

Gravitation versus Brownian motion

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Abstract. We investigate the motion of an inert (massive) particle being impinged from below by a particle performing (reflected) Brownian motion. The velocity of the inert particle increases in proportion to the local time of collisions and decreases according to a constant downward gravitational acceleration. We study fluctuations and strong laws of the motion of the particles. We further show that the joint distribution of the velocity of the inert particle and the gap between the two particles converges in total variation distance to a stationary distribution which has an explicit product form.

Résumé. Nous étudions le mouvement d'une particule inerte (massive) qui est frappé par en dessous par une particule effectuant un mouvement brownien (réfléchi). La vitesse de la particule inerte augmente proportionnellement au temps local des collisions et diminue en fonction d'une accélération gravitationnelle constante vers le bas. Nous étudions les fluctuations et les lois fortes du mouvement des particules. Enfin, nous montrons que la distribution conjointe de la vitesse de la particule inerte et de l'écart entre les deux particules converge pour la distance de variation totale vers une distribution stationnaire qui a une forme produit explicite.

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1. Introduction

We will investigate the motion of an inert (massive) particle that is impinged from below by a particle performing (reflected) Brownian motion. Whenever the two particles collide, the velocity of the inert particle increases in proportion to the local time of collisions. Furthermore, there is a gravitational field that pulls the inert particle downwards by giving it a constant acceleration. The Brownian particle is reflected on the trajectory of the inert particle according to the usual Skorokhod recipe.

Formally, the motion of the two particles will be defined by a system of SDE's. We will denote the driving Brownian motion by B . We will use X and S to denote the trajectories of the reflecting Brownian particle and the inert particle, respectively, and V to denote the velocity of the inert particle. Gravitation will be represented by a constant acceleration $g > 0$. We will write L to denote the intersection local time between the two particles, defined as the unique continuous non-decreasing process increasing only when $S_t = X_t$ (i.e., $L_t - L_0 = \int_0^t \mathbb{I}_{\{S_u = X_u\}} dL_u$ for all $t \geq 0$). The SDE's are

$$\begin{cases} dX_t = dB_t - dL_t, \\ dV_t = dL_t - g dt, \\ dS_t = V_t dt. \end{cases} \quad (1.1)$$

There is also an extra condition that $S_t \geq X_t$ for all $t \geq 0$, that is, the Brownian particle and the inert particle can collide but their trajectories cannot cross (this applies, in particular, to the initial condition, i.e., $S_0 \geq X_0$). We will show existence and uniqueness of the strong solution to (1.1) in Theorem 3.4.

The model without the gravitational component was originally introduced in [9]. The motivation came from trying to mathematically model the joint motion of a Brownian particle in a liquid and a semi-permeable membrane (thought of as the inert particle) which is permeable to the microscopic liquid molecules but not to the macroscopic Brownian particle. Without gravitation, the inert particle moves with constant velocity in the absence of collisions and the velocity increases (in proportion to the local time of collisions) only when the particles collide. Thus, it is clear that there will be a random time after which the particles never collide and the inert particle “escapes” the Brownian particle with constant velocity. The laws of the inverse velocity process $V^{(-1)}$ and the “escape velocity” were explicitly computed in [9].

A more realistic situation arises when we take into account the effect of gravitation which exerts a constant downward force on the inert particle (membrane) but barely affects the fluid molecules or the Brownian particle. We use the term “gravitation” as a representative of any constant force on the inert particle due to a potential or mechanical pressure. The gravitation component significantly changes the behavior of the model as the velocity of the inert particle is no longer an increasing process and the inert particle can never escape the Brownian particle (they keep colliding). The joint behavior of the two particles is thus, a priori, far from clear. Among other things, we will show that in this battle between the gravitational pull and the Brownian push, gravitation “wins” as both particles eventually “fall” with asymptotic velocity $-g$.

A number of related models were studied in [2,4,15]. In [2], reflecting diffusions were considered in bounded smooth domains in \mathbb{R}^d , that acquired drift proportional to the local time spent on the boundary of the domain. Product form stationary distributions were derived for the joint law of the position of the reflecting process and its drift. In [15], general classes of processes with inert drift were constructed and recurrence, transience and stationary distributions were investigated for some particular examples. In [4], some Markov processes with discrete state spaces were studied as approximations to processes with inert drift. Necessary and sufficient conditions in order to have a stationary distribution in product form were given for these discrete state space Markov processes and it was conjectured that these conditions carry over to some models with continuous state space via appropriate limiting operations.

Besides its initial motivation from physics, some important diffusion limits arising from heavy traffic asymptotics of ‘join the shortest queue’ systems with many servers (see the diffusion described in Theorem 2 of [5]) can be approximated by inert drift systems with gravitation and damping. The techniques introduced in this paper turn out to be crucial in understanding the stationary behavior of these diffusions. This is an ongoing project of the first author with Debankur Mukherjee (see [1]).

We will now discuss a few aspects of the model that we find intriguing. The constant g enters the model as the acceleration but ends up as the asymptotic velocity for both inert and Brownian particles, S and X (see Theorem 2.3). With the hindsight, one could provide the following “explanation” for this strange transformation of the role of g . Since the local time represents the change of position for X and the change of velocity for S , it is perhaps not so surprising that the acceleration of S becomes the velocity for X . Because of the parabolic drift, excursions of S above X are not very large, which makes the two particles remain close on large time scales, so their asymptotic velocity must be the same. We study the “zero-noise case” (i.e. with $B_t \equiv 0$) in Remark 2.4 and show that an analogous result holds for this deterministic system, which provides further evidence as to why this result might be true even in the presence of noise.

The product form of the stationary distribution for $(V, S - X)$ (see Theorem 2.1) came as a surprise to us but, with the hindsight, we see that the model (1.1) fits into the framework of [4, Section 3]. In other words, an appropriate discretized version of (1.1) should satisfy [4, Cor. 2.3], and it might be possible to perform a limiting operation on that discretized model as conjectured in [4] and deduce the product form stationary distribution for the original model, although we do not prove this in this paper.

The variance of the first component of the stationary distribution, representing V , does not depend on g . Once again, this is not surprising with the hindsight, since the stationary distribution for the local time in a related model in [2, Thm. 6.2] does not depend on the state space (Euclidean domain) for the Brownian particle.

The rest of the article is organized as follows. Our main results are stated in Section 2. Existence and uniqueness of the strong solution to (1.1) is proved in Section 3. Section 4 is devoted to some technical estimates. These estimates are used to obtain universal fluctuation results for V and $S - X$ and laws of large numbers in Section 7. Finally, the stationary distribution for the velocity and gap processes is derived in Section 6.

2. Main results

This section contains statements of those of our main results that are non-technical.

The first theorem states that $Z := (V, S - X)$ has a unique stationary distribution which is the product of a Gaussian distribution and an exponential distribution. We will prove in Section 6 that the laws of Z_t converge in the total variation distance to the stationary distribution.

Theorem 2.1. *The process $Z := (V, S - X)$ has a unique stationary distribution with the density with respect to Lebesgue measure given by*

$$\xi(v, h) = \frac{2g}{\sqrt{\pi}} e^{-2gh} e^{-(v+g)^2}, \quad v \in \mathbb{R}, h \geq 0. \tag{2.1}$$

Furthermore, Z_t converges to this distribution in total variation distance as $t \rightarrow \infty$.

The next result shows that the fluctuations of the velocity process are of the order $\sqrt{\log t}$ while the fluctuations of the ‘‘gap process’’ $S_t - X_t$ are of the order $\log t$.

Theorem 2.2. *For any $Z_0 = z$, almost surely,*

$$-\liminf_{t \rightarrow \infty} \frac{V_t}{\sqrt{\log t}} = \limsup_{t \rightarrow \infty} \frac{V_t}{\sqrt{\log t}} = 1, \tag{2.2}$$

$$\limsup_{t \rightarrow \infty} \frac{S_t - X_t}{\log t} = \frac{1}{2g}, \tag{2.3}$$

$$\liminf_{t \rightarrow \infty} \frac{S_t - X_t}{\log t} = 0. \tag{2.4}$$

We will show that both the ‘‘inert particle’’ S_t and the reflected Brownian particle X_t behave like Brownian motion with constant negative drift $-g$ on the large scale and, therefore, they satisfy the same Strong Law of Large Numbers as Brownian motion with drift. The next theorem gives precise estimates on the oscillations of S and X from Brownian motion with drift $-g$. The strong law follows as a consequence.

Theorem 2.3. *For any $Z_0 = z$, almost surely,*

$$-\liminf_{t \rightarrow \infty} \frac{X_t - (B_t - gt)}{\sqrt{\log t}} = \limsup_{t \rightarrow \infty} \frac{X_t - (B_t - gt)}{\sqrt{\log t}} = 1,$$

$$\limsup_{t \rightarrow \infty} \frac{S_t - (B_t - gt)}{\log t} = \frac{1}{2g},$$

$$\liminf_{t \rightarrow \infty} \frac{S_t - (B_t - gt)}{\log t} = 0.$$

It follows that, a.s.,

$$\lim_{t \rightarrow \infty} \frac{X_t}{t} = \lim_{t \rightarrow \infty} \frac{S_t}{t} = -g.$$

Remark 2.4. To gain some insight into why one might expect $\lim_{t \rightarrow \infty} \frac{X_t}{t} = \lim_{t \rightarrow \infty} \frac{S_t}{t} = -g$ to hold almost surely, we consider the ‘‘zero-noise case’’, i.e., the deterministic two-particle system driven by (1.1) with $B_t \equiv 0$. Suppose we start from the initial conditions $S_0 = X_0, V_0 < 0$. Let $\tau = \inf\{t > 0 : V_t = 0\}$. Then on $[0, \tau]$, S_t is decreasing and one can conclude from the Skorohod equation (see [8, Lem. 6.14, Ch. 3]) that $L_t = \sup_{u \leq t} (S_0 - S_u) = S_0 - S_t$. Using this in (1.1), we obtain

$$S_t = X_t, \quad V_t = -g + (V_0 + g)e^{-t} \quad \text{for all } t \leq \tau. \tag{2.5}$$

The above implies $\tau = \infty$ and $V_t \rightarrow -g$ as $t \rightarrow \infty$. Thus,

$$\lim_{t \rightarrow \infty} \frac{X_t}{t} = \lim_{t \rightarrow \infty} \frac{S_t}{t} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t V_u du = -g \tag{2.6}$$

holds for the zero-noise case when $S_0 = X_0, V_0 < 0$. If $S_0 > X_0$ and $V_0 \in \mathbb{R}$, it follows from (1.1) that if $\sigma = \inf\{t \geq 0 : S_t = X_t\}$, then $\sigma < \infty$ and $V_\sigma < 0$, and thus, we can perform the same computations for $t > \sigma$ to deduce that (2.6) holds.

Finally, suppose $S_0 = X_0$ and $V_0 \geq 0$. If $S_t = X_t$ for all $t \geq 0$, we use (1.1) to obtain $dL_t = -dX_t = -dS_t = -V_t dt$ which gives us $dV_t = -V_t dt - g dt$. This yields (2.5) and consequently (2.6). Otherwise, there exists $t_0 > 0$ such that $S_{t_0} > X_{t_0}$ and the previous calculations again yield (2.6). Thus, we see that $\lim_{t \rightarrow \infty} \frac{X_t}{t} = \lim_{t \rightarrow \infty} \frac{S_t}{t} = -g$ always holds in the zero-noise case. As this is a ‘‘law of large numbers’’ type result, it is natural to expect that this result would also hold with the Brownian noise via some scaling properties of Brownian motion. In the proof of Theorem 2.3, however, we derive the fluctuation results quite differently via some technical estimates and derive (2.6) as a consequence of these results.

We will use the following notation: $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

3. Existence and uniqueness of the process

In this section, we prove the existence and pathwise uniqueness of solutions to (1.1). We also prove the well-posedness of the submartingale problem corresponding to the process. The latter fact will be an essential ingredient in proving the existence of the stationary distribution in Section 6.

For notational convenience, all vectors written in the form $v = (v_1, \dots, v_k)$ for some $k \in \mathbb{N}$ should be thought of as column vectors. We will use v^T to denote the corresponding row vector.

Remark 3.1. The following translation invariance property follows immediately from the form of equations (1.1). Consider real numbers x, s, v, l with $s \geq x$ and $l \geq 0$. If $\{(X_t, S_t, V_t, L_t), t \geq 0\}$ solves (1.1) with the initial conditions $(X_0, S_0, V_0, L_0) = (0, s - x, v, 0)$ then $\{(\widehat{X}_t, \widehat{S}_t, \widehat{V}_t, \widehat{L}_t), t \geq 0\} := \{(x + X_t, x + S_t, V_t, \ell + L_t), t \geq 0\}$ is a solution to (1.1) with the initial conditions $(\widehat{X}_0, \widehat{S}_0, \widehat{V}_0, \widehat{L}_0) = (x, s, v, \ell)$. Because of this, we will always assume in our technical estimates that $B_0 = X_0 = L_0 = 0$, unless explicitly stated otherwise.

The process given by $H_t = S_t - X_t$ will be called the gap process. If we know both V and H , we can recover the movement of the individual particles by first integrating V to obtain S , and then computing $X_t = S_t - H_t$.

Thus, existence and uniqueness of a strong solution to the system (1.1) are equivalent to those of the following system of equations:

$$\begin{cases} dV_t = dL_t - g dt, \\ dH_t = -dB_t + V_t dt + dL_t, \end{cases} \tag{3.1}$$

where B_t is a standard one dimensional Brownian motion, $H_t \geq 0$ for all $t \geq 0$, and L_t is a continuous, non-decreasing process satisfying $dL_t = \mathbb{I}_{\{0\}}(H_t) dL_t$. As before, we will write $Z_t = (V_t, H_t)$.

If B_t and Z_t are given, then L_t can be computed from the equation $L_t = L_0 + V_t - V_0 + gt$. Thus, the complete description of the strong solution to (3.1) can be given in terms of only Z and B .

Even though reflected diffusions in $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+$ are well studied and many classical results are available, we have not found a direct reference for existence and uniqueness of equations (3.1), mainly due to two technical issues: (i) Z_t is not a strictly elliptic diffusion, and (ii) the drift vector $(-g, V_t)$ is unbounded in the v -component. We will split our proof of existence and uniqueness into two lemmas: one that shows that a local solution exists, and another that extends local solutions to global ones. The second lemma will also be used to show that the submartingale problem is well posed.

In the following, we will write

$$\mathcal{L} = \frac{1}{2} \partial_{hh} + v \partial_h - g \partial_v$$

for the second order differential operator associated to the generator of the Markov process Z_t satisfying (3.1) (provided it exists) in the interior of the upper half plane \mathbb{H} .

Lemma 3.2. *For each $N \geq 0$, there is a weak solution $Z_t^N = (V_t^N, H_t^N)$ of (3.1) up to time $T^N = \inf\{t > 0 : |Z_t^N| > N\}$. Also, (3.1) satisfies pathwise uniqueness up to time T^N .*

Proof. Note that equation (3.1) can be written as

$$dZ_t = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} dB_t^* + \begin{pmatrix} -g \\ V_t \end{pmatrix} dt + \begin{pmatrix} 1 \\ 1 \end{pmatrix} dL_t, \tag{3.2}$$

where $B_t^* = (W_t, B_t)$ and W_t is a standard, one dimensional Brownian motion independent of B , that has no effect on the paths of Z_t , that is, if we replace W_t with another Brownian motion W_t' then the same process Z_t will solve (3.2). We will now apply a standard localization technique. Note that if a weak solution Z_t^N exists for a modified version of (3.2) with the drift vector replaced by $(-g, V_t \wedge N)$, then Z_t^N will also be a weak solution to the original equation (3.2) up to time T^N . The drift and diffusion coefficients for the modified version of (3.2) are Lipschitz and bounded and thus, the equation fits into the setup of Theorem 1 in [14]: the diffusion matrix $a = \sigma\sigma^T$ has component $a_{22} = 1$, the drift vector is Lipschitz and bounded, and the reflection vector $\gamma := (1, 1)$ is constant with a unit length component in the direction of the normal to the boundary. Even though the statement of Theorem 1 in [14] is about weak existence and weak uniqueness, its proof actually shows pathwise uniqueness. This implies the two claims made in the lemma. \square

Lemma 3.3. *For each $N \geq 1$, let Z_t^N be a weak solution up to time $T^N = \inf\{t > 0 : |Z_t^N| > N\}$ of (3.1), with $Z_0^N = z$. Then*

- (a) *There are constants $C_1, C_2 > 0$, independent of N , such that $\mathbb{E}(|Z_{t \wedge T^N}^N|^2) \leq C_1 e^{C_2 t}$.*
- (b) *All processes Z_t^N can be chosen to be strong solutions of (3.1). They can be constructed so that $Z_t^N = Z_t^{N'}$ for $t \leq T^N$ if $N \leq N'$. Hence, they can be extended to a strong solution Z_t up to time $T^* := \sup_N T^N$. Pathwise uniqueness holds for Z_t up to time T^* .*
- (c) *$T^* = \infty$ a.s.*

Proof.

To prove (a), set $\eta(v, h) = 2h^2 + v^2 - 2hv$, and recall that $\mathcal{L}\eta = \frac{1}{2}\partial_{hh}\eta + v\partial_h\eta - g\partial_v\eta$ and $\gamma = (1, 1)$. It is elementary to check that $\frac{1}{3}(h^2 + v^2) \leq \eta(v, h) \leq 3(h^2 + v^2)$. Since $|Z_t^N| < N$ for $t < T^N$, we have by Itô's formula

$$\begin{aligned} \frac{1}{3} \mathbb{E}(|Z_{t \wedge T^N}^N|^2) &\leq \mathbb{E}\eta(Z_{t \wedge T^N}^N) = \eta(z) + \mathbb{E} \int_0^{t \wedge T^N} \mathcal{L}\eta(Z_u^N) du + \mathbb{E} \int_0^{t \wedge T^N} \nabla\eta(Z_u^N)^T \gamma dL_u^N \\ &= \eta(z) + \mathbb{E} \int_0^{t \wedge T^N} \mathcal{L}\eta(Z_u^N) du, \end{aligned}$$

since $\nabla\eta(Z_u^N)^T \gamma dL_u^N = 2H_u^N dL_u^N = 0$. To bound the integral on the right hand side, we note that there exist constants $K_1, K_2 > 0$ such that $|\mathcal{L}\eta(z)| \leq K_1 + K_2|z|^2$. Putting all these inequalities together we obtain

$$\mathbb{E}(|Z_{t \wedge T^N}^N|^2) \leq 9|z|^2 + 3K_1t + 3K_2 \int_0^t \mathbb{E}(|Z_{u \wedge T^N}^N|^2) du.$$

Now (a) follows from Gronwall's inequality.

Since pathwise uniqueness holds for (3.1) up to time T^N , we can apply a well-known argument by Yamada and Watanabe (see [6, Ch. IV, Thm. 11] or [8, Ch. 5, Corollary 3.23]) with minor modifications to show that Z_t^N can be chosen as a strong solution up to time T^N . By pathwise uniqueness, if $M > N$ we have that $Z_t^M = Z_t^N$ for $t < T^N$. This allows us to define $Z_t = Z_t^N$ for $t < T^N$ in a consistent way, and thus obtain a strong solution for $t < T^* = \sup_N T^N$. It is clear that pathwise uniqueness also holds up to time T^* . This shows (b).

