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On intermediate level sets of two-dimensional discrete Gaussian free field

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Abstract. We consider the discrete Gaussian Free Field (DGFF) in scaled-up (square-lattice) versions of suitably regular continuum domains $D \subset \mathbb{C}$ and describe the scaling limit, including local structure, of the level sets at heights growing as a λ-multiple of the height of the absolute maximum, for any $\lambda \in (0, 1)$. We prove that, in the scaling limit, the scaled spatial position of a typical point x sampled from this level set is distributed according to a Liouville Quantum Gravity (LQG) measure in D at parameter equal λ-times its critical value, the field value at x has an exponential intensity measure and the configuration near x reduced by the value at x has the law of a pinned DGFF reduced by a suitable multiple of the potential kernel. In particular, the law of the total size of the level set, properly-normalized, converges to that of the total mass of the LQG measure. This sharpens considerably an earlier conclusion by Daviaud (*Ann. Probab.* **34** (2006) 962–986).

Résumé. Nous considérons le champs Gaussien libre discret (DGFF) sur des versions renormalisées sur le réseau carré de domaines continus suffisamment réguliers $D \subset \mathbb{C}$ et décrivons la limite d'échelle, incluant la structure locale, des lignes de niveau lorsque que la hauteur croît comme λ-fois la hauteur du maximum absolu, pour tout $\lambda \in (0, 1)$. Nous montons que, dans la limite d'échelle, la position normalisée d'un point typique x tiré aléatoirement sur cette ligne de niveau a la loi de la mesure de Gravité Quantique de Liouville (LQG) dans D avec paramètre égal à λ-fois sa valeur critique, la valeur du champs en x ayant une mesure d'intensité exponentielle et la configuration près de x, réduite par la valeur en x, ayant la loi d'un champ libre épinglé DGFF réduit par un multiple adéquat du noyau potentiel. En particulier, la loi de la taille totale de la ligne de niveau, proprement normalisée, converge vers celle de la masse totale de la mesure LQG. Ceci améliore considérablement les résultats précédents de Daviaud (*Ann. Probab.* **34** (2006) 962–986).

MSC: 60G15; 82B41; 60G70; 62G30; 60G55; 60G57

Keywords: Gaussian Free Field; Level set; Point process; Liouville Quantum Gravity; Scaling limit; Conformal invariance

1. Introduction

It has long been recognized that the two-dimensional *continuum* Gaussian Free Field (CGFF) offers a variety of constructions of random fractals with an underlying conformally-invariant structure. This has been used fruitfully in the work of Schramm and Sheffield [30] on the convergence to SLE_4 of the level lines at specific heights of order unity and the ensuing coupling of the whole field to the Conformal Loop Ensemble by Sheffield and Werner [32,33]. Other examples include the study of the Gaussian Multiplicative Chaos associated with the CGFF, known widely as the Liouville Quantum Gravity measure. This goes back, in greater generality, to the works of Kahane [21] (see Section 2.3 below) and is also a part of the recent research programs of Miller and Sheffield on its connection to the Brownian Map [24,28] and on imaginary geometry [23,25–27].

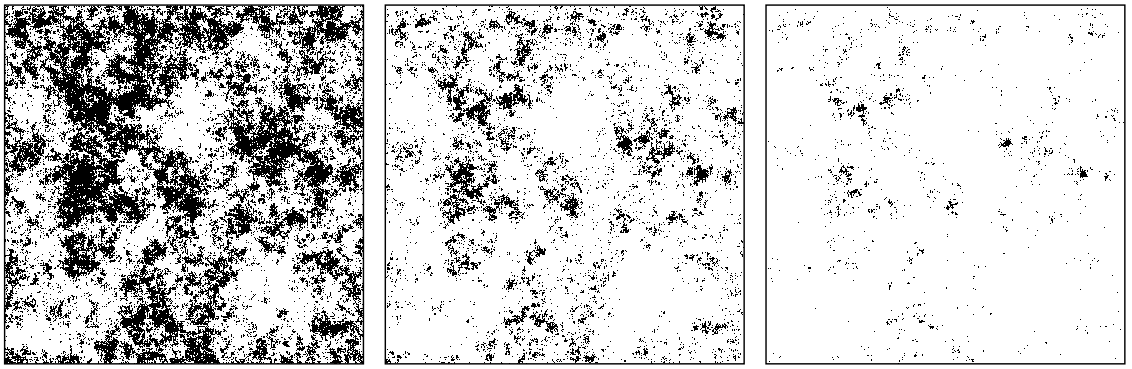

Fig. 1. Plots of the level sets (1.3) for a sample of the DGFF on a square domain with $N := 500$ and λ taking values (as labeled left to right) 0.1, 0.3 and 0.5, respectively. The clustering (and fractal) nature of these sets is quite apparent.

A parallel, and largely independent, line of recent research has focused on various quantitative aspects of the extremal values associated with the *discrete* Gaussian Free Field (DGFF). This is a Gaussian process $\{h_x : x \in \mathbb{Z}^2\}$ marked by a proper (typically finite) subset V of the square lattice (other infinite graphs can be considered as well) with the law determined by

$$E(h_x) = 0 \quad \text{and} \quad E(h_x h_y) = G^V(x, y), \tag{1.1}$$

where G^V denotes the Green function of the simple symmetric random walk in V killed upon exit from V. (In particular, h vanishes outside of V almost surely.) Here an early paper of Bolthausen, Deuschel and Giacomin [12] showed that the maximum of the DGFF in square boxes $V_N := (0, N)^2 \cap \mathbb{Z}^2$ grows as

$$\max_{x \in V_N} h_x \sim 2\sqrt{g} \log N, \quad N \to \infty, \tag{1.2}$$

where "$\sim$" designates that the ratio of the two quantities tends to one in the stated limit and $g := 2/\pi$ is a constant such that the Green function obeys $G^{V_N}(x, x) = g \log N + O(1)$ as $N \to \infty$ for x "deep" inside V_N. Daviaud [17] subsequently finessed the approach of [12] to capture some geometric aspects of the *intermediate* level sets

$$\{x \in V_N : h_x \geq 2\sqrt{g}\lambda \log N\} \quad \text{for } \lambda \in (0, 1). \tag{1.3}$$

Specifically, he showed that this set contains $N^{2(1-\lambda^2)+o(1)}$ points, where $o(1) \to 0$ in probability as $N \to \infty$, and thus demonstrated a fractal nature of this set. A continuum version of Daviaud's result dealing with thick points of the CGFF has subsequently been proved by Hu, Peres and Miller [20]. As explained in Chatterjee, Dembo and Ding [16], the structure of the exponent is quite universal. Multifractal aspects of more general chaos measures have been studied extensively in the past (see, e.g., the review by Barral and Mandelbrot [4]).

The objective of the present paper is to show that the intermediate level set (1.3) admits a non-trivial *scaling limit* which can be quite explicitly characterized. A number of issues need to be addressed when setting the problem up mathematically. The first one is a proper formulation of the limit. Indeed, after scaling the space by N, the set (1.3) is increasingly dense everywhere in the unit square $[0, 1]^2$ and so taking its limit directly (e.g., in the topology of Hausdorff convergence) does not seem useful. We resolve this by encoding the level set into the point measure

$$\sum_{x \in V_N} \delta_{x/N} \otimes \delta_{h_x - a_N}, \tag{1.4}$$

where a_N is a scale sequence such that, in light of (1.2),

$$\frac{a_N}{\log N} \underset{N \to \infty}{\longrightarrow} 2\sqrt{g}\lambda \tag{1.5}$$

for some $\lambda \in (0,1)$. The next issue is unbounded mass, and that even under integration against compactly-supported continuous functions. We resolve this by showing that (1.4) can be normalized by a *deterministic* quantity so that a non-degenerate distributional limit becomes possible. Another issue is dependence on the underlying domain; we resolve this by working in a class of lattice approximations D_N of a "nice" continuum set $D \subset \mathbb{C}$. This will be useful technically and will permit discussion of behavior under conformal maps.

2. Main results

Throughout the rest of the paper we will write $h^V(x)$ to denote the DGFF in V evaluated at x. The presentation of our results opens up with the existence of the scaling limit.

2.1. *Existence of scaling limit*

We start by fixing the class of admissible domains. Let $\mathfrak{D}$ be the class of all bounded open sets $D \subset \mathbb{C}$ with a finite number of connected components and with boundary ∂D that has only a finite number of connected components with each having a positive (Euclidean) diameter. Given $D \in \mathfrak{D}$, we will restrict to sequences $\{D_N\}$ of lattice domains such that

$$D_N \subseteq \Big\{ x \in \mathbb{Z}^2 \colon \mathrm{d}_\infty\big(x/N, D^{\mathrm{c}}\big) > \frac{1}{N} \Big\} \tag{2.1}$$

and, for each $\delta > 0$ and all N sufficiently large, also

$$D_N \supseteq \big\{ x \in \mathbb{Z}^2 \colon \mathrm{d}_\infty\big(x/N, D^{\mathrm{c}}\big) > \delta \big\}, \tag{2.2}$$

where d_∞ denotes the ℓ^∞-distance on $\mathbb{Z}^2$. Note that $x \in D_N$ implies $x/N \in D$.

Next let us consider the DGFF ϕ on $\mathbb{Z}^2$ pinned to zero at the origin or, equivalently, the DGFF on $\mathbb{Z}^2 \setminus \{0\}$. This is a Gaussian process $\{\phi_x \colon x \in \mathbb{Z}^2\}$ with law to be denoted by ν^0 which is of mean zero and covariance given by

$$E_{\nu^0}(\phi_x \phi_y) = \mathfrak{a}(x) + \mathfrak{a}(y) - \mathfrak{a}(x-y), \tag{2.3}$$

where $\mathfrak{a} \colon \mathbb{Z}^2 \to [0,\infty)$ is the potential kernel, i.e., the unique function that obeys $\mathfrak{a}(0) = 0$, is discrete harmonic on $\mathbb{Z}^2 \setminus \{0\}$ and has the asymptotic form $\mathfrak{a}(x) = g \log |x| + O(1)$ as $|x| \to \infty$, with $|x|$ denoting the Euclidean norm of x. Our main result is then:

Theorem 2.1. *For each $\lambda \in (0,1)$ and each $D \in \mathfrak{D}$, there is a random Borel measure Z^D_λ on $\overline{D}$ with $E[Z^D_\lambda(\overline{D})] \in (0,\infty)$ such that the following holds for each sequence a_N satisfying (1.5) and each sequence D_N of scaled-up versions of D obeying (2.1)–(2.2): Set*

$$K_N := \frac{N^2}{\sqrt{\log N}} \mathrm{e}^{-\frac{a_N^2}{2g\log N}} \tag{2.4}$$

and, for each sample h^{D_N} of the DGFF *in D_N, define the point measure*

$$\eta^D_N := \frac{1}{K_N} \sum_{x \in D_N} \delta_{x/N} \otimes \delta_{h^{D_N}(x) - a_N} \otimes \delta_{\{h^{D_N}(x) - h^{D_N}(x+z) \colon z \in \mathbb{Z}^2\}}. \tag{2.5}$$

Then, relative to the topology of vague convergence of measures on $\overline{D} \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^2}$,

$$\eta^D_N \xrightarrow[N\to\infty]{\text{law}} Z^D_\lambda(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha\lambda h}\,\mathrm{d}h \otimes \nu_\lambda(\mathrm{d}\phi), \tag{2.6}$$

where $\alpha := 2/\sqrt{g}$ *and* ν_λ *is the probability measure on* $\mathbb{R}^{\mathbb{Z}^2}$ *defined by*

$$\nu_\lambda(\cdot) := \nu^0\Bigl(\phi + \frac{2}{\sqrt{g}}\lambda\mathfrak{a} \in \cdot\Bigr), \tag{2.7}$$

with ν^0 *and* $\mathfrak{a}$ *as specified above.*

As it turns out, the convergence in (2.6) actually holds in (a somewhat larger) space of Radon measures on $\overline{D} \times (\mathbb{R} \cup \{+\infty\}) \times \mathbb{R}^{\mathbb{Z}^2}$. As an immediate consequence we thus get:

Corollary 2.2. *Let* $D \in \mathfrak{D}$. *For any* D_N *related to* D *as in* (2.1)–(2.2), *any sequence* a_N *such that* (1.5) *holds with some* $\lambda \in (0,1)$, *and* K_N *as in* (2.4),

$$\frac{1}{K_N}\#\bigl\{x \in D_N\colon h^{D_N}(x) \geq a_N\bigr\} \xrightarrow[N\to\infty]{\text{law}} (\alpha\lambda)^{-1} Z^D_\lambda(\overline{D}), \tag{2.8}$$

where $\alpha := 2/\sqrt{g}$. *Moreover, the random variables on the left are uniformly integrable and the convergence thus holds also under expectation.*

As is readily checked, (1.5) yields $K_N = N^{2(1-\lambda^2)+o(1)}$ as $N \to \infty$. Since $Z^D_\lambda(\overline{D}) \in (0,\infty)$ a.s. (see Theorem 2.3(7)), (2.8) determines the asymptotic size of the level set (1.3). This strengthens considerably the aforementioned conclusion of Daviaud [17]. The asymptotic positivity of the size of the level set normalized by its expectation appears already in the recent work by the first author, Ding and Goswami [8].

2.2. *Properties of* Z_λ*-measures*

In order to make our description of the limit law in (2.6) complete, we have to characterize the law of the random measure Z^D_λ. For this, we will first note a number of properties of these measures that will in fact be proved jointly with the above convergence theorem.

We first need some additional notations. For each $D \in \mathfrak{D}$ and each $x \in D$, let $\Pi^D(x,\cdot)$ denote the harmonic measure on ∂D relative to x. As is well known (see, e.g., [9, Lemma 2.3]), if $\widetilde{D} \subseteq D$ are two admissible domains, then

$$C^{D,\widetilde{D}}(x,y) := g\int_{\partial D}\Pi^D(x,\mathrm{d}z)\log|y-z| - g\int_{\partial\widetilde{D}}\Pi^{\widetilde{D}}(x,\mathrm{d}z)\log|y-z| \tag{2.9}$$

defines a symmetric, positive semi-definite function $C^{D,\widetilde{D}}\colon \widetilde{D}\times\widetilde{D} \to \mathbb{R}$ which is smooth in both variables. We may thus define $\{\Phi^{D,\widetilde{D}}(x)\colon x \in \widetilde{D}\}$ to be a Gaussian field with mean zero and covariance $C^{D,\widetilde{D}}$; this field has smooth sample paths a.s. For $\lambda \geq 0$, we define $\psi^D_\lambda\colon D \to [0,\infty)$ by

$$\psi^D_\lambda(x) := \exp\biggl\{2\lambda^2\int_{\partial D}\Pi^D(x,\mathrm{d}z)\log|x-z|\biggr\}. \tag{2.10}$$

For D simply connected, $\psi^D_\lambda(x)$ is the $2\lambda^2$-th power of the conformal radius of D from x. Setting $\psi^D_\lambda(x) := 0$ for $x \in \partial D$, the resulting function is continuous on $\overline{D}$. Writing $\mathrm{Leb}(A)$ to denote the Lebesgue measure of a (measurable) set $A \subset \mathbb{R}^2$, we then have:

Theorem 2.3. *Let* $\lambda \in (0,1)$ *and recall that* $\alpha := 2/\sqrt{g}$. *Then the family of* (*laws of*) *random measures* $\{Z^D_\lambda\colon D \in \mathfrak{D}\}$ *obeys the following properties*:

(1) *For each* $D \in \mathfrak{D}$, *the measure* Z^D_λ *is supported on* D; *i.e.*, $Z^D_\lambda(\mathbb{R}^2 \setminus D) = 0$ *a.s.*
(2) *If* $A \subset D \in \mathfrak{D}$ *is measurable with* $\mathrm{Leb}(A) = 0$, *then* $Z^D_\lambda(A) = 0$ *a.s.*

(3) *There is* $c = c(\lambda) \in (0,\infty)$ *such that for each* $D \in \mathfrak{D}$ *and each measurable* $A \subset D$,

$$E Z_\lambda^D(A) = c \int_A \psi_\lambda^D(x)\,\mathrm{d}x. \tag{2.11}$$

(4) *If* $D, \widetilde{D} \in \mathfrak{D}$ *obey* $D \cap \widetilde{D} = \varnothing$, *then*

$$Z_\lambda^{D\cup\widetilde{D}}(\mathrm{d}x) \overset{\text{law}}{=} Z_\lambda^D(\mathrm{d}x) + Z_\lambda^{\widetilde{D}}(\mathrm{d}x), \tag{2.12}$$

with the measures Z_λ^D *and* $Z_\lambda^{\widetilde{D}}$ *on the right regarded as independent.*

(5) (*Gibbs–Markov*) *If* $D, \widetilde{D} \in \mathfrak{D}$ *obey* $\widetilde{D} \subseteq D$ *and* $\mathrm{Leb}(D \setminus \widetilde{D}) = 0$, *then*

$$Z_\lambda^D(\mathrm{d}x) \overset{\text{law}}{=} \mathrm{e}^{\alpha\lambda\Phi^{D,\widetilde{D}}(x)} Z_\lambda^{\widetilde{D}}(\mathrm{d}x), \tag{2.13}$$

where $\{\Phi^{D,\widetilde{D}}(x) \colon x \in \widetilde{D}\}$ *is independent of* $Z_\lambda^{\widetilde{D}}$ *with the law as above.*

(6) *The law of* Z_λ^D *is translation invariant*; $Z_\lambda^{a+D}(a + \mathrm{d}x) \overset{\text{law}}{=} Z_\lambda^D(\mathrm{d}x)$ *for each* $a \in \mathbb{R}^2$.

(7) *For each* $A \subset D$ *non-empty and open*, $Z_\lambda^D(A) > 0$ *a.s.*

The properties (1)–(6), *for a given* $c > 0$ *in* (2.11), *determine the laws of* $\{Z_\lambda^D \colon D \in \mathfrak{D}\}$ *uniquely.*

Obviously, (2) follows from (3) although we prefer to state these separately. The constant c in (2.11) can be computed explicitly; just compare (2.8) with (3.6). Perhaps the most important property of all is (5). Here we note that the measure on the right of (2.13) is well defined due to the fact that $Z_\lambda^{\widetilde{D}}(D \setminus \widetilde{D}) = 0$ a.s. thanks to property (1), and this carries no loss on the left-hand side because also $Z_\lambda^D(D \setminus \widetilde{D}) = 0$ a.s. thanks to property (2). We will refer to property (5) – sometimes also in conjunction with (4) – as the Gibbs–Markov property. This is because properties (4)–(5) arise directly from the Gibbs–Markov decomposition of the DGFF; see (A.6).

By property (6), the law of Z_λ^D transforms canonically under the spatial shifts. The behavior of Z_λ^D under scaling of D is more subtle as it is intimately tied to the existence of the limit (2.6) and its independence of the sequence of discrete domains D_N *and* of how the centering sequence a_N achieves the overall asymptotic (1.5). Once a suitable scaling relation is established, the Gibbs–Markov property and (3) yield also rotation invariance and, in fact, lead to:

Theorem 2.4. *Let* $\lambda \in (0,1)$. *Under any conformal bijection* $f \colon D \to f(D)$ *between the admissible domains* D, $f(D) \in \mathfrak{D}$, *the laws of the above measures transform as*

$$Z_\lambda^{f(D)} \circ f(\mathrm{d}x) \overset{\text{law}}{=} \bigl|f'(x)\bigr|^{2+2\lambda^2} Z_\lambda^D(\mathrm{d}x). \tag{2.14}$$

2.3. *Connection to Liouville quantum gravity*

Although the above properties already determine the law of $\{Z_\lambda^D \colon D \in \mathfrak{D}\}$ uniquely, we are able to make even a more explicit connection with the so called Liouville Quantum Gravity measures. These are natural examples of Gaussian Multiplicative Chaos considered already by Kahane [21] who was in turn inspired by earlier work of Mandelbrot and others (see, e.g., Barral and Mandelbrot [4] for a review). They owe their name to a recent work of Duplantier and Sheffield [19] who proved that these measures obey the KPZ relation for scaling exponents. (As mentioned to us by a referee, these measures are only "baby" versions of actual Liouville Quantum Gravity. Notwithstanding, the name seems to have stuck in the probability community.) We refer to Rhodes and Vargas [29] or Berestycki [6] for recent reviews of this subject area.

We start again with some definitions. Let $\mathsf{H}_0^1(D)$ denote the closure of the set of smooth, functions with compact support in D with respect to the norm induced by the Dirichlet inner product $\langle f, g\rangle_\nabla := \frac{1}{4}\int_D \nabla f(x)\cdot\nabla g(x)\,\mathrm{d}x$. Given a sequence $\{X_n \colon n \geq 1\}$ of i.i.d. standard normal random variables and an orthonormal basis $\{f_n \colon n \geq 1\}$ in $\mathsf{H}_0^1(D)$, define

$$\varphi_n(x) := \sum_{k=1}^n X_k f_k(x). \tag{2.15}$$

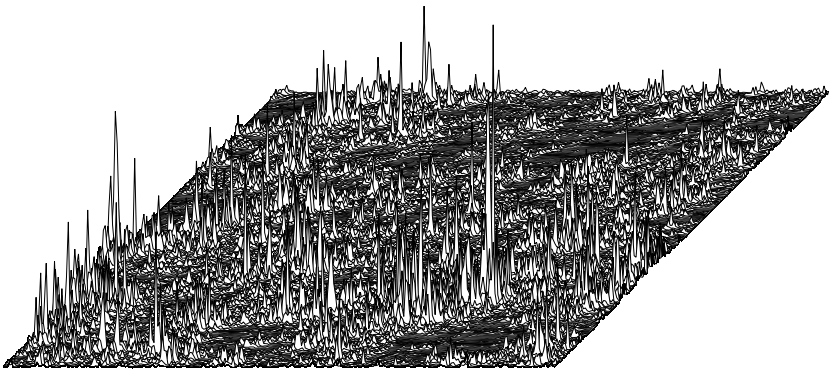

Fig. 2. A sample of the LQG measure $\psi_\lambda^D(x)\mu_\infty^{D,\beta}(\mathrm{d}x)$ for D a unit square and parameters $\beta = \lambda\alpha$ and $\lambda = 0.3$. The high points indicate places of high local intensity.

