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Couplings, attractiveness and hydrodynamics for conservative particle systems

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Abstract. Attractiveness is a fundamental tool to study interacting particle systems and the basic coupling construction is a usual route to prove this property, as for instance in simple exclusion. The derived Markovian coupled process $(\xi_t, \zeta_t)_{t\geq 0}$ satisfies:

(A) if $\xi_0 \leq \zeta_0$ (coordinate-wise), then for all $t \geq 0$, $\xi_t \leq \zeta_t$ a.s.

In this paper, we consider generalized misanthrope models which are conservative particle systems on $\mathbb{Z}^d$ such that, in each transition, k particles may jump from a site x to another site y, with $k \geq 1$. These models include simple exclusion for which $k = 1$, but, beyond that value, the basic coupling construction is not possible and a more refined one is required. We give necessary and sufficient conditions on the rates to insure attractiveness; we construct a Markovian coupled process which both satisfies (A) and makes discrepancies between its two marginals non-increasing. We determine the extremal invariant and translation invariant probability measures under general irreducibility conditions. We apply our results to examples including a two-species asymmetric exclusion process with charge conservation (for which $k \leq 2$) which arises from a solid-on-solid interface dynamics, and a stick process (for which k is unbounded) in correspondence with a generalized discrete Hammersley–Aldous–Diaconis model. We derive the hydrodynamic limit of these two one-dimensional models.

Résumé. L'attractivité est un outil fondamental d'étude des systèmes à une infinité de particules en interaction; la construction du couplage de base est la méthode habituelle pour démontrer cette propriété (par exemple pour l'exclusion simple). Le processus couplé markovien $(\xi_t, \zeta_t)_{t\geq 0}$ obtenu vérifie:

(A) si $\xi_0 \leq \zeta_0$ (coordonnée par coordonnée), alors pour tout $t \geq 0$, $\xi_t \leq \zeta_t$ p.s.

Nous considérons dans cet article des systèmes de particules conservatifs sur $\mathbb{Z}^d$ qui généralisent le processus des misanthropes en ce que, à chaque transition, k particules peuvent sauter d'un site x vers un autre site y, avec $k \geq 1$. Ces modèles incluent l'exclusion simple, où $k = 1$, mais, au-delà de cette valeur, le couplage de base n'est plus valide et il faut une autre construction. Nous obtenons des conditions nécessaires et suffisantes pour l'attractivité sur les taux de transition; nous construisons un processus couplé markovien qui à la fois satisfait (A), et fait décroitre les discrépances entre ses deux marginales. Nous déterminons les probabilités invariantes et invariantes par translation extrémales sous des conditions générales d'irréductibilité. Nous appliquons nos résultats à des exemples incluant un modèle d'exclusion asymétrique à deux espèces avec conservation de la charge (où $k \leq 2$) issu d'une dynamique d'interfaces 'solid-on-solid', et un modèle de batons (où k n'est pas borné) en correspondance avec un processus de Hammersley–Aldous–Diaconis discret généralisé. Nous obtenons la limite hydrodynamique de ces deux modèles unidimensionnels.

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1. Introduction

Attractiveness is a fundamental property to study interacting particle systems: combined with coupling, it is a key tool to determine the set of invariant (and translation invariant) probability measures for the dynamics; it is also essential in obtaining hydrodynamic limits for conservative systems under Euler scaling (see [16,18]). Most widely known examples are simple exclusion (SEP), zero range (ZRP) or misanthropes (MP) processes (see [1,8]). Actually, these examples share two features: first, their infinitesimal transitions consist in the jump of a single particle (from a site x to another site y); they are also of *misanthrope type*, in the sense that their transition rates depend at most on x, y and on occupation numbers at x and y. These limitations on the possible variations of the conserved quantity are at the root of the construction of basic coupling, through which attractiveness is shown to hold.