It remains to show that $T^* = \infty$ almost surely. For any $\alpha > 0$ and $R > 0$, by (a), we have

$$\mathbb{P}(T^* \leq \alpha) \leq \mathbb{E} \left(\frac{|Z_\alpha^R|^2}{R^2} \mathbb{I}_{\{T^R \leq \alpha\}} \right) \leq \frac{C_1}{R^2} e^{C_2 \alpha}.$$

Taking $R \rightarrow \infty$ on the right hand side, we conclude that $\mathbb{P}(T^* \leq \alpha) = 0$ for each $\alpha > 0$, and thus $T^* = \infty$ a.s. □

As a direct consequence of the two previous lemmas, we are able to show existence and pathwise uniqueness for equation (3.1), which we record in the following theorem.

Theorem 3.4. *The system of stochastic equations (3.1) has a square integrable, strong solution (V_t, H_t) , and satisfies pathwise uniqueness.*

Our next theorem will be used in the derivation of the stationary distribution of Z_t . Recall that $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+$. We will denote by \mathcal{C} the set of continuous functions from $[0, \infty)$ to $\overline{\mathbb{H}}$. We denote by $C_0^k(A)$ the set of compactly supported functions on A with continuous derivatives up to order k , allowing $k = \infty$. We will use \mathcal{B} to denote the Borel σ -algebra of \mathcal{C} , and $\{\mathcal{F}_t\}$ will denote the natural filtration on \mathcal{C} . Recall that $\gamma = (1, 1)$ and $\mathcal{L}f(v, h) = \frac{1}{2} \partial_{hh} f + v \partial_h f - g \partial_v f$.

Theorem 3.5. *The submartingale problem for (\mathcal{L}, γ) in \mathbb{H} is well-posed, that is, there is a unique family of measures $\{\mathbb{P}_z : z \in \overline{\mathbb{H}}\}$ on $(\mathcal{C}, \mathcal{B})$ such that, for each $z \in \overline{\mathbb{H}}$, the following properties hold*

1. $\mathbb{P}_z(\omega(0) = z) = 1$.
2. For $t \geq 0$, and each $f \in C_0^2(\mathbb{R}^2)$ such that $\nabla f(y)^T \gamma \geq 0$ for all $y \in \partial \mathbb{H}$, the process

$$S_t[f] := f(\omega(t)) - \int_0^t \mathcal{L}f(\omega(u)) du \tag{3.3}$$

is a submartingale in $(\mathbb{P}_z, \mathcal{C}, \mathcal{B}, \{\mathcal{F}_t\})$.

Moreover, the unique solution to the submartingale problem corresponds to the law of the process $Z_t = (V_t, H_t)$ solving (3.1).

Before proceeding to the proof of Theorem 3.5, we will prove an important property about the amount of time the process associated to a solution to the submartingale problem spends on the boundary of the domain.

Lemma 3.6. *Let $Z_t^* = (V_t^*, H_t^*)$ be a solution of the submartingale problem for (\mathcal{L}, γ) in \mathbb{H} . Then*

$$\int_0^t \mathbb{I}_{\partial \mathbb{H}}(Z_u^*) du = 0 \quad a.s.$$

Proof. We will write $z = (v, h)$ to simplify the notation. For $n > 0$, define $q_n(h) = h^2 \exp(-nh)$. Note that $q_n(h) = \partial_h q_n(h) = 0$ on $\partial \mathbb{H}$. For $N > 0$, let φ_N be a $C_0^2(\mathbb{R}^2)$ function satisfying: $0 \leq \varphi_N(z) \leq 1$, $\varphi_N(z) = 1$ for $|z| \leq N$, and φ_N has uniformly (in N) bounded derivatives up to the second order. Let $\tilde{q}_{n,N}(z) = q_n(h) \varphi_N(z)$. Since $q_n(h)$ is bounded above by $4e^{-2}/n^2$, it is clear that

$$\lim_{n \rightarrow \infty} \tilde{q}_{n,N}(z) = 0, \quad \lim_{n \rightarrow \infty} \mathbb{E}(\tilde{q}_{n,N}(Z_t^*)) = 0 \quad \text{for any } t \geq 0. \tag{3.4}$$

We have,

$$\int_0^t \mathcal{L} \tilde{q}_{n,N}(Z_u^*) du = \int_0^t (\varphi_N(Z_u^*) \mathcal{L} q_n(H_u^*) + q_n(H_u^*) \mathcal{L} \varphi_N(Z_u^*) + \partial_h q_n(H_u^*) \partial_h \varphi_N(Z_u^*)) du.$$

We will argue that the integrals of the second and third terms on the right hand side go to zero as $n \rightarrow \infty$, for every fixed N . The claim holds for the second term because $q_n(h)$ is bounded above by $4e^{-2}/n^2$ and $\mathcal{L} \varphi_N(z)(h)$ is

uniformly bounded. The function $\partial_h q_n(h)$ converges to zero, and is uniformly bounded in n . The function $\partial_h \varphi_N(Z_u^*)$ is uniformly bounded. These observations prove the claim for the third term.

We now turn our attention to the first term on the right hand side. The function $\varphi_N(z) \mathcal{L}q_n(h)$ is uniformly bounded and converges to $\varphi_N(z) \mathbb{I}_{\{0\}}(h)$ as n goes to infinity. Therefore, applying the dominated convergence theorem, we see that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(\int_0^t \mathcal{L} \tilde{q}_{n,N}(Z_u^*) du \right) = \mathbb{E} \left(\int_0^t \varphi_N(Z_u^*) \mathbb{I}_{\partial \mathbb{H}}(Z_u^*) du \right). \tag{3.5}$$

Since $(\nabla \tilde{q}_{n,N}(z))^T \gamma = 0$ on $\partial \mathbb{H}$ we have that $S_t[\tilde{q}_{n,N}] - S_0[\tilde{q}_{n,N}]$ is a martingale (this can be proved by considering the submartingale problem applied to $\tilde{q}_{n,N}$ and $-\tilde{q}_{n,N}$). Therefore,

$$\mathbb{E}(\tilde{q}_{n,N}(Z_t^*)) = \tilde{q}_{n,N}(z) + \mathbb{E} \left(\int_0^t \mathcal{L} \tilde{q}_{n,N}(Z_u^*) du \right).$$

Letting $n \rightarrow \infty$ in the above equation and using (3.4) and (3.5), we obtain

$$\mathbb{E} \left(\int_0^t \varphi_N(Z_u^*) \mathbb{I}_{\partial \mathbb{H}}(Z_u^*) du \right) = 0.$$

The lemma now follows from the monotone convergence theorem upon taking $N \rightarrow \infty$. □

Proof of Theorem 3.5. Using Itô’s formula, it is straightforward to check that solutions Z to (3.1), for different initial values $Z_0 = z$, constitute a family that solves the submartingale problem for (\mathcal{L}, γ) in \mathbb{H} . It only remains to prove the uniqueness in law of this solution. To this end, we will show that any solution Z^* to the submartingale problem is a weak solution of (3.1).

Consider a solution $Z_t^* = (V_t^*, H_t^*) = \omega(t)$ to the submartingale problem (3.3) with $Z_0^* = z \in \overline{\mathbb{H}}$. We will use Theorem 2.4 of [12]. That paper is concerned with processes whose diffusion coefficients a_{ij} are strictly elliptic and drift coefficients b_i are bounded (see page 147, and (i’) and (ii’) on page 159). Our diffusion coefficients are not elliptic and our drift coefficients are not bounded, so, the results from the cited paper do not apply directly to our setting. However, these assumptions are used neither in the definition of the class F in [12, page 161], nor in the statements and proofs of Lemmas 2.3, 2.4, and 2.5, and Theorem 2.4 in that paper. Also, the definition of the submartingale problem in [12] is slightly different than ours, because, in their setting, the integral in (3.3) has the indicator function of \mathbb{H} as a factor in the integrand, which is not an issue in view of Lemma 3.6 above. In order to use Theorem 2.4 of [12], we will show that all functions $f \in C_0^2(\mathbb{R}^2)$ belong to the class F (whose definition we provide next), by modifying an argument from [12].

The class F consists of those function $f : [0, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ which satisfy:

- (i) f is bounded and continuous, and all its first order partial derivatives (both with respect to t and x_i) exist and are bounded and continuous on $[0, \infty) \times \partial \mathbb{H}$,
- (ii) there is a bounded, continuous function Kf such that

$$N_t[f] = f(t, Z_t^*) - \int_0^t \mathbb{I}_{\mathbb{H}} Kf(u, Z_u^*) du$$

is a local martingale in \mathbb{H} .

- (iii) there is a real valued, continuous, non-anticipating process $L_t[f]$ such that:
 - (a) $L_0[f] = 0$, $L_t[f]$ is of locally bounded variation, and $\mathbb{E}(|L_t[f]|) < \infty$ for $t \geq 0$,
 - (b) $N_t[f] - L_t[f]$ is a martingale.
- (iv) if $g \in C_b^{1,2}([0, \infty) \times \mathbb{R}^2)$, that is, partial derivatives in t , and second partial derivatives in x_i are continuous and bounded, then $\bar{f} = f + g$, satisfies (i), (ii) and (iii), and if $(\nabla \bar{f})^T \gamma \geq 0$ on $[0, \infty) \times \mathbb{R}^2$, then $L_t[\bar{f}]$ can be chosen to be non-decreasing.

We will briefly sketch the underlying idea of the proof. By setting $Kf = \mathcal{L}f$, Theorem 2.4 of [12] shows (without using ellipticity or boundedness of drift) that there is a unique, continuous, non-decreasing, non-anticipating “local time” L^* , such that

$$M_t[f] = f(Z_t^*) - \int_0^t \mathcal{L}f(Z_u^*) du - \int_0^t (\nabla f(Z_u^*))^T \gamma dL_u^*$$

is a martingale for each $f \in F$. We will first show that every $f \in C_0^2(\mathbb{R}^2)$ belongs to the class F . Then, by appropriate choices of $f \in C_0^2(\mathbb{R}^2)$, it will be shown that any solution Z^* to the submartingale problem is also a weak solution of (3.1) with L taken as L^* determined by the class of functions F as described above.

We will first show that (i) and (ii) in the definition of F are satisfied by any $f \in C_0^2(\mathbb{R}^2)$. To see this, fix an arbitrary $f \in C_0^2(\mathbb{R}^2)$. First, suppose that f has support in \mathbb{H} , so that $(\nabla f)^T \gamma = 0$ on $\partial\mathbb{H}$. Hence, $S_t[f]$ is a martingale. For general $f \in C_0^2(\mathbb{R}^2)$, let $\mathbb{H}_n = \{(v, h) : h > n^{-1}, |v| < n\}$ and consider $\eta_n \in C_0^\infty(\mathbb{R}^2)$ such that $\eta_n = 1$ in \mathbb{H}_n and $\eta_n = 0$ outside of \mathbb{H}_{n+1} . Define the “smooth localization” $f_n \in C_0^2(\mathbb{R}^2)$ of f as $f_n = f\eta_n$. For any stopping time τ , set $\tau_n = \tau \wedge n$, and $\tau'_n = \inf\{t \geq \tau_n : Z_t^* \notin \mathbb{H}_n\} \wedge n$. It follows that $S_{t \wedge \tau'_n}[f_n] - S_{t \wedge \tau_n}[f_n]$ is a martingale. But

$$S_{t \wedge \tau'_n}[f] - S_{t \wedge \tau_n}[f] = S_{t \wedge \tau'_n}[f_n] - S_{t \wedge \tau_n}[f_n].$$

By taking $n \uparrow \infty$, we obtain that $S_t[f]$ is a local martingale in \mathbb{H} , in the sense of [12, page 158]. Since $Kf = \mathcal{L}f$ is bounded for any $f \in C_0^2(\mathbb{R}^2)$, we see that (i) and (ii) in the definition of F are satisfied.

Next, we will show that (iii) and (iv) are satisfied by any $f \in C_0^2(\mathbb{R}^2)$. If $f \in C_0^2(\mathbb{R}^2)$ satisfies $(\nabla f)^T \gamma \geq 0$ on $\partial\mathbb{H}$, then from the statement of the submartingale problem above, $S_t[f]$ is a locally bounded, continuous submartingale. By the Doob-Meyer decomposition theorem, it follows that there is an integrable, non-decreasing, non-anticipating continuous function $L_t[f]$ such that $L_0[f] = 0$ and $S_t[f] - L_t[f]$ is a martingale. For general $f \in C_0^2(\mathbb{R}^2)$, consider $\phi(v, h) = \arctan(h)$, which is a defining function for \mathbb{H} in the sense of [12] (see page 158), set $\alpha = -\inf\{(\nabla f(y))^T \gamma : y \in \partial\mathbb{H}\}$, and define $f_\alpha = f + \alpha\phi$. It is clear that $(\nabla f_\alpha)^T \gamma \geq 0$, and we can define $L_t[f] = L_t[f_\alpha] - \alpha L_t[\phi]$. We obtain that $L_t[f]$ is a non-anticipating continuous function of bounded variation such that $L_0[f] = 0$, $\mathbb{E}(|L_t[f]|) \leq \mathbb{E}(L_t[f_\alpha]) + \alpha \mathbb{E}(L_t[\phi]) < \infty$, and $S_t[f] - L_t[f]$ is a martingale. This shows (iii) and (iv) in the definition of F , and proves that f belongs to the class F , for any $f \in C_0^2(\mathbb{R}^2)$.

Hence, we can apply [12, Thm. 2.4] to see that there exists a unique, continuous, non-decreasing, non-anticipating process $t \mapsto L_t^*$, such that $L_0^* = 0$, $\mathbb{E}(L_t^*) < \infty$, $dL_t^* = \mathbb{I}_{\partial\mathbb{H}}(Z_t^*) dL_t^*$, and

$$M_t[f] = f(Z_t^*) - \int_0^t \mathcal{L}f(Z_u^*) du - \int_0^t (\nabla f(Z_u^*))^T \gamma dL_u^* \tag{3.6}$$

is a martingale for each $f \in F$, and in particular, for $f \in C_0^2(\mathbb{R}^2)$.

Using that $\mathcal{L}f^2(z) = 2f(z)\mathcal{L}f(z) + |\partial_h f(z)|^2$, we obtain

$$M_t[f^2] = f(Z_t^*)^2 - \int_0^t |\partial_h f(Z_u^*)|^2 du - 2 \int_0^t f(Z_u^*)\mathcal{L}f(Z_u^*) du - 2 \int_0^t f(Z_u^*)(\nabla f(Z_u^*))^T \gamma dL_u^*.$$

Using (3.6) to compute $df(Z_t^*)$, we see that

$$\int_0^t f(Z_u^*) df(Z_u^*) = \int_0^t f(Z_u^*) dM_u[f] + \int_0^t f(Z_u^*)\mathcal{L}f(Z_u^*) du + \int_0^t f(Z_u^*)(\nabla f(Z_u^*))^T \gamma dL_u^*.$$

It follows that

$$f(Z_t^*)^2 - 2 \int_0^t f(Z_u^*) df(Z_u^*) = M_t[f^2] - 2 \int_0^t f(Z_u^*) dM_u[f] + \int_0^t |\partial_h f(Z_u^*)|^2 du.$$

Itô’s formula shows that the left hand side in the equation above equals to $f(Z_0^*)^2 + \langle f(Z^*) \rangle_t$, where the bracket $\langle \cdot \rangle$ stands for quadratic variation. The right hand side has two continuous martingales plus a continuous process of

bounded variation. By uniqueness of the decomposition of continuous semimartingales, we conclude that

$$\langle f(Z^*) \rangle_t = \int_0^t |\partial_h f(Z_u^*)|^2 du. \tag{3.7}$$

Since $f(Z_t^*) - M_t[f]$ is continuous with bounded variation, we see that $\langle M[f] \rangle_t = \langle f(Z^*) \rangle_t$ is also given by (3.7). This formula suggests that there is a Brownian motion B^* such that $dM_t[f] = \partial_h f(Z_t^*) dB_t^*$ for all $f \in C_0^2(\mathbb{R}^2)$. We proceed to prove this by localization.

Let $T^N = \inf\{t \geq 0 : |Z_t^*| > N\}$, and $Z_t^N = Z_{t \wedge T^N}^*$. For $\mu, \lambda \in \mathbb{R}$, let f be a $C_0^2(\mathbb{R}^2)$ function such that $f(v, h) = \mu h + \lambda v$ for $|(v, h)| < N$. From (3.7) we obtain $\langle \mu H^* + \lambda V^* \rangle_{t \wedge T^N} = \mu^2 \cdot (t \wedge T^N)$. By Levy's characterization theorem, there is a Brownian motion B_t^N (in a possibly enlarged probability space) such that $M_t[f] = M_0[f] - \mu B_t^N$ for $t < T^N$. Unravelling our definitions and using (3.6) for f at time $t \wedge T^N$ we obtain

$$\mu H_{t \wedge T^N}^* + \lambda V_{t \wedge T^N}^* = \mu H_0^* + \lambda V_0^* - \mu B_{t \wedge T^N}^N + \int_0^{t \wedge T^N} (\mu V_u^* - g\lambda) du + (\mu + \lambda) dL_{t \wedge T^N}^*.$$

From this, it is direct to see that for each $N \geq 0$, Z_t^N is a weak solution to (3.1) up to time T^N . It follows from Lemma 3.3 that Z^* is a weak solution to (3.1). Since this equation satisfies pathwise uniqueness by Theorem 3.4, it also satisfies uniqueness in law ([8, Ch. 5, Proposition 3.20] with minor modifications for the reflected case), which shows that there is a unique solution to the submartingale problem. \square

4. Hitting time estimates

In this section, we derive some preliminary estimates for hitting times of V and $S - X$. These will be essential in most of the calculations leading to fluctuation results, strong laws and convergence to stationarity.

We will use $[\cdot]$ to denote the greatest integer function. We will write C, C', C'', \dots for finite positive constants, whose values might change from line to line.

Recall that $H_t = S_t - X_t$. Let

$$\begin{aligned} \tau_a^V &= \inf\{t \geq 0 : V_t = a\}, \\ \tau_a^H &= \inf\{t \geq 0 : H_t = a\}, \\ \tau_a^{B,c} &= \inf\{t \geq 0 : B_t + ct = a\}, \\ \sigma(u) &= \inf\{t \geq u : S_t = X_t\}, \end{aligned}$$

where $a, c \in \mathbb{R}, u \geq 0$, with the convention that $\inf \emptyset = \infty$.

Remark 4.1. If $a > V_0$, then $S_{\tau_a^V} = X_{\tau_a^V}$, as otherwise, by path continuity of S and X , there will exist a small time interval $[\tau_a^V - \delta, \tau_a^V]$ for some $\delta > 0$ such that $S_u > X_u$ for all $u \in [\tau_a^V - \delta, \tau_a^V]$ and consequently, the velocity will be strictly decreasing in this interval, which is a contradiction to τ_a^V being the first hitting time of level a by the velocity process V . It is not necessarily true that $S_{\tau_a^V} = X_{\tau_a^V}$ for $a < V_0$.

