Schatte [26] is the following result. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with values in $\mathbb{R}^d$ such that $\mathbb{E}U_1 = 0$ and $\Sigma_{U_1} = I_d$. Then,

$$\mathbb{P}\text{-a.s} \qquad \frac{1}{\ln n}\sum_{k=1}^{n}\frac{1}{k}\delta_{1/\sqrt{k}\sum_{i=1}^{k}U_i} \Longrightarrow \mathcal{N}(0, I_d).$$

This result is obviously connected with the central limit theorem which expresses the fact that every square-integrable centered random variable is in the domain of normal attraction of the normal law. When the square-integrability no longer holds, Berkes, Horvath and Khoshnevisan [3] obtained an extension of this result connected with the nonsquare-integrable attractive laws which are stable laws [with index $\alpha \in (0,2)$]. We are going to show that we can deduce this extension from Theorem 2.

Let c be a positive number and denote by $(Z^{\alpha,c}_t)_{t\ge 0}$ a symmetrical one-dimensional α-stable process such that the characteristic function ϕ of $Z^{\alpha,c}_1$ satisfies $\phi(u) = e^{-\rho|u|^\alpha}$, where $\rho = 2c\int_0^{+\infty} y^{-\alpha}\sin y\,dy$. Consider a sequence $(V_n)_{n\in\mathbb{N}}$ of symmetrical i.i.d. random variables such that, for $x>0$,

$$\mathbb{P}(V_1 \ge x) = \frac{c}{x^\alpha} + \delta(x)(x^{-\alpha}(\ln x)^{-\gamma}) \quad \text{with } \gamma > 0 \text{ and } \delta(x) \xrightarrow{x\to+\infty} 0. \tag{73}$$

By a result of Gnedenko and Kolmogorov (see [10]), we know that

$$\frac{V_1 + \cdots + V_n}{n^{1/\alpha}} \Longrightarrow Z^{\alpha,c}_1.$$

Then, the following a.s. CLT holds:

THEOREM 4. *Let $(\eta_k)_{k\in\mathbb{N}}$ be a nonincreasing sequence with infinite sum such that $(k\eta_k)_{k\in\mathbb{N}}$ is nonincreasing and set $\nu = \mathcal{L}(Z^{\alpha,c}_1)$. Then, if $\gamma > \frac{1}{\alpha}$, a.s.,*

$$\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\delta_{(V_1+\cdots+V_k)/k^{1/\alpha}} \xRightarrow{(\mathbb{R})} \nu.$$

In particular,

$$\frac{1}{\ln n}\sum_{k=1}^{n}\frac{1}{k}\delta_{(V_1+\cdots+V_k)/k^{1/\alpha}} \overset{(\mathbb{R})}{\Longrightarrow} \nu \qquad a.s.$$

In order to prove this theorem, we first need an almost sure invariance principle due to Stout (see [27] or [3]).

PROPOSITION 6. *Let $(V_n)_{n\geq 1}$ and $(\zeta_n)_{n\geq 1}$ be sequences of i.i.d. random variables such that $\zeta_1 \overset{\mathcal{L}}{=} Z_1^{\alpha,c}$ and V_1 is defined as above. Then, if $\gamma > \frac{1}{\alpha}$, there exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and sequences of i.i.d. random variables $(\hat{V}_n)_{n\geq 1}$ and $(\hat{\zeta}_n)_{n\geq 1}$ such that $\hat{V}_1 \overset{\mathcal{L}}{=} V_1$, $\hat{\zeta}_1 \overset{\mathcal{L}}{=} \zeta_1$ and*

$$\sum_{i=1}^{n}\hat{\zeta}_i - \sum_{i=1}^{n}\hat{V}_i \overset{n\to+\infty}{=} o(n^{1/\alpha}(\ln n)^{-\rho}) \qquad a.s.\ \forall \rho \in \left(0, \gamma - \frac{1}{\alpha}\right). \tag{74}$$

PROOF OF THEOREM 4. First, we assume that $V_1 = \zeta_1 \overset{\mathcal{L}}{=} Z_1^{\alpha,c}$. Set

$$S_n = \frac{\zeta_1 + \cdots + \zeta_{n+1}}{(n+1)^{1/\alpha}} \qquad \forall n \geq 0.$$