In this paper, we consider conservative particle systems of misanthrope type but allowing more than one particle to jump at a time from a given site to another one. This seemingly slight generalization requires to restart almost from scratch to check attractiveness through a coupling construction. We obtain necessary and sufficient conditions on the rates for attractiveness, giving an explicit construction of an increasing coupling (that is, which shows that the process is attractive). Using these results, we determine the extremal invariant and translation invariant probability measures of the process. Finally, we illustrate our approach on a few examples for which we derive hydrodynamics.

The conservative particle systems we study are Markov processes $(\eta_t)_{t\geq 0}$ defined on a state space $\Omega = X^S$, with X a subset of $\mathbb{Z}$, and $S = \mathbb{Z}^d$; η_t is the configuration at time t and $\eta_t(z)$ its value at site $z \in S$. Depending on X, one of the two following interpretations can be used: for $X \subset \mathbb{N}$, $\eta_t(z)$ is the number of particles at site z; if $X \subset \mathbb{Z}$, $|\eta_t(z)|$ is the number of unit charges at site z (either positive for $\eta_t(z) > 0$ or negative for $\eta_t(z) < 0$). The evolution of the process can be informally described as follows. Given a pair $(x, y) \in S^2$ and an integer $k > 0$, with a rate $\Gamma^k_{\eta(x),\eta(y)}(y - x)$, k particles (resp. k positive charges) 'attempt a jump' from site x to site y. By this, we mean that the values at sites x and y are changed to $\eta(x) - k$ and $\eta(y) + k$ as long as the resulting configuration belongs to the state space. The conserved quantity is $\eta(x) + \eta(y)$, the total number of particles (resp. the total charge) involved in the transition.

The SEP–ZRP–MP are examples of such processes, for which $X \subset \mathbb{N}$, $k = 1$ is the only allowed move and the rates take the form $\Gamma^1_{\alpha,\beta}(y - x) = p(y - x)b(\alpha, \beta)$ for $\alpha, \beta \in X$, $x, y \in S$, $p(\cdot)$ a translation invariant probability kernel on S, and $b(\cdot, \cdot)$ a function of the configuration values at the two involved sites. These processes (see [1,8,18] for details) are *attractive*, that is, the coordinate-wise (partial) order on configurations is preserved (stochastically) by the dynamics (see Section 2 for precise definitions), if and only if

(A) The rate $b(\alpha, \beta)$ is a function of α, β non-decreasing in α and non-increasing in β.

Attractiveness for interacting particle systems is derived by constructing an *increasing Markovian coupling*. A *coupled* Markov process $(\xi_t, \zeta_t)_{t\geq 0}$ is a process on $\Omega \times \Omega$ such that each marginal is a copy of the original process $(\eta_t)_{t\geq 0}$. The coupling is *increasing* if partial order between marginal configurations is preserved under the coupled dynamics.

The simplest candidate for this purpose is the *basic* (or Vasershtein) *coupling*. It is designed to make the two coupled processes $(\xi_t)_{t\geq 0}$, $(\zeta_t)_{t\geq 0}$ 'as similar as possible,' and its construction can be obtained by relying on three rules:

(i) coupled transitions have the same departure and arrival sites for the two marginals,
(ii) an identical change in both processes is given the largest possible rate,
(iii) non coupled transitions are supplemented to get the correct left and right marginals.

In the context of SEP–ZRP–MP, these rules give: by (i), the same probability $p(y - x)$ is taken for coupled jumps from site x to site y. Then, if the values at sites x and y are $\xi_{t^-}(x) = \alpha$, $\xi_{t^-}(y) = \beta$, $\zeta_{t^-}(x) = \gamma$ and $\zeta_{t^-}(y) = \delta$, three changes may occur (remember that at most one particle can jump in a transition):

- with rate $(b(\alpha, \beta) \wedge b(\gamma, \delta))p(y - x)$, a particle jumps from x to y simultaneously in both processes;
- with rate $(b(\alpha, \beta) - b(\alpha, \beta) \wedge b(\gamma, \delta))p(y - x)$, only a ξ-particle jumps from x to y;
- with rate $(b(\gamma, \delta) - b(\alpha, \beta) \wedge b(\gamma, \delta))p(y - x)$, only a ζ-particle jumps from x to y.