For each $\beta \in [0, \infty)$, define the random measure

$$\mu_n^{D,\beta}(\mathrm{d}x) := 1_D(x)\mathrm{e}^{\beta\varphi_n(x)-\frac{\beta^2}{2}E[\varphi_n(x)^2]}\,\mathrm{d}x. \tag{2.16}$$

As goes back to Kahane [21] there exists a random, a.s. finite (albeit possibly trivial) Borel measure $\mu_\infty^{D,\beta}$ – called the *Gaussian multiplicative chaos* associated with the continuum Gaussian Free Field – which is concentrated on D and such that, for each measurable set A,

$$\mu_n^{D,\beta}(A) \underset{n\to\infty}{\longrightarrow} \mu_\infty^{D,\beta}(A) \quad \text{a.s.} \tag{2.17}$$

It is also known (cf. a remark after Rhodes and Vargas [29, Theorem 5.5]) that for each $\beta \in (0, \beta_\mathrm{c})$, where (in our normalization) $\beta_\mathrm{c} := \alpha = 2/\sqrt{g}$, we have $\mu_\infty^{D,\beta}(D) > 0$ a.s. Moreover, as was shown in [29, Theorem 5.5], the law of the limit measure does not depend on the choice of the above orthonormal basis. (In fact, thanks to Shamov [31, Corollary 5], the law of $\mu_\infty^{D,\beta}$ is determined solely by its expectation and the way the measure transforms under the Cameron–Martin shifts of the underlying CGFF.) With this stated, we now claim:

Theorem 2.5. *Let $\lambda \in (0, 1)$, $\alpha := 2/\sqrt{g}$ and consider the family of measures $\{Z_\lambda^D : D \in \mathfrak{D}\}$ as above. Then, for $c \in (0, \infty)$ as in (2.11) and for each $D \in \mathfrak{D}$,*

$$Z_\lambda^D(\mathrm{d}x) \overset{\text{law}}{=} c\psi_\lambda^D(x)\mu_\infty^{D,\lambda\alpha}(\mathrm{d}x), \tag{2.18}$$

where ψ_λ^D is as in (2.10). In particular, Z_λ^D has the law of the Liouville Quantum Gravity measure in D corresponding to (subcritical) parameter $\beta := \lambda\alpha$.

2.4. *Remarks and open problems*

We proceed by a series of remarks and questions left to be studied.

(1) *General Gaussian processes*: We believe that the *form* of the limit measure in (2.6) is actually quite universal. For instance, for i.i.d. Gaussians indexed by the vertices in D_N (see Figure 3) with variance $g \log N$ with same K_N we get the same limit statement with Z_λ^D replaced by (a multiple of) the Lebesgue measure on D and ν_λ by the point mass concentrated on ϕ defined by $\phi_0 := 0$ and $\phi_x := -\infty$ for $x \neq 0$. That Z_λ^D is itself random in the case of the DGFF is a reflection of long-range correlations.

(2) *Simultaneous limit for all* λ: Our proofs are technically based on the computation of the first two moments of the measure η_N^D integrated against compactly-supported, continuous functions. (This is literally true when $\lambda < 1/\sqrt{2}$ with a truncation needed for complementary λ.) One could use similar techniques to study the level sets for several

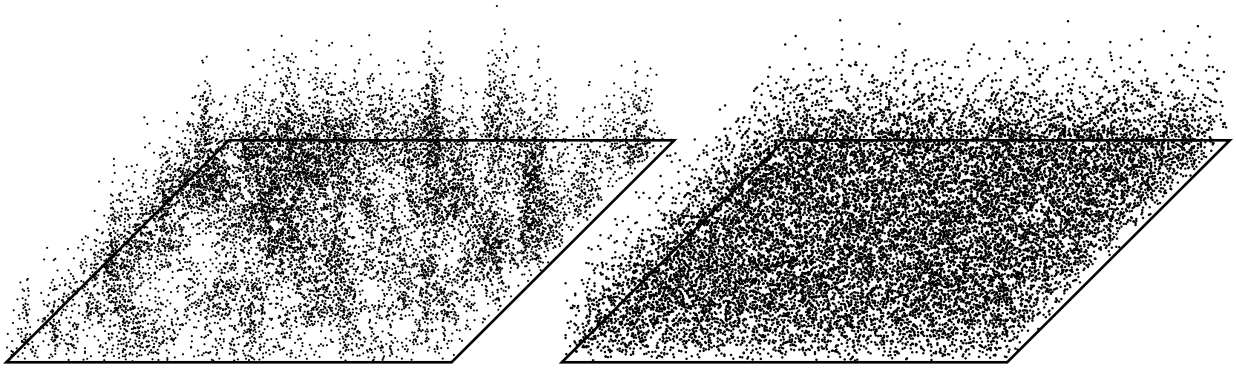

Fig. 3. Left: A sample of the measure in (1.4) on a square of side-length $N := 300$ with $\lambda := 0.2$. Right: A corresponding sample for i.i.d. normals with mean zero and variance $g \log N$. Only the points with positive vertical coordinate are depicted.

values of λ simultaneously but the number of required moments seems to increase with the number of the levels to be controlled. In particular, we presently do not see a way to solve:

Problem 2.6. Find a way to extract a joint distributional limit of the level sets (1.3), or their associated point measures (2.5), simultaneously for all $\lambda \in (0, 1)$.

Our belief that a joint limit should exist is supported by the fact that (2.15)–(2.17), and a suitable continuity argument, define the LQG measure for all $\beta \in (0, \beta_c)$ at the same time. This is because the LQG measure is a measurable function of the underlying CGFF.

(3) *Connection to Liouville Quantum Gravity*: A referee of of the first submission of this paper wondered why it does not seem possible to derive the existence of the limit of discrete LQG measures

$$\frac{1}{K'_N} \sum_{x \in D_N} \mathrm{e}^{\beta h_x^{D_N}} \delta_{x/N}, \tag{2.19}$$

for a suitable sequence K'_N, from Theorem 2.1, and *vice versa*. We note that this is in spite of the fact that this measure is, for $\beta = \lambda\alpha$, supported near the level set $h_x^{D_N} \geq 2\sqrt{g}\lambda \log N$. A formal reason is that the growth of $h \mapsto \mathrm{e}^{\beta h}$ matches the decay of the measure on the right of (2.6) and so the limit cannot be applied to the test function $f(x, h, \phi) := \mathrm{e}^{\beta h}$. A deeper (albeit related) reason is that the measure (2.19) is actually not supported on the level set where $h_x^{D_N} \approx 2\sqrt{g}\lambda \log N$ but over a range of values within distance order $\sqrt{\log N}$ from $2\sqrt{g}\lambda \log N$, and so K'_N needs to be taken larger by factor $\sqrt{\log N}$ than K_N in (2.4).

(4) *Relation to extremal process*: Our point process-based approach is strongly motivated by recent advances in the understanding of the extremal values of the DGFF; i.e., roughly speaking, the set (1.3) with $\lambda := 1$. Here, first, building on the work of Bolthausen, Deuschel and Zeitouni [13] and Bramson and Zeitouni [15], Bramson, Ding and Zeitouni [14] showed that for

$$m_N := 2\sqrt{g} \log N - \frac{3}{4}\sqrt{g} \log\log N, \tag{2.20}$$

the centered maximum, $\max_{x \in V_N} h_x - m_N$, converges to a non-degenerate distributional limit as $N \to \infty$. Then, in [9–11], the present authors described the limit of the full extremal process for the DGFF expressed in terms of the "structured" point process,

$$\eta_{N,r}^D := \sum_{x \in D_N} 1_{\{h_x = \max_{z \in \Lambda_r(x)} h_z\}} \delta_{x/N} \otimes \delta_{h_x - m_N} \otimes \delta_{\{h_x - h_{x+z} \colon z \in \mathbb{Z}^2\}}, \tag{2.21}$$

where $\Lambda_r(x) := \{z \in \mathbb{Z}^2 \colon |z - x| \leq r\}$ and the indicator thus effectively restricts the sum to the points where the field has an r-local maximum. The main result of [9–11] is that, for any r_N with $r_N \to \infty$ and $N/r_N \to \infty$, relative to the

topology of vague convergence,

$$\eta^D_{N,r_N} \xrightarrow[N\to\infty]{\text{law}} \text{PPP}\bigl(Z^D_1(\mathrm{d}x)\otimes \mathrm{e}^{-\alpha h}\,\mathrm{d}h\otimes \nu_1(\mathrm{d}\phi)\bigr), \tag{2.22}$$

where PPP stands for Poisson point process, Z^D_1 is a random a.s.-finite Borel measure on D, and ν is a (deterministic) probability measure on $[0,\infty)^{\mathbb{Z}^2}$; cf. (2.27). In a recent update of [9], the Z^D_1 measure was identified with a version of the critical Liouville Quantum Gravity constructed in Duplantier, Rhodes, Sheffield and Vargas [18].

As part of the proof in [9] it was shown that

$$Z^{f(D)}_1 \circ f(\mathrm{d}x) \overset{\text{law}}{=} \bigl|f'(x)\bigr|^4 Z^D_1(\mathrm{d}x). \tag{2.23}$$

This corresponds, at least formally, to $\lambda\uparrow 1$ limit of (2.14). The $\lambda\downarrow 0$ limit reduces Z^D_λ to (a multiple of) the Lebesgue measure on D.

(5) *Conformal invariance and uniqueness of LQG measure*: The previous remark brings us to the formulation of the conformal transformation rule in Theorem 2.4. In Duplantier and Sheffield [19, Proposition 1.2], a version of this rule is stated for the LQG measure in the following form: *Supposing* that the CGFF h^D in a domain D transforms under a conformal map $f\colon D\to f(D)$ (in our parametrization) as follows

$$h^{f(D)} \overset{\text{law}}{=} h^D\circ f + \left(\frac{2}{\beta}+\frac{2\beta}{\alpha^2}\right)\log\bigl|f'\bigr|, \tag{2.24}$$

which, we note, requires working with CGFF of non-zero mean and/or Cameron–Martin shifts, the LQG measure in D for parameter β transforms into the corresponding LQG measure in $f(D)$. Leaving aside the somewhat peculiar fact that the transformation (2.24) is tied to β and changes the mean of the field, the statement harbors a technical caveat: The measure must be realized as a *unique* function of the CGFF or, at least, one that is independent of the approximation scheme used to define it. This is in fact a subtle issue that has been fully settled only quite recently (cf. the aforementioned references to [29, Theorem 5.5] or [31, Corollary 5]).

Our approach to Theorem 2.4 has the advantage that it works solely with the family of random measures $\{Z^D_\lambda\colon D\in\mathfrak{D}\}$ and, in particular, avoids dealing with the uniqueness of LQG and/or its dependence on the underlying CGFF. In fact, Theorem 2.4 could concisely be stated as:

Theorem 2.7. *Every family of measures $\{Z^D_\lambda\colon D\in\mathfrak{D}\}$ satisfying properties* (1)–(6) *in Theorem* 2.3 *obeys also* (2.14), *for each conformal bijection $f\colon D\to f(D)$ with $D, f(D)\in\mathfrak{D}$.*

Our proof of the transformation rule (2.14) reduces, after some minor amount of preparation, to the same argument as the proof of (2.23) in Theorem 7.2 in [9]. The only time when we need to invoke uniqueness of the LQG measure is, quite naturally, when we identify Z^D_λ with the (unique) LQG measure in the proof of Theorem 2.5.

(6) *Fluctuations around random limit*: Our next remark concerns going beyond the limit statement (2.6). The point is that the limit measure, albeit random, captures only the leading-order growth of the level set. We thus pose:

Problem 2.8. Characterize the limit law of the (suitably scaled) fluctuations in the limit (2.6).

To make the formulation easier, one may choose to work in the setting when all the DGFFs are defined on the same probability space as the limit LQG measure. Perhaps the easiest underlying graph for this is the triangular lattice.

(7) *Crossover to critical regime*: Of quite some interest is how the behavior for $\lambda<1$ blends with that at $\lambda=1$. Our proofs only apply for a_N such that $a_N/\log N$ is, in the limit, strictly less than $2\sqrt{g}$. This is for a good reason: When $a_N/\log N\to 2\sqrt{g}$, the growth rate of the requisite normalizing sequence should be slower than (2.4). This stems from a subtle entropic-repulsion effect that lies at the heart of the paper [11] and can be seen by noting that plugging $a_N := m_N$ for m_N as in (2.20) results in $K_N\sim\log N$ (and not in K_N of order unity!). We thus pose:

Problem 2.9. Suppose that $a_N/\log N\to 2\sqrt{g}$ yet $m_N - a_N\to\infty$. Prove that, for a suitably re-defined K_N, we still have (2.5) with Z^D_λ replaced by the critical LQG measure Z^D from (2.22).

An affirmative resolution of this problem may in fact require that $m_N - a_N$ tends to infinity at some minimal rate. A reasonable guess is that $m_N - a_N$ of order $\sqrt{\log N}$ should already be enough. This scale appears naturally as it marks the level where the discrete approximation to the critical LQG measure is typically supported. We refer to the last chapter of the recent review (Biskup [7]) for more discussion and specific conjectures.

We note that, as suggested by a referee, if a solution to Problem 2.6 includes a statement of continuity in λ as $\lambda \uparrow 1$, we could perhaps solve Problem 2.9 by taking a suitable limit. For the family of LQG measures, continuity as the critical value is approached from below has been shown quite recently (Aru, Powell and Sepúlveda [3]). For the Branching Random Walk this has been proved by Madaule [22].

(8) *Beyond two-dimensional DGFF*: A natural question is of course whether the above results are in any sense *universal* for other models that are, at least at large spatial scales, well captured by CGFF. These include general logarithmically-correlated Gaussian fields, gradient models and local time of a two-dimensional simple random walk. Some progress on these has already been made (e.g., Belius and Wu [5], Abe [1]).

2.5. *Proof strategy*

The overall strategy of our proofs is rather simple. Through moment calculations for the size of the level set, we establish tightness and asymptotic non-triviality of the measures $\{\eta_N^D\colon N \geq 1\}$ relative to the vague topology. This permits extraction of subsequential weak limits. We then proceed to derive various relations that such limits have to satisfy which ultimately characterize them uniquely. This proves existence of the limit as well as its desired properties.

The specific "characterization" steps are as follows. First we focus only on the measures (2.5) restricted to the first two coordinates. One more (subtle) second moment calculation shows that every subsequential limit η^D of such two-coordinate measures admits the decomposition

$$\eta^D(\mathrm{d}x\,\mathrm{d}h) = Z_\lambda^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha\lambda h}\,\mathrm{d}h, \tag{2.25}$$

with Z_λ^D a non-degenerate, a.s. finite measure whose law possibly depends on the subsequence, the sequence of approximation domains D_N as well as the way a_N approaches the limit (1.5).

Next we demonstrate that the measures Z_λ^D, with D restricted to a suitable countable collection of domains (this is the best one can hope to have when extracting limits by subsequences), obey properties (1)–(7) in Theorem 2.3. Property (5) is then particularly important, as it yields a representation of Z_λ^D, for D a dyadic square, in terms of a multiplicative chaos measure. From here we get uniqueness of the law of Z_λ^D on dyadic squares; one more use of property (5) then extends this to all $D \in \mathfrak{D}$. The existence of the limit of η_N^D, and its independence of the approximation domains D_N or the specific way a_N achieves the limit $a_N/\log N \to 2\sqrt{g}\lambda$, follow. This pretty much completes the proof of Theorems 2.1, 2.3 and Theorem 2.5 for the two-coordinate measures.

The sheer existence of the limit (and translation invariance of the DGFF) now implies the transformation rule for shifts and scaling of the underlying domain:

$$Z_\lambda^{a+rD}(a + r\,\mathrm{d}x) \overset{\text{law}}{=} r^{2+2\lambda^2} Z_\lambda^D(\mathrm{d}x), \quad a \in \mathbb{R}^2, r > 0. \tag{2.26}$$

The representation using multiplicative chaos adds rotation invariance to this as well. One more use of property (5) for a decomposition of a given D using a myriad of tiny dyadic squares then permits us to apply these symmetries "infinitesimally" thus proving, with the help of conformal invariance of the "binding" fields $\Phi^{D,\tilde{D}}$, Theorem 2.4.

As a final step, we extend control to the full three-coordinate process (2.5). This boils down to yet another moment calculation, which yields factorization of the limit into the product measure on the right of (2.6). It is easy to see why the limit law of $h^{D_N}(x) - h^{D_N}(x+\cdot)$ should be described by (2.7): Conditioning on $h^{D_N}(x)$ to be roughly $2\sqrt{g}\lambda\log N$ changes the mean of the field at y by, roughly, $\frac{2}{\sqrt{g}}\lambda\mathfrak{a}(y-x)$ while the variance tends to that of the DGFF on $\mathbb{Z}^2 \setminus \{0\}$. We note that a similar reasoning applies also to the extremal process limit (2.22) except there the "cluster law" ν_1 requires an extra conditioning (to ensure a local maximum at x) and taking a limit,

$$\nu_1(\cdot) = \lim_{r\to\infty} \nu^0\biggl(\phi + \frac{2}{\sqrt{g}}\mathfrak{a} \in \cdot \;\Big|\; \phi(x) + \frac{2}{\sqrt{g}}\mathfrak{a}(x) \geq 0\colon |x| \leq r\biggr). \tag{2.27}$$

This limit is singular, which is a source of much headache in the proofs of [11].

Remark 2.10. The above strategy – extract a subsequential limit and then prove its uniqueness – also lies at the core of our earlier work [9–11] on the extrema of the DGFF. However, the two approaches are technically quite different, both in the proof of the factorization (which, in [9–11], relies on a connection with particle systems) and in the proof of uniqueness (which, for the extremal values, relies on the existence of the limit of the centered absolute maximum).

Since we are dealing with scaling limits of the DGFF, it is no surprise that we will need to invoke bounds on, as well as limits of, the Green function in various lattice domains of interest. The limit statements in particular require weak convergence of the harmonic measure on $\frac{1}{N}\partial D_N$ to that on ∂D. This is where the containment $D \in \mathfrak{D}$ and relations (2.1)–(2.2) are required. To make referencing easier, we collect the needed statements in the Appendix.

3. Proofs in the second moment regime

We are now ready to commence the exposition of our proofs. As noted above, the starting point are calculations of the first two moments of the size of the level set. These are straightforward for $\lambda \in (0, 1/\sqrt{2})$ but harder in the complementary regime of λ, where additional (albeit standard) truncations are required to keep the second moment comparable to the square of the first. For ease of exposition as well as pedagogical appeal, we will first deal with the former regime leaving the latter to a subsequent section.

Throughout this section, we thus assume that a_N is a sequence such that (1.5) holds for some $\lambda \in (0, 1/\sqrt{2})$. We suppose that, for each $D \in \mathfrak{D}$, a sequence $\{D_N\}$ of approximating lattice domains is given satisfying (2.1)–(2.2). Unless stated otherwise, all estimates will depend on the choice of D and the sequences a_N and D_N.

3.1. *Level-set size moments*

For each $b \in \mathbb{R}$, define

$$\Gamma_N^D(b) := \bigl\{x \in D_N : h^{D_N}(x) \geq a_N + b\bigr\}. \tag{3.1}$$

We begin by a bound on the overall size of $\Gamma_N^D(b)$:

Lemma 3.1. *For each $D \in \mathfrak{D}$ and each $\delta \in (0,1)$ there is $c = c(\delta, D) \in (0,\infty)$ such that for all $b \in \mathbb{R}$ with $|b| \leq \log N$, all sequences a_N satisfying $\delta \log N \leq a_N \leq \delta^{-1}\sqrt{g}\log N$, all sequences $\{D_N\}$ satisfying (2.1)–(2.2), all $A \subset D_N$ and all $N \geq 1$, we have*

$$E\bigl|\Gamma_N^D(b) \cap A\bigr| \leq cK_N \frac{|A|}{N^2} \mathrm{e}^{-\frac{a_N}{g\log N}b}. \tag{3.2}$$

Proof. The claim will follow by summing over $x \in D_N$ once we prove that, for some constant c depending only on δ and the diameter of D, we have

$$P\bigl(h^{D_N}(x) \geq a_N + b\bigr) \leq c\frac{1}{\sqrt{\log N}} \mathrm{e}^{-\frac{a_N^2}{2g\log N}} \mathrm{e}^{-\frac{a_N}{g\log N}b} \tag{3.3}$$

uniformly in $x \in D_N$ and in $b \in [-\log N, \log N]$. To this end we first invoke the Gibbs–Markov property of the DGFF (cf. (A.6)) to note that, if $U \subset V$ are finite lattice domains, then by writing $h^V(x) \overset{\text{law}}{=} h^U(x) + \varphi^{V,U}(x)$ and requiring $\varphi^{V,U}(x) \geq 0$,

$$P\bigl(h^U(x) \geq a\bigr) \leq 2P\bigl(h^V(x) \geq a\bigr). \tag{3.4}$$

By enlarging D_N to, say, a square domain $\widetilde{D}_N$ of comparable diameter, we may thus assume that all $x \in D_N$ lie deep inside $\widetilde{D}_N$. The variance of $h^{\widetilde{D}_N}(x)$ is then within a constant of $g\log N$ uniformly in $x \in D_N$ and so we get, for some $c > 0$ independent of N, and all N larger than some constant (which suffices for the full claim),

$$P\bigl(h^{\widetilde{D}_N}(x) \geq a_N + b\bigr) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{g\log N - c}} \int_b^\infty \mathrm{e}^{-\frac{(a_N+s)^2}{2g\log N + c}} \,\mathrm{d}s. \tag{3.5}$$

Bounding $(a_N+s)^2 \geq a_N^2 + 2a_N s$, we can bound the exponent from below by the quantity $a_N^2/(2g\log N)^2 + (a_N/g\log N)s - c'$, where c' depends only on c and the assumed upper bound on a_N. The integral over s is then performed with the result as stated. □

With the overall scale under control, we can now calculate the leading-order asymptotic of the above expectation for nice-enough sets A.

Lemma 3.2. *There is a constant $c_0 \in \mathbb{R}$ such that for each $b \in \mathbb{R}$ and each open set $A \subseteq D$,*

$$E\bigl|\{x \in \Gamma_N^D(b)\colon x/N \in A\}\bigr| = \frac{\mathrm{e}^{2c_0\lambda^2/g}}{\lambda\sqrt{8\pi}}\mathrm{e}^{-\alpha\lambda b}\biggl[\int_A \psi_\lambda^D(x)\,\mathrm{d}x + o(1)\biggr]K_N, \tag{3.6}$$

where $o(1) \to 0$ as $N \to \infty$ uniformly on compact sets of b.