One easily checks that $S_{n+1} = S_n - \frac{1}{\alpha}\gamma_{n+1}S_n + \gamma_{n+1}^{\frac{1}{\alpha}}\zeta_{n+2} + R_{n+1}$ with $\gamma_n = \frac{1}{n+1}$ and $R_{n+1} = O(\gamma_{n+1}^2|S_n|)$. The idea of the proof is to compare $(S_n)_{n\geq 0}$ with the exact Euler scheme with initial value ζ_1 and step sequence (γ_n) associated with the SDE $(\mathbf{E}_{\alpha,\mathbf{c}})$ defined by $dX_t = -\frac{1}{\alpha}X_{t^-}\,dt + dZ_t^{\alpha,c}$. Since $(Z_t^{\alpha,c})_{t\geq 0}$ is a self-similar process with index $1/\alpha$ (see, e.g., [25]), its Euler scheme can be written

$$\bar{X}_0 = \zeta_1 \quad \text{and} \quad \bar{X}_{n+1} = \bar{X}_n - \frac{1}{\alpha}\gamma_{n+1}\bar{X}_n + \gamma_{n+1}^{\frac{1}{\alpha}}\zeta_{n+2}.$$

As an Ornstein–Uhlenbeck process driven by a symmetric stable law, (X_t) admits a unique invariant measure ν and $\nu = \mathcal{L}(Z_1^{\alpha,c})$ (see [25], page 188). Since κ is bounded, assumptions of Theorem 2 are clearly fulfilled with $V(x) = 1 + x^2$, $a = 1$ and for any $p \in (0, \alpha/2)$ and $q \in (\alpha/2, 1)$. (In the rest of the paper the initial value of the Euler scheme is supposed to be constant, but it is obvious that Theorem 2 is still true when $\bar{X}_0$ is a random variable satisfying $\mathbb{E}\{|\bar{X}_0|^{2p}\} < +\infty$.) Hence, it follows from Theorem 2 that

$$\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\delta_{\bar{X}_{k-1}} \overset{n\to+\infty}{\Longrightarrow} \nu \qquad \text{a.s.} \tag{75}$$

Then, by using that $|f(S_k) - f(\bar{X}_k)| \leq C(|S_k - \bar{X}_k| \wedge 1)$ for every Lipschitz bounded function f, one easily checks that Theorem 4 holds with $V_1 = \zeta_1$ if

$$\Delta_n := S_n - \bar{X}_n \overset{n\to+\infty}{\longrightarrow} 0 \qquad \text{a.s.} \tag{76}$$

Let us show (76). One first checks that

$$\Delta_0 = 0 \quad \text{and} \quad \Delta_n = \left(1 - \frac{1}{\alpha(n+1)}\right)\Delta_{n-1} + R_n \qquad \forall n \geq 1.$$

Setting $k_0 = \inf\{k \geq 0, k - 1/\alpha > 0\}$, we deduce that, for every $n \geq k_0 + 1$,

$$\Delta_n = \frac{\Delta_{k_0}}{c_n} + \frac{1}{c_n}\sum_{k=k_0+1}^{n} c_k R_k \qquad \text{with } c_n = \prod_{k=k_0+1}^{n}\left(1 - \frac{1}{\alpha(k+1)}\right)^{-1}.$$

One observes that

$$c_n = \exp\left(-\sum_{k=k_0+2}^{n+1} \ln\left(1 - \frac{1}{\alpha k}\right)\right) = \exp\left(\frac{1}{\alpha}\sum_{k=k_0+2}^{n+1} \frac{1}{k} + O\left(\frac{1}{k^2}\right)\right) \overset{n\to+\infty}{\sim} C' n^{1/\alpha}.$$

Then, $\Delta_{k_0}/c_n \to 0$ a.s. Hence, (76) holds if we check that $1/c_n \sum_{k=k_0+1}^{n} c_k R_k \to 0$ a.s. First, if $\alpha > 1$, as ζ_1 is integrable and $R_k = O(S_{k-1}/(k+1)^2)$, we have

$$\begin{aligned}\sum_{k\geq 1} \mathbb{E}\{|R_k|\} &\leq C\sum_{k\geq 1}\frac{\mathbb{E}\{|S_{k-1}|\}}{(k+1)^2} \leq C\sum_{k\geq 1}\frac{k\mathbb{E}\{|\zeta_1|\}}{(k+1)^{1/(\alpha+2)}}\\ &\leq C\sum_{k\geq 1}\frac{1}{(k+1)^{1+1/\alpha}} < +\infty.\end{aligned}$$