By rule (ii), the last two changes are mutually exclusive, and their rates are fixed by rule (iii). Notice that the resulting Markovian coupling is not necessarily increasing. In fact, there are exactly two instances in which the coordinate-wise order $\xi_{t^-} \leq \zeta_{t^-}$ might be destroyed, namely ($\alpha \leq \gamma$ and $\beta = \delta$) or ($\alpha = \gamma$ and $\beta \leq \delta$). In the first (resp. second) one, the isolated jump of a ξ-particle (resp. a ζ-particle) would break the order between configurations. Attractiveness requires that the corresponding rates are equal to zero, or equivalently

$$b(\alpha,\delta) \le b(\gamma,\delta) \quad \text{for all } \delta \text{ and } \alpha \le \gamma,$$
$$b(\gamma,\beta) \le b(\gamma,\delta) \quad \text{for all } \gamma \text{ and } \beta \ge \delta,$$

which is exactly the content of condition (A).

Beyond attractiveness, the basic coupling construction turns out to be essential to characterize the set $(\mathcal{I}\cap\mathcal{S})_e$ of extremal invariant and translation invariant probability measures of $(\eta_t)_{t\ge 0}$, through a control of the evolution of *discrepancies* between the marginals of $(\xi_t,\zeta_t)_{t\ge 0}$. There is a discrepancy at $x\in S$ at time t if $\xi_t(x)\ne\zeta_t(x)$. In SEP–ZRP–MP, in a coupled transition the number of discrepancies on the involved sites remains constant whenever the values of the two marginal configurations are ordered, but may decrease otherwise. This is a consequence of the fact that condition (A) (applied twice) leads to inequalities between rates on unordered configurations,

$$b(\alpha,\beta) \le b(\gamma,\delta) \quad \text{for all } \alpha<\gamma \text{ and } \beta>\delta.$$

This decreasing property is crucial: under suitable conditions on $p(y-x)$ and $b(\alpha,\beta)$, it permits the identification of $(\mathcal{I}\cap\mathcal{S})_e$ as a one parameter family $\{\mu_\rho\}_\rho$ of product probability measures, where the parameter ρ fixes the average particle density per site.

Beyond these classical examples, the requirements needed for attractiveness through rules (i)–(iii) might be so drastic that the basic coupling construction is not possible. We therefore have to weaken those rules in order to get attractiveness through a coupling construction imposing less restrictive conditions on the rates. At the same time, we have to ask for a decrease of discrepancies, which came for free within basic coupling but could be less obvious in a larger setting.

Our starting point will be the more investigated case of pure jump processes with denumerable state space, which provides us with a set of necessary conditions on the rates for attractiveness. Restricting ourselves to the class of conservative systems we consider in this paper, it turns out that it is possible to construct a Markovian coupling, which is increasing whenever these necessary conditions are fulfilled, hence proving also their sufficiency. Another property of this coupling is that discrepancies are non increasing. However, a decrease of discrepancies requires additional irreducibility conditions. Under them, we prove that $(\mathcal{I}\cap\mathcal{S})_e$ is a one parameter family of probability measures. This gives the first step in the derivation of hydrodynamic limits under Euler scaling, which we explicitely achieve for two one-dimensional examples, where an alternative proof of macroscopic stability is needed in order to follow the approach of [4].

It is important to notice that the restriction to single particle jumps ($k=1$) is essential in the basic coupling construction and in the derivation of attractiveness, in particular in the two following instances: it induces implicitely a total order on S and a partial lexicographic order on the state space; it drastically restricts the ways by which the order between configurations can be broken. These two points make the weakening of basic coupling rules useless and hide the interplay between partial order, conservation rule and optimization of coupling rates, though its understanding seems necessary to deal with more complex systems. Working with $k>1$ gives some hints in these directions. In a forthcoming paper, we will consider attractiveness for conservative particle systems with speed change, a problem which requires a fine tuning of these (yet mostly hidden) properties.