Remark 4.2. For any initial values $V_0 = v, X_0 = x$ and $S_0 = y, \sigma(0) < \infty$, a.s. To see this, note that on the event $\{\sigma(0) = \infty\}$, the trajectory of S is a downward parabola and the trajectory of X is the trajectory of Brownian motion B shifted by a constant, and staying forever under the parabola. This event has zero probability because $B_t/t \rightarrow 0$, a.s.

It is elementary to check that if $S_0 \geq B_0 = X_0$, then the local time satisfies the usual Skorohod equation (see [8, Lem. 6.14, Ch. 3]),

$$L_t = 0 \vee \sup_{u \leq t} (B_u - S_u), \quad t \geq 0. \tag{4.1}$$

We will be frequently approximating the local time L by using the local time of standard Brownian motion B reflected, via the Skorohod equation, downward on a line of slope a passing through the origin. We will use the following notation,

$$L_t^{(a)} = \sup_{u \leq t} (B_u - au). \tag{4.2}$$

We will use the following well known formulas (see [8, Ch. 2, (9.20); Ch. 3, (5.12) and (5.13)]). If $B_0 = 0$ then

$$\mathbb{P}\left(\sup_{s \leq t} B_s \geq x\right) = 2 \int_x^\infty \frac{1}{\sqrt{2\pi t}} e^{-u^2/(2t)} du \leq \frac{2\sqrt{t}}{\sqrt{2\pi x}} e^{-x^2/(2t)}, \quad t, x > 0, \tag{4.3}$$

$$\mathbb{P}(\tau_a^{B,m} \in dt) = \frac{|a|}{\sqrt{2\pi t^3}} \exp(-(a - mt)^2/(2t)) dt, \quad t \geq 0, \tag{4.4}$$

$$\mathbb{P}(\tau_a^{B,m} < \infty) = \exp(ma - |ma|). \tag{4.5}$$

The following two lemmas contain estimates for the hitting times of different levels by the velocity process V_t , for starting points in different ranges of values.

Lemma 4.3. *Assume that $H_0 = 0$. Then for $0 < a_1 < a_2$, and $t \geq 2(a_2 - a_1)/a_1$,*

$$\mathbb{P}(\tau_{-g-a_1}^V > t \mid V_0 = -g - a_2) \leq \frac{4(a_2 - a_1)}{((a_2 - a_1) + a_1 t)a_1 \sqrt{2\pi t}} e^{-a_1^2 t/8}. \tag{4.6}$$

Proof. It follows easily from (4.1)–(4.2) that, assuming that $V_0 = -g - a_2$ and $t < \tau_{-g-a_1}^V$, we have $L_t \geq L_t^{(-g-a_1)} = \sup_{u \leq t} (B_u + (g + a_1)u)$. We will use similar inequalities between L_t and $L_t^{(m)}$ later in the paper a number of times, without explicitly referring to (4.1)–(4.2). We have,

$$\begin{aligned} &\mathbb{P}(\tau_{-g-a_1}^V > t \mid V_0 = -g - a_2) \\ &= \mathbb{P}(L_s - gs < a_2 - a_1 \text{ for } s \leq t) \\ &\leq \mathbb{P}\left(\sup_{u \leq s} (B_u + (g + a_1)u) - gs < a_2 - a_1 \text{ for } s \leq t\right) \\ &\leq \mathbb{P}((B_s + (g + a_1)s) - gs < a_2 - a_1 \text{ for } s \leq t) \\ &= \mathbb{P}(B_s + a_1 s < a_2 - a_1 \text{ for } s \leq t) \\ &= \mathbb{P}(\tau_{a_2-a_1}^{B,a_1} > t). \end{aligned}$$

This and (4.4) imply that,

$$\mathbb{P}(\tau_{-g-a_1}^V > t \mid V_0 = -g - a_2) \leq \int_t^\infty \frac{a_2 - a_1}{\sqrt{2\pi u^3}} \exp\left(-\frac{((a_2 - a_1) - a_1 u)^2}{2u}\right) du.$$

Making a change of variable from u to $z = \frac{a_2 - a_1}{\sqrt{u}} - a_1 \sqrt{u}$, and using the fact that $t \leq u$, we see that

$$\begin{aligned} &\mathbb{P}(\tau_{-g-a_1}^V > t \mid V_0 = -g - a_2) \\ &\leq \int_t^\infty \frac{a_2 - a_1}{\sqrt{2\pi u^3}} \exp\left(-\frac{((a_2 - a_1) - a_1 u)^2}{2u}\right) du \\ &= (a_2 - a_1) \int_{-\infty}^{\frac{a_2 - a_1}{\sqrt{t}} - a_1 \sqrt{t}} u^{-3/2} \left(\frac{1}{2}(a_2 - a_1)u^{-3/2} + \frac{1}{2}a_1 u^{-1/2}\right)^{-1} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \end{aligned}$$

$$\begin{aligned}
 &= (a_2 - a_1) \int_{-\infty}^{\frac{a_2 - a_1}{\sqrt{t}} - a_1 \sqrt{t}} \left(\frac{1}{2}(a_2 - a_1) + \frac{1}{2}a_1 u \right)^{-1} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
 &\leq \frac{2(a_2 - a_1)}{(a_2 - a_1) + a_1 t} \int_{-\infty}^{\frac{a_2 - a_1}{\sqrt{t}} - a_1 \sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz.
 \end{aligned}$$

Note that, when $t \geq 2(a_2 - a_1)/a_1$, $\frac{a_2 - a_1}{\sqrt{t}} \leq \frac{a_1 \sqrt{t}}{2}$. Thus, the last estimate and (4.3) yield

$$\begin{aligned}
 \mathbb{P}(\tau_{-g-a_1}^V > t \mid V_0 = -g - a_2) &\leq \frac{2(a_2 - a_1)}{(a_2 - a_1) + a_1 t} \int_{-\infty}^{\frac{a_2 - a_1}{\sqrt{t}} - a_1 \sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
 &\leq \frac{2(a_2 - a_1)}{(a_2 - a_1) + a_1 t} \int_{-\infty}^{-a_1 \sqrt{t}/2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
 &\leq \frac{2(a_2 - a_1)}{(a_2 - a_1) + a_1 t} \cdot \frac{2}{\sqrt{2\pi} a_1 \sqrt{t}} e^{-a_1^2 t/8},
 \end{aligned}$$

which proves the lemma. □

Lemma 4.4. Assume that $H_0 = 0$. Then for $0 < a_1 < a_2$,

(i) If $a_1 > g$, then for $t \geq 2(a_2 - a_1)/g$,

$$\mathbb{P}(\tau_{-g+a_1}^V > t \mid V_0 = -g + a_2) \leq e^{-g(a_1-g)t}. \quad (4.7)$$

(ii) If $a_1 \leq g$, then for $t \geq 2(a_2 - a_1)/a_1$,

$$\mathbb{P}(\tau_{-g+a_1}^V > t \mid V_0 = -g + a_2) \leq \frac{4}{a_1 \sqrt{2\pi t}} e^{-a_1^2 t/8}. \quad (4.8)$$

Proof. For $t < \tau_{-g+a_1}^V$, the following inequality holds, $L_t \leq L_t^{(-g+a_1)} = \sup_{u \leq t} (B_u - (a_1 - g)u)$. Therefore for $a_1 > g$ and $t \geq 2(a_2 - a_1)/g$,

$$\begin{aligned}
 \mathbb{P}(\tau_{-g+a_1}^V > t \mid V_0 = -g + a_2) &= \mathbb{P}(L_s - gs > -(a_2 - a_1) \text{ for } s \leq t) \\
 &\leq \mathbb{P}\left(\sup_{u \leq s} (B_u - (a_1 - g)u) - gs > -(a_2 - a_1) \text{ for } s \leq t\right) \\
 &\leq \mathbb{P}\left(\sup_{u \leq t} (B_u - (a_1 - g)u) - gt > -(a_2 - a_1)\right) \\
 &\leq \mathbb{P}\left(\sup_{u < \infty} (B_u - (a_1 - g)u) > gt - (a_2 - a_1)\right) \\
 &= \mathbb{P}(\tau_{gt-(a_2-a_1)}^{B, -(a_1-g)} < \infty).
 \end{aligned}$$

We use (4.5) and the assumption that $a_2 - a_1 \leq gt/2$ to conclude that

$$\mathbb{P}(\tau_{-g+a_1}^V > t \mid V_0 = -g + a_2) \leq \exp(-2(a_1 - g)(gt - (a_2 - a_1))) \leq e^{-g(a_1-g)t}.$$

This proves (i).

For $a_1 \leq g$ and $t \geq 2(a_2 - a_1)/a_1$,

$$\begin{aligned}
 \mathbb{P}(\tau_{-g+a_1}^V > t \mid V_0 = -g + a_2) &= \mathbb{P}(L_s - gs > -(a_2 - a_1) \text{ for } s \leq t) \\
 &\leq \mathbb{P}\left(\sup_{u \leq s} (B_u + (g - a_1)u) - gs > -(a_2 - a_1) \text{ for } s \leq t\right)
 \end{aligned}$$

$$\begin{aligned} &\leq \mathbb{P}\left(\sup_{u \leq t} B_u + (g - a_1)t - gt > -(a_2 - a_1)\right) \\ &\leq \mathbb{P}\left(\sup_{u \leq t} B_u > a_1 t/2\right). \end{aligned}$$

This and (4.3) show that

$$\mathbb{P}(\tau_{-g+a_1}^V > t \mid V_0 = -g + a_2) \leq \frac{4}{a_1 \sqrt{2\pi t}} e^{-a_1^2 t/8},$$

which proves (ii). □

The following lemma gives a uniform control over $\sigma(t)$ (the first time the Brownian particle and the inert particle meet after time t) over all times in a large interval.

Lemma 4.5. *For every $\delta > 0$ and C_0 , we can find positive constants C_1, C_2, a_0 such that for all $a \geq a_0$ and all $v \in [-g - \delta^2 a^2/8, -g + \sqrt{g}\delta a/4]$, the following holds for $V_0 = v, H_0 = 0$, and any $m \geq 1$:*

$$\mathbb{P}\left(\sigma(t) > t + 3\frac{\delta a}{\sqrt{g}} \text{ for some } t \leq C_0 a^m \wedge \tau_{-g+\sqrt{g}\delta a/4}^V \wedge \tau_{-g-\delta^2 a^2/8}^V\right) \leq C_1 a^{m-1} e^{-C_2 a^3}.$$

Proof. Fix any $\delta > 0$ and $v \in [-g - \frac{\delta^2 a^2}{8}, -g + \frac{\sqrt{g}\delta a}{4}]$. Assume that $V_0 = v$ and $H_0 = 0$. Note that for $t \leq \tau_{-g+\sqrt{g}\delta a/4}^V$,

$$\begin{aligned} L_t - gt &\leq \frac{\delta^2 a^2}{8} + \frac{\sqrt{g}\delta a}{4}, \\ S_t &\leq \left(-g + \frac{\sqrt{g}\delta a}{4}\right)t, \end{aligned}$$

where we used $V_0 = v \geq -g - \frac{\delta^2 a^2}{8}$ to obtain the first inequality.

These together yield

$$S_t - X_t = S_t - B_t + L_t \leq -B_t + \frac{\sqrt{g}\delta a}{4}t + \frac{\delta^2 a^2}{8} + \frac{\sqrt{g}\delta a}{4}.$$

Thus we have

$$\begin{aligned} \mathbb{P}\left(S_{\delta a/\sqrt{g}} - X_{\delta a/\sqrt{g}} > \delta^2 a^2, \frac{\delta a}{\sqrt{g}} \leq \tau_{-g+\sqrt{g}\delta a/4}^V\right) &\leq \mathbb{P}\left(-B_{\delta a/\sqrt{g}} + \frac{3\delta^2 a^2}{8} + \frac{\sqrt{g}\delta a}{4} > \delta^2 a^2\right) \\ &\leq \mathbb{P}\left(-B_{\delta a/\sqrt{g}} > \frac{9}{16}\delta^2 a^2\right) \leq e^{-\sqrt{g}\delta^3 a^3/8}. \end{aligned} \tag{4.9}$$

The second inequality holds for large enough a (depending on δ) and the last inequality follows from (4.3).

Suppose that the following event holds

$$\left\{ \sigma\left(\frac{\delta a}{\sqrt{g}}\right) > 3\frac{\delta a}{\sqrt{g}}, S_{\delta a/\sqrt{g}} - X_{\delta a/\sqrt{g}} \leq \delta^2 a^2, \frac{\delta a}{\sqrt{g}} \leq \tau_{-g+\sqrt{g}\delta a/4}^V \right\}. \tag{4.10}$$

Then S is a parabola on the interval $[\frac{\delta a}{\sqrt{g}}, \frac{3\delta a}{\sqrt{g}}]$. If $w = V_{\delta a/\sqrt{g}}$ then the parabola increment over this interval is

$$-\frac{1}{2}g\left(\frac{2\delta a}{\sqrt{g}}\right)^2 + w\frac{\delta a}{\sqrt{g}} \leq -\frac{1}{2}g\left(\frac{2\delta a}{\sqrt{g}}\right)^2 + \left(-g + \frac{\sqrt{g}\delta a}{4}\right)\frac{\delta a}{\sqrt{g}} \leq -\frac{3}{2}\delta^2 a^2.$$

Since $S_{\delta a/\sqrt{g}} - X_{\delta a/\sqrt{g}} \leq \delta^2 a^2$ and X stays below S on the interval $[\frac{\delta a}{\sqrt{g}}, \frac{3\delta a}{\sqrt{g}}]$, the following event must hold,

$$\{B_{3\delta a/\sqrt{g}} - B_{\delta a/\sqrt{g}} \leq -\delta^2 a^2/2\}.$$

Recalling (4.10), we conclude that

$$\mathbb{P}\left(\sigma\left(\frac{\delta a}{\sqrt{g}}\right) > 3\frac{\delta a}{\sqrt{g}}, S_{\delta a/\sqrt{g}} - X_{\delta a/\sqrt{g}} \leq \delta^2 a^2, \frac{\delta a}{\sqrt{g}} \leq \tau_{-g+\sqrt{g}\delta a/4}^V\right) \leq \mathbb{P}(B_{\frac{3\delta a}{\sqrt{g}}} - B_{\frac{\delta a}{\sqrt{g}}} \leq -\delta^2 a^2/2).$$

This, (4.9) and (4.3) yield for large a ,

$$\begin{aligned} & \mathbb{P}\left(\sigma\left(\frac{\delta a}{\sqrt{g}}\right) > 3\frac{\delta a}{\sqrt{g}}, \frac{\delta a}{\sqrt{g}} \leq \tau_{-g+\sqrt{g}\delta a/4}^V\right) \\ & \leq \mathbb{P}\left(S_{\delta a/\sqrt{g}} - X_{\delta a/\sqrt{g}} > \delta^2 a^2, \frac{\delta a}{\sqrt{g}} \leq \tau_{-g+\sqrt{g}\delta a/4}^V\right) \\ & \quad + \mathbb{P}\left(\sigma\left(\frac{\delta a}{\sqrt{g}}\right) > 3\frac{\delta a}{\sqrt{g}}, S_{\delta a/\sqrt{g}} - X_{\delta a/\sqrt{g}} \leq \delta^2 a^2, \frac{\delta a}{\sqrt{g}} \leq \tau_{-g+\sqrt{g}\delta a/4}^V\right) \\ & \leq e^{-\sqrt{g}\delta^3 a^3/8} + \mathbb{P}(B_{3\delta a/\sqrt{g}} - B_{\delta a/\sqrt{g}} \leq -\delta^2 a^2/2) \\ & \leq e^{-\sqrt{g}\delta^3 a^3/8} + e^{-\sqrt{g}\delta^3 a^3/16}. \end{aligned} \tag{4.11}$$

Define stopping times $T_0 = 0$ and

$$T_{k+1} = \inf\{t \geq T_k + \delta a/\sqrt{g} : H_t = 0\},$$

for $k \geq 0$. Then by (4.11), the strong Markov property applied at T_k , and Remark 3.1, for large a ,

$$\begin{aligned} & \mathbb{P}\left(T_{k+1} - T_k > 3\frac{\delta a}{\sqrt{g}}, \tau_{-g+\sqrt{g}\delta a/4}^V \geq T_k + \delta a/\sqrt{g}, \tau_{-g-\delta^2 a^2/8}^V \geq T_k\right) \\ & \leq \mathbb{P}\left(T_{k+1} - T_k > 3\frac{\delta a}{\sqrt{g}}, \tau_{-g+\sqrt{g}\delta a/4}^V \geq T_k + \delta a/\sqrt{g}\right) \\ & \leq e^{-\sqrt{g}\delta^3 a^3/8} + e^{-\sqrt{g}\delta^3 a^3/16}. \end{aligned} \tag{4.12}$$

Consider any $m \geq 1$. Suppose that there is $t_1 \in [0, C_0 a^m \wedge \tau_{-g+\sqrt{g}\delta a/4}^V \wedge \tau_{-g-\delta^2 a^2/8}^V]$ such that $\sigma(t_1) > t_1 + 3\frac{\delta a}{\sqrt{g}}$. Then $t_1 \in [0, C_0 a^m]$ and, therefore, we can find $0 \leq k_1 \leq C_0 \sqrt{g} a^{m-1}/\delta$ with $T_{k_1} \leq t_1 \leq T_{k_1+1}$, because $T_{k+1} - T_k \geq \delta a/\sqrt{g}$ for all k .

It follows from the definition of T_{k+1} that $\sigma(t_1) - t_1 \leq T_{k_1+1} - T_{k_1}$. Since $V_0 \leq -g + \frac{\sqrt{g}\delta a}{4}$ and $H_0 = 0$, the processes S and X must take the same value at the time $\tau_{-g+\sqrt{g}\delta a/4}^V$, by Remark 4.1. Hence, if $\tau_{-g+\sqrt{g}\delta a/4}^V \in [t_1, T_{k_1+1} + \delta a/\sqrt{g}]$, then

$$\sigma(t_1) \leq \tau_{-g+\sqrt{g}\delta a/4}^V \leq T_{k_1} + \delta a/\sqrt{g} \leq t_1 + \delta a/\sqrt{g},$$

which contradicts the assumption that $\sigma(t_1) > t_1 + 3\frac{\delta a}{\sqrt{g}}$. Thus, if $t_1 \leq \tau_{-g+\sqrt{g}\delta a/4}^V$ and $\sigma(t_1) > t_1 + 3\frac{\delta a}{\sqrt{g}}$ then $\tau_{-g+\sqrt{g}\delta a/4}^V \geq T_{k_1} + \delta a/\sqrt{g}$. These observations and (4.12) imply that

$$\mathbb{P}\left(\sigma(t) > t + 3\frac{\delta a}{\sqrt{g}} \text{ for some } t \leq a^m \wedge \tau_{-g+\sqrt{g}\delta a/4}^V \wedge \tau_{-g-\delta^2 a^2/8}^V\right)$$

$$\begin{aligned} &\leq \sum_{k=0}^{\lfloor C_0\sqrt{g}a^{m-1}/\delta \rfloor} \mathbb{P}\left(\sigma(t) > t + 3\frac{\delta a}{\sqrt{g}}, t \leq \tau_{-g+\sqrt{g}\delta a/4}^V \wedge \tau_{-g-\delta^2 a^2/8}^V \text{ for some } t \in [T_k, T_{k+1}]\right) \\ &\leq \sum_{k=0}^{\lfloor C_0\sqrt{g}a^{m-1}/\delta \rfloor} \mathbb{P}\left(T_{k+1} - T_k > 3\frac{\delta a}{\sqrt{g}}, \tau_{-g+\sqrt{g}\delta a/4}^V \geq T_k + \delta a/\sqrt{g}, \tau_{-g-\delta^2 a^2/8}^V \geq T_k\right) \\ &\leq (\lfloor C_0\sqrt{g}a^{m-1}/\delta \rfloor + 1)(e^{-\sqrt{g}\delta^3 a^3/8} + e^{-\sqrt{g}\delta^3 a^3/16}). \end{aligned}$$

This proves the lemma. □

The following lemma tells us that the probability of the velocity staying inside the interval $[-g - a, -g + a]$ for all times up to t decays exponentially with t .