Proof (sketch). Thanks to the uniform control from Lemma 3.1, we may assume that the closure of A lies in D. We will need the asymptotic

$$G^{D_N}\bigl(\lfloor xN\rfloor, \lfloor xN\rfloor\bigr) = g\log N + g\int_{\partial D}\Pi^D(x,\mathrm{d}z)\log|x-z| + c_0 + o(1), \tag{3.7}$$

with c_0 the constant from the asymptotic of the potential kernel in (A.2) and $o(1) \to 0$ as $N \to \infty$ uniformly on compact sets in D, and thus in $x \in A$. (This is where the conditions on D_N are relevant, see (A.4)–(A.5) or Biskup [7, Theorem 1.17].) Now we repeat the calculation from the proof of Lemma 3.1 while keeping careful track of all non-vanishing terms. The boundedness and continuity of ψ_λ^D finally permit us to replace a Riemann sum by the integral in (3.6). □

Our next lemma concerns the second moment estimate for the size of $\Gamma_N^D(b)$. It is here where we need to limit the range of possible λ:

Lemma 3.3. *Suppose $0 < \lambda < 1/\sqrt{2}$. For each $b_0 \in \mathbb{R}$ and each $D \in \mathfrak{D}$ there is $c_1 = c_1(\lambda, b_0, D) \in (0,\infty)$ such that for each $b \in [-b_0, b_0]$ and each $N \geq 1$,*

$$E\bigl(\bigl|\Gamma_N^D(b)\bigr|^2\bigr) \leq c_1 K_N^2 \tag{3.8}$$

Moreover, there is an absolute constant $c_2 \in (0,\infty)$ such that for all $D \in \mathfrak{D}$,

$$\limsup_{N\to\infty}\frac{1}{K_N^2}E\bigl(\bigl|\Gamma_N^D(0)\bigr|^2\bigr) \leq c_2\int_{D\times D}\biggl(\frac{[\operatorname{diam} D]^2}{|x-y|}\biggr)^{4\lambda^2}\mathrm{d}x\,\mathrm{d}y, \tag{3.9}$$

where diam *D is the diameter of D in the Euclidean norm.*

Proof of (3.8). Thinking, without much loss of generality, of b as absorbed into a_N, we can assume $b := 0$ in the following. Writing

$$E\bigl(\bigl|\Gamma_N^D(0)\bigr|^2\bigr) = \sum_{x,y\in D_N} P\bigl(h^{D_N}(x) \geq a_N, h^{D_N}(y) \geq a_N\bigr) \tag{3.10}$$

we will need a good estimate on the probability on the right-hand side. First we again take $\widetilde{D}_N$ to be a neighborhood of D_N of diameter twice the diameter of D_N and note that, by the argument leading to (3.4) and the FKG inequality for $\varphi^{V,U}$ (implied by $\operatorname{Cov}(\varphi^{V,U}(x), \varphi^{V,U}(y)) \geq 0$)

$$P\bigl(h^{D_N}(x) \geq a_N, h^{D_N}(y) \geq a_N\bigr) \leq 4P\bigl(h^{\widetilde{D}_N}(x) \geq a_N, h^{\widetilde{D}_N}(y) \geq a_N\bigr). \tag{3.11}$$

Next we invoke the Gibbs–Markov decomposition (see (A.6))

$$h^{\widetilde{D}_N}(y) = \mathfrak{g}_x(y)h^{\widetilde{D}_N}(x) + \hat{h}^{\widetilde{D}_N\setminus\{x\}}(y), \tag{3.12}$$

where $h^{\widetilde{D}_N}(x)$ and $\hat{h}^{\widetilde{D}_N\setminus\{x\}}$ on the right-hand side are independent with $\hat{h}^{\widetilde{D}_N\setminus\{x\}}$ having the law of the DGFF in $\widetilde{D}_N \setminus \{x\}$ and where $\mathfrak{g}_x$ is a function that is harmonic in $\widetilde{D}_N \setminus \{x\}$, vanishing outside $\widetilde{D}_N$ and normalized such that $\mathfrak{g}_x(x) = 1$. Using this decomposition, the above probability is recast as

$$\begin{aligned} &P\big(h^{\widetilde{D}_N}(x) \geq a_N, h^{\widetilde{D}_N}(y) \geq a_N\big) \\ &\quad = \int_0^\infty P\big(\hat{h}^{\widetilde{D}_N\setminus\{x\}}(y) \geq a_N\big(1-\mathfrak{g}_x(y)\big) - s\mathfrak{g}_x(y)\big)P\big(h^{\widetilde{D}_N}(x) - a_N \in \mathrm{d}s\big). \end{aligned} \tag{3.13}$$

We will pick $\delta > 0$ and bound the right-hand side by $P(h^{\widetilde{D}_N} \geq a_N)$ when $|x-y| \leq \delta\sqrt{K_N}$ so let us suppose that $|x-y| > \delta\sqrt{K_N}$ from now on.

Observe that since x, y lie "deep" inside $\widetilde{D}_N$ and $|x-y| > \delta\sqrt{K_N} = N^{1-\lambda^2+o(1)}$, we have

$$\mathfrak{g}_x(y) = \frac{G^{\widetilde{D}_N}(x,y)}{G^{\widetilde{D}_N}(x,x)} \leq \frac{\log\frac{N}{|x-y|}+c}{\log N - c} \leq 1 - \big(1-\lambda^2\big) + o(1) = \lambda^2 + o(1), \tag{3.14}$$

where $o(1) \to 0$ uniformly in $x, y \in D_N$. Assuming $s \in [0, a_N]$, from $\lambda < 1/\sqrt{2}$ we then have

$$a_N\big(1-\mathfrak{g}_x(y)\big) - s\mathfrak{g}_x(y) > \epsilon a_N \tag{3.15}$$

for some $\epsilon > 0$ as soon as N is large enough, uniformly in $x, y \in D_N$. The argument in Lemma 3.1 in conjunction with $\mathfrak{g}_x(y) \in [0, 1]$ and the asymptotic (1.5) then show

$$\begin{aligned} &P\big(\hat{h}^{\widetilde{D}_N\setminus\{x\}}(y) \geq a_N\big(1-\mathfrak{g}_x(y)\big) - s\mathfrak{g}_x(y)\big) \\ &\quad \leq \frac{c}{\sqrt{\log N}}\mathrm{e}^{-\frac{[a_N(1-\mathfrak{g}_x(y))-s\mathfrak{g}_x(y)]^2}{2G(y,y)}} \leq c\frac{K_N}{N^2}\mathrm{e}^{\mathfrak{g}_x(y)\frac{a_N^2}{g\log N}+\frac{a_N}{G(y,y)}\mathfrak{g}_x(y)s}, \end{aligned} \tag{3.16}$$

where we wrote $G(y,y)$ for $G^{\widetilde{D}_N\setminus\{x\}}(y,y)$ to reduce clutter of indices and then used that $|G(y,y) - g\log N| \leq c$ uniformly in $y \in D_N$. The explicit form of the law of h^{D_N} with respect to the Lebesgue measure readily shows

$$P\big(h^{\widetilde{D}_N}(x) - a_N \in \mathrm{d}s\big) \leq c\frac{K_N}{N^2}\mathrm{e}^{-\frac{a_N}{G(x,x)}s}\,\mathrm{d}s. \tag{3.17}$$

Since $G(x,x)/G(y,y) = 1 + o(1)$ and $\mathfrak{g}_x(y) \leq \lambda^2 + o(1) < 1$, the integral in (3.13) over $s \in [0, a_N]$ yields a harmless multiplicative factor. Also, the middle inequality in (3.14) implies

$$\mathrm{e}^{\mathfrak{g}_x(y)\frac{a_N^2}{g\log N}} \leq c\bigg(\frac{N}{|x-y|}\bigg)^{4\lambda^2+o(1)} \tag{3.18}$$

with $o(1) \to 0$ uniformly in $x, y \in D_N$ with $|x-y| > \delta\sqrt{K_N}$. From (3.11) we thus get

$$P\big(h^{D_N}(x) \geq a_N, h^{D_N}(y) \geq a_N\big) \leq 4P\big(h^{\widetilde{D}_N}(x) \geq 2a_N\big) + c\bigg(\frac{K_N}{N^2}\bigg)^2\bigg(\frac{N}{|x-y|}\bigg)^{4\lambda^2+o(1)} \tag{3.19}$$

uniformly in $x, y \in D_N$ with $|x-y| > \delta\sqrt{K_N}$.

In order to finish the proof, we now write

$$\big(\big|\Gamma_N^D(0)\big|^2\big) \leq \sum_{\substack{x,y\in D_N\\ |x-y|\leq\delta\sqrt{K_N}}} P\big(h^{D_N}(x) \geq a_N\big) + \sum_{\substack{x,y\in D_N\\ |x-y|>\delta\sqrt{K_N}}} P\big(h^{D_N}(x) \geq a_N, h^{D_N}(y) \geq a_N\big). \tag{3.20}$$

Summing over y and invoking Lemma 3.1 bounds the first term by a factor of order $(\delta K_N)^2$. The contribution of the first term on the right of (3.19) to the second sum is bounded via Lemma 3.1 as well:

$$P\bigl(h^{\widetilde{D}_N}(x)\ge 2a_N\bigr)\le \frac{c}{\sqrt{\log N}}\mathrm{e}^{-2\frac{a_N^2}{g\log N}}=c\biggl(\frac{K_N}{N^2}\biggr)^2\mathrm{e}^{-\frac{a_N^2}{g\log N}}\sqrt{\log N}\le c\delta\biggl(\frac{K_N}{N^2}\biggr)^2. \tag{3.21}$$

Plugging in also the second term on the right of (3.19), we thus get

$$E\bigl(\bigl|\Gamma_N^D(0)\bigr|^2\bigr)\le 8c\delta(K_N)^2+c\biggl(\frac{K_N}{N^2}\biggr)^2\sum_{\substack{x,y\in D_N\\ |x-y|>\delta\sqrt{K_N}}}\biggl(\frac{N}{|x-y|}\biggr)^{4\lambda^2+o(1)}. \tag{3.22}$$

The standard domination by integrals bounds the sum by $c(N^2)^2\int_{D\times D}|x-y|^{-4\lambda^2+o(1)}\,\mathrm{d}x\,\mathrm{d}y$ regardless of δ, with the integral convergent since $\lambda<1/\sqrt{2}$ implies $4\lambda^2<2$. Hence, also the second term on the right is of order $(K_N)^2$, thus proving (3.8). □

Proof of (3.9). For the limit statement (3.9), we will have to reveal the D-dependence of certain constants in the above derivation. The bound (3.22) – and the fact that the sum therein is dominated by pairs of vertices in distances of order N from each other – shows that we need to consider only pairs $x, y\in D_N$ with $|x-y|\ge\epsilon N$ as the contribution of the complementary pairs is negligible in the limit $N\to\infty$ and $\epsilon\downarrow 0$. We only need to refine the bounds (3.17) and (3.18). Concerning (3.17), the asymptotic (3.7) along with the fact that $\operatorname{diam}(\widetilde{D}_N)\le 2\operatorname{diam}(D_N)$ gives

$$G^{\widetilde{D}_N}(x,x)\le g\log N+g\log(2\operatorname{diam}D)+c_0+o(1) \tag{3.23}$$

and so, in light of $a_N=O(\log N)$, the constant c in (3.17) is bounded by a numerical constant (which comes from bounding such constants in the probability density of $h^{\widetilde{D}_N}(x)$) times

$$\mathrm{e}^{\frac{a_N^2}{g(\log N)^2}\log(\operatorname{diam}D)+o(1)}=(\operatorname{diam}D)^{4\lambda^2+o(1)} \tag{3.24}$$

with $o(1)\to 0$ as $N\to\infty$. Concerning (3.18), the asymptotic of the Green function (A.5) at points of distance order N in turns gives

$$\mathfrak{g}_x(y)\le\frac{1}{\log N}\biggl[-\log\frac{|x-y|}{N}+\log(2\operatorname{diam}D)+o(1)\biggr] \tag{3.25}$$

which then implies

$$\mathrm{e}^{\mathfrak{g}_x(y)\frac{a_N^2}{g\log N}}\le c\biggl(\frac{N\operatorname{diam}D}{|x-y|}\biggr)^{4\lambda^2+o(1)}. \tag{3.26}$$

Using (3.24) and (3.26) in the derivation of (3.19) and taking $N\to\infty$ followed by $\epsilon\downarrow 0$, we then readily get (3.9) as well. □

3.2. *Subsequential limits and factorization*

We will now start deriving consequences of the above lemmas for the random measures η_N^D from (2.5). Since our strategy is to first deal only with events/functions that are trivial in the third "coordinate," we will temporarily abuse notation and set

$$\eta_N^D:=\frac{1}{K_N}\sum_{x\in D_N}\delta_{x/N}\otimes\delta_{h^{D_N}(x)-a_N} \tag{3.27}$$

instead of the full definition in (2.5). As a direct consequence of the above lemmas and the fact that $\overline{D}\times(\mathbb{R}\cup\{\infty\})$ is a separable metric space, we then get:

Corollary 3.4. *Suppose* $\lambda \in (0, 1/\sqrt{2})$. *Then* $\{\eta_N^D\colon N \geq 1\}$ *is tight with respect to the vague topology on the space of Radon measures on* $\overline{D} \times (\mathbb{R} \cup \{\infty\})$. *Moreover, every subsequential weak limit* η^D *of these measures satisfies, for each* $b \in \mathbb{R}$,

$$P\bigl(\eta^D\bigl(D \times [b, \infty)\bigr) < \infty\bigr) = 1 \tag{3.28}$$

and, for each non-empty open $A \subset D$ *and each* $b \in \mathbb{R}$,

$$P\bigl(\eta^D\bigl(A \times [b, \infty)\bigr) > 0\bigr) > 0. \tag{3.29}$$

Furthermore, we have $\eta^D(A \times \mathbb{R}) = 0$ *a.s. for each measurable* A *with* $\mathrm{Leb}(A) = 0$ *and, in particular,* $\eta^D(\partial D \times \mathbb{R}) = 0$ *a.s.*

Proof. The first part of the statement requires showing that, for any continuous compactly-supported function $f\colon \overline{D} \times (\mathbb{R} \cup \{\infty\}) \to \mathbb{R}$, the family of random variables $\{\langle \eta_N^D, f\rangle\colon N \geq 1\}$ is tight. For this it suffices to show that the family $\{\eta_N^D(\overline{D} \times [b, \infty))\colon N \geq 1\}$ is tight for each $b \in \mathbb{R}$. This is a consequence of Lemma 3.1 and the fact that $\eta_N^D(\overline{D} \times [b, \infty)) = |\Gamma_N^D(b)|$.

Let now η^D be a subsequential weak limit of the measures $\{\eta_N^D\colon N \geq 1\}$. Fatou's lemma, a straightforward approximation argument and Lemma 3.1 then show $E\eta^D(\overline{D} \times [b, \infty)) < \infty$ for each $b \in \mathbb{R}$. Lemma 3.1 also gives $\eta^D(A \times \mathbb{R}) = 0$ a.s. whenever $\mathrm{Leb}(A) = 0$. It remains to show that η^D is non-trivial in the sense stated in (3.29). Let $A \subset D$ be non-empty and open and pick $b \in \mathbb{R}$. Denote $X_N := \eta_N^D(A \times [b, \infty))$. Lemma 3.3 shows that $\sup_{N\geq 1} E(X_N^2) < \infty$ and so the family $\{X_N\colon N \geq 1\}$ is uniformly integrable. Since $\inf_{N\geq 1} E(X_N) > 0$ by Lemma 3.2 and the fact that $\psi_\lambda^D > 0$, any distributional limit of X_N has positive expectation as well. □

Given a function $f\colon \overline{D} \times (\mathbb{R} \cup \{\infty\}) \to \mathbb{R}$ and $b \in \mathbb{R}$, define

$$f_b(x, h) := f(x, h + b)\mathrm{e}^{-\alpha\lambda b}. \tag{3.30}$$

A key step is now the proof of:

Proposition 3.5. *Suppose* $\lambda \in (0, 1/\sqrt{2})$. *Any subsequential limit* η^D *of* $\{\eta_N^D\colon N \geq 1\}$ *obeys the following*: *For each* $b \in \mathbb{R}$ *and each* $f\colon \overline{D} \times (\mathbb{R} \cup \{\infty\}) \to \mathbb{R}$ *of the form* $f(x, h) = \mathsf{1}_A(x)\mathsf{1}_{[0,\infty)}(h)$ *with* $A \subset D$ *open*,

$$\langle \eta^D, f_b\rangle = \langle \eta^D, f\rangle \in \mathbb{R} \tag{3.31}$$

holds with probability one.

The proof of this proposition relies on a calculation that is formalized as:

Lemma 3.6. *For any* $\lambda \in (0, 1/\sqrt{2})$, *any open* $A \subset D$, *any* $b \in \mathbb{R}$, *and* $A_N := \{x \in \mathbb{Z}^2\colon x/N \in A\}$,

$$\lim_{N\to\infty} \frac{1}{K_N} E\bigl|\bigl|\Gamma_N^D(0) \cap A_N\bigr| - \mathrm{e}^{\alpha\lambda b}\bigl|\Gamma_N^D(b) \cap A_N\bigr|\bigr| = 0. \tag{3.32}$$

Proof. Since any open set $A \subset D$ can be written as the union of an increasing sequence of open sets whose closure lies inside D, in light of Lemma 3.1 we can assume that A has positive Euclidean distance to D^{c}. Then, using Cauchy–Schwarz, we may as well show

$$\lim_{N\to\infty} \frac{1}{K_N^2} E\Bigl(\bigl(\bigl|\Gamma_N^D(0) \cap A_N\bigr| - \mathrm{e}^{\alpha\lambda b}\bigl|\Gamma_N^D(b) \cap A_N\bigr|\bigr)^2\Bigr) = 0. \tag{3.33}$$

Invoking (3.10), the calculation in the proof of Lemma 3.3 shows that the second moment of $|\Gamma_N^D(b)|$ is dominated by the sum of pairs $x, y \in D_N$ that are within distance of order N. It thus suffices to prove that, for any $\epsilon > 0$,

$$\begin{aligned}\max_{\substack{x,y\in A_N\\ |x-y|\ge \epsilon N}} \Big(& P\big(h^{D_N}(x) \ge a_N, h^{D_N}(y) \ge a_N\big)\\ &- \mathrm{e}^{\alpha\lambda b} P\big(h^{D_N}(x) \ge a_N + b, h^{D_N}(y) \ge a_N\big)\\ &- \mathrm{e}^{\alpha\lambda b} P\big(h^{D_N}(x) \ge a_N, h^{D_N}(y) \ge a_N + b\big)\\ &+ \mathrm{e}^{2\alpha\lambda b} P\big(h^{D_N}(x) \ge a_N + b, h^{D_N}(y) \ge a_N + b\big)\Big) = o\Big(\frac{K_N^2}{N^4}\Big)\end{aligned} \tag{3.34}$$

as $N \to \infty$. For this we need to compute a sharp leading order asymptotic of the probability $P(h^{D_N}(x) \ge a_N + b_1, h^{D_N}(y) \ge a_N + b_2)$ for the four possible choices $b_1, b_2 \in \{0, b\}$.

We will invoke the decomposition (3.12) and the representation (3.13). Writing again $G(y, y)$ for $G^{D_N\setminus\{x\}}(y, y)$, for any $s \in [0, a_N + b_1]$ we then get

$$\begin{aligned}&P\big(\hat h^{D_N\setminus\{x\}}(y) \ge a_N\big(1 - \mathfrak{g}_x(y)\big) - s\mathfrak{g}_x(y) + b_2\big)\\ &= \frac{c + o(1)}{\sqrt{\log N}} \mathrm{e}^{-\frac{[a_N(1-\mathfrak{g}_x(y)) - s\mathfrak{g}_x(y) + b_2]^2}{2G(y,y)}} = \frac{c + o(1)}{\sqrt{\log N}} \mathrm{e}^{-\frac{[a_N(1-\mathfrak{g}_x(y)) - s\mathfrak{g}_x(y)]^2}{2G(y,y)}} \mathrm{e}^{-\alpha\lambda b_2}\\ &= \big(\mathrm{e}^{-\alpha\lambda b_2} + o(1)\big) P\big(\hat h^{D_N\setminus\{x\}}(y) \ge a_N\big(1 - \mathfrak{g}_x(y)\big) - s\mathfrak{g}_x(y)\big),\end{aligned} \tag{3.35}$$

where $c > 0$ is a numerical constant and where we used that $\mathfrak{g}_x(y) = O(1/\log N)$ when $|x - y| \ge \epsilon N$ and then applied the asymptotic of a_N and $G^{D_N\setminus\{x\}}(y, y) = g \log N + O(1)$ enabled by the fact that now y is "deep" inside $D_N \setminus \{x\}$ as implied by the assumptions on A and x, y. In addition, writing $G(x, x)$ for $G^{D_N}(x, x)$, we also get

$$\begin{aligned}P\big(h^{D_N}(x) - a_N - b_1 \in \mathrm{d}s\big) &= \frac{c + o(1)}{\sqrt{\log N}} \mathrm{e}^{-\frac{[a_N + b_1 + s]^2}{2G(x,x)}}\,\mathrm{d}s\\ &= \mathrm{e}^{-\alpha\lambda b_1} \frac{c + o(1)}{\sqrt{\log N}} \mathrm{e}^{-\frac{[a_N + s]^2}{2G(x,x)}}\,\mathrm{d}s\\ &= \big(\mathrm{e}^{-\alpha\lambda b_1} + o(1)\big) P\big(h^{D_N}(x) - a_N \in \mathrm{d}s\big),\end{aligned} \tag{3.36}$$

where c is again a positive constant and $o(1) \to 0$ as $N \to \infty$ uniformly in $s \in [0, a_N]$. Putting (3.35)–(3.36) together and integrating over $s \in [0, a_N]$ we get

$$\begin{aligned}&P\big(h^{D_N}(x) \ge a_N + b_1, h^{D_N}(y) \ge a_N + b_2\big)\\ &\quad= \big(\mathrm{e}^{-\alpha\lambda(b_1+b_2)} + o(1)\big) P\big(h^{D_N}(x) \ge a_N, h^{D_N}(y) \ge a_N\big),\end{aligned} \tag{3.37}$$

where we used that, by the FKG inequality and Lemma 3.2,

$$P\big(h^{D_N}(x) \ge 2a_N\big) = o(1) P\big(h^{D_N}(x) \ge a_N, h^{D_N}(y) \ge a_N\big) \tag{3.38}$$

with $o(1) \to 0$ as $N \to \infty$ uniformly in $x, y \in A_N$ with $|x - y| \ge \epsilon N$. Plugging (3.37) into (3.34), the desired claim follows. □

We are now ready to give:

Proof of Proposition 3.5. Let $f(x, h) := \mathsf{1}_A(x)\mathsf{1}_{[0,\infty)}(h)$ with $A \subset D$ closed. Lemma 3.6 can be rephrased as

$$\lim_{N\to\infty} E\big|\langle \eta_N^D, f\rangle - \langle \eta_N^D, f_b\rangle\big| = 0, \quad b \in \mathbb{R}. \tag{3.39}$$

Taking the distributional limit (choosing further subsequence if necessary) of $\langle \eta_N^D, f - f_b\rangle$ then shows, by Fatou's lemma, $\langle \eta^D, f - f_b\rangle = 0$ a.s. which is the desired claim. (The finiteness follows from the tightness proved in Lemma 3.1.) □

The identity (3.31) now readily yields the desired factorization property:

Lemma 3.7 (Factorization). *Suppose η^D is a Borel measure on $\overline{D} \times \mathbb{R}$ with $\eta^D(\partial D \times \mathbb{R}) = 0$ a.s. such that (3.31) holds for each function $f\colon \overline{D} \times \mathbb{R} \to \mathbb{R}$ of the form $f(x,h) := \mathsf{1}_A(x)\mathsf{1}_{[0,\infty)}(h)$ with $A \subset D$ open and each $b \in \mathbb{R}$. Then, with probability one, η^D takes the form*

$$\eta^D(\mathrm{d}x\,\mathrm{d}h) = Z_\lambda^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha\lambda h}\,\mathrm{d}h \tag{3.40}$$

for some finite random Borel measure Z_λ^D on $\overline{D}$.