In the last part of this paper, we will develop our results on various examples: First, in the well-known cases where only $k=1$ is possible (thus in particular SEP–ZRP–MP) our construction reduces to basic coupling. Moreover, our irreducibility conditions relax restrictions on the rates for ZRP–MP and allow to determine $(\mathcal{I}\cap\mathcal{S})_e$ in more general cases than previously treated.

As soon as we depart from $k\le 1$, both constructions show very different features: we focus on two one-dimensional examples, a discrete nearest neighbor stick process in correspondence with a generalized (that is, with nearest neighbor jumps) discrete Hammersley–Aldous–Diaconis (HAD) model, for which k is unbounded, and a two-species asymmetric exclusion process with charge conservation, for which $k\le 2$. A continuous stick process with totally asymmetric jumps was introduced in [20] in relation with the (continuous) HAD model, and the discrete HAD process was studied in particular in [10]. The two-species asymmetric exclusion process was introduced in [9] in the context of solid-on-solid interface dynamics and independently in [22,23]. Particular versions of this dynamics have been investigated in [11,12] in order to derive hydrodynamics in non-attractive cases.

The paper is organised as follows. In Section 2, we first recall the definition and properties of attractiveness. Then we introduce the conservative models $(\eta_t)_{t\geq 0}$ we are interested in, and state our main results: the set of necessary and sufficient conditions on the transition rates of $(\eta_t)_{t\geq 0}$ for attractiveness; the construction of the Markovian increasing coupling it involves, and under which discrepancies may decrease. Proofs are given in Section 3. In Section 5, we obtain $(\mathcal{I}\cap\mathcal{S})_e$ for $(\eta_t)_{t\geq 0}$ under additional irreducibility conditions. In Section 7, we analyze the hydrodynamic behavior of $(\eta_t)_{t\geq 0}$ when the dynamics is one-dimensional and nearest neighbor. We have illustrated our results on examples, the classical ones, SEP–ZRP–MP, as well as on the stick process and the two-species asymmetric exclusion process: In Section 4 and the Appendix, the coupling construction is presented in some detail and in Section 6 we determine $(\mathcal{I}\cap\mathcal{S})_e$ for each example. In Section 7 we obtain the hydrodynamic limit of the stick process and the two species asymmetric exclusion; for the latter we prove heuristic results given in [9] and rederived more recently in [22,23].

2. Preliminaries and results

2.1. *Background*

We first recall the general definitions and theorems about attractiveness and couplings that we use in this paper.

Let $(\eta_t)_{t\geq 0}$ be an interacting particle system of state space $\Omega = X^S$, with X a subset of $\mathbb{Z}$, and $S=\mathbb{Z}^d$. We denote by $T(t)$ the semi-group of this Markov process and by $\mathcal{L}$ its infinitesimal generator. We mainly refer to [18], which deals with Feller processes on compact state spaces. Since we also consider other cases, we assume that

$$\begin{aligned}&(\eta_t)_{t\geq 0} \text{ is a well defined Markov process on a subset } \Omega_0\subset\Omega\\ &\text{such that for any bounded local function } f \text{ on } \Omega_0,\\ &\forall\eta\in\Omega_0\quad \lim_{t\to 0}\frac{T(t)f(\eta)-f(\eta)}{t}=\mathcal{L}f(\eta)<+\infty.\end{aligned}\tag{2.1}$$

A *coupled process* $(\xi_t,\zeta_t)_{t\geq 0}$ of two copies $(\xi_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ of $(\eta_t)_{t\geq 0}$ is a Markov process with state space $\Omega_0\times\Omega_0$, such that each marginal is a copy of the original process $(\eta_t)_{t\geq 0}$.