We deduce that $\sum_{k\geq 1}|R_k| < +\infty$ a.s. Since $c_n \overset{n\to+\infty}{\longrightarrow} +\infty$, we derive from Kronecker's lemma that

$$\frac{1}{c_n}\sum_{k=k_0+1}^{n} c_k R_k \overset{n\to+\infty}{\longrightarrow} 0 \quad \Longrightarrow \quad \Delta_n \overset{n\to+\infty}{\longrightarrow} 0 \qquad \text{a.s. if } \alpha > 1.$$

Second, if $\alpha \leq 1$, ζ_1 has a moment of order θ for every $\theta < \alpha$. It follows from inequality (25) that

$$\mathbb{E}\{|R_k|^\theta\} \leq C\frac{k}{(k+1)^{2\theta+\theta/\alpha}}\mathbb{E}\{|\zeta_1|^\theta\} \leq \frac{C}{(k+1)^{\theta(2+1/\alpha)-1}}.$$

Therefore, if θ satisfies $\theta(2+\frac{1}{\alpha}) - 1 > 1$, that is, if $\frac{2\alpha}{2+\alpha} < \theta < \alpha$, we have $\sum_{k\geq 1}|R_k|^\theta < +\infty$ a.s. Hence, by inequality (25) and Kronecker's lemma, it follows that

$$\left|\frac{1}{c_n}\sum_{k=k_0+1}^{n} c_k R_k\right|^\theta \leq \frac{1}{c_n^\theta}\sum_{k=k_0+1}^{n} c_k^\theta |R_k|^\theta \overset{n\to+\infty}{\longrightarrow} 0 \qquad \text{a.s.}$$

and the theorem is proved when $V_1 = \zeta_1$. Now, consider a sequence $(V_n)_{n\geq 0}$ of i.i.d. symmetric random variables satisfying (73). Since Theorem 4 is true for $(\zeta_n)_{n\geq 1}$, it is also true for every sequence $(\hat{\zeta}_n)_{n\geq 1}$ of i.i.d. random variables satisfying $\hat{\zeta}_1 \overset{\mathcal{L}}{=} \zeta_1$. By taking $(\hat{\zeta}_n)_{n\geq 1}$ such that Proposition 6 holds, we derive from (74) that there exists a sequence of i.i.d. random variables $(\hat{V}_n)_{n\geq 1}$ such that $V_1 \overset{\mathcal{L}}{=} \hat{V}_1$

and

$$\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\delta_{(\hat{V}_1+\cdots+\hat{V}_k)/k^{1/\alpha}} \overset{n\to+\infty}{\Longrightarrow} \nu \qquad \text{a.s.} \tag{77}$$

As $(V_n)_{n\geq 1}$ and $(\hat{V}_n)_{n\geq 1}$ are sequences of i.i.d. random variables such that $V_1 \overset{\mathcal{L}}{=} \hat{V}_1$, (77) is also true for $(V_n)_{n\geq 1}$. $\square$

7. Simulations.

EXAMPLE 4. Denote by $(Z_t)_{t\geq 0}$ a Cauchy process with parameter 1 [with Lévy measure defined by $\pi(dy) = 1/y^2\,dy$] and consider the Ornstein–Uhlenbeck process solution to $dX_t = -X_{t^-}\,dt + dZ_t$ corresponding to $(E_{1,1})$ defined in the previous subsection. The unique invariant measure of $(X_t)_{t\geq 0}$ is the Cauchy law (see [25], page 188) and the assumptions of Theorem 2 are fulfilled with $V(x) = 1 + x^2$, $a = 1$ and every $p \in (0, 1/2)$ and $q \in (1/2, 1)$. Therefore,

$$\bar{\nu}_n(f), \bar{\nu}_n^B(f), \bar{\nu}_n^C(f) \overset{n\to+\infty}{\longrightarrow} \int \frac{f(x)}{\pi(1+x^2)}\,dx \qquad \text{a.s.}$$

for every f satisfying $f = O(|x|^{1/2-\varepsilon})$ with $\varepsilon > 0$. In Figures 1, 2 and 3, one