Proof. For $A \subset D$ Borel, define $Z_\lambda^D(A) := \alpha\lambda\eta^D(A \times [0,\infty))$. This is automatically a finite Borel measure on D. The condition (3.31) now shows that, almost surely if A is open then

$$\begin{aligned}\eta^D\big(A \times [b,\infty)\big) &= \mathrm{e}^{-\alpha\lambda b}\langle \eta^D, f_b\rangle \overset{\text{a.s.}}{=} \mathrm{e}^{-\alpha\lambda b}\langle \eta^D, f\rangle \\ &= (\alpha\lambda)^{-1}\mathrm{e}^{-\alpha\lambda b}Z_\lambda^D(A) = \int_{A\times[b,\infty)} Z_\lambda^D(\mathrm{d}x)\mathrm{e}^{-\alpha\lambda h}\,\mathrm{d}h.\end{aligned} \tag{3.41}$$

The (implicit) null event in this statement may depend on A and b but we can choose a common null event for all sets in the class $\{A \times [b,\infty)\colon A \subset D \text{ open dyadic square}, b \in \mathbb{Q}\}$ as it is countable. The equality of the measures (3.40) on D then follows from the fact that this class is a π-system (in the sense of Dynkin) which generates the product Borel σ-algebra on $D \times \mathbb{R}$. As $\eta^D(\partial D \times \mathbb{R}) = Z_\lambda^D(\partial D) = 0$, the equality in (3.40) extends to all of D. □

3.3. *Uniqueness of subsequential limit*

At this point we have shown that, assuming $\lambda \in (0, 1/\sqrt{2})$, every subsequential limit η^D of the measures $\{\eta_N^D\colon N \geq 1\}$, with η_N^D as in (3.27), factors into the form (3.40). The goal of this subsection is to show that the measure Z_λ^D, and thus also the subsequential limit η^D, is in fact unique. This will in particular show that η_N^D converges in distribution to the same limit, regardless of the approximating sequence D_N or the way a_N achieves the asymptotic (1.5).

As our first lemma we will check that Z_λ^D obeys the properties listed in Theorem 2.3. However, these require extracting subsequential limits for multiple domains at the same time. Cantor's diagonal argument makes this possible provided we restrict ourselves to a countable class $\mathfrak{D}_0$ of domains in $\mathfrak{D}$. We will assume that $\mathfrak{D}_0$ contains all open squares of the form

$$\big(k2^{-n}, (k+1)2^{-n}\big) \times \big(\ell 2^{-n}, (\ell+1)2^{-n}\big), \quad k,\ell,n \in \mathbb{Z}, \tag{3.42}$$

and all finite unions thereof. For each domain $D \in \mathfrak{D}_0$ we then fix a sequence $\{D_N\}$ of lattice approximations satisfying (2.1)–(2.2). All (simultaneous) subsequential limits will naturally pertain to the specific choice of $\mathfrak{D}_0$ as well as the lattice approximations $\{D_N\}$.

Proposition 3.8. *Assume $\lambda \in (0, 1/\sqrt{2})$ and let $\{\eta^D\colon D \in \mathfrak{D}_0\}$ be subsequential limits (along the same subsequence) of $\{\eta_N^D\colon N \geq 1\}$ for $D \in \mathfrak{D}_0$. For each $D \in \mathfrak{D}_0$, let Z_λ^D be the measure associated with η^D as in (3.40). Then $\{Z_\lambda^D\colon D \in \mathfrak{D}_0\}$ obeys properties (1)–(7) in Theorem 2.3, with all domains restricted to be contained in $\mathfrak{D}_0$ or translates thereof.*

Proof of properties (1)–(6). Properties (1), (2) are direct consequences of Lemma 3.1. Property (3) holds for all open $A \subset D$ thanks to Lemma 3.2; the equality for general measurable $A \subset D$ is then obtained by realizing that (2.11) represents equality of two Borel measures. Property (4) is a consequence of the representation of $\eta_N^{D\cup\widetilde{D}}$ as the sum of independent copies of η_N^D and $\eta_N^{\widetilde{D}}$ – which itself follows by representing $h^{D_N\cup\widetilde{D}_N}$ as the sum of independent fields h^{D_N} and $h^{\widetilde{D}_N}$. The translation invariance in property (6) is immediate.

Concerning property (5), let $D, \widetilde{D} \in \mathfrak{D}_0$ with $\widetilde{D} \subset D$ and $\mathrm{Leb}(D \setminus \widetilde{D}) = 0$. The Gibbs–Markov decomposition of the DGFF (see (A.6)) then yields

$$h^{D_N} \stackrel{\text{law}}{=} h^{\widetilde{D}_N} + \varphi^{D_N, \widetilde{D}_N}. \tag{3.43}$$

This means that if $f\colon \overline{D} \times \mathbb{R} \to \mathbb{R}$ is continuous with compact support in $\widetilde{D}$, then

$$\langle \eta_N^D, f \rangle \stackrel{\text{law}}{=} \langle \eta_N^{\widetilde{D}}, f_\varphi \rangle \tag{3.44}$$

where

$$f_\varphi(x,h) := f\bigl(x, h + \varphi^{D_N, \widetilde{D}_N}(\lfloor xN \rfloor)\bigr) \tag{3.45}$$

with $\varphi^{D_N, \widetilde{D}_N}$ independent of $\eta_N^{\widetilde{D}}$ on the right-hand side of (3.44). As shown in [11, Lemma B.14], for each $N \geq 1$ and each $\delta > 0$, there is a coupling of $\varphi^{D_N, \widetilde{D}_N}(\lfloor \cdot N \rfloor)$ with $\Phi^{D, \widetilde{D}}$ such that the supremum of the difference on D^δ tends to zero in probability; see (A.8). Thanks to continuity and restriction on the support of f, we thus have

$$\langle \eta_N^{\widetilde{D}}, f_\varphi \rangle \stackrel{\text{law}}{=} \langle \eta_N^{\widetilde{D}}, f_\Phi \rangle + o(1), \tag{3.46}$$

where $o(1) \to 0$ in probability (as $N \to \infty$) and where

$$f_\Phi(x,h) := f\bigl(x, h + \Phi^{D, \widetilde{D}}(x)\bigr) \tag{3.47}$$

with $\Phi^{D, \widetilde{D}}$ independent of $\eta_N^{\widetilde{D}}$ on the right-hand side of (3.46). Since $x \mapsto \Phi^{D, \widetilde{D}}(x)$ is continuous on $\widetilde{D}$ a.s., for any simultaneous subsequential limits η^D of $\{\eta_N^D \colon N \geq 1\}$ and $\eta^{\widetilde{D}}$ of $\{\eta_N^{\widetilde{D}} \colon N \geq 1\}$, we thus obtain

$$\langle \eta^D, f \rangle \stackrel{\text{law}}{=} \langle \eta^{\widetilde{D}}, f_\Phi \rangle, \tag{3.48}$$

where $\Phi^{D, \widetilde{D}}$ (implicitly contained in f_Φ) is independent of $\eta^{\widetilde{D}}$ on the right-hand side. But the representation (3.40) now permits us to write

$$\begin{aligned} \langle \eta^{\widetilde{D}}, f_\Phi \rangle &= \int_{D \times \mathbb{R}} Z_\lambda^{\widetilde{D}}(\mathrm{d}x) \mathrm{e}^{-\alpha\lambda h} \, \mathrm{d}h f\bigl(x, h + \Phi^{D, \widetilde{D}}(x)\bigr) \\ &= \int_{D \times \mathbb{R}} Z_\lambda^{\widetilde{D}}(\mathrm{d}x) \mathrm{e}^{-\alpha\lambda (h - \Phi^{D, \widetilde{D}}(x))} \, \mathrm{d}h f(x,h). \end{aligned} \tag{3.49}$$

As this holds for any continuous $f\colon D \times \mathbb{R} \to \mathbb{R}$ with support in $\widetilde{D}$, and since both $Z_\lambda^{\widetilde{D}}$ and Z_λ^D assign zero mass to $D \setminus \widetilde{D}$ due to the fact that $\mathrm{Leb}(D \setminus \widetilde{D}) = 0$, property (5) follows. □

For property (7), and also later use, we will need:

Lemma 3.9. *For each $\lambda \in (0, 1/\sqrt{2})$ there is $c \in (0, \infty)$ such that for any open square $S \subset \mathbb{C}$*

$$E\bigl[Z_\lambda^S(S)^2\bigr] \leq c\bigl[E Z_\lambda^S(S)\bigr]^2. \tag{3.50}$$

Proof. Suppose S is a translation (and rotation) of $(0,r)^2$. Then (3.9) in Lemma 3.3 along with Fatou's lemma, the inequality $4\lambda^2 < 2$ and a scaling argument show that $E[Z_\lambda^S(S)^2] \leq c r^{4+4\lambda^2}$ for some constant c independent of r. On the other hand, Lemma 3.2 along with uniform integrability of normalized level-set sizes and the fact that $\psi_\lambda^{rD}(rx) = r^{2\lambda^2} \psi_\lambda^D(x)$ show $E[Z_\lambda^S(S)] \geq \tilde{c} r^{2+2\lambda^2}$ for some absolute $\tilde{c} > 0$. The claim follows. □

Proof of property (7). It suffices to prove this for all squares of the form (3.42) as each open set contains at least one such square. (Note that $\mathfrak{D}_0$ contains all these squares.) For $n \in \mathbb{Z}$, let $p_n := P(Z_\lambda^{S_n} > 0)$ where (appealing to

translation invariance), $S_n := (0, 2^{-n})^2$. The second moment estimate in conjunction with (the second moment estimate in) Lemma 3.9 show that $p := \inf_{n\in\mathbb{Z}} p_n > 0$. Decomposing S_n into 4^m translates $S_{n+m,1}, \dots, S_{n+m,4^m}$ of the square S_{n+m}, the Gibbs–Markov property yields

$$Z_\lambda^{S_n}(S_n) \overset{\text{law}}{=} \sum_{i=1}^{4^m} \int_{S_{n+m,i}} Z_\lambda^{S_{n+m,i}}(\mathrm{d}x) \mathrm{e}^{\alpha\lambda\Phi^{S_n,\widetilde{S}_{n,m}}(x)}, \tag{3.51}$$

where $\widetilde{S}_{n,m} := \bigcup_{i=1}^{4^m} S_{n+m,i}$. Then $Z_\lambda^{S_n}(S_n) = 0$ forces $Z_\lambda^{S_{n+m,i}}(S_{n+m,i}) = 0$ for all $i = 1, \dots, 4^m$. Since the latter measures are independent of one another as well as of the field $\Phi^{S_n,\widetilde{S}_{n,m}}$, we have

$$1 - p_n \le (1 - p_{n+m})^{4^m} \le (1 - p)^{4^m}. \tag{3.52}$$

Taking $m \to \infty$ we get $p_n = 1$ for each $n \in \mathbb{Z}$ and so property (6) follows. □

Using the same notation as in the previous proof, in order to prove uniqueness, we will first characterize $Z_\lambda^{S_n}$ as the limit of the measures

$$Y_m^{S_n}(\mathrm{d}x) := c \sum_{i=1}^{4^m} \mathrm{e}^{\alpha\lambda\Phi^{S_n,\widetilde{S}_{n,m}}(x)} \psi_\lambda^{S_{n+m,i}}(x) \mathsf{1}_{S_{n+m,i}}(x)\, \mathrm{d}x, \tag{3.53}$$

where c is the constant from (2.11). Indeed, we have:

Lemma 3.10. *For each $\lambda \ge 0$, there exists an a.s. finite random measure $Y_\infty^{S_n}$ (possibly degenerate to zero), such that for each bounded, measurable $f : \overline{D} \to \mathbb{R}$,*

$$\langle Y_m^{S_n}, f\rangle \underset{m\to\infty}{\longrightarrow} \langle Y_\infty^{S_n}, f\rangle, \quad \text{a.s.} \tag{3.54}$$

Proof. Thanks to the structure of the covariances (2.9), we can write $\Phi^{S_n,\widetilde{S}_{n,m}}(x)$ as the sum of independent fields

$$\Phi^{S_n,\widetilde{S}_{n,m}}(x) = \sum_{j=1}^{m} \Phi^{\widetilde{S}_{n,j-1},\widetilde{S}_{n,j}}(x), \tag{3.55}$$

where $\widetilde{S}_{n,0} := S_n$. In light of the fact that

$$\sum_{j=1}^{4^m} \psi_\lambda^{S_{n+m,j}}(x) \mathsf{1}_{S_{n+m,j}}(x) = \psi_\lambda^{\widetilde{S}_{n,m}}(x), \quad x \in \widetilde{S}_{n,m} \tag{3.56}$$

and that, for any $\widetilde{D} \subset D$,

$$E\mathrm{e}^{\alpha\lambda\Phi^{D,\widetilde{D}}(x)} = \mathrm{e}^{\frac{1}{2}\alpha^2\lambda^2 C^{D,\widetilde{D}}(x)} = \frac{\psi_\lambda^D(x)}{\psi_\lambda^{\widetilde{D}}(x)}, \quad x \in \widetilde{D}, \tag{3.57}$$

a straightforward calculation shows that $\{\langle Y_m^{S_n}, f\rangle : m \ge 1\}$ is a martingale with respect to the filtration

$$\mathcal{F}_m := \sigma\big(\Phi^{\widetilde{S}_{n,j-1},\widetilde{S}_{n,j}}(x) : x \in S_n', j = 1, \dots, m\big) \quad \text{where } S_n' := \bigcap_{m\ge1} \widetilde{S}_{n,m}. \tag{3.58}$$

(Note that, as $\mathrm{Leb}(S_n \setminus S_n') = 0$, restricting the measures to S_n' carries no loss.) Since for $f \ge 0$ the martingale is non-negative, the Martingale Convergence Theorem shows that

$$L(f) := \lim_{m\to\infty} \langle Y_m^{S_n}, f\rangle \tag{3.59}$$

exists almost surely for each bounded measurable f (we treat the positive and the negative parts of f separately). The null event in this statement may depend on f.

In order to show that the limit is an integral of f with respect to a random measure, we follow a standard argument from multiplicative chaos theory: Fix a countable dense subset $A \subset C(\overline{D})$. Fatou's lemma yields

$$E\big|L(f)\big| \le c \int_{S_n} \big|f(x)\big|\psi_\lambda^D(x)\,\mathrm{d}x, \quad f \in C(\overline{D}), \tag{3.60}$$

and so by the Markov inequality, on an event of full probability, the linear functional $f \mapsto L(f)$ is well-defined for all $f \in A$ simultaneously and bounded on A by a multiple of the supremum norm. It follows that $f \mapsto L(f)$ extends uniquely to an almost-everywhere defined continuous linear functional $f \mapsto \overline{L}(f)$ on $C(\overline{D})$ such that $L(f) = \overline{L}(f)$ holds almost surely for each $f \in C(\overline{D})$. (The null event may still depend on f.) The Riesz Representation Theorem then readily gives existence of a Borel measure $Y_\infty^{S_n}$ on $\overline{D}$ such that $L(f) = \langle Y_\infty^{S_n}, f\rangle$ holds almost surely for each $f \in C(\overline{D})$. The measure is finite a.s. (albeit possibly trivially zero) thanks to (3.60). □

The desired uniqueness of the law of Z_λ^D will now follow from:

Proposition 3.11. *Suppose $\lambda \in (0, 1/\sqrt{2})$. Then for each $n \in \mathbb{Z}$,*

$$Z_\lambda^{S_n}(\mathrm{d}x) \overset{\text{law}}{=} Y_\infty^{S_n}(\mathrm{d}x). \tag{3.61}$$

For the proof of this proposition, let $f\colon S_n \to [0,\infty)$ be a bounded, measurable function. Our aim is to prove that

$$E\mathrm{e}^{-\langle Z_\lambda^{S_n}, f\rangle} = E\mathrm{e}^{-\langle Y_\infty^{S_n}, f\rangle}. \tag{3.62}$$

We will do this by separately proving $\ge$ and $\le$.

Proof of $\ge$ in (3.62). Thanks to the Gibbs–Markov property, we may represent $Z_\lambda^{S_n}$ as

$$Z_\lambda^{S_n}(\mathrm{d}x) = \sum_{i=1}^{4^m} \mathrm{e}^{\alpha\lambda\Phi^{S_n,\widetilde{S}_{n,m}}(x)} \mathsf{1}_{S_{n+m,i}}(x) Z_\lambda^{S_{n+m,i}}(\mathrm{d}x), \tag{3.63}$$

where $Z_\lambda^{S_{n+m,i}}$, $i = 1,\dots,4^m$, are independent of one another as well as of $\Phi^{S_n,\widetilde{S}_{n,m}}$ on the right-hand side. In light of (2.11) we have

$$E\big(\big\langle Z_\lambda^{S_n}, f\big\rangle | \sigma\big(\Phi^{S_n,\widetilde{S}_{n,m}}\big)\big) = \big\langle Y_m^{S_n}, f\big\rangle. \tag{3.64}$$

Jensen's inequality then shows

$$E\mathrm{e}^{-\langle Z_\lambda^{S_n}, f\rangle} \ge E\mathrm{e}^{-\langle Y_m^{S_n}, f\rangle} \underset{m\to\infty}{\longrightarrow} E\mathrm{e}^{-\langle Y_\infty^{S_n}, f\rangle}, \tag{3.65}$$

where the limit uses Lemma 3.10 and the Bounded Convergence Theorem. □

For the opposite bound, we will need the following "reverse Jensen" inequality:

Lemma 3.12. *If $X_1,\dots,X_n$ are non-negative independent random variables, then for each $\epsilon > 0$,*

$$E\left(\exp\left\{-\sum_{i=1}^n X_i\right\}\right) \le \exp\left\{-\mathrm{e}^{-\epsilon}\sum_{i=1}^n E(X_i; X_i \le \epsilon)\right\}. \tag{3.66}$$

Proof. Thanks to independence, it suffices to prove this for $n = 1$. This is done by bounding $E(\mathrm{e}^{-X}) \le E(\mathrm{e}^{-\widetilde{X}})$, where $\widetilde{X} := X1_{\{X\le\epsilon\}}$, writing

$$-\log E\big(\mathrm{e}^{-\widetilde{X}}\big) = \int_0^1 \mathrm{d}s\, \frac{E(\widetilde{X}\mathrm{e}^{-s\widetilde{X}})}{E(\mathrm{e}^{-s\widetilde{X}})} \tag{3.67}$$

and invoking the bounds $E(\widetilde{X}\mathrm{e}^{-s\widetilde{X}}) \ge \mathrm{e}^{-\epsilon}E(\widetilde{X})$ and $E(\mathrm{e}^{-s\widetilde{X}}) \le 1$. □

We will also need to invoke an additional truncation: For $\delta \in (0, 1/2)$, let S_k^δ be the translate of $(\delta 2^{-k}, (1-\delta)2^{-k})$ centered at the same point as S_k. Analogously, let $S_{n+m,i}^\delta$ be the corresponding truncation of $S_{n+m,i}$. Then set $\widetilde{S}_{n,m}^\delta := \bigcup_{i=1}^{4^m} S_{n+m,i}^\delta$ and let

$$f_{m,\delta}(x) := 1_{\widetilde{S}_{n,m}^\delta}(x) f(x). \tag{3.68}$$

Thanks to $f \ge 0$ we have $\langle Y_m^{S_n}, f_{m,\delta}\rangle \le \langle Y_m^{S_n}, f\rangle \le \langle Y_m^{S_n}, f_{m,\delta}\rangle + \|f\| Y_m^{S_n}(S_n \setminus \widetilde{S}_{n,m}^\delta)$ where here and henceforth $\|f\|$ denotes the supremum norm of f. Markov's inequality and the definition of $Y_m^{S_n}$ then show $P(Y_m^{S_n}(S_n \setminus \widetilde{S}_{n,m}^\delta) > \epsilon) \le \epsilon^{-1}\,\mathrm{Leb}(S_n \setminus \widetilde{S}_{n,m}^\delta) \le c\epsilon^{-1}\delta$ with c independent of m. In conjunction with Lemma 3.10, it follows that

$$\lim_{\delta\downarrow 0} \limsup_{m\to\infty} P\big(\big|\big\langle Y_m^{S_n}, f_{m,\delta}\big\rangle - \big\langle Y_\infty^{S_n}, f\big\rangle\big| > \epsilon\big) = 0. \tag{3.69}$$

This will permit us to work with $f_{m,\delta}$ instead of f. Next we will need:

Lemma 3.13. *Suppose $\lambda \in (0, 1/\sqrt{2})$. Given $\delta \in (0, 1/2)$, for each $i = 1, \dots, 4^m$ abbreviate*

$$X_i := \int_{S_{n+m,i}^\delta} \mathrm{e}^{\alpha\lambda\Phi^{S_n,\widetilde{S}_{n,m}}(x)} f_{m,\delta}(x) Z_\lambda^{S_{n+m,i}}(\mathrm{d}x). \tag{3.70}$$

Then for each $\epsilon > 0$,

$$\sum_{i=1}^{4^m} E(X_i; X_i > \epsilon) \underset{m\to\infty}{\longrightarrow} 0. \tag{3.71}$$

Proof. Abbreviate $L := 2^m$ throughout this proof. Since $E(X_i; X_i > \epsilon) \le \frac{1}{\epsilon}E(X_i^2)$, we can bound the sum in (3.71) by

$$\frac{1}{\epsilon}\sum_{i=1}^{L^2} E\big(X_i^2\big) \le \frac{\|f\|^2}{\epsilon}\sum_{i=1}^{L^2} E\int_{S_{n+m,i}^\delta \times S_{n+m,i}^\delta} \mathrm{e}^{\alpha\lambda[\Phi^{S_n,\widetilde{S}_{n,m}}(x)+\Phi^{S_n,\widetilde{S}_{n,m}}(y)]} Z_\lambda^{S_{n+m,i}}(\mathrm{d}x) Z_\lambda^{S_{n+m,i}}(\mathrm{d}y). \tag{3.72}$$

Since $Z_\lambda^{S_{n+m,i}}$, $i = 1, \dots, 4^m$, are independent of the field, we will now take conditional expectation given these measures and invoking that

$$\begin{aligned} E\mathrm{e}^{\alpha\lambda[\Phi^{S_n,\widetilde{S}_{n,m}}(x)+\Phi^{S_n,\widetilde{S}_{n,m}}(y)]} &= \mathrm{e}^{\frac{1}{2}\alpha^2\lambda^2\,\mathrm{Var}(\Phi^{S_n,\widetilde{S}_{n,m}}(x)+\Phi^{S_n,\widetilde{S}_{n,m}}(y))} \\ &\le c\mathrm{e}^{\frac{1}{2}\alpha^2\lambda^2 4g\log(L)} = cL^{8\lambda^2} \end{aligned} \tag{3.73}$$

for some constant c that arises from the uniform bound $\mathrm{Var}(\Phi^{S_n,\widetilde{S}_{n,m}}(x)) \le c' + g\log(2^m)$ valid with the same constant c' for all $x \in \widetilde{S}_{n,m}^\delta$. The right-hand side of (3.72) is thus at most

$$c\frac{\|f\|^2}{\epsilon}L^{8\lambda^2}\sum_{i=1}^{L^2} E\big(Z_\lambda^{S_{n+m,i}}(S_{n+m,i})^2\big). \tag{3.74}$$

Under the condition $\lambda \in (0, 1/\sqrt{2})$, Lemma 3.9 implies

$$E\bigl(Z_\lambda^{S_{n+m,i}}(S_{n+m,i})^2\bigr) \le c\bigl[E\bigl(Z_\lambda^{S_{n+m,i}}(S_{n+m,i})\bigr)\bigr]^2 \tag{3.75}$$

and an elementary scaling argument applied to (2.11) shows

$$E\bigl(Z_\lambda^{S_{n+m,i}}(S_{n+m,i})\bigr) \le cL^{-(2+2\lambda^2)}, \tag{3.76}$$

where c depends on n, which is fixed throughout the proof. These observations yield

$$\sum_{i=1}^{L^2} E(X_i; X_i > \epsilon) \le \frac{c}{\epsilon}\|f\|^2 L^{8\lambda^2-2-4\lambda^2} = \frac{c}{\epsilon}\|f\|^2 L^{-2(1-2\lambda^2)}. \tag{3.77}$$

As $L = 2^m$, this tends to zero as $m \to \infty$ for all $\lambda < (0, 1/\sqrt{2})$ as claimed. □

Proof of $\le$ in (3.62). Consider the σ-algebra $\mathcal{F} := \sigma(\Phi^{S_n,\widetilde{S}_{n,m}}(x)\colon x \in \widetilde{S}_{n,m})$ and note that, for X_i as in (3.70), we have $\langle Y_m^{S_n}, f_{m,\delta}\rangle = \sum_{i=1}^{4^m} E(X_i|\mathcal{F})$. Fix $\epsilon > 0$. The "reverse Jensen" inequality (3.66) for the conditional expectation given $\mathcal{F}$ then yields

$$E\mathrm{e}^{-\langle Z_\lambda^{S_n}, f\rangle} \le E\mathrm{e}^{-\langle Z_\lambda^{S_n}, f_{m,\delta}\rangle} \le E\left(\exp\left\{-\mathrm{e}^{-\epsilon}\left[\bigl\langle Y_m^{S_n}, f_{m,\delta}\bigr\rangle - \sum_{i=1}^{4^m} E(X_i 1_{\{X_i>\epsilon\}}|\mathcal{F})\right]\right\}\right). \tag{3.78}$$

From (3.69) we know that $\langle Y_m^{S_n}, f_{m,\delta}\rangle \to \langle Y_\infty^{S_n}, f\rangle$ in probability as $m \to \infty$ followed by $\delta \downarrow 0$, while Lemma 3.13 shows

$$\sum_{i=1}^{4^m} E(X_i 1_{\{X_i>\epsilon\}}|\mathcal{F}) \underset{m\to\infty}{\longrightarrow} 0, \quad \text{in probability.} \tag{3.79}$$

Since the square bracket on the right-hand side of (3.78) is non-negative, taking $m \to \infty$ and $\delta \downarrow 0$ with the help of the Bounded Convergence Theorem yields

$$E\mathrm{e}^{-\langle Z_\lambda^{S_n}, f\rangle} \le E\mathrm{e}^{-\mathrm{e}^{-\epsilon}\langle Y_\infty^{S_n}, f\rangle}. \tag{3.80}$$

From here $\le$ in (3.62) follows by taking $\epsilon \downarrow 0$. □

Proof of Proposition 3.11. Since the Laplace transform determines the law for non-negative random variables, (3.62) implies $\langle Z_\lambda^{S_n}, f\rangle \overset{\text{law}}{=} \langle Y_\infty^{S_n}, f\rangle$ for each bounded, measurable f. This is what is represented by (3.61). □

We are now ready to summarize our conclusions in:

Theorem 3.14. *For each $\lambda \in (0, 1/\sqrt{2})$ and each $D \in \mathfrak{D}$ there is a random Borel measure Z_λ^D on D such that the following holds for each a_N satisfying (1.5) and each sequence $\{D_N\}$ of lattice domains satisfying (2.1)–(2.2): The family of measures $\{\eta_N^D\colon N \ge 1\}$ from (3.27) obeys*

$$\eta_N^D \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} Z_\lambda^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha\lambda h}\,\mathrm{d}h. \tag{3.81}$$

The measures $\{Z_\lambda^D\colon D \in \mathfrak{D}\}$ obey conditions (1-7) from Theorem 2.3 and these identify their laws uniquely. In particular, on dyadic squares we have $Z_\lambda^{S_n}(\mathrm{d}x) \overset{\text{law}}{=} Y_\infty^{S_n}(\mathrm{d}x)$.

Proof. That subsequential limits of η_N^D take the form on the right of (3.81) has been shown in Lemma 3.7 and that the resulting Z_λ^D measures obey properties (1)–(7) from Theorem 2.3 is the content of Proposition 3.8. Thanks to the representation in Proposition 3.11, the law of Z_λ^D is determined for D being any dyadic square. In order to prove the theorem, it thus suffices to show that the law of Z_λ^D is similarly determined for all $D \in \mathfrak{D}$.

Let $D \in \mathfrak{D}$. Since we may assume that D belongs to the distinguished set $\mathfrak{D}_0$ of domains, the properties (1)–(7) from Theorem 2.3 apply to D as well. Thus, in particular, $Z_\lambda^D(\overline{D} \setminus D^n) \to 0$ for any sequence of measurable sets D^n such that $D^n \uparrow D$. Thanks to property (1) in Theorem 2.3, the same holds even if we take D^n to be the union of all open dyadic squares $S_{n,i}$, $i = 1, \dots, m(n)$ of side length 2^{-n} whose closure is contained in D. However, in light of properties (4)–(5) in Theorem 2.3, in this case we may write

$$1_{D^n}(x) Z_\lambda^D(\mathrm{d}x) \overset{\text{law}}{=} \mathrm{e}^{\alpha\lambda\Phi^{D,D^n}(x)} \sum_{i=1}^{m(n)} Y_\infty^{S_{n,i}}(\mathrm{d}x), \tag{3.82}$$

where $\{Y_\infty^{S_{n,i}} : i = 1, \dots, m(n)\}$ are independent of Φ^{D,D^n} and of one another and are equidistributed, modulo a shift, to $Y_\infty^{S_n}$. The law of Z_λ^D is thus determined solely by those of $\{Y_\infty^{S_n} : n \geq 1\}$ and the Gaussian fields $\{\Phi^{D,D^n} : n \geq 1\}$. We conclude that limit (3.81) exists for all $D \in \mathfrak{D}$ and is the same regardless of the approximating sequence of lattice domains D_N and/or the way a_N approaches the limit (1.5). □

4. Beyond untruncated second moments

Our next goal is to eliminate the restriction to $\lambda < 1/\sqrt{2}$ assumed throughout the proofs in the previous section. There were three specific steps where this restriction was crucially used: the non-triviality of the subsequential limits of $\{\eta_N^D : N \geq 1\}$, the factorization property in Lemma 3.6 and the estimate (3.77) of expectations of integrals against Z_λ-measures in Lemma 3.13 based on Lemma 3.9. This is because all three rely on the second moment estimate on the size of the level-set in Lemma 3.3 which fails when $\lambda \geq 1/\sqrt{2}$. (Lemmas 3.1 and 3.2 hold for all $\lambda \in (0, 1)$.)

It turns out, and this is no surprise in this subject area, that the lack of the second moment is remedied by introducing a suitable truncation. This will help us fix the above three second-moment calculations while preserving the overall strategy of the proof.

4.1. *Truncated measures*

Let us start with a truncated version of the measures in (3.27). Pick a sequence of domains $\{D_N\}$ approximating, via (2.1)–(2.2), a given continuum domain $D \in \mathfrak{D}$. Recall our earlier notation $\Lambda_r(x) := \{z \in \mathbb{Z}^2 : |z - x| \leq r\}$ and, for each $N \geq 1$ and each $x \in D_N$, let

$$n(x) := \max\bigl\{n \geq 0 : \Lambda_{\mathrm{e}^{n+1}}(x) \subseteq D_N\bigr\}. \tag{4.1}$$

Observe that $\log N - c \leq n(x) \leq \log N + c'$ for all $x \in D_N$ such that $\operatorname{dist}(x, D_N^{\mathrm{c}}) > \epsilon N$, with the first constant depending (for large N) only on the choice of $\epsilon > 0$ and the second only on D. Define now the sequence of domains

$$\Delta^k(x) := \begin{cases} \varnothing & \text{for } k = 0, \\ \Lambda_{\mathrm{e}^k}(x) & \text{for } k = 1, \dots, n(x) - 1, \\ D_N & \text{for } k = n(x). \end{cases} \tag{4.2}$$

In accord with (A.6), for $V \subseteq U$ let us write $\varphi^{U,V}$ for the conditional field $E(h^U | \sigma(h^U(z) : z \in U \setminus V))$. We now set

$$S_k(x) := \varphi^{D_N, \Delta^k(x)}(x), \quad k = 0, \dots, n(x). \tag{4.3}$$

Observe that, by definition, $S_0(x) = h^{D_N}(x)$ while $S_{n(x)}(x) = 0$.

Next, for a given sequence a_N such that (1.5) holds for some $\lambda \in (0,1)$ and $M > 0$, define the truncation event

$$T_{N,M}(x) := \bigcap_{k=k_N}^{n(x)} \biggl\{ \Bigl| S_k(x) - a_N \frac{n(x)-k}{n(x)} \Bigr| \le M\bigl(n(x)-k\bigr)^{3/4} \biggr\}, \tag{4.4}$$

where

$$k_N := \biggl\lfloor \frac{1}{8}\log(K_N) \biggr\rfloor = \frac{1}{4}\bigl[\bigl(1-\lambda^2\bigr) + o(1)\bigr]\log(N) \tag{4.5}$$

and the intersection is regarded as empty when $k_N > n(x)$. Consider the point measure

$$\widehat{\eta}_N^{D,M} := \frac{1}{K_N} \sum_{x \in D_N} \mathsf{1}_{T_{N,M}(x)} \delta_{x/N} \otimes \delta_{h^{D_N}(x) - a_N}. \tag{4.6}$$

Obviously, $\langle \widehat{\eta}_N^{D,M}, f\rangle \le \langle \eta_N^D, f\rangle$ for any measurable $f \ge 0$. We will now re-run the arguments from the previous section replacing the key second-moment lemmas by their suitable truncated versions. These are the content of Lemmas 4.1–4.3 below which will be proved in the next subsection.

Our starting point is that the difference between the measures $\widehat{\eta}_N^{D,M}$ and η_N^D disappears when the truncation is removed by taking $M \to \infty$. For this we introduce the truncated level set

$$\widehat{\Gamma}_N^{D,M}(b) := \bigl\{ x \in D_N : h^{D_N}(x) \ge a_N + b, T_{N,M}(x) \text{ occurs} \bigr\}. \tag{4.7}$$

Then we have:

Lemma 4.1. *For each $\lambda \in (0,1)$ and each $b_0 > 0$ there are constants $c, \tilde{c} \in (0,\infty)$ such that for all $D \in \mathfrak{D}$, all $b \in [-b_0, b_0]$, all $M \ge 1$ and all N sufficiently large,*

$$E\bigl|\Gamma_N^D(b) \setminus \widehat{\Gamma}_N^{D,M}(b)\bigr| \le c\mathrm{e}^{-\tilde{c}M^2}(\operatorname{diam} D)^{2+2\lambda^2} K_N. \tag{4.8}$$

Using this lemma we immediately get that for any continuous, compactly-supported function $f : \overline{D} \times \mathbb{R} \to \mathbb{R}$,

$$\lim_{M\to\infty} \limsup_{N\to\infty} E\bigl|\langle \widehat{\eta}_N^{D,M}, f\rangle - \langle \eta_N^D, f\rangle\bigr| = 0. \tag{4.9}$$

Since Lemma 3.1 and the aforementioned domination of $\widehat{\eta}_N^{D,M}$ by η_N^D show that the family of measures $\{\widehat{\eta}_N^{D,M} : N \ge 1\}$ is tight in the topology of vague convergence, we can extract a subsequential weak limit $\widehat{\eta}^{D,M}$ and study its properties.

The next point to address is non-triviality of the limit. Here we need an analogue of Lemma 3.3, now without restrictions on λ. For $b, b' \in \mathbb{R}$ with $b < b'$, abbreviate

$$\widehat{\Gamma}_N^{D,M}\bigl(b,b'\bigr) := \widehat{\Gamma}_N^{D,M}(b) \setminus \widehat{\Gamma}_N^{D,M}\bigl(b'\bigr). \tag{4.10}$$

Then we have:

Lemma 4.2. *Let $\lambda \in (0,1)$. For all $\epsilon > 0$, all $M \ge 0$ and all $b, b' \in \mathbb{R}$ with $b < b'$, there is $c = c(M,b,b',\epsilon) \in (1,\infty)$ such that for all $D \in \mathfrak{D}$ and all N large enough,*

$$E\bigl(\bigl|\widehat{\Gamma}_N^{D,M}\bigl(b,b'\bigr) \cap D_N^{\epsilon}\bigr|^2\bigr) \le c(\operatorname{diam} D)^{4+4\lambda^2} K_N^2. \tag{4.11}$$

The second moment calculation spelled out in the proof of Corollary 3.4 together with Lemma 4.1 then show that every subsequential weak limit $\widehat{\eta}^{D,M}$ of measures $\{\widehat{\eta}_N^{D,M} : N \ge 1\}$ has positive total mass with positive probability, provided M is chosen large enough. In light of the domination of $\widehat{\eta}_N^{D,M}$ by η_N^D, the same applies to any subsequential limit of the measures $\{\eta_N^D : N \ge 1\}$.

The next, and technically hardest, point to be addressed is factorization. This is the subject of the following lemma which effectively replaces Lemma 3.6:

Lemma 4.3. *Let $\lambda \in (0,1)$. Using the notation $A_N := \{x \in \mathbb{Z}^2 : x/N \in A\}$, for each open $A \subseteq D$ and each $b \in \mathbb{R}$, we have*

$$\lim_{N\to\infty} \frac{1}{K_N} E\bigl|\bigl|\widehat{\Gamma}_N^{D,M}(0) \cap A_N\bigr| - \mathrm{e}^{\alpha\lambda b}\bigl|\widehat{\Gamma}_N^{D,M}(b) \cap A_N\bigr|\bigr| = 0. \tag{4.12}$$

Lastly, we will need one more lemma dealing with the maximum of the field $\Phi^{S_n,\widetilde{S}_{n,m}}$, where S_n is a dyadic square of side-length 2^{-n} and $S_{n,m}$ is the disjoint union of 4^m dyadic squares of side-length $2^{-(n+m)}$ that just barely fit into S_n. Recall that $\widetilde{S}_{n+m}^{\delta}$ is the union of "shrunk" dyadic squares $S_{n,m}^{\delta}$ centered at the same points as $S_{n,m}$, respectively. See the paragraph before (3.69).

Lemma 4.4. *For each $\delta > 0$ there is a constant $c = c(\delta)$ such that*

$$P\Bigl(\sup_{x\in\widetilde{S}_{n,m}^{\delta}} \Phi^{S_n,\widetilde{S}_{n,m}}(x) > 2\sqrt{g}\log\bigl(2^m\bigr) + c\sqrt{\log\bigl(2^m\bigr)}\Bigr) \underset{m\to\infty}{\longrightarrow} 0. \tag{4.13}$$

Deferring (as mentioned above) the proofs of Lemmas 4.1–4.3 to the next subsection and that of Lemma 4.4 to the Appendix, we now use them to prove:

Theorem 4.5. *The statement of Theorem 3.14 applies to all $\lambda \in (0,1)$.*

Proof. Consider a countable family $\mathfrak{D}_0$ of domains in $\mathfrak{D}$ which include all dyadic squares and all finite unions thereof. A diagonal argument permits us to to extract a subsequence along which $\widehat{\eta}_N^{D,M}$ tends in law to a limit $\widehat{\eta}^{D,M}$ for every $D \in \mathfrak{D}_0$ and every integer $M \geq 1$. Applying monotonicity in M, we can then define $\eta^D := \lim_{M\to\infty} \widehat{\eta}^{D,M}$. By (4.9), η^D is the limit of η_N^D along the chosen subsequence. Lemma 4.3 implies that η^D obeys (3.31) for every f of the stated form. By Lemma 3.7 we then have

$$\eta^D(\mathrm{d}x\,\mathrm{d}h) = Z_\lambda^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha\lambda h}\,\mathrm{d}h, \quad D \in \mathfrak{D}_0, \tag{4.14}$$

for some a.s.-finite random Borel measure Z_λ^D which has positive mass with positive probability. By the same reasoning as in the proof of Proposition 3.8, the measures $\{Z_\lambda^D : D \in \mathfrak{D}_0\}$ obey properties (1)–(7) in Theorem 2.3. In particular, Z_λ^D charges every non-empty open set a.s.

In order to determine the law of Z_λ^D uniquely, and thus prove the existence of the limit (3.81), we claim that, on the dyadic square S_n, we have the representation

$$Z_\lambda^{S_n}(\mathrm{d}x) \overset{\text{law}}{=} Y_\infty^{S_n}(\mathrm{d}x), \tag{4.15}$$

where $Y_\infty^{S_n}$ is the measure constructed in Lemma 3.10. As the Gibbs–Markov property for Z_λ^D was already proved as part of the properties of Theorem 2.3 above, the starting equation (3.63) is valid and the argument thereafter applies. We just need to replace Lemma 3.13 with a suitable analogue that does not rely on the existence of the second moment of $Z_\lambda^{S_n}(S_n)$.

The idea is to reintroduce the truncation while sticking with the $N \to \infty$ limit measures. Indeed, using the above subsequential limit point $\widehat{\eta}^{D,M}$ of $\{\widehat{\eta}_N^{D,M} : N \geq 1\}$, we define

$$\widehat{Z}_\lambda^{D,M}(A) := \alpha\lambda\widehat{\eta}^{D,M}\bigl(A \times [0,\infty)\bigr) \tag{4.16}$$

for each Borel measurable A. These measures are not expected to obey most of the properties in Theorem 2.3. Notwithstanding, we have

$$\widehat{Z}_\lambda^{D,M}(A) \leq Z_\lambda^D(A) \quad \text{and} \quad \widehat{Z}_\lambda^{D,M}(A) \uparrow Z_\lambda^D(A) \quad \text{as } M \to \infty. \tag{4.17}$$

We will refer to $\widehat{Z}_\lambda^{D,M}$ as a "truncated measure" although this not very accurate.

Remark 4.6. Note that Lemma 4.3 gives us a factorization property (4.14) for $\widehat{\eta}^{D,M}$ as well; just replace Z_λ^D by $\widehat{Z}_\lambda^{D,M}$. One might thus be tempted to think that $\widehat{\eta}^{D,M}$ also satisfies the Gibbs–Markov property. However, this is false because the addition of the "binding field" $\Phi^{D,\widetilde{D}}$ changes the truncation events on the subdomain $\widetilde{D}$. In any case, if the properties (1)–(7) of Theorem 2.3 were true for $\widehat{Z}_\lambda^{D,M}$, our argument from the previous section would represent this measure using a derivative martingale and, later, by the LQG measure. This would lead to a contradiction because the LQG measure is known to lack the second moment for all $\lambda \in [1/\sqrt{2}, 1)$ yet (by Fatou) $\widehat{Z}_\lambda^{D,M}$ is square integrable for all $\lambda \in (0, 1)$.