There is a partial order (coordinate-wise) on the state space Ω, defined by

$$\forall\xi,\zeta\in\Omega\quad \xi\leq\zeta\quad\Longleftrightarrow\quad\big(\forall x\in S,\xi(x)\leq\zeta(x)\big).\tag{2.2}$$

Definition 2.1 (See [15]). *Let W be a set endowed with a partial order relation. A subset $V\subset W$ is* increasing *if*

$$\forall l\in V, m\in W\quad l\leq m\quad\Longrightarrow\quad m\in V.\tag{2.3}$$

A subset $V\subset W$ is decreasing *if*

$$\forall l\in V, m\in W\quad l\geq m\quad\Longrightarrow\quad m\in V.\tag{2.4}$$

A function $f:W\to\mathbb{R}$ is monotone *if*

$$\forall l,m\in W\quad l\leq m\quad\Longrightarrow\quad f(l)\leq f(m).\tag{2.5}$$

Example 2.2 ([14]). *Let $l\in W$. Then $I_l=\{m\in W\colon l\leq m\}$ is an increasing set, and $D_l=\{m\in W\colon l\geq m\}$ a decreasing one.*

Remark 2.3. *For any subset $V\subset W$,*

$$V \text{ is increasing}\quad\Longleftrightarrow\quad W\setminus V\text{ is decreasing}\quad\Longleftrightarrow\quad \mathbf{1}_V\text{ is monotone}.$$

We denote by $\mathcal{M}$ the set of all bounded, monotone continuous functions on Ω. The partial order (2.2) induces a *stochastic order* (see [15]) on the set $\mathcal{P}$ of probability measures on Ω endowed with the weak topology:

$$\forall \nu, \nu' \in \mathcal{P} \quad \nu \le \nu' \quad \Longleftrightarrow \quad \big(\forall f \in \mathcal{M}, \nu(f) \le \nu'(f)\big). \tag{2.6}$$

Remark 2.4. *$\nu \le \nu' \Longleftrightarrow \nu(V) \le \nu'(V)$ for all increasing subsets $V \subset \Omega$.*

Indeed, on the one hand use Remark 2.3, and on the other hand, notice that for any non-negative $f \in \mathcal{M}$ and all $s \ge 0$, the set $\{f \ge s\}$ is increasing, and $\nu(f) = \int_0^\infty \nu\{f \ge s\}\,\mathrm{d}s$.

Theorem 2.5 ([18], Theorem II.2.2). *For the particle system $(\eta_t)_{t\ge 0}$, the following two statements are equivalent*:

(a) *$f \in \mathcal{M}$ implies $T(t)f \in \mathcal{M}$ for all $t \ge 0$.*
(b) *For $\nu, \nu' \in \mathcal{P}$, $\nu \le \nu'$ implies $\nu T(t) \le \nu' T(t)$ for all $t \ge 0$.*

Definition 2.6 ([18], Definition II.2.3). *The particle system $(\eta_t)_{t\ge 0}$ is* attractive *if the equivalent statements of Theorem 2.5 are satisfied.*

By Remarks 2.3 and 2.4, it is enough to check Theorem 2.5(a) for indicator functions of all increasing sets.

In the following our main object is the infinitesimal generator $\mathcal{L}$ of the process since we look for necessary and sufficient conditions on the transition rates that yield attractiveness of the particle system. As mentioned in [15], this was investigated by many authors for pure jump processes (see also [13]; [21], Chapter 4) on a countable state space. In such cases, sufficiency consists in writing the semigroup as an exponentiation of the generator to check Theorem 2.5(b). This follows from Strassen's theorem, which links stochastic order to coupling (for a thorough analysis of this link, see [15]).