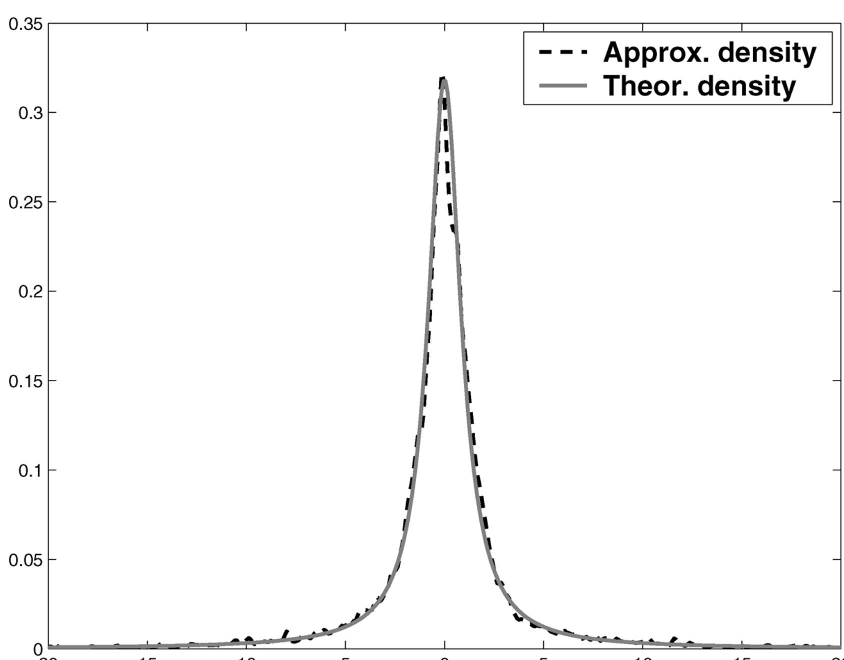

FIG. 1. *Scheme* (A), $t = 12.5$.

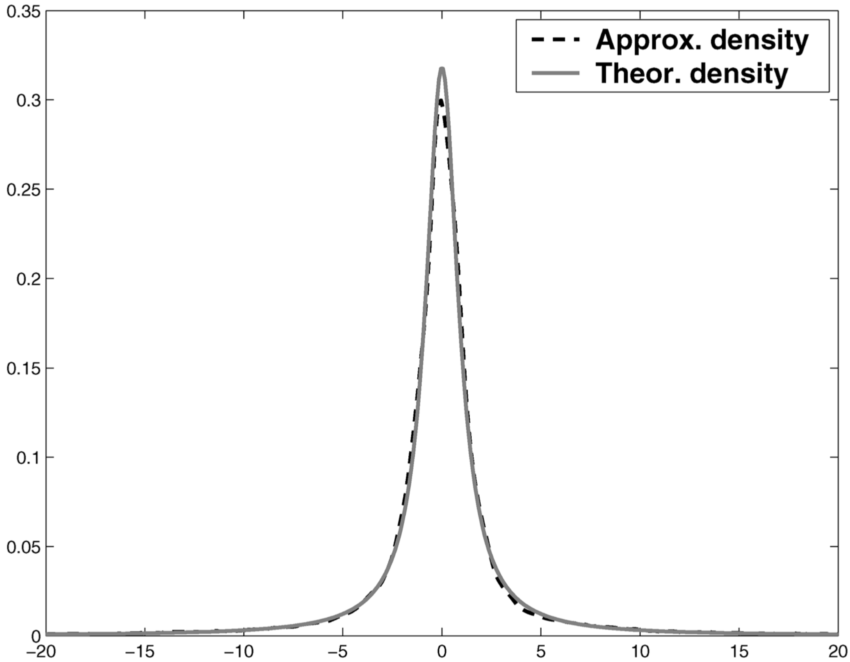

FIG. 2. *Scheme* (B), $t = 16.6$.