Moving back to the proof of Theorem 4.5, we now define a measure $\widetilde{Z}_m^{S_n,M}$ by (3.63) with the $Z_\lambda^{S_{n+m,i}}$ on the right-hand side now replaced by their truncated analogues,

$$\widetilde{Z}_\lambda^{S_n,M}(\mathrm{d}x) := \sum_{i=1}^{4^m} \mathrm{e}^{\alpha\lambda\Phi^{S_n,\widetilde{S}_{n,m}}(x)} \mathsf{1}_{S_{n+m,i}}(x) \widehat{Z}_\lambda^{S_{n+m,i},M}(\mathrm{d}x). \tag{4.18}$$

For each bounded, measurable $f : \overline{D} \mapsto [0,\infty)$ and each $\delta > 0$ we then have

$$E\bigl(\mathrm{e}^{-\langle Z_\lambda^{S_n}, f\rangle}\bigr) \le E\bigl(\mathrm{e}^{-\langle \widetilde{Z}_\lambda^{S_n,M}, f\rangle}\bigr) \le E\bigl(\mathrm{e}^{-\langle \widetilde{Z}_\lambda^{S_n,M}, f_{m,\delta}\rangle}\bigr), \tag{4.19}$$

where we $f_{m,\delta}$ is as defined in (3.68). Let $\Phi^{S_n,\widetilde{S}_{n,m}}$ be independent of the truncated measures $\{\widehat{Z}_\lambda^{S_{n+m,i},M} : i = 1,\dots,4^m\}$, which are themselves regarded as independent, and set

$$\widetilde{X}_i := \int_{S_{n+m,i}} \mathrm{e}^{\alpha\lambda\Phi^{S_n,\widetilde{S}_{n,m}}(x)} f_{m,\delta}(x) \widehat{Z}_\lambda^{S_{n+m,i},M}(\mathrm{d}x). \tag{4.20}$$

Noting that $\langle \widetilde{Z}_\lambda^{S_n,M}, f_{m,\delta}\rangle = \sum_{i=1}^{4^m} \widetilde{X}_i$, we then get for each $\epsilon > 0$,

$$E\bigl(\mathrm{e}^{-\langle \widetilde{Z}_\lambda^{S_n,M}, f_{m,\delta}\rangle}\bigr) \le E\biggl(\exp\biggl\{-\mathrm{e}^{-\epsilon}\sum_{i=1}^{4^m} E\bigl(\widetilde{X}_i \mathsf{1}_{\{\widetilde{X}_i \le \epsilon\}} | \Phi^{S_n,\widetilde{S}_{n,m}}\bigr)\biggr\}\biggr) \tag{4.21}$$

from the "reverse Jensen" inequality in Lemma 3.12.

To replace Lemma 3.13, we claim that, for each $\epsilon > 0$,

$$\sum_{i=1}^{4^m} E\bigl(\widetilde{X}_i \mathsf{1}_{\{\widetilde{X}_i > \epsilon\}} | \Phi^{S_n,\widetilde{S}_{n,m}}\bigr) \underset{m\to\infty}{\longrightarrow} 0 \quad \text{in probability.} \tag{4.22}$$

For this abbreviate $L := 2^m$ and let $A_{n,m}$ denote the event in (4.13). Restricting to the complement of $A_{n,m}$, we first bound $\widetilde{X}_i \mathsf{1}_{\{\widetilde{X}_i > \epsilon\}} \le \epsilon^{-1}\widetilde{X}_i^2$ and then bound the field $\Phi^{S_n,\widetilde{S}_{n,m}}$ in one of the terms using the bound on the maximum. In light of the independence of $\Phi^{S_n,\widetilde{S}_{n,m}}$ and $Z_\lambda^{S_{n+m,i},M}$ we can then take expectation over the remaining occurrence of $\Phi^{S_n,\widetilde{S}_{n,m}}$ and apply the uniform bound $\mathrm{Var}(\Phi^{S_n,\widetilde{S}_{n,m}}(x)) \le c + g\log(L)$ with the result

$$\begin{aligned}
&E\bigl(1_{A_{n,m}^{\mathrm{c}}} E\bigl(\widetilde{X}_i \mathsf{1}_{\{\widetilde{X}_i > \epsilon\}} | \Phi^{S_n,\widetilde{S}_{n,m}}\bigr)\bigr) \\
&\quad\le \frac{1}{\epsilon} E\bigl(1_{A_{n,m}^{\mathrm{c}}} E\bigl(\widetilde{X}_i^2 | \Phi^{S_n,\widetilde{S}_{n,m}}\bigr)\bigr) \\
&\quad\le \frac{\|f\|^2}{\epsilon} \mathrm{e}^{2\alpha\lambda\sqrt{g}\log(L)+c(\delta)\sqrt{\log(L)}} E\biggl(\widehat{Z}_\lambda^{S_{n+m,i},M}\bigl(S_{n+m,i}^\delta\bigr) \int_{S_{n+m,i}^\delta} \mathrm{e}^{\alpha\lambda\Phi^{S_n,\widetilde{S}_{n,m}}(x)} \widehat{Z}_\lambda^{S_{n+m,i},M}(\mathrm{d}x)\biggr) \\
&\quad\le c\frac{\|f\|^2}{\epsilon} \mathrm{e}^{2\alpha\lambda\sqrt{g}\log(L)+c(\delta)\sqrt{\log(L)}} \mathrm{e}^{\frac{1}{2}\alpha^2\lambda^2 g\log(L)} E\bigl(\widehat{Z}_\lambda^{S_{n+m,i},M}\bigl(S_{n+m,i}^\delta\bigr)^2\bigr).
\end{aligned} \tag{4.23}$$

Here we will finally benefit from using the truncated measures $\widehat{Z}_\lambda^{S_{n+m,i},M}$. Indeed, Lemma 4.2, monotonicity and Lemma 3.2 together with the scaling of the integral in (3.6) ensure that, for some $c, c' \in (0,\infty)$ depending on M, n and δ but not on m or i,

$$E\big(\widetilde{Z}_\lambda^{S_{n+m,i},M}\big(S^\delta_{n+m,i}\big)^2\big) \le c\big[E\big(Z_\lambda^{S_{n+m,i}}(S_{n+m,i})\big)\big]^2 \le c' L^{-2(2+2\lambda^2)}, \tag{4.24}$$

where the last inequality follows from (3.76). Since $2\alpha\lambda\sqrt{g} = 4\lambda$ while $\frac{1}{2}\alpha^2\lambda^2 g = 2\lambda^2$, the right-hand side of (4.23) decays as $L^{-2-2(1-\lambda)^2}$. For any $\zeta > 0$ we thus get

$$P\left(\sum_{i=1}^{4^m} E\big(\widetilde{X}_i 1_{\{\widetilde{X}_i>\epsilon\}}|\Phi^{S_n,\widetilde{S}_{n,m}}\big) > \zeta\right) \le P(A_{n,m}) + \frac{c''}{\epsilon\zeta} L^{-2(1-\lambda)^2} \mathrm{e}^{c(\delta)\sqrt{\log L}} \tag{4.25}$$

for some c'' independent of m. Hereby (4.22) follows from (4.13).

Using (4.22) in (4.21) then gives

$$\limsup_{m\to\infty} E\big(\mathrm{e}^{-\langle \widetilde{Z}_\lambda^{S_n,M}, f_{m,\delta}\rangle}\big) \le \limsup_{m\to\infty} E\big(\mathrm{e}^{-\mathrm{e}^{-\epsilon} E(\langle \widetilde{Z}_\lambda^{S_n,M}, f_{m,\delta}\rangle|\Phi^{S_n,\widetilde{S}_{n,m}})}\big). \tag{4.26}$$

Our next task is to prove

$$E\big(\langle Z_\lambda^{S_n}, f_{m,\delta}\rangle|\Phi^{S_n,\widetilde{S}_{n,m}}\big) - E\big(\langle \widetilde{Z}_\lambda^{S_n,M}, f_{m,\delta}\rangle|\Phi^{S_n,\widetilde{S}_{n,m}}\big) \longrightarrow 0 \tag{4.27}$$

in probability in the limit as $m\to\infty$ followed by $M\to\infty$. Since the difference of expectations is non-negative, it thus suffices to show convergence to 0 in the mean.

As $\mathrm{Var}(\Phi^{S_n,\widetilde{S}_{n,m}}(x)) \le g\log(2^m) + c$ uniformly on $\widetilde{S}^\delta_{n,m}$, we have

$$E\big(\text{l.h.s. of (4.27)}\big) \le c\big(2^m\big)^{2\lambda^2} 4^m \|f\|\big[E\big(Z_\lambda^{S_{n+m}}(S_{n+m})\big) - E\big(\widehat{Z}_\lambda^{S_{n+m},M}(S_{n+m})\big)\big]. \tag{4.28}$$

Lemma 4.1 now implies

$$E\big(Z_\lambda^{S_{n+m}}(S_{n+m})\big) - E\big(\widehat{Z}_\lambda^{S_{n+m},M}(S_{n+m})\big) \le c\mathrm{e}^{-\tilde{c}M^2}\big(2^m\big)^{-2-2\lambda^2}. \tag{4.29}$$

This bounds the expectation of (4.27) by $c\mathrm{e}^{-\tilde{c}M^2}\|f\|$ which tends to zero as $M\to\infty$.

Combining (4.19) with (4.26)–(4.27), using (3.64), Lemma 3.10 and the limits $\epsilon\downarrow 0$ and $\delta\downarrow 0$ (with (3.69)) thus show

$$E\big(\mathrm{e}^{-\langle Z_\lambda^{S_n}, f\rangle}\big) \le E\big(\mathrm{e}^{-\langle Y_\infty^{S_n}, f\rangle}\big). \tag{4.30}$$

Jointly with (3.65), we then get (4.15). The same argument as in the proof of Theorem 3.14 now gives uniqueness of the law of Z_λ^D for all $D\in\mathfrak{D}$. □

4.2. *Truncated moment calculations*

We now move to the technical statements (Lemmas 4.1–4.3) in the proof of Theorem 4.5 whose proof was deferred from the previous subsection to here. For small enough $\epsilon > 0$, we write $D_N^\epsilon := \{x\in D_N\colon \mathrm{dist}(x, D_N^c) > \epsilon N\}$. We start by some observations concerning the law of the random variables S_k defined in (4.3).

Lemma 4.7. *Recall that $g := 2/\pi$. For each $\epsilon > 0$ and each $r > 0$, there is $c = c(\epsilon, r) \in (0,\infty)$ such that for all $D\in\mathfrak{D}$ with $\mathrm{diam}\, D \le r$ and all N large enough, we have*:

(1) *For all $x\in D_N$ and all $k_N \le k \le m < n(x)$,*

$$\mathrm{Var}\big(S_k(x) - S_m(x)\big) = (m-k)g + o(1), \tag{4.31}$$

where $o(1)\to 0$ when $N\to\infty$ uniformly in k and $x\in D_N^\epsilon$.

(2) *For all $x \in D_N^\epsilon$ and all k with $k_N \le k \le n(x)$,*

$$\bigl|\mathrm{Var}\bigl(S_k(x)\bigr) - \bigl(n(x)-k\bigr)g\bigr| \le c. \tag{4.32}$$

Moreover, for all $\ell \ge 1$ there is $c' = c'(\epsilon,\ell) > 0$ such that for all $x \in D_N^\epsilon$, all k with $k_N \le k \le n(x)$, all m satisfying $k - \ell \le m \le k$ and all $y \in D_N$ such that $\Delta^{m+1}(y) \subseteq \Delta^k(x) \setminus \{x\}$, we have

$$E\bigl(S_k(x)S_m(y)\bigr) \le \bigl(n(x)-k\bigr)g + c' \quad \text{and} \quad \mathrm{Var}\bigl(S_m(y) - S_k(x)\bigr) \in [g/4, c] \tag{4.33}$$

as soon as N is sufficiently large.

Proof. Fix $r > 0$ and consider any domain $D \in \mathfrak{D}$ with $\operatorname{diam} D \le r$. By the Gibbs–Markov property, translation invariance and Green function asymptotics (A.5),

$$\begin{aligned}\mathrm{Var}\bigl(S_k(x) - S_m(x)\bigr) &= \mathrm{Var}\bigl(\varphi^{\Lambda_{\mathrm{e}^m}(0),\Lambda_{\mathrm{e}^k}(0)}(0)\bigr)\\ &= G_{\Lambda_{\mathrm{e}^m}(0)}(0,0) - G_{\Lambda_{\mathrm{e}^k}(0)}(0,0) = g(m-k) + o(1).\end{aligned} \tag{4.34}$$

This gives the first statement. For the second, we assume that $k < n(x)$ since otherwise it is trivially true. Then, since $n(x) \ge \log N - c$ for $c = c(\epsilon) > 0$, we may find $\tilde c = \tilde c(\epsilon, r) > 0$ such that $D_N \subseteq \Lambda_{\mathrm{e}^{n(x)+\tilde c}}(x)$. Monotonicity of the Green function with respect to inclusion and similar considerations as above now show

$$\mathrm{Var}\bigl(S_k(x)\bigr) \le G_{\Lambda_{\mathrm{e}^{n(x)+\tilde c}(0)}}(0,0) - G_{\Lambda_{\mathrm{e}^k}(0)}(0,0) \le \bigl(n(x)-k\bigr)g + g\tilde c + o(1). \tag{4.35}$$

On the other hand, by definition $D_N \supseteq \Lambda_{\mathrm{e}^{n(x)+1}}(x)$ and consequently

$$\mathrm{Var}\bigl(S_k(x)\bigr) \ge G_{\Lambda_{\mathrm{e}^{n(x)+1}(0)}}(0,0) - G_{\Lambda_{\mathrm{e}^k}(0)}(0,0) \ge \bigl(n(x)-k\bigr)g + g/2 + o(1). \tag{4.36}$$

This completes the second statement.

Turning to the third statement, here we first observe that the expectation there can be written explicitly as $E[\varphi^{D_N,\Delta^k(x)}(x)\varphi^{D_N,\Delta^k(x)}(y)]$. By the Gibbs–Markov property, this expectation equals

$$E\bigl(h^{D_N}(x)h^{D_N}(y)\bigr) - E\bigl(h^{\Delta^k(x)}(x)h^{\Delta^k(x)}(y)\bigr) \le \bigl(n(x)-k\bigr)g + c', \tag{4.37}$$

where we have used the Green function asymptotics again. The constant $c' > 0$ above depends on the distance of y to the boundary of $\Delta^k(x)$ relative to its diameter. This in turn is governed by the choice of ℓ.

Finally, the upper bound on the variance follows from the above bounds together with

$$\mathrm{Var}\bigl(S_m(y) - S_k(x)\bigr) = \mathrm{Var}\bigl(S_m(y)\bigr) + \mathrm{Var}\bigl(S_k(x)\bigr) - 2E\bigl(S_m(y)S_k(x)\bigr) \tag{4.38}$$

and the relation between m and k. As for the lower bound,

$$\begin{aligned}\mathrm{Var}\bigl(S_m(y) - S_k(x)\bigr) &= \mathrm{Var}\bigl(\varphi^{D,\Delta^k(x)}(y) - \varphi^{D,\Delta^k(x)}(x) + \varphi^{\Delta^k(x),\Delta^m(y)}(y)\bigr)\\ &\ge \mathrm{Var}\bigl(\varphi^{\Delta^k(x),\Delta^m(y)}(y)\bigr) \ge \mathrm{Var}\bigl(\varphi^{\Delta^{m+1}(x),\Delta^m(y)}(y)\bigr) \ge g + o(1)\end{aligned} \tag{4.39}$$

thus proving the third statement as well. □

The next lemma notes that the dependency structure of the process $S_k(x)$ is tree-like.

Lemma 4.8. *If $x, y \in D_N$ and $0 \le k_1 < k_2 \le n(x)$, $0 \le m_1 < m_2 \le n(y)$ are such that $\Delta^{k_2}(x) \subseteq \Delta^{m_1}(y)$, then the increments $S_{k_1}(x) - S_{k_2}(x)$ and $S_{m_1}(y) - S_{m_2}(y)$ are independent. In particular, for any $x \in D_N$, the process $(S_k(x))_{k=0}^{n(x)}$ has independent increments.*

Proof. This is a direct consequence of the definition and the Gibbs–Markov property. □

Finally, we will also need the following simple fact:

Lemma 4.9. *Suppose* $X \overset{\text{law}}{=} \mathcal{N}(0, \sigma_X^2)$ *and* $Y \overset{\text{law}}{=} \mathcal{N}(0, \sigma_Y^2)$ *are independent. Then* $(X|X+Y)$*, i.e., X conditional on* $X+Y$*, obeys*

$$(X|X+Y) \overset{\text{law}}{=} \mathcal{N}\biggl(\frac{\sigma_X^2}{\sigma_X^2+\sigma_Y^2}(X+Y), \frac{\sigma_X^2\sigma_Y^2}{\sigma_X^2+\sigma_Y^2}\biggr). \tag{4.40}$$

Proof. By way of a simple manipulation we get

$$X = \frac{\sigma_X^2}{\sigma_X^2+\sigma_Y^2}(X+Y) + \frac{\sigma_Y^2 X - \sigma_X^2 Y}{\sigma_X^2+\sigma_Y^2}. \tag{4.41}$$

The second expression on the right is a Gaussian random variable that is independent of $X+Y$, has mean zero and variance as the random variable on the right of (4.40). □

We are now ready to control the defect to the level set size caused by the truncation:

Proof of Lemma 4.1. Pick $b, b' \in \mathbb{R}$ with $b < b'$. Let $\epsilon > 0$ and note that

$$\begin{aligned} &E\bigl|\Gamma_N^D(b) \setminus \widehat{\Gamma}_N^{D,M}(b)\bigr| \\ &\quad\le E\bigl|\Gamma_N^D(b) \setminus D_N^\epsilon\bigr| + E\bigl|\Gamma_N^D\bigl(b'\bigr)\bigr| \\ &\qquad + \sum_{x\in D_N^\epsilon} \sum_{k=k_N}^{n(x)} P\biggl(h^{D_N}(x) - a_N \in \bigl[b,b'\bigr), \biggl|S_k(x) - a_N\frac{n(x)-k}{n(x)}\biggr| > M\bigl(n(x)-k\bigr)^{3/4}\biggr). \end{aligned} \tag{4.42}$$

By Lemma 3.1, the sum of the first two expectations are bounded by the desired estimate provided we take ϵ small and b' sufficiently large (proportional to M). We thus have to show the bound for the double sum regardless of $\epsilon > 0$ and $b' \in [b, \infty)$.

Fix $x \in D_N^\epsilon$ and let $k \in \{0, \dots, n(x)\}$. We will estimate the probability on the right-hand side of (4.42) by conditioning on the value of $h^{D_N}(x)$. For this we note that, by Lemma 4.8, $h^{D_N}(x)$ is the sum of independent random variables $S_k(x)$ and $S_0(x) - S_k(x)$. Applying Lemma 4.9 for $X := S_k(x)$ and $Y := S_0(x) - S_k(x)$, we thus get for $s \in [b, b']$

$$\bigl(S_k(x)|h^{D_N}(x) = a_N + s\bigr) \overset{\text{law}}{=} \mathcal{N}\biggl(\frac{\operatorname{Var}(S_k(x))}{\operatorname{Var}(S_0(x))}(a_N+s), \frac{\operatorname{Var}(S_k(x))\operatorname{Var}(S_0(x)-S_k(x))}{\operatorname{Var}(S_0(x))}\biggr). \tag{4.43}$$

Invoking the variance estimates in Lemma 4.7, we obtain

$$\biggl|\frac{\operatorname{Var}(S_k(x))}{\operatorname{Var}(S_0(x))} - \frac{n(x)-k}{n(x)}\biggr| \le \frac{c_1}{n(x)} \tag{4.44}$$

and

$$\frac{\operatorname{Var}(S_k(x))\operatorname{Var}(S_0(x)-S_k(x))}{\operatorname{Var}(S_0(x))} \le c_2\bigl(n(x)-k\bigr), \tag{4.45}$$

where the constants c_1 and c_2 are independent of k, x and $n(x)$ as chosen above. Plugging these in (4.43) and using that a_N is proportional to $n(x)$, a standard Gaussian estimate yields

$$P\biggl(\biggl|S_k(x) - a_N\frac{n(x)-k}{n(x)}\biggr| > M\bigl(n(x)-k\bigr)^{3/4} | h^{D_N}(x) = a_N + s\biggr) \le c\mathrm{e}^{-\tilde{c}M^2(n(x)-k)^{1/2}}. \tag{4.46}$$

Thanks to the uniformity in s of (4.46), the last term in (4.42) is bounded by

$$c\sum_{x\in D_N^\epsilon}\sum_{k=1}^{\infty}\mathrm{e}^{-\tilde{c}M^2k^{1/2}}P\bigl(h^{D_N}(x)-a_N\in[b,b')\bigr)\le c'\mathrm{e}^{-\tilde{c}M^2}E\bigl|\Gamma_N^D(b)\bigr|. \tag{4.47}$$

By Lemma 3.2, this obeys the desired bound as soon as N is sufficiently large. □

Next we move to the proof of the second moment estimate for truncated level sets:

Proof of Lemma 4.2. Pick $b,b'\in\mathbb{R}$ with $b<b'$ and fix $\epsilon>0$ and $M>0$. Given $N\ge1$ and $x,y\in D_N^\epsilon$, we will first estimate the probability that $x,y\in\widehat{\Gamma}_N^D(b,b')$ for $|x-y|>K_N^{1/4}$. Denote

$$k:=\bigl(\lceil\log^+|x-y|\rceil+1\bigr)\wedge n(x), \tag{4.48}$$

and let $\ell\ge1$ be the minimal such that

$$\Delta^{k-\ell}(x)\cap\Delta^{k-\ell}(y)=\varnothing\quad\text{and}\quad\Delta^{k-\ell+1}(x)\cup\Delta^{k-\ell+1}(y)\subseteq\Delta^k(x). \tag{4.49}$$

Observe that since $n(x)\le\log N+c$ and $n(y)\ge\log N-c'$ for $c=c(D)>0$ and $c'=c(\epsilon)>0$ we must have $\ell\le\tilde{c}$ with $\tilde{c}=\tilde{c}(\epsilon,D)>0$. Also note that

$$h^{D_N}(x)=S_k(x)+\bigl(S_{k-\ell}(x)-S_k(x)\bigr)+\bigl(S_0(x)-S_{k-\ell}(x)\bigr) \tag{4.50}$$

and

$$h^{D_N}(y)=S_k(x)+\bigl(S_{k-\ell}(y)-S_k(x)\bigr)+\bigl(S_0(y)-S_{k-\ell}(y)\bigr). \tag{4.51}$$

By (4.49) and Lemma 4.8, the three terms on the right of (4.50) are independent of each other, while for the terms on the right of (4.51) we get that the last one is independent of the first two as well as of of the last term on the right of (4.50). For any $t\in[-M(n(x)-k)^{3/4},M(n(x)-k)^{3/4}]$, any $s_1,s_2\in[b,b')$ and any $u_1,u_2\in[-n(x)^{3/4},n(x)^{3/4}]$ and employing the convention that $P(X\in\mathrm{d}x)=f(x)\,\mathrm{d}x$ abbreviates $P(X\in A)=\int_A f(x)\,\mathrm{d}x$ we then write

$$\begin{aligned}
&P\biggl(h(x)-a_N\in\mathrm{d}s_1,h(y)-a_N\in\mathrm{d}s_2\bigl|S_k(x)-a_N\frac{n(x)-k}{n(x)}=t,\\
&\qquad S_{k-\ell}(x)-S_k(x)=u_1,S_{k-\ell}(y)-S_k(x)=u_2\biggr)\\
&=P\biggl(S_0(x)-S_{k-\ell}(x)-a_N\frac{k}{n(x)}+t+u_1\in\mathrm{d}s_1\biggr)\\
&\qquad\times P\biggl(S_0(y)-S_{k-\ell}(y)-a_N\frac{k}{n(x)}+t+u_2\in\mathrm{d}s_2\biggr)\\
&\le\frac{c}{k}\exp\biggl\{-\frac{(\frac{a_N}{n}(x)k-t-u_1+s_1)^2+(\frac{a_N}{n}(x)k-t-u_2+s_2)^2}{2gk}\biggr\}\mathrm{d}s_1\,\mathrm{d}s_2\\
&\le\frac{c}{k}\exp\biggl\{-\frac{a_N^2}{gn^2}k+\frac{a_N}{gn}(2t+u_1+u_2-s_1-s_2)\biggr\}\mathrm{d}s_1\,\mathrm{d}s_2.
\end{aligned}\tag{4.52}$$

Here in the first inequality we used Lemma 4.7 to replace variances of the random variables $S_0(x)-S_{k-\ell}(x)$ and $S_0(y)-S_{k-\ell}(y)$ by gk. This causes only a change in the multiplicative constant because, by our assumptions on t, u_1, u_2, s_1 and s_2, the quantities in the squares in the exponent are both at most order k. In the second inequality we opened up the squares and retained, through a bound, only the quantities that depend on a_N.