Theorem 2.7 ([18], Theorem II.2.4). *For $\nu, \nu' \in \mathcal{P}$, a necessary and sufficient condition for $\nu \le \nu'$ is the existence of a probability measure μ on $\Omega \times \Omega$, called a coupled probability measure, which satisfies*:

(a) *$\mu\{(\xi,\zeta)\colon \xi \in A\} = \nu(A)$, $\mu\{(\xi,\zeta)\colon \zeta \in A\} = \nu'(A)$, for all Borel sets $A \in \Omega$, and*
(b) *$\mu\{(\xi,\zeta)\colon \xi \le \zeta\} = 1$.*

Here we take the usual but longer route to derive attractiveness for interacting particle systems (see [18]), and check Theorem 2.5(a) by constructing a *Markovian increasing coupling*, that is a coupled process $(\xi_t, \zeta_t)_{t\ge 0}$ with the property that $\xi_0 \le \zeta_0$ implies

$$P^{(\xi_0,\zeta_0)}\{\xi_t \le \zeta_t\} = 1 \tag{2.7}$$

for all $t \ge 0$, where $P^{(\xi_0,\zeta_0)}$ denotes the distribution of $(\xi_t, \zeta_t)_{t\ge 0}$ with initial state (ξ_0, ζ_0).

Our strategy for proving and using attractiveness will consist in three steps:

(a) obtain necessary conditions on the transition rates of the model;
(b) construct a Markovian coupling and show that it is increasing under conditions obtained in (a), thus proving that they are also sufficient;
(c) verify that discrepancies cannot increase. When supplemented with some conditions on the rates, this will allow to determine $(\mathcal{I} \cap \mathcal{S})_e$.

2.2. *The model, attractiveness and coupling*

From now on, we restrict our analysis to the following class of models.

Let $S = \mathbb{Z}^d$ be the set of sites and $X \subset \mathbb{Z}$ be the set of admissible values on each site. In our examples, X will be either a finite subset of $\mathbb{Z}$ or $\mathbb{N}$. The infinitesimal generator $\mathcal{L}$ of the process $(\eta_t)_{t\ge 0}$ on $\Omega = X^S$ is given, for a local function f, by

$$\mathcal{L}f(\eta) = \sum_{x,y\in S} \sum_{\alpha,\beta\in X} \chi_{x,y}^{\alpha,\beta}(\eta) \sum_{k\in\mathbb{N}} \Gamma_{\alpha,\beta}^{k}(y-x)\big(f\big(S_{x,y}^{k}\eta\big) - f(\eta)\big), \tag{2.8}$$

where $\chi_{x,y}^{\alpha,\beta}$ is the indicator function of configurations with values (α,β) on (x,y),

$$\chi_{x,y}^{\alpha,\beta}(\eta)=\begin{cases}1 & \text{if } \eta(x)=\alpha \text{ and } \eta(y)=\beta,\\ 0 & \text{otherwise,}\end{cases} \tag{2.9}$$

$S_{x,y}^k$ is a local operator performing the transformation whenever possible (the value $k=0$ is not excluded)

$$\left(S_{x,y}^k\eta\right)(z)=\begin{cases}\eta(x)-k & \text{if } z=x \text{ and } \eta(x)-k\in X, \eta(y)+k\in X,\\ \eta(y)+k & \text{if } z=y \text{ and } \eta(x)-k\in X, \eta(y)+k\in X,\\ \eta(z) & \text{otherwise.}\end{cases} \tag{2.10}$$

This particle system is conservative, with $\eta(x)+\eta(y)$ the *conserved quantity* in a transition between sites x and y.

Unless X is finite and the rates $\Gamma_{\cdot,\cdot}^{\cdot}(\cdot)$'s have finite range (in which case we refer to [18], Chapter 1), additional conditions on the transition rates, and/or a reduction of the state space, are required to ensure that (2.8) is the infinitesimal generator of a well defined Markov process. Since such conditions differ for the specific model one deals with, we assume (2.1) and state here only a common necessary restriction on the rates (complete precise assumptions being given on examples). For all $z\in S$, $\alpha,\beta\in X$, the rates $\Gamma_{\alpha,\beta}^k(z)$ satisfy

$$\sum_{k\in\mathbb{N}}\Gamma_{\alpha,\beta}^k(z)<\infty. \tag{2.11}$$

For notational convenience, we will often drop the explicit dependence on z of $\Gamma_{\alpha,\beta}^k(z)$; we also set

$$\Gamma_{\alpha,\beta}^k=0 \quad \text{if } \alpha-k\notin X, \beta+k\notin X \text{ or } k=0. \tag{2.12}$$