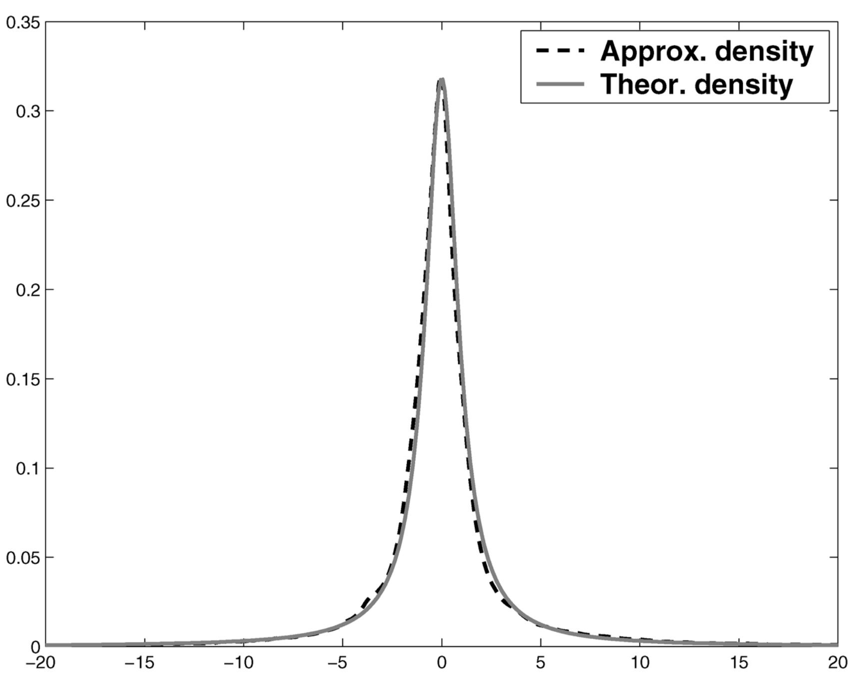

FIG. 3. *Scheme* (C), $t = 16.4$.

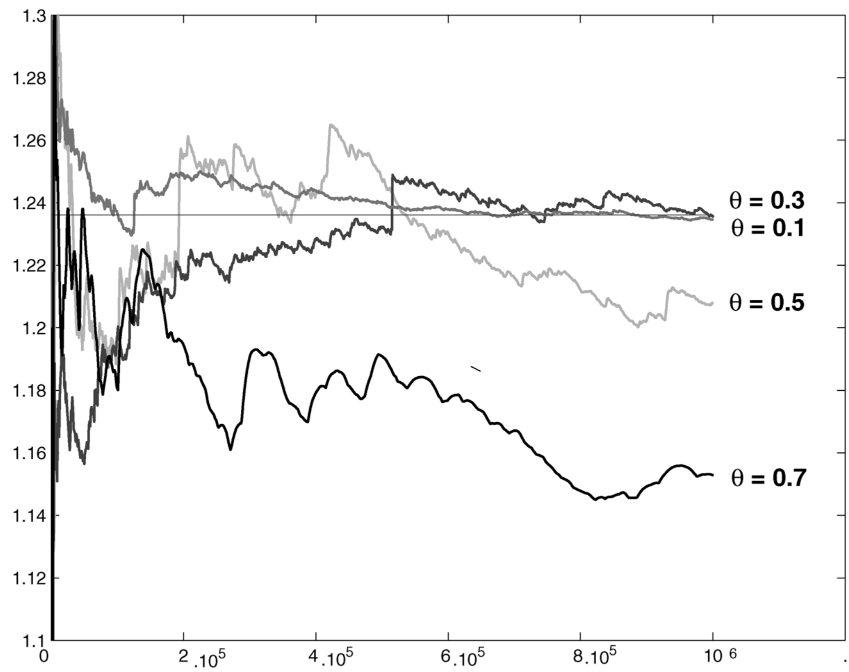

FIG. 4. *Scheme* (A).

compares the theoretical density of the invariant measure with the density obtained by convolution of each of the empirical measures by a Gaussian kernel for $N = 5.10^4$. We choose $\eta_n = \gamma_n = 1/\sqrt{n}$, $u_n = \sqrt{\gamma_n}$ [so that $\pi(D_n)\gamma_n \to 0$] and t indicates the CPU time. In order to have a more precise idea of the differences between the three Euler schemes, we simulate and represent on Figures 4, 5 and 6 the sequence $(\bar{\nu}_n(f))$ with $f(x) = |x|^{0.4}$, for several choices of polynomial steps. We set $\gamma_n = \eta_n = 1/n^\theta$ and $u_n = \gamma_n$ (resp. $u_n = \sqrt{\gamma}_n$) for Scheme (B) [resp. for Scheme (C)]. We observe that, among the tested steps, the best rate seems to be obtained for $\theta = 0.3$. Notably, in Schemes (B) and (C), we see that, on the one hand, if the step decreases too slowly (e.g., $\theta = 0.7$), so is the stabilization and, on the other hand, if the steps decreases too fast (e.g., when $\theta = 0.1$), there are not sufficient variations to correct the error.