Our next step is to integrate the above conditional probability with respect to the conditional law of $S_{k-\ell}(y)-S_k(x)$ and $S_{k-\ell}(x)-S_k(x)$ given $S_k(x)$. For this we will need to examine the dependency of $S_{k-\ell}(y)-S_k(x)$ on $S_k(x)$. By Lemma 4.7, there are $c>0$ and $c''>c'>0$ such that $E((S_{k-\ell}(y)-S_k(x))S_k(x))\le c$ and $\mathrm{Var}(S_{k-\ell}(y)-S_k(x))\in[c',c'']$. Consequently for all t with $|t|\le M(n(x)-k)^{3/4}$,

$$\left|E\left(S_{k-\ell}(y)-S_k(x)|S_k(x)-a_N\frac{n(x)-k}{n(x)}=t\right)\right|\le ct/\big(n(x)-k+1\big)\le cM \tag{4.53}$$

and

$$\mathrm{Var}\big(S_{k-\ell}(y)-S_k(x)|S_k(x)\big)\le\mathrm{Var}\big(S_{k-\ell}(y)-S_k(x)\big)\le c'', \tag{4.54}$$

with the conditional expectation vanishing and the conditional variance bounded similarly for $S_{k-\ell}(x)-S_k(x)$. Since $a_N/n(x)$ is bounded, the Cauchy–Schwarz inequality shows

$$E\left(\mathrm{e}^{\frac{a_N}{gn}(S_{k-\ell}(y)-S_k(x))+\frac{a_N}{gn}(S_{k-\ell}(x)-S_k(x))}|S_k(x)-a_N\frac{n(x)-k}{n(x)}=t\right)\le\tilde c, \tag{4.55}$$

uniformly in t as above. In conjunction with (4.52), this yields

$$\begin{aligned}
&P\left(h^{D_N}(x)-a_N\in\big[b,b'\big),h^{D_N}(y)-a_N\in\big[b,b'\big)|S_k(x)-a_N\frac{n(x)-k}{n(x)}=t\right)\\
&\quad\le P\left(\big|S_{k-l}(x)-S_k(x)\big|\vee\big|S_{k-l}(y)-S_k(x)\big|>n^{3/4}|S_k(x)-a_N\frac{n(x)-k}{n(x)}=t\right)\\
&\qquad+\frac{c}{k}\int_{[b,b')}\mathrm{d}s_1\int_{[b,b')}\mathrm{d}s_2\exp\left\{-\frac{a_N^2}{gn^2}k+\frac{a_N}{gn}(2t-s_1-s_2)\right\}\\
&\quad\le\frac{c'}{k}\exp\left\{-\frac{a_N^2}{gn^2}k+2\frac{a_N}{gn}t\right\}
\end{aligned}\tag{4.56}$$

uniformly in above t above, where c' depends on b, b' and M and we have used the fact that the right hand side is at least e^{-cn} for some $c>0$.

Now if $k=n(x)$ then $S_k(x)=t=0$ and therefore the right hand side above is also a bound on the unconditional probability. Otherwise, we integrate the left-hand side of (4.56) with respect to the distribution of the random variable $S_k(x)-a_N\frac{n(x)-k}{n(x)}$ to get (abbreviating $\theta_n(k):=(n(x)-k)^{3/4}$)

$$\begin{aligned}
&P\big(x,y\in\widehat\Gamma_N^{D,M}\big(b,b'\big)\big)\\
&\quad\le P\left(h^{D_N}(x)-a_N\in\big[b,b'\big),h^{D_N}(y)-a_N\in\big[b,b'\big),\left|S_k(x)-a_N\frac{n(x)-k}{n(x)}\right|\le M\theta_n(k)\right)\\
&\quad\le\frac{c}{k(n(x)-k)^{1/2}}\mathrm{e}^{-\frac{a_N^2}{gn^2}k}\int_{|t|\le M\theta_n(k)}\exp\left\{2\frac{a_N}{gn}t-\frac{(a_N\frac{n(x)-k}{n}(x)+t)^2}{2g(n(x)-k)}\right\}\mathrm{d}t\\
&\quad\le\frac{c}{k(n(x)-k)^{1/2}}\mathrm{e}^{-\frac{a_N^2}{gn^2}k-\frac{a_N^2}{2gn^2}(n(x)-k)}\int_{|t|\le M\theta_n(k)}\mathrm{e}^{+\frac{a_N}{gn}t}\,\mathrm{d}t\\
&\quad\le\frac{c'}{k(n(x)-k)^{1/2}}\mathrm{e}^{-\frac{a_N^2}{2gn^2}k-\frac{a_N^2}{2gn^2}n(x)+\tilde cM\theta_n(k)}\\
&\quad\le c''\frac{K_N}{N^2}\frac{n(x)^{1/2}}{k(n(x)-k+1)^{1/2}}\mathrm{e}^{-\frac{a_N^2}{2gn^2}k+\tilde cM\theta_n(k)}
\end{aligned}\tag{4.57}$$

for some constants $c,c',c'',\tilde c\in(0,\infty)$. The latter bound applies also to the case $k=n(x)$.

The desired expectation is now obtained by summing over $x, y \in D_N^\epsilon$. This yields

$$E\bigl|\widehat{\Gamma}_N^{D,M}(b,b') \cap D_n^\epsilon\bigr|^2 = \sum_{x,y\in D_N^\epsilon} P\bigl(x, y \in \widehat{\Gamma}_N^{\epsilon,M}(b,b')\bigr)$$
$$\leq \sum_{\substack{x,y\in D_N^\epsilon \\ |x-y|\leq K_N^{1/4}}} P\bigl(x \in \widehat{\Gamma}_N^{\epsilon,M}(b,b')\bigr) + \sum_{\substack{x,y\in D_N^\epsilon \\ |x-y|> K_N^{1/4}}} P\bigl(x, y \in \widehat{\Gamma}_N^{\epsilon,M}(b,b')\bigr). \tag{4.58}$$

The first term on the right-hand side is bounded by $K_N^{1/2} E|\widehat{\Gamma}_N^{D,M}(b,b')| = O(K_N^{3/2})$. For the second term we partition the pairs (x, y) further depending on which annulus $\Delta^k(x) \setminus \Delta^{k-1}(x)$ the vertex y belongs to. As there are order $N^2 \mathrm{e}^{2k}$ such pairs for a given k, the bound (4.57) gives

$$\sum_{\substack{x,y\in D_N^\epsilon \\ |x-y|> K_N^{1/4}}} P\bigl(x, y \in \widehat{\Gamma}_N^{\epsilon,M}(b,b')\bigr) \leq c' \frac{K_N}{N^2} \sum_{k=k_N}^{n} \frac{n^{1/2}}{k(n-k+1)^{1/2}} N^2 \mathrm{e}^{2k - \frac{a_N^2}{2gn^2}k + \tilde{c}M(n(x)-k)^{3/4}}, \tag{4.59}$$

where we set n to be the maximum of $n(x)$ over all $x \in D_N^\epsilon$. (Here we note that a change in n by a additive constant changes (4.57) only by a multiplicative constant.)

Now $\frac{a_N^2}{2gn^2}$ is asymptotic to $2\lambda^2 < 2$ in the limit as $N \to \infty$ and so the exponent on the right of (4.59) grows linearly with k. The sum is thus dominated by the $k = n$ term. Since $n = \log N + O(1)$, simple algebra shows that the expression on the right of (4.59) is $O(K_N^2)$. Since all bounds above were uniform in $D \in \mathfrak{D}$ with a given diameter, say, diam $D \leq 1$, we only need to show how to get the diameter dependence explicitly.

A key point is that the bounds were also independent of the approximating sequence of domains D_N, nor of the centering sequence a_N as long as it obeyed (2.22) and not even much on the sequence k_N in the cutoff for the event $T_{N,M}$ as long as N is large enough. Fix $D \in \mathfrak{D}$ with $r := \operatorname{diam} D \leq 1$, let D_N be a sequence of approximating domains obeying (2.1)–(2.2) and set $D' := r^{-1}D$. Fix $j \in \{0, 1, \ldots, \lfloor r^{-1} \rfloor\}$ and set

$$D'_N := D_{\lfloor N/r \rfloor - j}, \qquad a'_N := a_{\lfloor N/r \rfloor - j} \quad \text{and} \quad k'_N := k_{\lfloor N/r \rfloor - j}. \tag{4.60}$$

Then $\{D'_N\}$ is a sequence of domains approximating, in the sense of (2.1)–(2.2), domain D'. Using a'_N as the centering sequence and k'_N as the cutoff in the event $T_{N,M}$ in the definition of $\widehat{\Gamma}_N^{D',M}(b)$, we then get

$$\widehat{\Gamma}^{D,M}_{\lfloor N/r \rfloor - j}(b) = \widehat{\Gamma}_N^{D',M}(b). \tag{4.61}$$

A calculation now shows that the normalizing sequence K'_N defined using a'_N above obeys

$$K'_N = \bigl(r^{2+2\lambda^2} + o(1)\bigr) K_{\lfloor N/r \rfloor - j}. \tag{4.62}$$

Since every integer can be cast to the form $\lfloor N/r \rfloor - j$ for some N and some j as above, the claim for D follows from the claim for D'. □

Our final task is the proof of the factorization property for the truncated level sets:

Proof of Lemma 4.3. Using the Cauchy–Schwarz inequality, we bound the expression inside the limit in (4.12) by the square root of

$$\frac{1}{K_N^2} \sum_{x,y\in A_N} E\Bigl(\bigl(1_{\{h^{D_N}(x)\geq a_N\}} - \mathrm{e}^{\alpha\lambda b} 1_{\{h^{D_N}(x)\geq a_N + b\}}\bigr) 1_{T_{N,M}(x)}$$
$$\times \bigl(1_{\{h^{D_N}(y)\geq a_N\}} - \mathrm{e}^{\alpha\lambda b} 1_{\{h^{D_N}(y)\geq a_N + b\}}\bigr) 1_{T_{N,M}(y)}\Bigr). \tag{4.63}$$

To bound the sum above, we first consider pairs x, y for which $m = \lfloor \log \|x-y\|_\infty \rfloor$ obeys

$$m \geq \frac{3}{2}k_N \quad \text{and} \quad \|x-y\|_\infty \in \bigl[\mathrm{e}^m + 2\mathrm{e}^{k_N}, \mathrm{e}^{m+1} - 2\mathrm{e}^{k_N}\bigr]. \tag{4.64}$$

For such x, y, we have

$$\Delta^k(x) \cap \Delta^{k_N}(y) = \varnothing, \quad k = k_N, \dots, n(x), \tag{4.65}$$

and similarly

$$\Delta^k(y) \cap \Delta^{k_N}(x) = \varnothing, \quad k = k_N, \dots, n(y). \tag{4.66}$$

Consequently, letting $\mathcal{F} := \sigma(h_z^{D_N(x)} : z \in D_N \setminus (\Delta^{k_N}(x) \cup \Delta^{k_N}(y)))$ the term corresponding to such x and y in (4.63) can be written as

$$\begin{aligned} &E\bigl[\bigl(P\bigl(h^{D_N}(x) \geq a_N|\mathcal{F}\bigr) - \mathrm{e}^{\alpha\lambda b}P\bigl(h^{D_N}(x) \geq a_N + b|\mathcal{F}\bigr)\bigr)1_{T_{N,M}(x)} \\ &\quad \times \bigl(P\bigl(h^{D_N}(y) \geq a_N|\mathcal{F}\bigr) - \mathrm{e}^{\alpha\lambda b}P\bigl(h^{D_N}(y) \geq a_N + b|\mathcal{F}\bigr)\bigr)1_{T_{N,M}(y)}\bigr]. \end{aligned} \tag{4.67}$$

We now write $h_N^D(x)$ as $h_N^D(x) = (S_0(x) - S_{k_N}(x)) + S_{k_N}(x)$ and note that the quantity in the parenthesis is independent of $S_{k_N}(x)$, due to the fact that this term is $\mathcal{F}$-measurable. Using a similar decomposition for $h^{D_N}(y)$, the expectation in (4.67) is bounded by

$$E\biggl[1_{T_{N,M}(x)}F\biggl(S_{k_N}(x) - a_N\frac{\log N - k_N}{\log N}\biggr)1_{T_{N,M}(y)}F\biggl(S_{k_N}(y) - a_N\frac{\log N - k_N}{\log N}\biggr)\biggr], \tag{4.68}$$

where

$$F(u) = \biggl|P\biggl(\tilde{h}(0) \geq a_N\frac{k_N}{\log N} - u\biggr) - \mathrm{e}^{\alpha\lambda b}P\biggl(\tilde{h}(0) \geq a_N\frac{k_N}{\log N} - u + b\biggr)\biggr| \tag{4.69}$$

with $\tilde{h}$ denoting the DGFF on $\Lambda_{\mathrm{e}^{k_N}}$.

Since $a_N k_N/\log N \sim 2\sqrt{g}\lambda k_N$ and $E\tilde{h}(0)^2 = gk_N + O(1)$, if we assume that $|u| < (k_N)^{7/8}$ then we can use tail asymptotics for the Gaussian density to estimate the quantities above. In particular, $P(\tilde{h}(0) \geq a_N\frac{k_N}{\log N} - u + b)/P(\tilde{h}(0) \geq a_N\frac{k_N}{\log N} - u)$ is asymptotic to

$$\frac{a_N\frac{k_N}{\log N} - u}{a_N\frac{k_N}{\log N} - u + b}\exp\biggl\{\frac{-2b(a_N\frac{k_N}{\log N} - u) - b^2}{2E[\tilde{h}(0)^2]}\biggr\} \sim \mathrm{e}^{-\alpha\lambda b}, \quad N \to \infty. \tag{4.70}$$

It follows that

$$F(u) = o(1)P\biggl(\tilde{h}(0) \geq a_N\frac{k_N}{\log N} - u\biggr) \tag{4.71}$$

with $o(1) \to 0$ as $N \to \infty$ uniformly in u satisfying $|u| < (k_N)^{7/8}$. Since

$$\biggl|S_{k_N}(x) - a_N\frac{\log N - k_N}{\log N}\biggr| < k_N^{7/8}, \quad \text{on } T_{N,M}(x), \tag{4.72}$$

whenever N is large enough, with a similar condition holding for y under $T_{N,M}(y)$, we can apply (4.71) in (4.68). Reversing the step (4.68), the expectation in (4.67) is thus bounded by

$$o(1)P\bigl(h^{D_N}(x) \geq a_N, h^{D_N}(y) \geq a_N, T_{N,M}(x), T_{N,M}(y)\bigr). \tag{4.73}$$

Proceeding as in (4.59), the contribution to the sum in (4.63) from terms where (4.64) holds is therefore at most $o((K_N)^2)$.

Turning to the remaining terms in the sum in (4.63), if x, y satisfy $m < \frac{3}{2}k_N$, then we bound the corresponding term by $4\mathrm{e}^{2\alpha\lambda(b\vee 0)}P(h^{D_N}(x) \geq a_N + b \wedge 0)$. As in (3.20) the contribution to the sum from all such terms is $o(K_N^2)$. For the pairs with $m \geq \frac{3}{2}k_N$ not satisfying the second restriction in (4.64), we can bound by $4\mathrm{e}^{2\alpha\lambda(b\vee 0)}P(h^{D_N}(x) \geq a_N - b \wedge 0, h^{D_N}(y) \geq a_N - b \wedge 0)$. Observing that the number of such pairs for a given m is at most order $N^2\mathrm{e}^{m+k_N} = o(1)N^2\mathrm{e}^{2m}$, the calculation in (4.59) again shows that such terms contribute $o((K_N)^2)$ as well. The claim follows. □

5. Proofs of main theorems

The goal of this short section is to give formal proofs of our main theorems. Before we do that, we still have to address one issue that has been excluded from the discussion so far: the third component of the point process that captures the local behavior of the field near a point of an intermediate level set.

5.1. *Local structure of intermediate level sets*

Henceforth, let η_N^D denote the full three-component process defined in (2.5) and let $\widehat{\eta}_N^{D,M}$ be its truncation to points x where $T_{N,M}(x)$ holds. In addition, define

$$\widetilde{\eta}_N^{D,M} := \frac{1}{K_N}\sum_{x\in D_N} \delta_{x/N} \otimes \delta_{h^{D_N}(x)-a_N} \otimes \delta_{\{h^{D_N}(x)-h^{D_N}(x+z)\colon z\in\mathbb{Z}^2\}}\mathsf{1}_{\widetilde{T}_{N,M}(x)}, \tag{5.1}$$

where

$$\widetilde{T}_{N,M}(x) := T_{N,M}(x) \cap \Bigl\{\max_{y\in\partial\Delta^{k_N}(x)} \bigl|h^{D_N}(x) - h^{D_N}(y)\bigr| \leq k_N^2\Bigr\} \tag{5.2}$$

with $\Delta^k(x)$ and k_N defined in (4.2) and (4.5), respectively. Obviously, η_N^D dominates $\widehat{\eta}_N^{D,M}$ which in turn dominates $\widetilde{\eta}_N^{D,M}$. First we note that the truncations do not really matter as soon as proper limits are taken:

Lemma 5.1. *For any continuous, compactly-supported function* $f\colon \overline{D}\times\mathbb{R}\times\mathbb{R}^{\mathbb{Z}^2}\to\mathbb{R}$,

$$\lim_{M\to\infty}\limsup_{N\to\infty} E\bigl|\langle\widetilde{\eta}_N^{D,M}, f\rangle - \langle\eta_N^D, f\rangle\bigr| = 0. \tag{5.3}$$

Proof. In light of Lemma 3.1 it suffices to show that for any $\epsilon > 0$, the expected number of $x \in D_N^\epsilon$ where $T_{N,M}(x) \setminus \widetilde{T}_{N,M}(x)$ occurs is $o(K_N)$. Invoking the union bound, this is at most

$$\sum_{x\in D_N^\epsilon}\sum_{y\in\partial\Delta^{k_N}(x)} P\bigl(h^{D_N}(x) - h^{D_N}(y) > k_N^2\bigr). \tag{5.4}$$

Since $h^{D_N}(x) - h^{D_N}(y)$ has mean zero and variance bounded by a constant times k_N uniformly for all such pairs whenever N is large enough, the probability on the right is at most $\mathrm{e}^{-ck_N^3}$. As $k_N \sim c'\log N$ and as the number of terms in the sum is only order $N^2\mathrm{e}^{k_N}$, the claim follows. □

The principal computation to be done in this section is now the content of:

Proposition 5.2. *Let* ν *be the measure in* (2.7) *and, given any continuous, compactly-supported function* $f\colon \overline{D}\times\mathbb{R}\times\mathbb{R}^{\mathbb{Z}^2}\to\mathbb{R}$, *let*

$$f_\nu(x,h) := E_\nu\bigl(f(x,h,\phi)\bigr) \tag{5.5}$$

with the expectation over ϕ. Then for any $M > 0$,

$$\lim_{N\to\infty} E\bigl|\langle\widetilde{\eta}_N^{D,M}, f\rangle - \langle\widetilde{\eta}_N^{D,M}, f_\nu\rangle\bigr| = 0. \tag{5.6}$$

For the proof we will need:

Lemma 5.3. *Let $\epsilon > 0$ and, for $x \in D_N^\epsilon$ and a sample of h^{D_N}, let φ_N denote the discrete-harmonic extension of the values of h^{D_N} on $\{x\} \cup \Delta^{k_N}(x)^{\mathrm{c}}$. Recall the notation $\mathfrak{a}$ for the potential kernel associated with the simple symmetric random walk on $\mathbb{Z}^2$. Then for any sequence a_N related to $\lambda \in (0,1)$ via the asymptotic (1.5),*

$$\max_{x\in D_N^\epsilon} \max_{z\in\Lambda_r(x)} \sup_{\substack{h^{D_N}\in\widetilde{T}_{N,M}(x)\\ |h^{D_N}(x)-a_N|\le\log\log N}} \Bigl|h^{D_N}(x) - \varphi_N(z) - \frac{2}{\sqrt{g}}\lambda\mathfrak{a}(z-x)\Bigr| \underset{N\to\infty}{\longrightarrow} 0. \tag{5.7}$$

Proof of Lemma 5.3. To show this, let $H_{N,x}(z,y)$ denote the probability that the simple random walk started at z first returns to $\{x\} \cup \Delta^{k_N}(x)^{\mathrm{c}}$ at y. (Note that $H_{N,x}(x,x) > 0$ in this case.) Then

$$h^{D_N}(x) - \varphi_N(z) = \sum_{y\in\partial\Delta^{k_N}(x)} H_{N,x}(z,y)\bigl[h^{D_N}(x) - h^{D_N}(y)\bigr]. \tag{5.8}$$

Since $S_k(x)$ is the average of $\{h^{D_N}(y)\colon y \in \partial\Delta^{k_N}(x)\}$ with respect to the exit distribution from $\Delta^{K_N}(x)$ of the simple random walk started at x, we have

$$S_{k_N}(x) \sum_{y\in\partial\Delta^{k_N}(x)} H_{N,x}(x,y) = \sum_{y\in\partial\Delta^{k_N}(x)} H_{N,x}(x,y)h^{D_N}(y). \tag{5.9}$$

Swapping $h^{D_N}(x)$ for $S_{k_N}(x)$ on the right hand side of (5.8) then gives

$$\begin{aligned} h^{D_N}(x) - \varphi_N(z) &= \bigl(1 - H_{N,x}(z,x)\bigr)\bigl[h^{D_N}(x) - S_{k_N}(x)\bigr] \\ &\quad + \sum_{y\in\partial\Delta^{k_N}(x)} \bigl(H_{N,x}(z,y) - H_{N,x}(x,y)\bigr)\bigl[S_{k_N}(x) - h^{D_N}(y)\bigr]. \end{aligned} \tag{5.10}$$

We claim that the second term on the right vanishes in the stated limits. Indeed, on $\widetilde{T}_{N,M}(x)$ we have $|h^{D_N}(x) - h^{D_N}(y)| \le k_N^2$ for each $y \in \partial\Delta^{k_N}(x)$ and so $|h^{D_N}(y) - S_{k_N}(x)| \le 2k_N^2$. The standard bounds on the regularity of the harmonic measure show $|H_{N,x}(z,y) - H_{N,x}(x,y)| \le cr\mathrm{e}^{-2k_N}$ for all $z \in \Lambda_r(x)$. The second term is thus of order $k_N^2\mathrm{e}^{-k_N}$.