Lemma 4.6. *For any $a > 0$, there exists $p_0 \in (0, 1)$ depending on a such that for any integer $m \geq 1$,*

$$\sup_{v \in [-g-a, -g+a]} \mathbb{P}(|V_t + g| \leq a \text{ for all } t \in [0, m(1 + 2a/g)] \mid V_0 = v, H_0 = 0) \leq p_0^m. \tag{4.13}$$

Proof. Assume without loss of generality that $B_0 = 0$. Note that $L_t = \sup_{u \leq t} (B_u - S_u) \geq B_t - S_t$. If $V_u \in [-g - a, -g + a]$ for all $0 \leq u \leq t$, then $S_t \leq (-g + a)t$. This gives $L_t \geq B_t + (g - a)t$. Thus,

$$V_t = V_0 + L_t - gt \geq -g - a + B_t + (g - a)t - gt = B_t - at - g - a,$$

and, therefore,

$$\{|V_t + g| \leq a \text{ for all } t \in [0, 1]\} \subset \{V_1 + g \leq a\} \subset \{B_1 - 2a \leq a\} = \{B_1 \leq 3a\}.$$

Let $p_0 = \mathbb{P}(B_1 \leq 3a) < 1$. Then,

$$\begin{aligned} &\sup_{v \in [-g-a, -g+a]} \mathbb{P}(|V_t + g| \leq a \text{ for all } t \in [0, 1] \mid V_0 = v, H_0 = 0) \\ &\leq \mathbb{P}(B_1 \leq 3a) = p_0. \end{aligned}$$

If $V_u \in [-g - a, -g + a]$ for all $0 \leq u \leq 1 + 3a/g$, then $\sigma(1) \leq 1 + 2a/g$, for otherwise there would be $t \in [1 + 2a/g, 1 + 3a/g)$ with $V_t < -g - a$. Thus, applying the strong Markov property at $\sigma(1)$, in view of Remark 3.1, we have for any $m \geq 2$,

$$\begin{aligned} &\sup_{v \in [-g-a, -g+a]} \mathbb{P}(|V_t + g| \leq a \text{ for all } t \in [0, m(1 + 2a/g)] \mid V_0 = v, H_0 = 0) \\ &\leq \sup_{v \in [-g-a, -g+a]} \mathbb{P}(|V_t + g| \leq a \text{ for all } t \in [0, 1] \mid V_0 = v, H_0 = 0) \\ &\quad \times \sup_{v \in [-g-a, -g+a]} \mathbb{P}(|V_t + g| \leq a \text{ for all } t \in [0, (m - 1)(1 + 2a/g)] \mid V_0 = v, H_0 = 0) \\ &\leq p_0 \times \sup_{v \in [-g-a, -g+a]} \mathbb{P}(|V_t + g| \leq a \text{ for all } t \in [0, (m - 1)(1 + 2a/g)] \mid V_0 = v, H_0 = 0). \end{aligned}$$

Recursively applying the same argument, we get (4.13). □

Lemma 4.7. *There exist positive constants a_0 and C_1, C_2 such that for any $a \geq a_0$:*

$$\mathbb{P}(\tau_{-g+a}^V < \tau_{-g+1}^V \mid V_0 = v, H_0 = 0) \leq C_1 e^{-C_2 a^2}$$

uniformly over all $v \in [-g + 2, -g + a/2]$.

Proof. For $a \geq 4$, applying the strong Markov property at time τ_{-g+a-1}^V and Remarks 3.1 and 4.1, we have for any $v \in [-g+2, -g+a-1]$,

$$\begin{aligned} & \mathbb{P}(\tau_{-g+a}^V < \tau_{-g+1}^V \mid V_0 = v, H_0 = 0) \\ &= \mathbb{P}(\tau_{-g+a-1}^V < \tau_{-g+1}^V \mid V_0 = v, H_0 = 0) \mathbb{P}(\tau_{-g+a}^V < \tau_{-g+1}^V \mid V_0 = -g+a-1, H_0 = 0). \end{aligned} \quad (4.14)$$

We estimate the second probability on the right hand side of the above equation as follows,

$$\begin{aligned} & \mathbb{P}(\tau_{-g+a}^V < \tau_{-g+1}^V \mid V_0 = -g+a-1, H_0 = 0) \\ &= \sum_{k=2}^{a-1} \mathbb{P}(\tau_{-g+k}^V < \tau_{-g+a}^V \leq \tau_{-g+k-1}^V \mid V_0 = -g+a-1, H_0 = 0). \end{aligned} \quad (4.15)$$

If $V_0 = -g+k$ and $t \leq \tau_{-g+k-1}^V$ then $L_t \leq L_t^{(-g+k-1)} = \sup_{u \leq t} (B_u + (g-k+1)u)$. Thus, for $2 \leq k \leq a-1$,

$$\begin{aligned} & \mathbb{P}(\tau_{-g+k}^V < \tau_{-g+a}^V \leq \tau_{-g+k-1}^V \mid V_0 = -g+a-1, H_0 = 0) \\ & \leq \mathbb{P}(\tau_{-g+a}^V \leq \tau_{-g+k-1}^V \mid V_0 = -g+k, H_0 = 0) \\ & = \mathbb{P}(\inf\{t \geq 0 : L_t - gt = a-k\} < \inf\{t \geq 0 : L_t - gt = -1\} \mid V_0 = -g+k, H_0 = 0) \\ & \leq \mathbb{P}(\inf\{t \geq 0 : L_t^{(-g+k-1)} - gt = a-k\} \\ & < \inf\{t \geq 0 : L_t^{(-g+k-1)} - gt = -1\} \mid V_0 = -g+k, H_0 = 0) \\ & \leq \mathbb{P}(\inf\{t \geq 0 : L_t^{(-g+k-1)} - gt = a-k\} < \infty \mid V_0 = -g+k, H_0 = 0) \\ & = \mathbb{P}(\inf\{t \geq 0 : B_t + (g-k+1)t - gt = a-k\} < \infty) \\ & = \mathbb{P}(\inf\{t \geq 0 : B_t - (k-1)t = a-k\} < \infty) \\ & = e^{-2(k-1)(a-k)}, \end{aligned}$$

for sufficiently large a . In the first step, we used the strong Markov property at the stopping time $U = \inf\{t \geq \tau_{-g+k}^V : V_t = -g+k, S_t = X_t\}$ which satisfies $\tau_{-g+k}^V \leq U < \tau_{-g+a}^V$ on the event $\{\tau_{-g+k}^V < \tau_{-g+a}^V \leq \tau_{-g+k-1}^V\}$, by Remark 4.1. We also used Remark 3.1. For the last step, we used (4.5). For $a \geq 4$, $(k-1)(a-k) \geq a/2$ for $2 \leq k \leq a-1$ and thus, substituting the above estimate back into (4.15), we get

$$\mathbb{P}(\tau_{-g+a}^V < \tau_{-g+1}^V \mid V_0 = -g+a-1, H_0 = 0) \leq (a-2)e^{-a}$$

for $a \geq 4$. This, along with (4.14), gives us for $v \in [-g+2, -g+a-1]$,

$$\mathbb{P}(\tau_{-g+a}^V < \tau_{-g+1}^V \mid V_0 = v, H_0 = 0) \leq \mathbb{P}(\tau_{-g+a-1}^V < \tau_{-g+1}^V \mid V_0 = v, H_0 = 0)(a-2)e^{-a}.$$

For $a \geq 10$, we apply the above estimate inductively with a replaced by $a/2, a/2+1, \dots, a/2+k_*$, where k_* is the largest integer such that $a/2+k_* \leq a$, to obtain

$$\sup_{v \in [-g+2, -g+a/2]} \mathbb{P}(\tau_{-g+a}^V < \tau_{-g+1}^V \mid V_0 = v, H_0 = 0) \leq C(a-2)^{a/2+1} \exp\left(-\sum_{k=\lfloor a/2 \rfloor}^{\lfloor a \rfloor} k\right),$$

for some positive constant C , which implies the bound claimed in the lemma. \square

Remark 4.8. We will sketch an argument showing that for any initial values $V_0 = v, X_0 = x$ and $S_0 = y$, and any $z \in \mathbb{R}$ we have $\tau_z^V < \infty$, a.s.

In view of Remark 4.2, we can assume that $H_0 = 0$. By Lemma 4.6, the process V cannot stay in any bounded interval forever, a.s. By Lemmas 4.3 and 4.4, the probability that V converges to ∞ or $-\infty$ is 0, a.s.

Suppose that $\limsup_{t \rightarrow \infty} V_t = \infty$ and $\liminf_{t \rightarrow \infty} V_t$ is finite with positive probability. Let $a \in \mathbb{R}$ and $\varepsilon > 0$ be such that $\mathbb{P}(\liminf_{t \rightarrow \infty} V_t \in (a, a + 1)) > \varepsilon$. The methods used in our proofs show easily that for some $p > 0$, all $v \in (a, a + 1)$, and all $x < y$, if $V_0 = v$, $S_0 = y$ and $X_0 = x$ then $\tau_{a-3}^V < \infty$ with probability greater than p . It is now standard to prove that on the event where V visits $(a, a + 1)$ infinitely often, it has to visit $(a - 2, a - 1)$ infinitely often as well, and, therefore, $\liminf_{t \rightarrow \infty} V_t$ cannot lie in $(a, a + 1)$ with positive probability, a contradiction. A similar argument shows that the event that $\liminf_{t \rightarrow \infty} V_t = -\infty$ and $\limsup_{t \rightarrow \infty} V_t$ is finite has probability 0.

5. Renewal times

We will define several sequences of stopping times and derive tail estimates for them that will help us estimate the fluctuations of the velocity process V_t and the gap process $S_t - X_t$. The path $\{Z_s : s \leq t\}$ will be decomposed into *cycles* between consecutive renewal times defined in this section. In Section 7, fluctuation results will be established for these cycles. These, in turn, will yield global fluctuation results and strong laws for S_t and X_t stated in Theorem 2.2 and Theorem 2.3.

We assume that the starting configuration is $V_0 = -g$ and $H_0 = 0$, although the results that follow will not depend on this choice. Fix $a_0 > g$. We define a sequence of *renewal times* $\{\zeta_k\}_{k \geq 0}$ as follows. Let $\zeta_0 = 0$ and for $k \geq 0$,

$$\eta_k = \inf\{t \geq \zeta_k : |V_t + g| = a_0 + 2\}, \quad (5.1)$$

$$\zeta_{k+1} = \inf\{t \geq \eta_k : V_t = -g \text{ and } S_t = X_t\}. \quad (5.2)$$

Define $\alpha_{-1} = 0$. If $\tau_{-g+a_0+2}^V < \tau_{-g-a_0-2}^V$, define $\alpha_k = 0$ for all $k \geq 0$ and let $N^- = 0$. On the event $\{\tau_{-g-a_0-2}^V < \tau_{-g+a_0+2}^V\}$, define $\alpha_0 = \tau_{-g-a_0-2}^V$. For $k \geq 0$, if $V_{\alpha_{3k}} = -g - a_0 - 2$, then define

$$\begin{aligned} \alpha_{3k+1} &= \inf\{t \geq \alpha_{3k} : S_t = X_t\}, \\ \alpha_{3k+2} &= \inf\{t \geq \alpha_{3k+1} : V_t = -g - a_0 - 1\}, \\ \alpha_{3k+3} &= \inf\{t \geq \alpha_{3k+2} : V_t = -g \text{ or } -g - a_0 - 2\}. \end{aligned} \quad (5.3)$$

If $V_{\alpha_{3k}} = -g$, then define $\alpha_j = \alpha_{3k}$ for all $j \geq 3k$. Define $N^- = \inf\{k \geq 1 : V_{\alpha_{3k}} = -g\}$. This corresponds to the first hitting of $-g$ by the velocity after time $\tau_{-g-a_0-2}^V$. By Remark 4.1, if $V_{\alpha_{3k}} = -g$ then $S_{\alpha_{3k}} = X_{\alpha_{3k}}$. Thus, on the event $\{\tau_{-g-a_0-2}^V < \tau_{-g+a_0+2}^V\}$, we have that $\zeta_1 = \alpha_{3N^-}$. Also, note that $V_{\alpha_{3k+1}} \leq -g - a_0 - 2$, and $H_{\alpha_{3k+2}} = 0$ for $k < N^-$.

Define $\beta_{-1} = 0$. If $\tau_{-g-a_0-2}^V < \tau_{-g+a_0+2}^V$, define $\beta_k = 0$ for all $k \geq 0$ and let $N^+ = 0$. On the event $\{\tau_{-g+a_0+2}^V < \tau_{-g-a_0-2}^V\}$, define $\beta_0 = \tau_{-g+a_0+2}^V$. For $k \geq 0$, if $V_{\beta_{3k}} = -g + a_0 + 2$, then define

$$\begin{aligned} \beta_{3k+1} &= \inf\{t \geq \beta_{3k} : V_t = -g + a_0 + 1\}, \\ \beta_{3k+2} &= \inf\{t \geq \beta_{3k+1} : S_t = X_t \text{ or } V_t = -g\}, \\ \beta_{3k+3} &= \inf\{t \geq \beta_{3k+2} : V_t = -g \text{ or } -g + a_0 + 2\}. \end{aligned}$$

Otherwise, if $V_{\beta_{3k}} = -g$, define $\beta_j = \beta_{3k}$ for all $j \geq 3k$. Define $N^+ = \inf\{k \geq 1 : V_{\beta_{3k}} = -g\}$. This corresponds to the first down-crossing of the velocity below the level $-g$ after $\tau_{-g+a_0+2}^V$. But with positive probability, $S_{\beta_{3N^+}} > X_{\beta_{3N^+}}$. Thus, to reach the renewal time ζ_1 , we will define a further set of stopping times $\{\tilde{\alpha}_k\}_{k \geq -1}$ till the first time the velocity hits $-g$ again from below and thus the processes S and X coincide.

If $\tau_{-g-a_0-2}^V < \tau_{-g+a_0+2}^V$, we define $\tilde{\alpha}_k = 0$ for all $k \geq -1$ and we let $\tilde{N}^- = 0$. On the event $\{\tau_{-g+a_0+2}^V < \tau_{-g-a_0-2}^V\}$, let

$$\begin{aligned} \tilde{\alpha}_{-1} &= \inf\{t \geq \beta_{3N^+} : S_t = X_t \text{ or } V_t = -g - a_0 - 2\}, \\ \tilde{\alpha}_0 &= \inf\{t \geq \tilde{\alpha}_{-1} : V_t = -g \text{ or } -g - a_0 - 2\}. \end{aligned}$$

For $k \geq 0$, if $V_{\tilde{\alpha}_{3k}} = -g - a_0 - 2$, define $\tilde{\alpha}_i$ for $i = 3k + 1, 3k + 2, 3k + 3$ exactly as in (5.3) replacing the α 's with $\tilde{\alpha}$'s. If $V_{\tilde{\alpha}_{3k}} = -g$, define $\tilde{\alpha}_j = \tilde{\alpha}_{3k}$ for all $j \geq 3k$. Let $\tilde{N}^- = \inf\{k \geq 0 : V_{\tilde{\alpha}_{3k}} = -g\}$. Thus, on the event $\{\tau_{-g+a_0+2}^V < \tau_{-g-a_0-2}^V\}$, we have $\zeta_1 = \tilde{\alpha}_{3\tilde{N}^-}$.

We will argue that all stopping times α_k are finite a.s. First of all, by Lemma 4.6,

$$\mathbb{P}(\{\tau_{-g-a_0-2}^V < \infty\} \cup \{\tau_{-g+a_0+2}^V < \infty\}) = 1,$$

so at least one of the sequences $\{\alpha_k\}$ or $\{\beta_k\}$ is non-trivial. Suppose that $\alpha_{3k} < \infty$, a.s. Then $\alpha_{3k+1} < \infty$, a.s., by Remark 4.2. If $\alpha_{3k+1} < \infty$, a.s. then $\alpha_{3k+2} < \infty$, a.s., by Remark 4.8. Since the argument in that remark was only sketched, note that we can alternatively apply Lemma 4.3 which has a detailed proof. Finally, if $\alpha_{3k+2} < \infty$, a.s. then $\alpha_{3k+3} < \infty$, a.s., by Lemma 4.6. A similar argument applies to β_k 's and $\tilde{\alpha}_k$'s.

In the following lemma, we will show that the distribution of ζ_1 has a rapidly decaying tail and thus has finite moments of all orders. We remark here that to prove the main results in this article, we only need the first moment of ζ_1 to be finite. Due to the degeneracy and high correlations in the system, we did not find a direct proof of this result and adopted a ‘‘hands-on’’ approach via controlling excursions. Nevertheless, besides being of independent interest, we believe that the tail estimate of ζ_1 will be useful in obtaining the rate of convergence to stationarity, which we hope to address in a subsequent article.

Lemma 5.1. *Suppose that $V_0 = -g$ and $H_0 = 0$. There exist constants $C, C' > 0$ such that for all $t \geq 0$,*

$$\mathbb{P}(\zeta_1 > t) \leq C e^{-C't^{1/2}}.$$

It follows that for any integer $n \geq 1$, $\mathbb{E}(\zeta_1^n) < \infty$.

Proof. In this proof, we will assume that t is sufficiently large without explicitly mentioning it every time.