REMARK 10. In [20] we study the rate of convergence of these procedures in terms of steps, weights and truncation thresholds. This enlightens these first numerical illustrations.

EXAMPLE 5. Now we deal with the following SDE:

$$dX_t = (1 - X_{t^-})\,dt - X_{t^-}\,dZ_t,$$

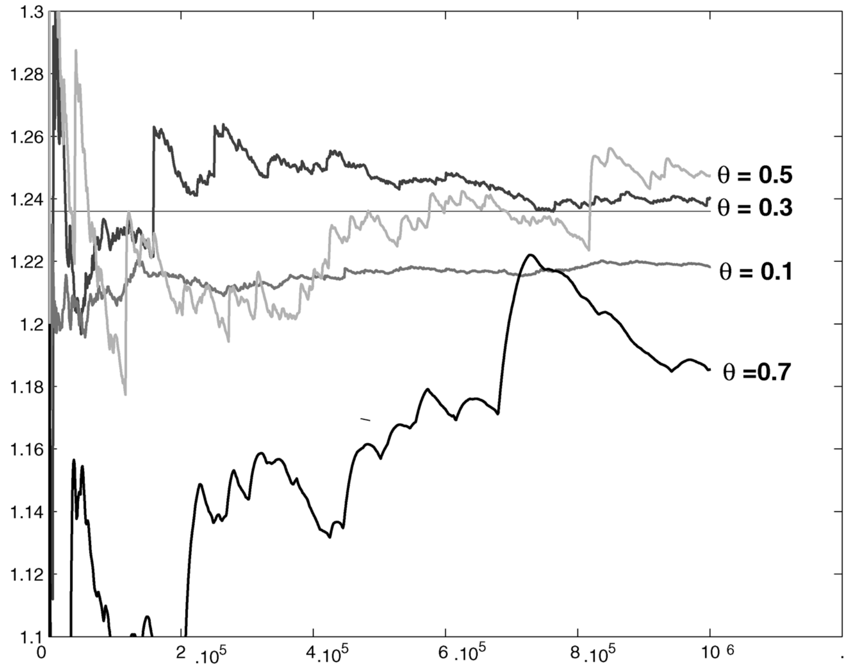

FIG. 5. *Scheme* (B).

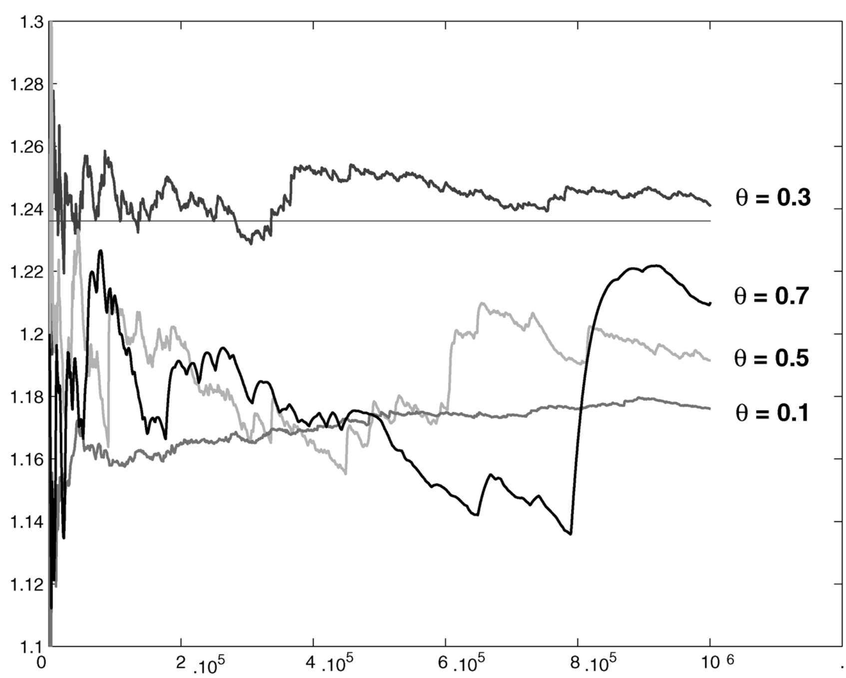

FIG. 6. *Scheme* (C).