Concerning the first term on the right of (5.10) we note that, on the event $T_{N,M}(x) \cap \{|h^{D_N}(x) - a_N| \le \log\log N\}$ we have

$$h^{D_N}(x) - S_{k_N}(x) = a_N\frac{k_N}{n(x)} + O\bigl(k_N^{3/4}\bigr), \tag{5.11}$$

where $n = \log N + O(1)$, while

$$gk_N\bigl(1 - H_{N,x}(z,x)\bigr) = \mathfrak{a}(z-x) + o(1) \tag{5.12}$$

uniformly in $z \in \Lambda_r(x)$. Using the asymptotic (1.5) for a_N, we then get (5.7). □

Proof of Proposition 5.2. By way of limit arguments, we may assume that f depends only on a finite number of coordinates of ϕ, say, those in $\Lambda_r(0)$, and that $f(x,h,\phi) \neq 0$ implies $\mathrm{dist}(x,\Delta^{\mathrm{c}}) > \epsilon$ and $h \in [b,b')$ for some $\epsilon > 0$ and $b < b'$. Abbreviate

$$F(x,h,\phi) := f(x,h,\phi) - f_\nu(x,h). \tag{5.13}$$

Using Cauchy–Schwarz, the expectation in (5.6) squared is bounded by

$$\frac{1}{K_N^2}\sum_{x,y\in D_N^\epsilon} E\biggl(F\Bigl(\frac{x}{N}, h^{D_N}(x)-a_N, \bigl(h^{D_N}(x)-h^{D_N}(x+z)\bigr)_{z\in\Lambda_r(0)}\Bigr)\mathsf{1}_{\widetilde{T}_{N,M}(x)} \\ \times F\Bigl(\frac{y}{N}, h^{D_N}(y)-a_N, \bigl(h^{D_N}(y)-h^{D_N}(y+z)\bigr)_{z\in\Lambda_r(0)}\Bigr)\mathsf{1}_{\widetilde{T}_{N,M}(y)}\biggr). \tag{5.14}$$

The argument at the end of the proof of Lemma 4.3 permits us to assume that x, y are such that (4.64) applies. We also assume that N is so large that $\mathrm{e}^{k_N} > r$. Conditioning on the sigma algebra

$$\mathcal{F} := \sigma\bigl(h^{D_N}(z)\colon z\notin\bigl[\Delta^{k_N}(x)\cup\Delta^{k_N}(y)\bigr]\setminus\{x,y\}\bigr) \tag{5.15}$$

then splits the expectation into a product of two parts, one for x and the other for y. Using the Gibbs–Markov decomposition to write h^{D_N} on $\Delta^{k_N}(x)\setminus\{x\}$ as $\varphi_N+\tilde{h}_N$, where $\tilde{h}_N$ is the DGFF on $\Delta^{k_N}(x)\setminus\{x\}$ and φ_N is as in Lemma 5.3, we now write the term corresponding to x as

$$E\biggl(\widetilde{F}_N\Bigl(\frac{x}{N}, h^{D_N}(x)-a_N, h^{D_N}(x)-\varphi_N(\cdot-x)\Bigr)\mathsf{1}_{\widetilde{T}_{N,M}(x)}\,\Big|\,\mathcal{F}\biggr), \tag{5.16}$$

where

$$\widetilde{F}_N(x,h,\phi) := EF(x,h,\phi+\tilde{h}_N) \tag{5.17}$$

with the expectation with respect to $\tilde{h}_N$. Our aim is to show that the random variable under expectation in (5.16) is small uniformly in $x\in D_N^\epsilon$ and the part of the configuration measurable with respect to $\mathcal{F}$.

Thanks to uniform continuity of f, the identity (5.7) permits us to replace $h^{D_N}-\varphi_N(\cdot-x)$ in (5.16) by $\frac{2}{\sqrt{g}}\lambda\mathfrak{a}$ at a cost that tends deterministically to zero. The random variable under expectation then depends on the conditional field only through $h^{D_N}(x)$. We now observe that, by the weak convergence of the DGFF on $\Lambda_{k_N}(0)\setminus\{0\}$ to the DGFF on $\mathbb{Z}^2\setminus\{0\}$ – which can be verified by comparing covariances – we get

$$E\widetilde{F}_N\Bigl(x,h,\frac{2}{\sqrt{g}}\lambda\mathfrak{a}\Bigr)\underset{N\to\infty}{\longrightarrow}0 \tag{5.18}$$

uniformly in x and h. We conclude that the conditional expectation in (5.16) is bounded by

$$o(1)1_{\{h^{D_N}(x)-a_n\in[b,b')\}}\mathsf{1}_{T_{N,M}(x)} \tag{5.19}$$

with $o(1)\to0$ uniformly in $x\in D_N^\epsilon$. In light of Lemma 4.3, the quantity in (5.14) tends to zero as $N\to\infty$. The claim follows. □

5.2. *Proofs of main results*

We are now ready to give the proofs of our main theorems:

Proof of Theorem 2.1. Let $f\colon\overline{D}\times\mathbb{R}\times\mathbb{R}^{\mathbb{Z}^2}\to\mathbb{R}$ be continuous with compact support and let f_ν be as in (5.5). Theorems 3.14 and 4.5 and ensure that $\langle\eta_N^D, f_\nu\rangle$ tends in distribution to

$$\int Z_\lambda^D(\mathrm{d}x)\otimes\mathrm{e}^{-\alpha\lambda h}\,\mathrm{d}hf_\nu(x,h) = \int Z_\lambda^D(\mathrm{d}x)\otimes\mathrm{e}^{-\alpha\lambda h}\,\mathrm{d}h\otimes\nu(\mathrm{d}\phi)\,f(x,h,\phi). \tag{5.20}$$

Lemma 5.1 and Proposition 5.2 then identify this with the distributional limit of $\langle\eta_N^D, f_\nu\rangle$. As this holds for all such f, the claim follows. □

Proof of Corollary 2.2. This follows from Theorem 2.1 with Lemma 3.1 used to reduce the problem to level sets between two values of the form $a_N + b$. □

Proof of Theorem 2.3. This was proved as part of the proofs of Theorems 3.14 and 4.5. □

Proof of Theorem 2.4. This is proved exactly as [9, Theorem 7.2]; one just needs to change the exponent 4 into $2 + 2\lambda^2$ in suitable places. We only verify the parts of this theorem where this change shows up. First off, the independence of the limit (2.6) on the particular sequence a_N permits us to assume $a_N := 2\sqrt{g}\lambda \log N$ for which we then have $K_{rN}/K_N \to r^{2+2\lambda^2}$ as $N \to \infty$. This implies the scaling relation (2.26). The representation of $Z^S_\lambda \overset{\text{law}}{=} Y^S_\infty$ for any square S then yields rotation invariance. This is because, under a conformal map $f\colon D \to f(D)$, we have

$$C^{f(D),f(\widetilde{D})}\bigl(f(x), f(y)\bigr) = C^{D,\widetilde{D}}(x,y) \tag{5.21}$$

for any admissible $\widetilde{D} \subset D$ and thus

$$\Phi^{f(D),f(\widetilde{D})} \circ f \overset{\text{law}}{=} \Phi^{D,\widetilde{D}}. \tag{5.22}$$

The rotation invariance of Y^S_∞ then follows from the rotation invariance of the function ψ^D_λ.

With these properties verified, the proof of [9, Proposition 7.2] can then be followed literally to yield, for any $u\colon D \to [0,\infty)$ bounded and measurable,

$$E\bigl(\mathrm{e}^{-\langle Z^{f(D)}_\lambda, u\circ f\rangle}\bigr) \geq E\bigl(\mathrm{e}^{-\langle Z^D_\lambda, |f'\circ f^{-1}|^{2+2\lambda^2} u\rangle}\bigr). \tag{5.23}$$

Iterating this with f replaced by f^{-1} then gives equality in (5.23). The claim follows. □

Proof of Theorem 2.5. Let $D \in \mathfrak{D}$ be such that it D fits an open dyadic square of side r. For any integer $k \geq 0$, let $S_{k,i}$, $i = 1,\dots,n(k)$ be open dyadic squares of side $r2^{-k}$ that lie entirely in S. Clearly, each $S_{k,i}$ has a non-empty intersection and contains exactly 4 squares of the form $S_{k+1,j}$ although there may be squares of the latter form that do not belong to any square of the form $S_{k,i}$. For each $k \geq 1$, let H_k be the set of functions in $\mathsf{H}^1_0(D)$ that are harmonic on each $S_{k,i}$, $i = 1,\dots,n(k)$, and vanish on $D \setminus \bigcup_{i=1}^{n(k)} S_{k,i}$. Then, as is checked by the Gauss-Green formula, $\{\mathsf{H}_k\colon k \geq 0\}$ are orthogonal subspaces of $\mathsf{H}^1_0(D)$ with

$$\mathsf{H}^1_0(D) = \bigoplus_{k\geq 0} \mathsf{H}_k. \tag{5.24}$$

A minor complication that arises in this setting is that each H_k is infinitely dimensional. Still, by separability of $\mathsf{H}^1_0(D)$, we can find an countable orthonormal basis $\{\widetilde{\varphi}_{k,j}\colon j \geq 1\}$ in each H_k.

Let $\{X_{k,j}\colon k, j \geq 1\}$ be i.i.d. standard normals and write $D^0 := D$ and $D^k := \bigcup_{i=1}^{n(k)} S_{k,i}$. A covariance calculation shows

$$\Phi^{D^{k-1},D^k} \overset{\text{law}}{=} \sum_{j\geq 1} X_{k,j}\widetilde{\varphi}_{k,j} \quad \text{on } D^k, k \geq 1. \tag{5.25}$$

We also have

$$\Phi^{D,D^m} \overset{\text{law}}{=} \sum_{k=1}^{m} \Phi^{D^{k-1},D^k} \tag{5.26}$$

with the fields on the right-hand side regarded as independent. Using a suitable coupling to realize these distributional identities as almost sure equalities, letting

$$Y^D_k(\mathrm{d}x) := c\psi^D_\lambda(x) \sum_{i=1}^{n(k)} \mathrm{e}^{\alpha\lambda\Phi^{D,D^k}(x) - \frac{1}{2}\alpha^2\lambda^2 E[\Phi^{D,D^k}(x)^2]} 1_{S_{k,i}}(x)\,\mathrm{d}x \tag{5.27}$$

and setting $\mathcal{F}_k := \sigma(X_{\ell,j}\colon \ell + j \le k\}$, then for any measurable $A \subset D$,

$$E\big(Y_m^D(A)|\mathcal{F}_k\big) = \int_A c\psi_\lambda^D(x)\mu^{D,\alpha\lambda}_{k(k-1)/2}(\mathrm{d}x), \quad m \ge k, \tag{5.28}$$

where $\mu_n^{D,\beta}$ is the measure defined in (2.16) for the basis $\{\varphi_n\colon n \ge 1\}$ in $\mathsf{H}_0^1(D)$ which is obtained by reordering $\{\widetilde{\varphi}_{k,j}\colon j \ge 1\}$ according to the complete order

$$(k,j) \preceq \big(k',j'\big) \quad \Leftrightarrow \quad k + j < k' + j' \quad \text{or} \quad k + j = k' + j' \quad \& \quad j \le j'. \tag{5.29}$$

Since $\lambda < 1$ and thus $\alpha\lambda < \beta_{\mathrm{c}}$, it is known (cf. a remark after Rhodes and Vargas [29, Theorem 5.5]) that $\mu_n^{D,\beta}$ converges to a non-trivial measure $\mu_\infty^{D,\beta}$ almost surely and in L^1. It follows that $\{Y_k^D(A), \mathcal{F}_k\colon k \ge 1\}$ is a uniformly integrable martingale. The Martingale Convergence Theorem then gives

$$Y_k^D(A) \underset{k\to\infty}{\longrightarrow} Y_\infty^D(A) \quad \text{a.s. and in } L^1. \tag{5.30}$$

Using this in (5.28) shows

$$E\big(Y_\infty^D(A)|\mathcal{F}_k\big) = \int_A c\psi_\lambda^D(x)\mu^{D,\alpha\lambda}_{k(k-1)/2}(\mathrm{d}x), \quad k \ge 1. \tag{5.31}$$

The Levy Backward Theorem and the convergence $\mu_n^{D,\beta} \to \mu_\infty^{D,\beta}$ along with the fact that $\bigcap_{k\ge1}\mathcal{F}_k$ is trivial now identify Y_∞^D with the LQG measure on the right of (2.18).

To link this to the law of Z_λ^D we note that, as part of the proofs of Theorems 3.14 and 4.5, we showed that $Z_\lambda^D \overset{\text{law}}{=} Y_\infty^D$ for D being a dyadic square. The Gibbs–Markov property and the construction (5.27) then readily extend this to all D. □

Appendix: Useful properties and bounds

Here we will review some of the needed facts concerning the DGFF as well as the Green function of the simple symmetric random walk on $\mathbb{Z}^2$. We begin with the latter.

Given $D \subsetneq \mathbb{Z}^2$, the Green function $G^D(x,y)$ is the expected number of visits to y of the simple random walk started at x before the walk exits D. Denoting $V_N' := (-N,N)^2 \cap \mathbb{Z}^2$, the potential kernel can be defined by the limit

$$\mathfrak{a}(x) := \lim_{N\to\infty}\big[G^{V_N'}(0,0) - G^{V_N'}(0,x)\big]. \tag{A.1}$$

The potential kernel admits the asymptotic form

$$\mathfrak{a}(x) = g\log|x| + c_0 + O\big(|x|^{-2}\big), \quad |x| \to \infty. \tag{A.2}$$

with c_0 a numerical constant. For D finite, the fact that $\mathfrak{a}$ is discrete harmonic away from 0 while $x \mapsto G^D(x,y)$ is harmonic on $D \setminus \{y\}$, we have

$$G^D(x,y) = -\mathfrak{a}(x-y) + \sum_{z\in\partial D} H^D(x,z)\mathfrak{a}(y-z), \tag{A.3}$$

where $H^D(x,z)$ is the probability that the simple random walk started at x exits D at z. As shown in [9, Lemma A.2], the class of domains $\mathfrak{D}$ is such that, for any $D \in \mathfrak{D}$ and any sequence D_N approximating D in the sense of (2.1)–(2.2), we have

$$\sum_{z\in\partial D_N} H^{D_N}\big(\lfloor xN\rfloor, z\big)\delta_{z/N}(\cdot) \overset{\text{vaguely}}{\underset{N\to\infty}{\longrightarrow}} \Pi^D(x,\cdot), \quad x \in D, \tag{A.4}$$

where $\Pi^D(x,\cdot)$ is the harmonic measure on (i.e., the hitting probability of the Brownian motion started from x to) the boundary ∂D of the continuum domain D. Using this in conjunction with (A.2), for $x, y \in D$ with $x \neq y$ we then get

$$G^{D_N}\big(\lfloor xN\rfloor, \lfloor yN\rfloor\big) = -g\log|x-y| + g\int_{\partial D}\Pi^D(x,\mathrm{d}z)\log|x-z| + o(1). \tag{A.5}$$

For $x = y$ we instead get (3.7) effectively replacing $-g\log|x-y|$ by $g\log N + c_0$.

Moving to the properties of the DGFF, one that is most fundamental is the Gibbs–Markov decomposition. If $V \subset U \subset \mathbb{Z}^2$ are finite domains and h^U and h^V the DGFFs on U, resp., V, then

$$h^U \overset{\text{law}}{=} h^V + \varphi^{U,V} \tag{A.6}$$

with $\varphi^{U,V}$ independent of h^V and having sample paths that are discrete harmonic on V and equidistributed to h^U on $U \setminus V$. The law of $\varphi^{U,V}$ can be alternatively prescribed by its covariance structure, which turns out to be the difference $G^U - G^V$. It is now easy to check from (A.3)–(A.4), this difference admits a scaling limit in the sense that, for any $D, \widetilde{D} \in \mathfrak{D}$ with $\widetilde{D} \subset D$ and locally uniformly in $x, y \in \widetilde{D}$

$$G^{D_N}\big(\lfloor xN\rfloor, \lfloor yN\rfloor\big) - G^{\widetilde{D}_N}\big(\lfloor xN\rfloor, \lfloor yN\rfloor\big) \underset{N\to\infty}{\longrightarrow} C^{D,\widetilde{D}}(x,y), \tag{A.7}$$

with $C^{D,\widetilde{D}}$ as in (2.9). Letting $\Phi^{D,\widetilde{D}}$ be the Gaussian process with covariance $C^{D,\widetilde{D}}$, for each $N \geq 1$ and each $\delta > 0$, there is a coupling of $\varphi^{D_N,\widetilde{D}_N}$ with $\Phi^{D,\widetilde{D}}$ such that

$$\lim_{N\to\infty} P\Big(\sup_{\substack{x\in\widetilde{D}\\ \operatorname{dist}(x,\widetilde{D}^{\mathrm{c}})>\delta}} \big|\varphi^{D_N,\widetilde{D}_N}\big(\lfloor xN\rfloor\big) - \Phi^{D,\widetilde{D}}(x)\big| > \delta\Big) = 0, \tag{A.8}$$

see [11, Lemma B.14].

As our final item of business, we will prove a lemma that was used in the proof of Theorem 4.5. The proof is standard; we include it merely for completeness of exposition.

Proof of Lemma 4.4. To lighten the notation suppose S is an open square of side r and let S_i, $i = 1,\dots,L^2$ be disjoint open squares of side r/L that just barely fit into S. Denote $\widetilde{S} := \bigcup_{i=1}^{L^2} S_i$ and let x_i be the center point of S_i, for each i. Then $\operatorname{Var}(\Phi^{S,\widetilde{S}}(x_i)) \leq g\log L + c$ for some constant c and so, by a straightforward union bound,

$$P\Big(\max_{i=1,\dots,L^2}\Phi^{S,\widetilde{S}}(x_i) > 2\sqrt{g}\log L\Big) \leq \frac{c'}{\sqrt{\log L}}. \tag{A.9}$$

Next let $S_i^\delta := \{z \in S_i :\ \operatorname{dist}(z, S_i^{\mathrm{c}}) > \delta\}$ and note that

$$\max_{i=1,\dots,L^2}\sup_{z\in S_i^\delta}\operatorname{Var}\big(\Phi^{S,\widetilde{S}}(z) - \Phi^{S,\widetilde{S}}(x_i)\big) \leq c \tag{A.10}$$

with c independent of L. Letting

$$M_L^\star := \max_{i=1,\dots,L^2}\sup_{z\in S_i^\delta}\big(\Phi^{S,\widetilde{S}}(z) - \Phi^{S,\widetilde{S}}(x_i)\big) \tag{A.11}$$

the Borell–Tsirelson inequality (see Adler [2, Theorem 2.1]) shows that $M_L^\star$ has a uniform Gaussian tail and so

$$P\big(M_L^\star - EM_L^\star > \sqrt{\log L}\big) \leq L^{-c} \tag{A.12}$$

for some $c > 0$ independent of L. It thus remains to control the growth rate of $EM_L^\star$. For this we consider the pseudometric space $(\mathcal{X}, \rho)$, where $\mathcal{X} := \{(i,z)\colon i = 1,\dots,L^2, z \in S_i^\delta\}$ and $\rho((i,z),(i',z')) := E[\Phi^{S,\widetilde{S}}(z) - \Phi^{S,\widetilde{S}}(z')]$

when $z \in S_i^\delta$ and $z' \in S_{i'}^\delta$. Writing $B_\rho((i,z),r)$ for the closed ball in $\mathcal{X}$ of radius r centered at (i,z) and using m for the normalized Lebesgue measure on $\bigcup_{i=1}^{L^2} S_i^\delta$, the Fernique criterion (cf. Adler [2, Theorem 4.1]) then gives

$$E M_L^\star \le c \int_0^\infty \mathrm{d}r \sqrt{\log \frac{1}{m(B_\rho(x,r))}} \tag{A.13}$$

for some universal constant c. The fact that $(x,y) \mapsto C^{S,\widetilde{S}}(x,y)$ is uniformly Lipschitz on each S_i^δ gives $m(B_\rho((i,z),r)) \ge c'(r \wedge L^{-1})^4$ with $c' > 0$ independent of L as soon as δ is sufficiently small. Plugging this into (A.13), we get $E M_L^\star \le c'' \sqrt{\log L}$. Combining this with (A.12) and (A.9), the claim follows. □

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