First, we consider the event $\{\tau_{-g-a_0-2}^V < \tau_{-g+a_0+2}^V\}$. Write

$$p^- = \mathbb{P}(\tau_{-g-a_0-2}^V < \tau_{-g}^V \mid V_0 = -g - a_0 - 1, H_0 = 0).$$

Note that if $V_0 = -g - a_0 - 1, H_0 = 0$, then for $t \leq \tau_{-g}^V$, we have $L_t \geq L_t^{(-g)}$. Note that

$$\mathbb{P}(\inf\{t \geq 0 : L_t^{(-g)} - gt = -1\} \leq 1/g) \leq \mathbb{P}\left(\sup_{t \leq 1/g} (B_t + gt) - g(1/g) < -1\right) = 0.$$

This implies that

$$\begin{aligned} p^- &\leq \mathbb{P}(\inf\{t \geq 0 : L_t^{(-g)} - gt = -1\} < \inf\{t \geq 0 : L_t^{(-g)} - gt = a_0 + 1\}) \\ &\leq \mathbb{P}\left(\sup_{t \leq 1/g} (B_t + gt) - g(1/g) \leq a_0 + 1\right) < 1. \end{aligned}$$

Hence, for any integer $n \geq 1$, we can apply the strong Markov property successively at $\alpha_{3n-1}, \alpha_{3n-4}, \dots$ to get

$$\mathbb{P}(N^- > n) \leq (p^-)^n. \tag{5.4}$$

For $t > 0$ and $n \geq 1$,

$$\mathbb{P}\left(1 \leq N^- \leq n, \sup_{0 \leq k \leq N^- - 1} (\alpha_{3(k+1)} - \alpha_{3k}) > 3t\right) \leq \sum_{k=0}^{n-1} \mathbb{P}(\alpha_{3(k+1)} - \alpha_{3k} > 3t, N^- \geq k + 1). \tag{5.5}$$

For all k and $t \in [\alpha_{3k-1}, \alpha_{3k}]$, we have $V_t \in [-g - a_0 - 2, -g + a_0 + 2]$. Therefore, by the strong Markov property applied at α_{3k-1} , and Lemma 4.6, there exist constants $C > 0$ and $p_0 \in (0, 1)$ such that for any integer $m \geq 1$,

$$\begin{aligned} \mathbb{P}(\alpha_{3k} - \alpha_{3k-1} > Cm, N^- \geq k + 1) &\leq \mathbb{P}(\alpha_{3k} - \alpha_{3k-1} > Cm) \\ &\leq \mathbb{P}(|V_s + g| \leq a_0 + 2 \text{ for all } s \in [0, Cm] \mid V_0 = -g - a_0 - 1, H_0 = 0) \leq p_0^m. \end{aligned} \tag{5.6}$$

If $V_{\alpha_{3k-3}} = -g$ then $\alpha_{3k+1} - \alpha_{3k} = 0$. If $V_{\alpha_{3k-3}} = -g - a_0 - 2$ then $V_{\alpha_{3k-1}} = -g - a_0 - 1$ and, by Remark 4.1, $S_{\alpha_{3k-1}} = X_{\alpha_{3k-1}}$. These remarks and the strong Markov property applied at time α_{3k-1} show that for any $\delta > 0$ and sufficiently large t ,

$$\begin{aligned} &\mathbb{P}(\alpha_{3k+1} - \alpha_{3k} > \delta t^{1/3}, \alpha_{3k} - \alpha_{3k-1} \leq t, N^- \geq k + 1) \\ &\leq \mathbb{P}(\alpha_{3k+1} - \alpha_{3k} > \delta t^{1/3}, \alpha_{3k} - \alpha_{3k-1} \leq t) \\ &= \mathbb{P}(\sigma(\alpha_{3k}) - \alpha_{3k} > \delta t^{1/3}, \alpha_{3k} - \alpha_{3k-1} \leq t) \\ &\leq \mathbb{P}(\sigma(u) > u + \delta t^{1/3} \text{ for some } u \leq t \wedge \tau_{-g-a_0-2}^V \wedge \tau_{-g+a_0+2}^V | V_0 = -g - a_0 - 1, H_0 = 0) \\ &\leq C e^{-C't}. \end{aligned} \tag{5.7}$$

The last estimate follows from Lemma 4.5 by applying it with $a = \sqrt{g}t^{1/3}/3$ and $m = 3$. We combine (5.6), taking $m = t/C$ there, and (5.7), to obtain,

$$\mathbb{P}(\alpha_{3k+1} - \alpha_{3k} > \delta t^{1/3}, N^- \geq k + 1) \leq C e^{-C't}. \tag{5.8}$$

We claim that for sufficiently large t ,

$$\begin{aligned} &\mathbb{P}(\alpha_{3k+2} - \alpha_{3k+1} > t, \alpha_{3k+1} - \alpha_{3k} \leq \delta t^{1/3}, N^- \geq k + 1) \\ &\leq \sup_{v \in [a_0+2, a_0+2+g\delta t^{1/3}]} \mathbb{P}(\tau_{-g-a_0-1}^V > t | V_0 = -g - v, H_0 = 0) \\ &\leq \sup_{v \in [a_0+2, a_0+2+g\delta t^{1/3}]} \frac{4(v - a_0 - 1)}{((v - a_0 - 1) + (a_0 + 1)t)(a_0 + 1)\sqrt{2\pi t}} e^{-(a_0+1)^2 t/8} \leq C e^{-C't}. \end{aligned} \tag{5.9}$$

The first inequality follows from the strong Markov property applied at α_{3k+1} . For the second inequality, we apply Lemma 4.3 with $a_1 = a_0 + 1$ and $a_2 = v$. Note that as $\frac{2(v-a_0-1)}{(a_0+1)} \leq \frac{2(1+g\delta t^{1/3})}{(a_0+1)} < t$ for sufficiently large t , the hypotheses of Lemma 4.3 are satisfied.

By Lemma 4.6,

$$\mathbb{P}(\alpha_{3k+3} - \alpha_{3k+2} > t, N^- \geq k + 1) \leq \mathbb{P}(\alpha_{3k+3} - \alpha_{3k+2} > t) \leq C e^{-C't}. \tag{5.10}$$

From (5.8), (5.9) and (5.10), we get

$$\mathbb{P}(\alpha_{3(k+1)} - \alpha_{3k} > 3t, N^- \geq k + 1) \leq C e^{-C't}.$$

Substituting this into (5.5), we obtain

$$\mathbb{P}\left(1 \leq N^- \leq t, \sup_{0 \leq k \leq N^- - 1} (\alpha_{3(k+1)} - \alpha_{3k}) > 3t\right) \leq C t e^{-C't}. \tag{5.11}$$

Recall that if $\tau_{-g-a_0-2}^V < \tau_{-g+a_0+2}^V$ then $\zeta_1 = \alpha_{3N^-}$. Thus, using (5.4) and (5.11),

$$\begin{aligned} \mathbb{P}(\zeta_1 > 3t^2, \tau_{-g-a_0-2}^V < \tau_{-g+a_0+2}^V) &= \mathbb{P}(\alpha_{3N^-} > 3t^2, N^- \geq 1) \\ &\leq \mathbb{P}(N^- > t) + \mathbb{P}\left(\sum_{k=0}^{N^- - 1} (\alpha_{3(k+1)} - \alpha_{3k}) > 3t^2, 1 \leq N^- \leq t\right) \\ &\leq \mathbb{P}(N^- > t) + \mathbb{P}\left(1 \leq N^- \leq t, \sup_{0 \leq k \leq N^- - 1} (\alpha_{3(k+1)} - \alpha_{3k}) > 3t\right) \\ &\leq (p^-)^t + C t e^{-C't}. \end{aligned}$$

After readjustment of constants we get

$$\mathbb{P}(\zeta_1 > t^2, \tau_{-g-a_0-2}^V < \tau_{-g+a_0+2}^V) \leq Ce^{-C't}. \tag{5.12}$$

We record a related estimate for later use. Note that $\inf\{t \geq \alpha_2 : V_t = -g\} \leq \zeta_1$. Hence our argument proving (5.12) also shows

$$\mathbb{P}(\tau_{-g}^V > t^2 \mid V_0 = -g - a_0 - 1, H_0 = 0) \leq Ce^{-C't}. \tag{5.13}$$

Next consider the event $\{\tau_{-g+a_0+2}^V < \tau_{-g-a_0-2}^V\}$. Note that if we start with $V_0 = v \in [-g, -g + a_0 + 1]$ and $H_0 = 0$, then for $t \leq \tau_{-g}^V$, it holds that $L_t \leq L_t^{(-g)}$. Therefore, for some $p^+ < 1$,

$$\begin{aligned} & \inf_{v \in [-g, -g+a_0+1]} \mathbb{P}(\tau_{-g}^V < \tau_{-g+a_0+2}^V \mid V_0 = v, H_0 = 0) \\ & \geq \mathbb{P}(\inf\{t \geq 0 : L_t^{(-g)} - gt = 1\} > \inf\{t \geq 0 : L_t^{(-g)} - gt = -a_0 - 1\}) \\ & \geq \mathbb{P}\left(\sup_{t \leq 2(a_0+1)/g} (B_t + gt) < 1\right) = 1 - p^+ > 0. \end{aligned}$$

Therefore, for any integer $n \geq 1$, we can apply the strong Markov property successively at times $\beta_{3n-1}, \beta_{3n-4}, \dots$ to get

$$\mathbb{P}(N^+ > n) \leq \left(\sup_{v \in [-g, -g+a_0+1]} \mathbb{P}(\tau_{-g}^V > \tau_{-g+a_0+2}^V \mid V_0 = v, H_0 = 0)\right)^n \leq (p^+)^n. \tag{5.14}$$

As before, we can write

$$\mathbb{P}\left(1 \leq N^+ \leq n, \sup_{0 \leq k \leq N^+-1} (\beta_{3(k+1)} - \beta_{3k}) > 3t\right) \leq \sum_{k=0}^{n-1} \mathbb{P}(\beta_{3(k+1)} - \beta_{3k} > 3t, N^+ \geq k + 1). \tag{5.15}$$

Let $0 \leq k \leq n - 1$. By Lemma 4.4, for large t ,

$$\mathbb{P}(\beta_{3k+1} - \beta_{3k} > t, N^+ \geq k + 1) \leq \mathbb{P}(\beta_{3k+1} - \beta_{3k} > t) \leq Ce^{-C't}. \tag{5.16}$$

For $N^+ \geq k + 1$, we have

$$-g < V_{\beta_{3k+2}} = -g + a_0 + 1 - g(\beta_{3k+2} - \beta_{3k+1})$$

yielding

$$\beta_{3k+2} - \beta_{3k+1} < \frac{a_0 + 1}{g}. \tag{5.17}$$

By Lemma 4.6,

$$\mathbb{P}(\beta_{3k+3} - \beta_{3k+2} > t, N^+ \geq k + 1) \leq \mathbb{P}(\beta_{3k+3} - \beta_{3k+2} > t) \leq Ce^{-C't}. \tag{5.18}$$

Combining (5.15), (5.16), (5.17) and (5.18), we get

$$\mathbb{P}\left(1 \leq N^+ \leq t, \sup_{0 \leq k \leq N^+-1} (\beta_{3(k+1)} - \beta_{3k}) > 3t\right) \leq Cte^{-C't}.$$

This and (5.14) can be combined as in the proof of (5.12) to show that for large t ,

$$\mathbb{P}(\beta_{3N^+} > t^2, \tau_{-g+a_0+2}^V < \tau_{-g-a_0-2}^V) \leq Ce^{-C't}. \tag{5.19}$$

The following estimate, needed later in the paper, can be derived just like the last estimate:

$$\mathbb{P}(\tau_{-g}^V > t^2 \mid V_0 = -g + a_0 + 1, H_0 = 0) \leq C e^{-C't}. \tag{5.20}$$

Next we will estimate the remaining time $\zeta_1 - \beta_{3N^+}$ before renewal happens. First consider the event $\{N^+ \geq 1, \tilde{N}^- = 0\}$ where the first renewal time ζ_1 is reached before the velocity hits level $-g - a_0 - 2$. Under this event, there are the following two possibilities. If $S_{\beta_{3N^+}} = X_{\beta_{3N^+}}$, then $\zeta_1 = \beta_{3N^+}$. Otherwise, $V_{\tilde{\alpha}_{-1}} \in (-g - a_0 - 2, -g)$, $V_{\tilde{\alpha}_0} = -g$ and $\zeta_1 = \tilde{\alpha}_0$. As the inert particle falls freely in the time interval $[\beta_{3N^+}, \tilde{\alpha}_{-1}]$, we have

$$\tilde{\alpha}_{-1} - \beta_{3N^+} \leq (a_0 + 2)/g. \tag{5.21}$$

By the strong Markov property applied at $\tilde{\alpha}_{-1}$ and Lemma 4.6,

$$\mathbb{P}(\tilde{\alpha}_0 - \tilde{\alpha}_{-1} > t, N^+ \geq 1, \tilde{N}^- = 0) \leq C e^{-C't}. \tag{5.22}$$

From (5.19), (5.21) and (5.22), it follows that

$$\mathbb{P}(\zeta_1 > t^2, N^+ \geq 1, \tilde{N}^- = 0) \leq C e^{-C't}. \tag{5.23}$$

We will next address the case when $\tilde{N}^- \geq 1$. Note that this implies that $N^+ \geq 1$. In this case the velocity V first reaches level $-g + a_0 + 2$ and then $-g - a_0 - 2$ before the renewal time ζ_1 . Under this event, $V_{\tilde{\alpha}_0} = -g - a_0 - 2$ and $\zeta_1 = \tilde{\alpha}_{3\tilde{N}^-}$. We need to control $\tilde{\alpha}_1 - \tilde{\alpha}_0$ and $\tilde{\alpha}_2 - \tilde{\alpha}_1$. At $\tilde{\alpha}_2$, we have $V_{\tilde{\alpha}_2} = -g - a_0 - 1$ and $S_{\tilde{\alpha}_2} = X_{\tilde{\alpha}_2}$ and we can apply the same analysis as in the case of the event $\{\tau_{-g-a_0-2}^V < \tau_{-g+a_0+2}^V\}$, replacing α 's with $\tilde{\alpha}$'s and N^- with \tilde{N}^- .

Note that for $\tilde{N}^- \geq 1$, $\tilde{\alpha}_0 = \tau_{-g-a_0-2}^V$ and $\tilde{\alpha}_1 = \sigma(\tau_{-g-a_0-2}^V)$. For any $\delta, \varepsilon > 0$, we have

$$\begin{aligned} &\mathbb{P}(\tilde{\alpha}_1 - \tilde{\alpha}_0 > \delta\sqrt{t}, \tilde{N}^- \geq 1) \\ &\leq \mathbb{P}(\tilde{\alpha}_1 - \tilde{\alpha}_0 > \delta\sqrt{t}, \tilde{\alpha}_0 \leq \tau_{\varepsilon\sqrt{t}}^V \wedge t^2, \tilde{N}^- \geq 1) + \mathbb{P}(\tau_{\varepsilon\sqrt{t}}^V < \tilde{\alpha}_0, N^+ \geq 1) + \mathbb{P}(\tilde{\alpha}_0 > t^2, \tilde{N}^- \geq 1). \end{aligned} \tag{5.24}$$

If we take $C_0 = 1$, $a = \sqrt{t}$ and $m = 4$ in Lemma 4.5 then we obtain for some C_1 and C_2 , and all $t > 0$,

$$\mathbb{P}\left(\sigma(s) > s + 3\frac{\delta\sqrt{t}}{\sqrt{g}} \text{ for some } s \leq t^2 \wedge \tau_{-g+\sqrt{g}\delta\sqrt{t}/4}^V \wedge \tau_{-g-\delta^2t/8}^V \mid V_0 = 0, H_0 = 0\right) \leq C_1 t^{3/2} e^{-C_2 t^{3/2}}.$$

Since $\tilde{\alpha}_0 = \tau_{-g-a_0-2}^V$, we have $\tilde{\alpha}_0 \leq \tau_{-g-\delta^2t/8}^V$ for large t . Adjusting the values of the constants in the last estimate, we obtain that for any $\delta > 0$, we can find $\varepsilon > 0$ such that for sufficiently large t ,

$$\mathbb{P}(\tilde{\alpha}_1 - \tilde{\alpha}_0 > \delta\sqrt{t}, \tilde{\alpha}_0 \leq \tau_{\varepsilon\sqrt{t}}^V \wedge t^2, \tilde{N}^- \geq 1) \leq C t^{3/2} e^{-C't^{3/2}}. \tag{5.25}$$

It follows from Lemma 4.7 that for sufficiently large $a > 0$,

$$\mathbb{P}(\tau_a^V < \tau_{-g+1}^V \mid V_0 = -g + a_0 + 2, H_0 = 0) \leq C e^{-C'a^2}. \tag{5.26}$$

Since Remark 4.1 implies that $S_{\beta_{3k}} = X_{\beta_{3k}}$, we can apply this inequality at $t = \beta_{3k}$, by the strong Markov property at β_{3k} . Note that $\sup_{s \in [\beta_{3k+1}, \beta_{3k+3}]} V_s \leq -g + a_0 + 2$ so for large a ,

$$\left\{ \sup_{s \in [\beta_{3k}, \beta_{3k+3}]} V_s > a \right\} = \left\{ \sup_{s \in [\beta_{3k}, \beta_{3k+1}]} V_s > a \right\}.$$

By definition, $\beta_{3k+1} < \inf\{s \geq \beta_{3k} : V_s = -g + 1\}$. These remarks and (5.26) imply that, for large a ,

$$\mathbb{P}\left(\sup_{s \in [\beta_{3k}, \beta_{3k+3}]} V_s > a, N^+ \geq k + 1\right) = \mathbb{P}\left(\sup_{s \in [\beta_{3k}, \beta_{3k+1}]} V_s > a, N^+ \geq k + 1\right) \leq C e^{-C'a^2}.$$

This and (5.14) show that for sufficiently large integer $a > 0$,

$$\begin{aligned} \mathbb{P}(\tau_a^V < \beta_{3N^+}, N^+ \geq 1) &\leq \sum_{k=0}^{a^2-1} \mathbb{P}\left(\sup_{s \in [\beta_{3k}, \beta_{3k+3}]} V_s > a, N^+ \geq k+1\right) + \mathbb{P}(N^+ > a^2) \\ &\leq Ca^2 e^{-C'a^2} + (p^+)^{a^2} \\ &\leq Ce^{-C'a^2}. \end{aligned} \tag{5.27}$$

Recall that on the event $\{N^+ \geq 1\}$, $V_u \in [-g, -g - a_0 - 2]$ for all $u \in [\beta_{3N^+}, \tilde{\alpha}_0]$. Therefore, for $a > -g$,

$$\mathbb{P}(\tau_a^V < \tilde{\alpha}_0, N^+ \geq 1) = \mathbb{P}(\tau_a^V < \beta_{3N^+}, N^+ \geq 1).$$

Thus, (5.27) with $a = \varepsilon\sqrt{t}$ gives us, for sufficiently large $t > 0$,

$$\mathbb{P}(\tau_{\varepsilon\sqrt{t}}^V < \tilde{\alpha}_0, N^+ \geq 1) \leq Ce^{-C'\varepsilon^2 t}. \tag{5.28}$$

To estimate the last probability in (5.24), note that under the event $\{\tilde{N}^- \geq 1\}$, $V_{\tilde{\alpha}_{-1}} \in [-g - a_0 - 2, -g]$ and $V_{\tilde{\alpha}_0} = -g - a_0 - 2$. Using (5.19), (5.21) and (5.22), we get

$$\mathbb{P}(\tilde{\alpha}_0 > t^2, \tilde{N}^- \geq 1) \leq Ce^{-C't}. \tag{5.29}$$

Combining the estimates (5.25), (5.28) and (5.29) with (5.24), we see that for any $\delta > 0$ there is $t_0 > 0$ such that for $t \geq t_0$,

$$\mathbb{P}(\tilde{\alpha}_1 - \tilde{\alpha}_0 > \delta\sqrt{t}, \tilde{N}^- \geq 1) \leq Ce^{-C't}, \tag{5.30}$$

where C, C' may depend on δ .