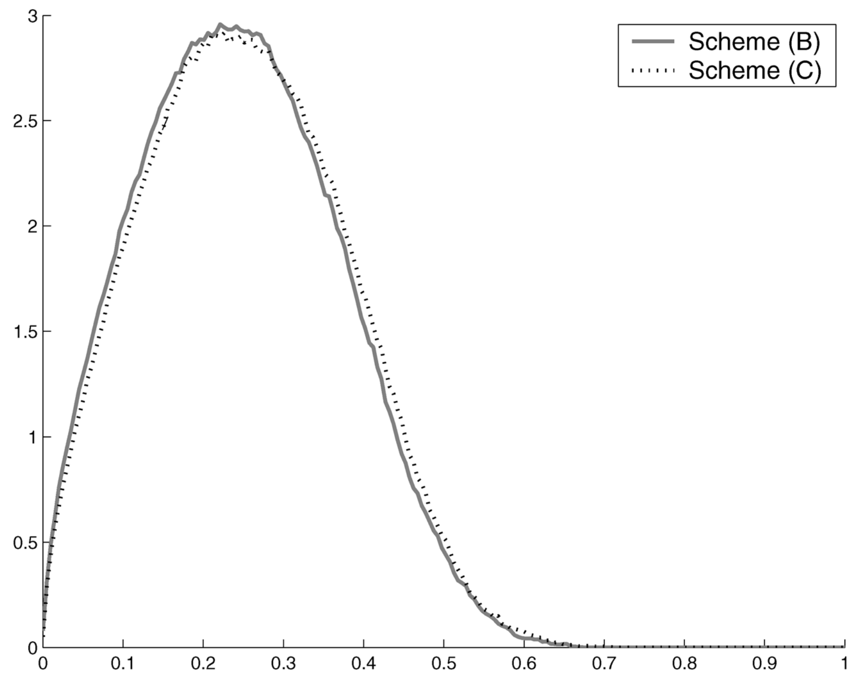

FIG. 7. *Approximated density*, $N = 10^6$.

where $(Z_t)_{t\geq 0}$ is a drift-free subordinator with Lévy measure π defined by

$$\pi(dy) = \frac{f_{3/2,1/2}(y)}{y^2}\, dy,$$

where $f_{a,b}$ is the density function of the $\beta(a, b)$-distribution. This SDE models the dust generated by a particular EFC process (see Introduction) whose sudden dislocations do not create dust, having parameters (according to the notation of [2]):

$$c_k = 0, \qquad c_e = 1, \qquad \nu_{coag}(dy) = f_{3/2,1/2}(y)\, dy.$$

One checks that $(\mathrm{S}_{1,1,1/2})$ is satisfied with $V(x) = 1 + x^2$. However, we do not have $\kappa(x) = o(|x|)$, but since $\mathrm{supp}(\pi)$ is restrained to $[0, 1]$ without singularities in 0 and 1, we are able to show that assumption $\kappa(x) = o(x)$ is no longer necessary in this case. In Figure 7 we represent the approximation of the invariant measure obtained for Schemes (B) and (C) [we are not able to simulate Scheme (A) in that case].

Acknowledgments. Thanks to Gilles Pagès for extensive discussions and suggestions.

REFERENCES

[1] BARNDORFF-NIELSEN, O., MIKOSH, T. and RESNICK, S. (2001). *Lévy Processes: Theory and Applications*. Birkhäuser, Boston. MR1833689

[2] BERESTYCKI, J. (2004). Exchangeable fragmentation-coalescence processes and their equilibrium measures. *Electron J. Probab.* **9** 770–824. MR2110018

[3] BERKES, I., HORVATH, L. and KHOSHNEVISAN, D. (1998). Logarithmic averages of stable random variables are asymptotically normal. *Stoch. Process. Appl.* **77** 35–51. MR1644606

[4] BERTOIN, J. (1996). *Lévy Processes*. Cambridge Univ. Press. MR1406564

[5] BOROVKOV, A. A. (1998). *Ergodicity and Stability of Stochastic Processes*. Wiley, New York. MR1658404

[6] BROSAMLER, G. (1988). An almost everywhere central limit theorem. *Math. Proc. Cambridge Phil. Soc.* **104** 561–574. MR0957261

[7] ETHIER, S. and KURTZ, T. (1986). *Markov Processes*, *Characterization and Convergence*. Wiley, New York. MR0838085

[8] DENG, S. (2000). Pricing electricity derivatives under alternative stochastic spot price models. In *Proceedings of the 33rd Hawaii International Conference on System Sciences* **4**. IEEE, Washington, DC.