We next control $\tilde{\alpha}_2 - \tilde{\alpha}_1$. Write

$$\mathbb{P}(\tilde{\alpha}_2 - \tilde{\alpha}_1 > t, \tilde{N}^- \geq 1) \leq \mathbb{P}(\tilde{\alpha}_2 - \tilde{\alpha}_1 > t, \tilde{\alpha}_1 - \tilde{\alpha}_0 \leq \sqrt{t}, \tilde{N}^- \geq 1) + \mathbb{P}(\tilde{\alpha}_1 - \tilde{\alpha}_0 > \sqrt{t}, \tilde{N}^- \geq 1). \tag{5.31}$$

Applying the strong Markov property at $\tilde{\alpha}_1$ and then Lemma 4.3, we obtain for large t ,

$$\begin{aligned} &\mathbb{P}(\tilde{\alpha}_2 - \tilde{\alpha}_1 > t, \tilde{\alpha}_1 - \tilde{\alpha}_0 \leq \sqrt{t}, \tilde{N}^- \geq 1) \\ &\leq \sup_{v \in [a_0+2, a_0+2+g\sqrt{t}]} \mathbb{P}(\tau_{-g-a_0-1}^V > t \mid V_0 = -g - v, H_0 = 0) \\ &\leq \sup_{v \in [a_0+2, a_0+2+g\sqrt{t}]} \frac{4(v - a_0 - 1)}{((v - a_0 - 1) + (a_0 + 1)t)(a_0 + 1)\sqrt{2\pi t}} e^{-(a_0+1)^2 t/8} \leq Ce^{-C't}. \end{aligned} \tag{5.32}$$

Note that the hypotheses of Lemma 4.3 hold because $t \geq 2(1 + g\sqrt{t})/(a_0 + 1)$ for large t .

Substituting the estimates obtained in (5.30) (with $\delta = 1$) and (5.32) into (5.31), we get

$$\mathbb{P}(\tilde{\alpha}_2 - \tilde{\alpha}_1 > t, \tilde{N}^- \geq 1) \leq Ce^{-C't}. \tag{5.33}$$

The strong Markov property applied at $\tilde{\alpha}_2$ and (5.13) imply that

$$\mathbb{P}(\tilde{\alpha}_{3\tilde{N}^-} - \tilde{\alpha}_2 > t^2, \tilde{N}^- \geq 1) \leq \mathbb{P}(\tau_{-g}^V > t^2 \mid V_0 = -g - a_0 - 1, H_0 = 0) \leq Ce^{-C't}.$$

Thus,

$$\mathbb{P}(\zeta_1 - \tilde{\alpha}_2 > t^2, \tilde{N}^- \geq 1) = \mathbb{P}(\tilde{\alpha}_{3\tilde{N}^-} - \tilde{\alpha}_2 > t^2, \tilde{N}^- \geq 1) \leq Ce^{-C't}. \tag{5.34}$$

Combining (5.29), (5.30), (5.33) and (5.34), we get

$$\mathbb{P}(\zeta_1 > t^2, \tilde{N}^- \geq 1) \leq C e^{-C't}. \tag{5.35}$$

From (5.23) and (5.35), we obtain

$$\mathbb{P}(\zeta_1 > t^2, \tau_{-g+a_0+2}^V < \tau_{-g-a_0-2}^V) \leq C e^{-C't}. \tag{5.36}$$

The lemma follows from (5.12) and (5.36). □

6. The stationary distribution for $Z = (V, S - X)$

We will use results from [7,13] to establish existence of a unique stationary distribution for $Z = (V, S - X)$ and convergence of Z to the stationary distribution in total variation distance. For this reason, we will refer the reader to the book [13] even for definitions of widely use terms. For example, the definition of the total variation distance can be found in [13, Ch. 1, Section 5.3]. We would like to point out that an alternative proof of the existence of the stationary distribution is contained in Corollary 6.3 and the calculations in the proof of Theorem 2.1. Hence, the arguments in this section are mostly needed for the proof of uniqueness and convergence in total variation distance.

The process $Z = (V, S - X)$ is a classical regenerative processes (see [13, Ch. 10, Section 3]) with regeneration times ζ_k defined in Section 5. This means that the process starts afresh at each ζ_k and the cycles of the process Z given by $\mathcal{C}_k = (Z_t)_{t \in [\zeta_{k-1}, \zeta_k)}$ for $k \geq 1$ form an i.i.d sequence. The gaps $\{\zeta_{k+1} - \zeta_k, k \geq 0\}$ between the regeneration times are called *inter-regeneration times*. If the process starts from some $z \in \mathbb{H} := \mathbb{R} \times [0, \infty)$ then we will write \mathbb{P}_z and \mathbb{E}_z for the law of Z and the corresponding expectation. If we are not assuming that Z starts from the renewal state, it will be convenient to redefine ζ_0 as

$$\zeta_0 = \inf\{t \geq 0 : Z_t = (-g, 0)\} \tag{6.1}$$

and then shift the definition of the sequence ζ_k by ζ_0 . The arguments used in the proof of Lemma 5.1 show that $\mathbb{P}(\zeta_0 < \infty) = 1$. We will use \mathbb{P}_R and \mathbb{E}_R to denote probability and expectation when Z starts from the renewal state, i.e., $Z_0 = (-g, 0)$. The following lemma will be used in the proof of Theorem 6.2.

Lemma 6.1. *The inter-regeneration time $\zeta_1 - \zeta_0$ has a density with respect to Lebesgue measure.*

Proof. Recall η_0 defined in (5.1) and the notation $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+$.

By the strong Markov property applied at ζ_0 , for any $z \in \mathbb{H}$ and any measurable set $A \subseteq [0, \infty)$, $\mathbb{P}_z(\zeta_1 - \zeta_0 \in A) = \mathbb{P}_R(\zeta_1 \in A)$. Thus, it is enough to prove that starting from $V_0 = -g, H_0 = 0$, the random variable ζ_1 has a density. Note that $\zeta_1 \geq \eta_0 > 0$, a.s., under \mathbb{P}_R . By the Radon–Nikodym theorem it suffices to show that $\mathbb{P}_R(\zeta_1 \in A) = 0$ for any $b < \infty$ and any set $A \subseteq [0, b]$ of Lebesgue measure zero.

Consider the stopping time

$$T = \inf\{t \geq \eta_0 : V_t \leq -g, S_t = X_t\}.$$

Since $T \leq \zeta_1$ a.s., by Remark 4.1 we have that $\zeta_1 = T + \tau_{-g}^V \circ \theta_T$, where θ is the standard shift operator for Markov processes. Applying the strong Markov property at T we obtain

$$\mathbb{P}_R(\zeta_1 \in A) = \int_{-\infty}^{-g} \int_0^b \mathbb{P}(\tau_{-g}^V \in A - t \mid V_0 = v, H_0 = 0) \mathbb{P}_R(T \in dt, V_T \in dv). \tag{6.2}$$

Suppose that $V_0 = v < -g$. Note that

$$\begin{aligned} \tau_{-g}^V &= \inf\left\{t \geq 0 : \sup_{u \leq t} (B_u - S_u) - gt = -g - v\right\} = \inf\{t \geq 0 : B_t - S_t - gt = -g - v\} \\ &= \inf\left\{t \geq 0 : B_t - \int_0^t (V_u + g) du = -g - v\right\}. \end{aligned}$$

Consider Brownian motion with drift

$$W_t = B_t - \int_0^t (V(u \wedge \tau_{-g}^V \wedge \tau_{v-g \sup A}^V) + g) du,$$

and let τ_a^W be the hitting time of a by W . The last displayed formula shows that $\{\tau_{-g}^V \in A - t\} = \{\tau_{-g-v}^W \in A - t\}$. Since the drift of W is bounded on finite time intervals, the Girsanov theorem implies that the laws of W and B are mutually absolutely continuous on finite time intervals. The event $\{\tau_{-g-v}^{B,0} \in A - t\} \in \mathcal{F}_{\sup A} \subseteq \mathcal{F}_b$ has probability zero since the law of $\tau_{-g-v}^{B,0}$ is absolutely continuous, and $A - t$ has zero Lebesgue measure. Thus, the event $\{\tau_{-g-v}^W \in A - t\}$ has probability zero. We conclude that $\mathbb{P}(\tau_{-g-v}^V \in A - t \mid V_0 = v, H_0 = 0) = 0$ for any $v < -g$.

The discussion above allows us to transform (6.2) into

$$\mathbb{P}_R(\zeta_1 \in A) = \int_0^b \mathbb{I}_A(t) \mathbb{P}_R(T \in dt, V_T = -g) \leq \mathbb{P}_R(V_T = -g) = \mathbb{P}_R(T = \zeta_1). \tag{6.3}$$

We will show that $T < \zeta_1$, a.s. This holds if $V_{\eta_0} = -g - a_0 - 2$ because then $V_t \leq -g - a_0 - 2$ for $t \in [\eta_0, T]$. Suppose that $V_{\eta_0} = -g + a_0 + 2$ and note that $S_{\eta_0} = X_{\eta_0}$, by Remark 4.1. Hence, by the strong Markov property, it is enough to show that starting from $V_0 = -g + a_0 + 2$ and $H_0 = 0$, we have $T = \sigma(\tau_{-g}^V) < \zeta_0$, a.s. (recall (6.1)). We will show that, with probability one, $B_{\tau_{-g}^V} - S_{\tau_{-g}^V} < \sup_{u \leq \tau_{-g}^V} (B_u - S_u)$, or, equivalently, in view of (4.1), $X_{\tau_{-g}^V} < S_{\tau_{-g}^V}$. This shows that, a.s., $\sigma(\tau_{-g}^V) > \tau_{-g}^V$ and, therefore, $V_{\sigma(\tau_{-g}^V)} < -g$, implying that $\sigma(\tau_{-g}^V) < \zeta_0$.

For any $r > -g + a_0 + 2$, the process $V_t^* := V_{t \wedge \tau_{-g}^V \wedge \tau_r^V}$ is bounded and thus the exponential local martingale

$$t \mapsto \exp \left\{ \int_0^t V_u^* dB_u - \frac{1}{2} \int_0^t (V_u^*)^2 du \right\} \tag{6.4}$$

is a positive martingale for all $t \geq 0$. Hence, by the Girsanov Theorem, the laws of processes $B_t - S_t$ and B_t are mutually absolutely continuous on finite time intervals. For $t \geq 0$, let us define

$$M_t = \sup_{u \leq t} B_u, \quad \tau_a^{M,-g} = \inf\{t \geq 0 : M_t - gt = a\}.$$

The mutual absolute continuity of the laws of $B_t - S_t$ and B_t implies that if

$$\mathbb{P} \left(B_{\tau_{-a_0-2}^{M,-g}} = \sup_{u \leq \tau_{-a_0-2}^{M,-g}} B_u \right) = 0, \tag{6.5}$$

then

$$\mathbb{P} \left(B_{\tau_{-g}^V} - S_{\tau_{-g}^V} = \sup_{u \leq \tau_{-g}^V} (B_u - S_u), \tau_{-g}^V \leq \tau_r^V \mid V_0 = -g + a_0 + 2, H_0 = 0 \right) = 0. \tag{6.6}$$

We will prove that (6.5) is true. We need more definitions. Set

$$M^{-1}(t) = \inf\{u > 0 : M_u > t\},$$

$$\lambda(x) = \inf\{t \geq 0 : M^{-1}(t) - t/g \geq x\}.$$

Note that

$$g \tau_{-a_0-2}^{M,-g} - a_0 - 2 = \lambda((a_0 + 2)/g). \tag{6.7}$$

Now we will use some facts from the theory of Levy processes. It is well known that M_t^{-1} is a stable process with index 1/2. By [3, Thm. 5(i), Ch. VIII], we have $\limsup_{t \downarrow 0} M^{-1}(t)t^{2-\varepsilon} = 0$, a.s., for every $\varepsilon > 0$. It follows that 0 is

an irregular point for $[0, \infty)$, i.e., if $M^{-1}(0) = 0$ then

$$\mathbb{P}(\inf\{t > 0 : M^{-1}(t) - t/g \in [0, \infty)\} = 0) = 0.$$

This fact implies that the ascending ladder height process K , which can roughly be viewed as the values taken by $M^{-1}(t) - t/g$ at its successive new maxima (see [10, Section 6.2] for a more precise definition), is a compound Poisson subordinator (see the last paragraph on p. 150 of [10]). This implies that the jump distribution of K does not have atoms and thus, a.s., K does not hit specified points (see the last paragraph in the proof of [10, Lem. 7.10]). Thus, a.s.,

$$M^{-1}(\lambda((a_0 + 2)/g)) - \frac{\lambda((a_0 + 2)/g)}{g} > (a_0 + 2)/g,$$

which, by the definition of $\tau_{-a_0-2}^{M,-g}$ and (6.7), gives, a.s.,

$$M^{-1}(M_{\tau_{-a_0-2}^{M,-g}}) = M^{-1}(g\tau_{-a_0-2}^{M,-g} - a_0 - 2) > \tau_{-a_0-2}^{M,-g}.$$

This means that at the stopping time $\tau_{-a_0-2}^{M,-g}$, Brownian motion B lies strictly below its running maximum, which gives (6.5) and thus (6.6). By letting $r \rightarrow \infty$ in (6.6), we get $B_{\tau_{-g}^V} - S_{\tau_{-g}^V} < \sup_{u \leq \tau_{-g}^V} (B_u - S_u)$, a.s., which in turn implies that $T < \zeta_1$, a.s., in view of the opening remarks of the proof. \square

Recall that $Z = (V, S - X)$ has the state space $\mathbb{H} = \mathbb{R} \times [0, \infty)$.

Theorem 6.2. *For any $z \in \mathbb{H}$, assuming that $Z_0 = z$, when $t \rightarrow \infty$, the law of Z_t converges in total variation distance to a unique stationary distribution π , given by*

$$\pi(A) = \frac{\mathbb{E}_R(\int_0^{\zeta_1} \mathbb{I}_A(Z_u) du)}{\mathbb{E}_R(\zeta_1)}, \tag{6.8}$$

for any measurable set $A \subseteq \mathbb{H}$. Furthermore, for every $z \in \mathbb{H}$, \mathbb{P}_z -a.s.,

$$\lim_{t \rightarrow \infty} \frac{\int_0^t \mathbb{I}_A(Z_u) du}{t} = \pi(A). \tag{6.9}$$

Proof. Recall the regeneration times ζ_k from the beginning of this section. Lemma 5.1 shows that $\mathbb{E}(\zeta_1 - \zeta_0) < \infty$. By Lemma 6.1, the distribution of an inter-regeneration time $\zeta_{k+1} - \zeta_k$ is spread out in the sense of [13, Ch. 10, Section 3.5]. The convergence in total variation distance and, hence, the uniqueness of the stationary distribution now follows from part (b) of [13, Ch. 10, Thm. 3.3] (see Section 2.4 in that chapter for the notation used in the cited theorem). The representation (6.8) follows from (2.1) in [13, Ch. 10, Thm. 2.1].

Let $N_t = \sup\{k \geq 0 : \zeta_k \leq t\}$ be the number of renewals up to time t . The arguments applied in the proof of Lemma 5.1 can be used to show that $\mathbb{P}(\zeta_0 < \infty) = 1$. To prove (6.9), note that for any measurable set $A \subseteq \mathbb{H}$, we have

$$\int_0^t \mathbb{I}_A(Z_u) du = \int_0^{\zeta_0} \mathbb{I}_A(Z_u) du + \sum_{k=1}^{N_t} \int_{\zeta_{k-1}}^{\zeta_k} \mathbb{I}_A(Z_u) du + \int_{\zeta_{N_t}}^t \mathbb{I}_A(Z_u) du.$$

Clearly $(1/t) \int_0^{\zeta_0} \mathbb{I}_A(Z_u) du \rightarrow 0$, a.s., as $t \rightarrow \infty$. Furthermore, $(1/t) \int_{\zeta_{N_t}}^t \mathbb{I}_A(Z_u) du \leq (1/t)(t - \zeta_{N_t}) \rightarrow 0$, a.s., by [11, Prop. 7.3]. The same proposition implies that

$$\frac{\sum_{k=1}^{N_t} \int_{\zeta_{k-1}}^{\zeta_k} \mathbb{I}_A(Z_u) du}{t} \rightarrow \frac{\mathbb{E}_R(\int_0^{\zeta_1} \mathbb{I}_A(Z_u) du)}{\mathbb{E}_R(\zeta_1)},$$

almost surely as $t \rightarrow \infty$, so (6.9) follows from (6.8). \square

The following corollary to Theorem 3.5 shows that we can use [7, Thm. 1] in our context, and will be the main tool in the proof of Theorem 2.1. Recall that $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+$.

Corollary 6.3. *Let π be a probability measure on $(\mathbb{H}, \mathcal{B}(\mathbb{H}))$, with $\pi(\partial\mathbb{H}) = 0$. Then, π is a stationary distribution for the solution to (3.1) if and only if, for every $f \in C_c^2(\overline{\mathbb{H}})$ such that $(\nabla f(z))^T \gamma \geq 0$ for all $z \in \partial\mathbb{H}$, the inequality*

$$\int_{\mathbb{H}} \mathcal{L}f(z)\pi(dz) \leq 0. \tag{6.10}$$

holds.

Proof. By Theorem 3.5, the law of the solution to (3.1) and the solution to the submartingale problem for (\mathcal{L}, γ) in \mathbb{H} are the same. We only need to show that [7, Thm. 1] applies.

Note that for any constant $c \in \mathbb{R}$ we have $\mathcal{L}c = 0$, thus, the set $C_c^2(\overline{\mathbb{H}})$ and the set \mathcal{H} from [7] coincide. Also, the set \mathcal{V} from [7] is empty in our case. We next check that Assumption 1 in [7] holds.

1. Routine arguments show that $C_c^2(\overline{\mathbb{H}})$ separates points.
2. We will use the notation from [7], including but not limited to Assumption 1. Since our reflection field $\gamma = (1, 1)$ is constant, we have $d(x) \cap S_1(0) = \frac{1}{\sqrt{2}}\gamma$. The function $f(v, h) = v + h$ satisfies $(\nabla f(v, h))^T \frac{1}{\sqrt{2}}\gamma = 1$. Let $\eta \in C^\infty(\overline{\mathbb{H}})$ be compactly supported and equal to 1 on an open neighborhood of x . The function $f_{r,s} := f\eta$ satisfies the condition stated in Assumption 1 of [7].

This shows that Assumption 1 holds so [7, Thm. 1] applies in our case. □

Proof of Theorem 2.1. The existence and uniqueness of the stationary distribution and convergence of Z_t to this distribution in total variation distance were proved in Theorem 6.2.

We will apply Corollary 6.3. It is readily verifiable from (2.1) that the proposed stationary distribution $\pi(dv, dh) = \xi(v, h) dv dh$ is indeed a probability distribution, and as it has a density, therefore trivially, we have $\pi(\partial\mathbb{H}) = 0$. We next show that (6.10) holds.

Recall that $\mathcal{L}f(v, h) = \frac{1}{2}\partial_{hh}f + v\partial_hf - g\partial_vf$, and $\gamma = (1, 1)$. Let $f \in C_c^2(\overline{\mathbb{H}})$ be a function satisfying

$$(\nabla f(v, 0))^T \gamma = \partial_h f(v, 0) + \partial_v f(v, 0) \geq 0 \tag{6.11}$$

for $v \in \mathbb{R}$. We proceed by direct computation. By Fubini’s Theorem,

$$\begin{aligned} \int_{\mathbb{H}} \mathcal{L}f\xi(v, h) dv dh &= \frac{2g}{\sqrt{\pi}} \int_0^\infty \int_{-\infty}^\infty \left(\frac{1}{2}\partial_{hh}f + v\partial_hf - g\partial_vf \right) e^{-2gh} e^{-(v+g)^2} dv dh \\ &= -\frac{2g^2}{\sqrt{\pi}} \int_0^\infty \int_{-\infty}^\infty \partial_v f(v, h) e^{-(v+g)^2} dv e^{-2gh} dh \\ &\quad + \frac{2g}{\sqrt{\pi}} \int_{-\infty}^\infty \int_0^\infty \left(\frac{1}{2}\partial_{hh}f + v\partial_hf \right) e^{-2gh} dh e^{-(v+g)^2} dv. \end{aligned}$$

Next, we integrate by parts the inner integral of each term; actually, we have to integrate by parts twice. We get

$$\begin{aligned} \int_{\mathbb{H}} \mathcal{L}f\xi(v, h) dv dh &= -\frac{4g^2}{\sqrt{\pi}} \int_0^\infty \int_{-\infty}^\infty f(v, h)(v+g) e^{-(v+g)^2} dv e^{-2gh} dh \\ &\quad - \frac{2g}{\sqrt{\pi}} \int_{-\infty}^\infty \left(\frac{1}{2}\partial_h f(v, 0) + (v+g)f(v, 0) \right) e^{-(v+g)^2} dv \\ &\quad + \frac{2g}{\sqrt{\pi}} \int_{-\infty}^\infty \int_0^\infty (2g^2 + 2vg) f(v, h) e^{-2gh} dh e^{-(v+g)^2} dv. \end{aligned}$$

The two double integrals above cancel each other so we are left with a single integral,

$$\begin{aligned} \int_{\mathbb{H}} \mathcal{L} f \xi(v, h) dv dh &= -\frac{2g}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{2} \partial_h f(v, 0) + (v+g) f(v, 0) \right) e^{-(v+g)^2} dv \\ &= -\frac{2g}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{2} \partial_h f(v, 0) e^{-(v+g)^2} - \frac{1}{2} f(v, 0) \partial_v e^{-(v+g)^2} \right) dv. \end{aligned}$$

Integrating by parts once again we obtain

$$\begin{aligned} \int_{\mathbb{H}} \mathcal{L} f \xi(v, h) dv dh &= -\frac{g}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\partial_h f(v, 0) e^{-(v+g)^2} + \partial_v f(v, 0) e^{-(v+g)^2}) dv \\ &= -\frac{g}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\partial_h f(v, 0) + \partial_v f(v, 0)) e^{-(v+g)^2} dv \\ &= -\frac{g}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\nabla f(v, 0))^T \gamma e^{-(v+g)^2} dv, \end{aligned}$$

which is nonpositive by (6.11). □

Remark 6.4. Recall that we showed existence and uniqueness of the stationary distribution in Theorem 6.2. The existence is also implied by [7, Thm. 1] via Corollary 6.3 since we showed that (6.10) holds for the distribution π with density (with respect to Lebesgue measure) given by (2.1). We also believe that the uniqueness of the stationary distribution follows from the Harris irreducibility of our process. But the main power of Theorem 6.2 comes from the fact that it uses Lemma 5.1 and Lemma 6.1 to show convergence to stationarity in total variation distance (the existence and uniqueness of the stationary distribution follow as a by-product of this convergence).

7. Fluctuations and strong laws of large numbers

This section is dedicated to the proofs of Theorem 2.2 and Theorem 2.3. We first derive fine estimates for a number of hitting times of the velocity process and the gap process. These, in turn, are used to establish precise estimates for the fluctuations of V and $S - X$ on the intervals $[\zeta_n, \zeta_{n+1}]$ for $n \geq 0$, where $(\zeta_n)_{n \geq 0}$ are the renewal times defined in Section 5. The global fluctuation results of Theorem 2.2 and Theorem 2.3 are then proved by decomposing the path $\{Z_s : s \leq t\}$ into random time intervals $[\zeta_n, \zeta_{n+1}]$ and applying the fluctuation results proved on these intervals.

Lemma 7.1.

(i) For some $C > 0$, all $0 \leq h \leq 1$ and $a \leq -g - 2$,

$$\mathbb{E}(\tau_{-g}^V \mid V_0 = a, H_0 = h) \leq C|a|.$$

(ii) For some $C > 0$, all $h \geq 0$ and $a \in \mathbb{R}$,

$$\mathbb{E}(\tau_{a+1}^V \mid V_0 = a, H_0 = h) \geq C \frac{1}{(-a) \vee 1}.$$

(iii) For some $C > 0$, all $h \geq 0$ and $a \geq 2$,

$$\mathbb{E}(\tau_{-g}^V \mid V_0 = a, H_0 = h) \leq Ca.$$

(iv) For any $h \geq 0$ and $a \in \mathbb{R}$,

$$\mathbb{E}(\tau_{a-1}^V \mid V_0 = a, H_0 = h) \geq 1/g.$$

(v) For some $C > 0$, any $h \geq 1$ and $-g - 2 \leq a \leq -g - 1$,

$$\mathbb{E}(\tau_{-g}^V \mid V_0 = a, H_0 = h) \leq C\sqrt{h}.$$

(vi) For some $C > 0$, any $h \geq 1$ and $a \in \mathbb{R}$,

$$\mathbb{E}(\tau_{h-1}^H \wedge \tau_{a-1}^V \mid V_0 = a, H_0 = h) \geq C \frac{1}{(-a) \vee 1}.$$

Proof. (i) If $V_0 = a \leq -g - 2$, then $\tau_{-g}^V = \sigma(0) + \tau_{-g}^V \circ \theta_{\sigma(0)}$ where θ is the standard shift operator for Markov processes. By applying the strong Markov property at time $\sigma(0)$ we obtain

$$\begin{aligned} &\mathbb{E}(\tau_{-g}^V \mid V_0 = a, H_0 = h) \\ &\leq \mathbb{E}(\sigma(0) \mid V_0 = a, H_0 = h) + \sum_{k=0}^{\infty} \mathbb{E}(\tau_{-g}^V \mid V_0 = a - k - 1, H_0 = 0) \mathbb{P}(V_{\sigma(0)} \leq a - k \mid V_0 = a, H_0 = 0) \\ &\leq \mathbb{E}(\sigma(0) \mid V_0 = a, H_0 = h) + \sum_{k=0}^{\infty} \mathbb{E}(\tau_{-g}^V \mid V_0 = a - k - 1, H_0 = 0) \mathbb{P}(\sigma(0) \geq k/g \mid V_0 = a, H_0 = 0). \end{aligned} \tag{7.1}$$

We will next estimate the terms in the sum.

Suppose that $B_0 = 0$, $a \leq 0$, $V_0 = a$, $h \geq 0$ and $H_0 = h$. If $\sigma(0) \geq s$ then $B_t \leq h + at - gt^2/2 \leq h - gt^2/2$ for all $t \leq s$. In particular, $B_s \leq h - gs^2/2$. Thus, using (4.3), for $s \geq 2\sqrt{h/g}$,

$$\mathbb{P}(\sigma(0) \geq s \mid V_0 = a, H_0 = h) \leq \mathbb{P}(B_s \leq h - gs^2/2) \leq C \frac{\sqrt{s}}{gs^2 - 2h} e^{-(h-g s^2/2)^2/(2s)}. \tag{7.2}$$

This implies the following two bounds, with $0 < C, C_1 < \infty$,

$$\mathbb{E}(\sigma(0) \mid V_0 = a, H_0 = h) \leq C, \quad 0 \leq h \leq 1 \tag{7.3}$$

$$\mathbb{P}(\sigma(0) \geq k/g \mid V_0 = a, H_0 = h) \leq Ck^{-1}e^{-C_1k^3}, \quad k \geq 1. \tag{7.4}$$

Next, we bound the expectations in (7.1). Consider $b \leq -g - 2$. According to Lemma 4.3 applied with $-g - a_1 = b + 1$ and $-g - a_2 = b$, for $t \geq 2/(-g - b - 1)$,

$$\mathbb{P}(\tau_{b+1}^V > t \mid V_0 = b, H_0 = 0) \leq \frac{4}{(1 + (-g - b - 1)t)(-g - b - 1)\sqrt{2\pi t}} e^{-(g-b-1)^2t/8}.$$

Since $-g - b - 1 \geq 1$, we obtain for $t \geq 2$,

$$\mathbb{P}(\tau_{b+1}^V > t \mid V_0 = b, H_0 = 0) \leq \frac{4}{(1 + t)\sqrt{2\pi t}} e^{-t/8}.$$

Hence,

$$\mathbb{E}(\tau_{b+1}^V \mid V_0 = b, H_0 = 0) \leq C.$$

Recall from Remark 4.1 that if $V_0 = b - k$ then $S_{\tau_{b-k+1}^V} = X_{\tau_{b-k+1}^V}$. Hence, the repeated application of the last estimate at the stopping times $\tau_{b-k+1}^V, \tau_{b-k+2}^V, \tau_{b-k+3}^V, \dots$, shows that if $b \leq -2g - 1$ then

$$\mathbb{E}(\tau_b^V \mid V_0 = b - k, H_0 = 0) \leq Ck. \tag{7.5}$$

We can take $a_0 = g + 1$ in (5.13) to obtain for $-2g - 2 \leq b \leq -2g - 1$,

$$\mathbb{E}(\tau_{-g}^V \mid V_0 = b, H_0 = 0) \leq \mathbb{E}(\tau_{-g}^V \mid V_0 = -2g - 2, H_0 = 0) \leq C.$$

This, the strong Markov property applied at τ_b^V where b satisfies $-2g - 2 \leq b \leq -2g - 1$, and (7.5) imply that for $a \leq -g - 2$,

$$\mathbb{E}(\tau_{-g}^V \mid V_0 = a, H_0 = 0) \leq C(-g - a). \tag{7.6}$$

Substituting (7.3), (7.4) and (7.6) into (7.1), we get

$$\mathbb{E}(\tau_{-g}^V \mid V_0 = a, H_0 = h) \leq C + \sum_{k=1}^{\infty} C(-g - a + k + 1)k^{-1}e^{-C_1k^3} \leq C|a|.$$

(ii) It is easy to see that for all $h \geq 0$,

$$\mathbb{E}(\tau_{a+1}^V \mid V_0 = a, H_0 = h) \geq \mathbb{E}(\tau_{a+1}^V \mid V_0 = a, H_0 = 0),$$

so we will assume that $V_0 = a$ and $H_0 = 0$.

For $t \leq 1/g$, we have $V_t \geq a - 1$, so $L_t \leq L_t^{(a-1)} = \sup_{u \leq t} (B_u - (a - 1)u)$. Suppose that $a \leq -g/2 + 1$ and, therefore, $0 \leq -1/(2(a - 1)) \leq 1/g$. If $\sup_{u \leq -1/(2(a-1))} B_u < 1/2$ then,

$$L_{-1/(2(a-1))} \leq L_{-1/(2(a-1))}^{(a-1)} = \sup_{u \leq -1/(2(a-1))} (B_u - (a - 1)u) \leq \sup_{u \leq -1/(2(a-1))} B_u + \frac{a - 1}{2(a - 1)} < 1,$$

and, therefore, $\tau_{a+1}^V \geq -1/(2(a - 1))$. It follows that,

$$\begin{aligned} \mathbb{E}(\tau_{a+1}^V \mid V_0 = a, S_0 - X_0 = 0) &\geq \frac{1}{-2(a - 1)} \mathbb{P}\left(\tau_{a+1}^V \geq \frac{1}{-2(a - 1)}\right) \\ &\geq \frac{1}{-2(a - 1)} \mathbb{P}\left(\sup_{u \leq -1/(2(a-1))} B_u < 1/2\right) \geq C/|a|. \end{aligned}$$

It is easy to check that for every $a \geq -g/2 + 1$, the same argument generates a bound at least as large as for $a = -g/2 + 1$. This easily translates onto the statement in part (ii) of the lemma.

(iii) Consider $a \geq 2$ and take any $v \in (a - 1, a]$. According to Lemma 4.4(i) applied with $-g + a_1 = a - 1$ and $-g + a_2 = v$, for $t \geq 2/g$,

$$\mathbb{P}(\tau_{a-1}^V > t \mid V_0 = v, H_0 = 0) \leq e^{-g(a-1)t} \leq e^{-gt}.$$

Hence,

$$\sup_{v \in (a-1, a]} \mathbb{E}(\tau_{a-1}^V \mid V_0 = v, H_0 = 0) \leq C. \tag{7.7}$$

We can write $\tau_{a-1}^V = \sigma(0) \wedge \tau_{a-1}^V + \tau_{a-1}^V \circ \theta_{\sigma(0) \wedge \tau_{a-1}^V}$ where θ is the standard shift map. Therefore, by the strong Markov property applied at the stopping time $\sigma(0)$, we get for any $h \geq 0$,

$$\begin{aligned} \mathbb{E}(\tau_{a-1}^V \mid V_0 = a, H_0 = h) &= \mathbb{E}(\sigma(0) \wedge \tau_{a-1}^V + \tau_{a-1}^V \circ \theta_{\sigma(0) \wedge \tau_{a-1}^V} \mid V_0 = a, H_0 = h) \\ &\leq \mathbb{E}(\sigma(0) \wedge \tau_{a-1}^V \mid V_0 = a, H_0 = h) + \sup_{v \in (a-1, a]} \mathbb{E}(\tau_{a-1}^V \mid V_0 = v, H_0 = 0). \end{aligned}$$

Note that if $\sigma(0) > 1/g$, then $\tau_{a-1}^V = 1/g$. Thus, $\mathbb{E}(\sigma(0) \wedge \tau_{a-1}^V \mid V_0 = a, H_0 = h) \leq 1/g$. Applying this and (7.7) to the above inequality yields

$$\sup_{h \geq 0} \mathbb{E}(\tau_{a-1}^V \mid V_0 = a, H_0 = h) \leq C.$$

The repeated application of the last estimate at the stopping times $\tau_{a-1}^V, \tau_{a-2}^V, \tau_{a-3}^V, \dots$, shows that if $a - k \geq 1$ and $h \geq 0$ then

$$\mathbb{E}(\tau_{a-k}^V \mid V_0 = a, H_0 = h) \leq Ck. \tag{7.8}$$

For $1 \leq a \leq 2$ and $h \geq 0$, we can write $\tau_{-g}^V = \sigma(0) \wedge \tau_{-g}^V + \tau_{-g}^V \circ \theta_{\sigma(0) \wedge \tau_{-g}^V}$. It is clear that $\sigma(0) \wedge \tau_{-g}^V \leq (g + 2)/g$. Thus, we get

$$\mathbb{E}(\tau_{-g}^V \mid V_0 = a, H_0 = h) \leq (g + 2)/g + \sup_{v \in (-g, 2]} \mathbb{E}(\tau_{-g}^V \mid V_0 = v, H_0 = 0). \tag{7.9}$$

By another application of the strong Markov property, we can write

$$\sup_{v \in (-g, 2]} \mathbb{E}(\tau_{-g}^V \mid V_0 = v, H_0 = 0) \leq \sup_{v \in (-g, 2]} \mathbb{E}(\tau_{-g}^V \wedge \tau_2^V \mid V_0 = v, H_0 = 0) + \mathbb{E}(\tau_{-g}^V \mid V_0 = 2, H_0 = 0).$$

Applying Lemma 4.6, we get $\sup_{v \in (-g, 2]} \mathbb{E}(\tau_{-g}^V \wedge \tau_2^V \mid V_0 = v, H_0 = 0) \leq C$. Further, taking $a_0 = g + 1$ in (5.20), we obtain $\mathbb{E}(\tau_{-g}^V \mid V_0 = 2, H_0 = 0) \leq C$. Substituting these estimates into (7.9), we get

$$\sup_{a \in [1, 2], h \geq 0} \mathbb{E}(\tau_{-g}^V \mid V_0 = a, H_0 = h) \leq C.$$

This, the strong Markov property applied at τ_{a-k}^V where k satisfies $1 \leq a - k \leq 2$, and (7.8) imply that for $a \geq 2$ and $h \geq 0$,

$$\mathbb{E}(\tau_{-g}^V \mid V_0 = a, H_0 = h) \leq Ca.$$

(iv) The estimate follows from the equation $dV_t = dL_t - g dt$ and the fact that L is non-decreasing.