[9] DUFLO, M. (1997). *Random Iterative Models*. Springer, Berlin. MR1485774

[10] GNEDENKO, B. and KOLMOGOROV, A. (1954). *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, Cambridge, MA. MR0062975

[11] HALL, P. and HEYDE, C. (1980). *Martingale Limit Theory and Its Application*. Academic Press, New York. MR0624435

[12] HAS'MINSKII, R. Z. (1981). *Stochastic Stability of Differential Equations*. Sijthoff and Noordhoff, Alphen aan de Rijn. MR0600653

[13] JACOD, J. and PROTTER, P. (1991). Une remarque sur les équations différentielles à solutions Markoviennes. *Séminaire de Probabilités XXV. Lecture Notes in Math.* **1485** 138–139. Springer, Berlin. MR1187777

[14] LAMBERTON, D. and PAGÈS, G. (2002). Recursive computation of the invariant distribution of a diffusion. *Bernoulli* **8** 367–405. MR1913112

[15] LAMBERTON, D. and PAGÈS, G. (2003). Recursive computation of the invariant distribution of a diffusion: The case of a weakly mean reverting drift. *Stoch. Dynamics* **4** 435–451. MR2030742

[16] LEMAIRE, V. (2005). Estimation numérique de la mesure invariante d'une diffusion. Ph.D. thesis, Univ. de Marne-La Vallée. Available at http://tel.ccsd.cnrs.fr/tel-00011281.

[17] LEMAIRE, V. (2007). An adaptative scheme for the approximation of dissipative systems. *Stochastic Process. Appl.* **117** 1491–1518.

[18] MEYN, S. P. and TWEEDIE, R. L. (1993). *Markov Chains and Stochastic Stability*. Springer, London. MR1287609

[19] PAGÉS, G. (2001). Sur quelques algorithmes récursifs pour les probabilités numériques. *ESAIM Probab. Statist.* **5** 141–170. MR1875668

[20] PANLOUP, F. (2007). Computation of the invariant measure of a Lévy driven SDE: Rate of convergence. *Stochastic Process. Appl.* To appear.

[21] PANLOUP, F. (2006). Approximation récursive du régime stationnaire d'une EDS avec sauts. Ph.D. thesis, Univ. de Paris VI. Available at http://tel.archives-ouvertes.fr/tel-00120508.

[22] PROTTER, P. (1990). *Stochastic Integration and Differential Equations*. Springer, Berlin. MR1037262

[23] PROTTER, P. and TALAY, D. (1997). The Euler scheme for Lévy driven stochastic differential equations. *Ann. Probab.* **25** 393–423. MR1428514

[24] ROGERS, L. C. G. and WILLIAMS, D. (1987). *Diffusions*, *Markov Processes*, *and Martingales*. **2**. Wiley, New York. MR0921238

[25] SATO, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Univ. Press. MR1739520

[26] SCHATTE, P. (1988). On strong versions of the central limit theorem. *Math. Nachr.* **137** 249–256. MR0968997

[27] STOUT, W. F. (1979). Almost sure invariance principles when $\mathbb{E}X_1^2 = \infty$. *Z. Wahrsch. Verw. Gebiete* **49** 23–32. MR0539661

[28] TALAY, D. (1990). Second order discretization schemes of stochastic differential systems for the computation of the invariant law. *Stoch. Stoch. Rep.* **29** 13–36.

LABORATOIRE DE PROBABILITÉS
ET MODÈLES ALÉATOIRES
UNIVERSITÉ PARIS 6
UMR 7599
BUREAU 4D1
175 RUE DU CHEVALERET
F-75013 PARIS
FRANCE
E-MAIL: fpanloup@insa-toulouse.fr