(v) Assume that $V_0 = a$, and $H_0 = h \geq 1$. By (7.2), we have for $s \geq 2/\sqrt{g}$:

$$\mathbb{P}(\sigma(0) \geq s\sqrt{h}) \leq C \frac{h^{1/4} \sqrt{s}}{ghs^2/2 - h} e^{-(h-ghs^2/2)^2/(2s\sqrt{h})} \leq C \frac{\sqrt{s}}{gs^2/2 - 1} e^{-(1-gs^2/2)^2/(2s)}.$$

This implies that

$$\mathbb{E}(\sigma(0) \mid V_0 = a, H_0 = h) \leq C\sqrt{h}. \tag{7.10}$$

Note that $V_{\sigma(0)} = a - g\sigma(0)$ so (7.2) yields for $k \geq 2\sqrt{hg}$,

$$\begin{aligned} \mathbb{P}(V_{\sigma(0)} \in [a - k - 1, a - k]) &\leq \mathbb{P}(V_{\sigma(0)} \leq a - k) = \mathbb{P}(\sigma(0) \geq k/g) \\ &\leq C \frac{\sqrt{k/g}}{g(k/g)^2 - 2h} e^{-(h-g(k/g)^2/2)^2/(2k/g)}. \end{aligned}$$

This, the strong Markov property applied at $\sigma(0)$, (7.6) and (7.10) imply that

$$\begin{aligned} &\mathbb{E}(\tau_{-g}^V \mid V_0 = a, H_0 = h) \\ &\leq \mathbb{E}(\sigma(0) \mid V_0 = a, H_0 = h) + \sum_{k=0}^{\infty} C(-g - a + k + 1) \mathbb{P}(V_{\sigma(0)} \in [a - k - 1, a - k]) \\ &\leq C\sqrt{h} + (-g - a + 2\sqrt{hg} + 1) \\ &\quad + \sum_{k=\lfloor 2\sqrt{hg} \rfloor + 1}^{\infty} C(-g - a + k + 1) \frac{\sqrt{k/g}}{g(k/g)^2 - 2h} e^{-(1-g(k/g)^2/2)^2/(2k/g)} \end{aligned}$$

$$\begin{aligned} &\leq C\sqrt{h} + (-g + g + 2 + 2\sqrt{hg} + 1) \\ &\quad + \sum_{k=\lfloor 2\sqrt{hg} \rfloor + 1}^{\infty} C(-g + g + 2 + k + 1) \frac{\sqrt{k/g}}{g(k/g)^2 - 2} e^{-(1-g(k/g)^2/2)^2/(2k/g)} \\ &\leq C\sqrt{h}. \end{aligned}$$

(vi) Suppose that $S_0 = h \geq 1$, $X_0 = 0$ and $V_0 = a \leq -\sqrt{g/8}$. Then, for $0 \leq t \leq -1/(4a)$,

$$S_t \geq h - gt^2/2 + at \geq h - \frac{g}{2} \cdot \frac{1}{16a^2} - a \frac{1}{4a} \geq h - 1/2.$$

Therefore, for any $t_0 \in (0, -1/(4a)]$, if $\sup_{u \leq t_0} B_u \leq 1/2$, then $X_t \leq 1/2$ for $0 \leq t \leq t_0$ and, therefore, $\tau_{h-1}^H \geq t_0$. The local time L does not increase on the interval $[0, \tau_{h-1}^H]$ because S and X do not meet on this interval. Hence, V decreases at the constant rate on this interval and, therefore, $\tau_{h-1}^H \wedge \tau_{a-1}^V = \tau_{h-1}^H \wedge 1/g$. Thus,

$$\begin{aligned} \mathbb{P}\left(\tau_{h-1}^H \wedge \tau_{a-1}^V \geq \frac{1}{(-4a) \vee g}\right) &= \mathbb{P}\left(\tau_{h-1}^H \wedge 1/g \geq \frac{1}{(-4a) \vee g}\right) = \mathbb{P}\left(\tau_{h-1}^H \geq \frac{1}{(-4a) \vee g}\right) \\ &\geq \mathbb{P}\left(\sup_{u \leq 1/((-4a) \vee g)} B_u \leq 1/2\right) \geq C, \end{aligned}$$

and, therefore,

$$\mathbb{E}(\tau_{h-1}^H \wedge \tau_{a-1}^V \mid V_0 = a, S_0 - X_0 = h) \geq C \cdot \frac{1}{(-4a) \vee g},$$

proving part (vi) of the lemma in the case $h \geq 1$ and $a \leq -\sqrt{g/8}$.

For $a \geq -\sqrt{g/8}$ and $0 \leq t \leq 1/(4\sqrt{g/8})$,

$$S_t \geq h - gt^2/2 + at \geq h - gt^2/2 - \sqrt{g/8}t \geq h - 1/2,$$

so the analogous argument gives

$$\mathbb{E}(\tau_{h-1}^H \wedge \tau_{a-1}^V \mid V_0 = a, H_0 = h) \geq C,$$

completing the proof of part (vi) of the lemma in the case $h \geq 1$ and $a \in \mathbb{R}$. □

Lemma 7.2. Assume that $V_0 = -g$, $H_0 = 0$ and let $\{\zeta_k\}_{k \geq 0}$ be the renewal times defined in (5.2). Then there exists $a_0 > 0$ and positive constants C_k , $k = 1, \dots, 6$, such that for $a, r \geq a_0$,

$$C_1 \frac{1}{a} e^{-a^2 + 2a(g-1)} \leq \mathbb{P}(\tau_{-a}^V < \zeta_1) \leq C_2 e^{-a^2 + 2a(g+1)}, \tag{7.11}$$

$$C_3 \frac{1}{a} e^{-a^2 - 2a(g+1)} \leq \mathbb{P}(\tau_a^V < \zeta_1) \leq C_4 \frac{1}{a} e^{-a^2 - 2a(g-1)}, \tag{7.12}$$

$$C_5 \frac{1}{\sqrt{r}} e^{-2gr} \leq \mathbb{P}(\tau_r^H < \zeta_1) \leq C_6 e^{-2gr}. \tag{7.13}$$

Proof. Recall from Lemma 5.1 that $\mathbb{E}(\zeta_1) < \infty$. We can apply (6.8) to a set $A \subset \mathbb{R} \times [0, \infty)$ and use (2.1) to see that

$$\mathbb{E}\left(\int_0^{\zeta_1} \mathbb{I}_A(Z_u) du\right) = C\pi(A) = C \iint_A \frac{2g}{\sqrt{\pi}} e^{-2gh} e^{-(v+g)^2} dv dh. \tag{7.14}$$

(i) Fix $a \geq g + 2$. We apply (7.14) to the set $A = (-a - 1, -a] \times [0, 1]$ to see that

$$\begin{aligned} \mathbb{E}\left(\int_0^{\zeta_1} \mathbb{I}(-a - 1 \leq V_u \leq -a, H_u \leq 1) du\right) &= C \int_0^1 \int_{-a-1}^{-a} \frac{2g}{\sqrt{\pi}} e^{-2gh} e^{-(v+g)^2} dv dh \\ &\geq C \int_{-a-1}^{-a} e^{-(v+g)^2} dv \\ &\geq C e^{-(a+1-g)^2} \\ &\geq C' e^{-a^2 - 2a(1-g)}. \end{aligned} \tag{7.15}$$

Let $T = \inf\{t \geq 0 : V_t \in [-a - 1, -a], H_t \leq 1\}$. Note that (7.15) only involves paths of (V_t, H_t) such that $T < \zeta_1$. It follows from Remark (4.1) that $T + \tau_{-g}^V \circ \theta_T = \zeta_1$ on the event $\{T < \zeta_1\}$. Lemma 7.1(i) and the strong Markov property applied at T imply that

$$\begin{aligned} \mathbb{E}\left(\int_0^{\zeta_1} \mathbb{I}(-a - 1 \leq V_u \leq -a, H_u \leq 1) du\right) &\leq \mathbb{E}(\mathbb{I}(T < \zeta_1)(\zeta_1 - T)) \\ &\leq \mathbb{P}(T < \zeta_1) \times \sup_{v \in [-a-1, -a], h \in [0, 1]} \mathbb{E}(\tau_{-g}^V \mid V_0 = v, H_0 = h) \\ &\leq Ca \mathbb{P}(T < \zeta_1). \end{aligned}$$

This and (7.15) yield

$$\mathbb{P}(\tau_{-a}^V < \zeta_1) \geq \mathbb{P}(T < \zeta_1) \geq C \frac{1}{a} e^{-a^2 - 2a(1-g)}.$$

This proves the lower bound in (7.11) for $a \geq g + 2$.

(ii) Fix $a \geq g + 2$. We apply (7.14) to the set $A = (-\infty, -a + 1] \times [0, \infty)$ and use (4.3) to see that

$$\begin{aligned} \mathbb{E}\left(\int_0^{\zeta_1} \mathbb{I}(V_u \leq -a + 1) du\right) &= C \int_0^\infty \int_{-\infty}^{-a+1} \frac{2g}{\sqrt{\pi}} e^{-2gh} e^{-(v+g)^2} dv dh \\ &= C \int_{-\infty}^{-a+1} \frac{1}{\sqrt{\pi}} e^{-(v+g)^2} dv \\ &\leq C' \frac{1}{a - 1 - g} e^{-(a-1-g)^2} \\ &\leq C'' \frac{1}{a} e^{-a^2 + 2a(g+1)}. \end{aligned} \tag{7.16}$$

Let $T = \inf\{t \geq \tau_{-a}^V : V_t = -a + 1\}$. Note that if $\tau_{-a}^V < \zeta_1$ then $T < \zeta_1$. Lemma 7.1(ii) implies that

$$\begin{aligned} \mathbb{E}\left(\int_0^{\zeta_1} \mathbb{I}(V_u \leq -a + 1) du\right) &\geq \mathbb{P}(\tau_{-a}^V < \zeta_1) \times \inf_{h \geq 0} \mathbb{E}(\tau_{-a+1}^V \mid V_0 = a, H_0 = h) \\ &\geq \frac{C}{a} \mathbb{P}(\tau_{-a}^V < \zeta_1), \end{aligned}$$

so (7.16) yields

$$\mathbb{P}(\tau_{-a}^V < \zeta_1) \leq C e^{-a^2 + 2a(g+1)}.$$

This proves the upper bound in (7.11) for $a \geq g + 2$.

(iii) Fix $a \geq 2$. We apply (7.14) to the set $A = [a, \infty) \times [0, \infty)$ to see that

$$\begin{aligned} \mathbb{E}\left(\int_0^{\zeta_1} \mathbb{I}(V_u \geq a) du\right) &= C \int_0^\infty \int_a^\infty \frac{2g}{\sqrt{\pi}} e^{-2gh} e^{-(v+g)^2} dv dh \\ &\geq C \int_a^\infty e^{-(v+g)^2} dv \geq C' e^{-(a+1+g)^2} \geq C'' e^{-a^2-2a(g+1)}. \end{aligned} \tag{7.17}$$

It follows from Remark (4.1) that on the event $\{\tau_{-g}^V < \zeta_1\}$, V will not take values greater than $-g$ on the interval $[\tau_{-g}^V, \zeta_1]$. Lemma 7.1(iii) and the strong Markov property applied at τ_a^V imply that

$$\mathbb{E}\left(\int_0^{\zeta_1} \mathbb{I}(V_u \geq a) du\right) \leq \mathbb{P}(\tau_a^V < \zeta_1) \mathbb{E}(\tau_{-g}^V \mid V_0 = a, H_0 = 0) \leq Ca \mathbb{P}(\tau_a^V < \zeta_1).$$

This and (7.17) yield

$$\mathbb{P}(\tau_a^V < \zeta_1) \geq C \frac{1}{a} e^{-a^2-2a(g+1)}.$$

This proves the lower bound in (7.12) for $a \geq 2$.

(iv) Fix $a \geq 2$. We apply (7.14) to the set $A = [a - 1, \infty) \times [0, \infty)$ and use (4.3) to see that

$$\begin{aligned} \mathbb{E}\left(\int_0^{\zeta_1} \mathbb{I}(V_u \geq a - 1) du\right) &= C \int_0^\infty \int_{a-1}^\infty \frac{2g}{\sqrt{\pi}} e^{-2gh} e^{-(v+g)^2} dv dh \\ &= C \int_{a-1}^\infty \frac{1}{\sqrt{\pi}} e^{-(v+g)^2} dv \\ &\leq C' \frac{1}{a - 1 + g} e^{-(a-1+g)^2} \\ &\leq C'' \frac{1}{a} e^{-a^2-2a(g-1)}. \end{aligned} \tag{7.18}$$

Let $T = \inf\{t \geq \tau_a^V : V_t = a - 1\}$. Note that if $\tau_a^V < \zeta_1$ then $T < \zeta_1$. Lemma 7.1(iv) implies that

$$\begin{aligned} \mathbb{E}\left(\int_0^{\zeta_1} \mathbb{I}(V_u \geq a - 1) du\right) &\geq \mathbb{P}(\tau_a^V < \zeta_1) \mathbb{E}(\tau_{a-1}^V \mid V_0 = a, H_0 = 0) \\ &\geq (1/g) \mathbb{P}(\tau_a^V < \zeta_1), \end{aligned}$$

so (7.18) yields

$$\mathbb{P}(\tau_a^V < \zeta_1) \leq C \frac{1}{a} e^{-a^2-2a(g-1)}.$$

This proves the upper bound in (7.12) for $a \geq 2$.

(v) Consider $r \geq 1$. We apply (7.14) to the set $A = [-g - 2, -g - 1] \times [r, \infty)$ to see that

$$\begin{aligned} \mathbb{E}\left(\int_0^{\zeta_1} \mathbb{I}(-g - 2 \leq V_u \leq -g - 1, H_u \geq r) du\right) \\ = C \int_r^\infty \int_{-g-2}^{-g-1} \frac{2g}{\sqrt{\pi}} e^{-2gh} e^{-(v+g)^2} dv dh \geq C e^{-2gr}. \end{aligned} \tag{7.19}$$

Let $T = \inf\{t \geq 0 : V_t \in [-g - 2, -g - 1], H_t \geq r\}$. It follows from Remark (4.1) that on the event $\{T < \zeta_1\}$, $\inf\{t \geq T : V_t = -g\} = \zeta_1$. Lemma 7.1(v) and the strong Markov property applied at T imply that

$$\begin{aligned} & \mathbb{E}\left(\int_0^{\zeta_1} \mathbb{I}(-g - 2 \leq V_u \leq -g - 1, H_u \geq r) du\right) \\ & \leq \mathbb{P}(T < \zeta_1) \times \sup_{v \in [-g-2, -g-1], h \geq r} \mathbb{E}(\tau_{-g}^V | V_0 = v, H_0 = h) \leq C\sqrt{r} \mathbb{P}(T < \zeta_1). \end{aligned}$$

This and (7.19) yield

$$\mathbb{P}(\tau_h^H < \zeta_1) \geq \mathbb{P}(T < \zeta_1) \geq C \frac{1}{\sqrt{r}} e^{-2gr}.$$

This proves the lower bound in (7.13) for $r \geq 1$.

(vi) Fix $r \geq 2$. We apply (7.14) to the set $A = [k - 2, k + 1) \times [r - 1, \infty)$ and use (4.3) to see that

$$\begin{aligned} & \mathbb{E}\left(\int_0^{\zeta_1} \mathbb{I}(k - 2 \leq V_u < k + 1, H_u \geq r - 1) du\right) \\ & = C \int_{r-1}^{\infty} \int_{k-2}^{k+1} \frac{2g}{\sqrt{\pi}} e^{-2gh} e^{-(v+g)^2} dv dh = C e^{-2g(r-1)} \int_{k-2}^{k+1} \frac{1}{\sqrt{\pi}} e^{-(v+g)^2} dv \\ & \leq C e^{-2g(r-1)} \frac{1}{|k| + g + 1} e^{-((|k|+g-2) \vee 0)^2}. \end{aligned} \tag{7.20}$$

Let

$$T_k = \inf\{t \geq \tau_r^H : H_t = r - 1 \text{ or } V_t = k - 2\}.$$

Note that if $\tau_r^H < \zeta_1$ then $T_k \leq \zeta_1$ for every integer k . Lemma 7.1(vi) implies that

$$\begin{aligned} & \mathbb{E}\left(\int_0^{\zeta_1} \mathbb{I}(k - 2 \leq V_u < k + 1, H_u \geq r - 1) du\right) \\ & \geq \mathbb{P}(\tau_r^H < \zeta_1, k - 1 \leq V_{\tau_r^H} < k) \times \inf_{v \in [k-1, k]} \mathbb{E}(\tau_{r-1}^H \wedge \tau_{v-1}^V | V_0 = v, H_0 = r) \\ & \geq C \frac{1}{(-k) \vee 1} \mathbb{P}(\tau_r^H < \zeta_1, k - 1 \leq V_{\tau_r^H} < k), \end{aligned}$$

so (7.20) yields

$$\mathbb{P}(\tau_r^H < \zeta_1, k - 1 \leq V_{\tau_r^H} < k) \leq C((-k) \vee 1) e^{-2g(r-1)} \frac{1}{|k| + g + 1} e^{-((|k|+g-2) \vee 0)^2}.$$

It follows that

$$\begin{aligned} \mathbb{P}(\tau_r^H < \zeta_1) & = \sum_{k=-\infty}^{\infty} \mathbb{P}(\tau_r^H < \zeta_1, k - 1 \leq V_{\tau_r^H} < k) \\ & \leq \sum_{k=-\infty}^{\infty} C((-k) \vee 1) e^{-2g(r-1)} \frac{1}{|k| + g + 1} e^{-((|k|+g-2) \vee 0)^2} \leq C' e^{-2gr}. \end{aligned}$$

This proves the upper bound in (7.13) for $r \geq 2$. □

Proof of Theorem 2.2. Fix arbitrarily small $\varepsilon > 0$. By (7.12) of Lemma 7.2, for $n \geq 0$,

$$\mathbb{P}\left(\sup_{t \in [\zeta_n, \zeta_{n+1}]} V_t > \sqrt{(1 + \varepsilon) \log n}\right) \leq \frac{C}{n^{1+\varepsilon/2}}$$

and

$$\mathbb{P}\left(\sup_{t \in [\zeta_n, \zeta_{n+1}]} V_t > \sqrt{(1 - \varepsilon) \log n}\right) \geq \frac{C}{n^{1-\varepsilon/2}}.$$

As the $\{\zeta_n\}_{n \geq 0}$ are renewal times, therefore by the Borel Cantelli lemma, a.s.,

$$\sqrt{1 - \varepsilon} \leq \limsup_{n \rightarrow \infty} \frac{\sup_{t \in [\zeta_n, \zeta_{n+1}]} V_t}{\sqrt{\log n}} \leq \sqrt{1 + \varepsilon}. \tag{7.21}$$

By Lemma 5.1, $\mathbb{E}(\zeta_1) < \infty$ and thus, by the Strong Law of Large Numbers, $\zeta_n/n \rightarrow \mathbb{E}(\zeta_1)$, a.s. From the lower bound in (7.21), with probability 1, there exists a subsequence $n_k \rightarrow \infty$ and $t_{n_k} \in [\zeta_{n_k}, \zeta_{n_k+1}]$ such that $V_{t_{n_k}}/\sqrt{\log n_k} \geq \sqrt{1 - \varepsilon}$. Moreover, the SLLN implies that, a.s., $\log t_{n_k} \leq (1 + \varepsilon) \log n_k$ for sufficiently large k . Therefore, a.s.,

$$\frac{V_{t_{n_k}}}{\sqrt{\log t_{n_k}}} \geq \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}}$$

for sufficiently large k . Since this holds for every $\varepsilon > 0$, we obtain, a.s.,

$$\limsup_{t \rightarrow \infty} \frac{V_t}{\sqrt{\log t}} \geq 1. \tag{7.22}$$

From the upper bound in (7.21) and the SLLN, we see that, a.s., there is a positive integer n_0 such that for all $n \geq n_0$, $V_t/\sqrt{\log n} \leq \sqrt{1 + \varepsilon}$ and $\log t \geq (1 - \varepsilon) \log n$ for all $t \in [\zeta_n, \zeta_{n+1}]$. These imply

$$\frac{V_t}{\sqrt{\log t}} \leq \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}}$$

for all $t \geq \zeta_{n_0}$. Since $\varepsilon > 0$ can be taken arbitrarily small, we have, a.s.,

$$\limsup_{t \rightarrow \infty} \frac{V_t}{\sqrt{\log t}} \leq 1.$$

This inequality and (7.22) prove the second equality in (2.2).

The proofs of the first equality in (2.2) and the equality in (2.3) follow similarly by using (7.11) and (7.13) of Lemma 7.2, respectively. The claim in (2.4) follows from the fact that ζ_k 's are i.i.d. with $\mathbb{E} \zeta_k > 0$, and $S_t - X_t = 0$ if t is a renewal time. □

Proof of Theorem 2.3. We have

$$X_t - (B_t - gt) = (X_0 + V_0) - V_t,$$

$$S_t - (B_t - gt) = X_t - (B_t - gt) + (S_t - X_t) = (X_0 + V_0) - V_t + (S_t - X_t).$$

These identities and Theorem 2.2 easily imply the assertions made in Theorem 2.3. □

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