Definition 2.8. *The notation $(\alpha,\beta)\le(\gamma,\delta)$ is equivalent to $\alpha\le\gamma,\beta\le\delta$; the two pairs $(\alpha,\beta),(\gamma,\delta)$ are* ordered *if $(\alpha,\beta)\le(\gamma,\delta)$ or $(\alpha,\beta)\ge(\gamma,\delta)$; they are* not ordered *otherwise, that is when $(\alpha<\gamma,\beta>\delta)$ or $(\alpha>\gamma,\beta<\delta)$.*

The main result of this section is the following theorem.

Theorem 2.9. *The particle system $(\eta_t)_{t\ge0}$ is attractive if and only if for all $(\alpha,\beta,\gamma,\delta)\in X^4$ with $(\alpha,\beta)\le(\gamma,\delta)$, for all $(x,y)\in S^2$,*

$$\forall l\ge0 \quad \sum_{k'>\delta-\beta+l}\Gamma_{\alpha,\beta}^{k'}(y-x)\le\sum_{l'>l}\Gamma_{\gamma,\delta}^{l'}(y-x), \tag{2.13}$$

$$\forall k\ge0 \quad \sum_{k'>k}\Gamma_{\alpha,\beta}^{k'}(y-x)\ge\sum_{l'>\gamma-\alpha+k}\Gamma_{\gamma,\delta}^{l'}(y-x). \tag{2.14}$$

Note that by (2.11) the above sums are finite. We now give the succession of steps which leads to this theorem. Proofs are postponed to Section 3.

First, we prove necessary conditions for attractiveness.

Proposition 2.10. *If the particle system $(\eta_t)_{t\ge0}$ is attractive, then for all $(\alpha,\beta,\gamma,\delta)\in X^4$ with $(\alpha,\beta)\le(\gamma,\delta)$, for all $(x,y)\in S^2$, inequalities (2.13)–(2.14) hold.*

The other part of Theorem 2.9 is harder and is obtained by constructing explicitly a Markovian coupling, which appears to be increasing under conditions (2.13)–(2.14). The evolution of the coupled process $(\xi_t,\zeta_t)_{t\ge0}$ we look for is defined through its infinitesimal generator $\overline{\mathcal{L}}$, whose rates will be derived from those of $\mathcal{L}$. Here, as in the basic coupling construction, *the same* departure and arrival sites x and y are chosen for jumps of coupled particles.

Proposition 2.11. *The operator $\overline{\mathcal{L}}$ defined on $\Omega_0 \times \Omega_0$ as*

$$\overline{\mathcal{L}} f(\xi,\zeta) = \sum_{x,y\in S} \sum_{\alpha,\beta\in X} \sum_{\gamma,\delta\in X} \chi_{x,y}^{\alpha,\beta}(\xi)\chi_{x,y}^{\gamma,\delta}(\zeta) \times \sum_{k,l} G_{\alpha,\beta;\gamma,\delta}^{k;l}(y-x)\big(f\big(S_{x,y}^k\xi, S_{x,y}^l\zeta\big) - f(\xi,\zeta)\big) \tag{2.15}$$

with coupling rates $G_{\alpha,\beta;\gamma,\delta}^{k;l}$ given by (2.16)–(2.19) below for all $(\alpha,\beta,\gamma,\delta)\in X^4$ and all non-negative k,l as functions of the initial rates $\Gamma_{\alpha,\beta}^{k'}$ and $\Gamma_{\gamma,\delta}^{l'}$, is the generator of a Markovian coupling between two copies of the Markov process defined by (2.8).

For all positive k,l, we set

$$G_{\alpha,\beta;\gamma,\delta}^{k;l} = \big(\Gamma_{\alpha,\beta}^k - \Gamma_{\alpha,\beta}^k \wedge \big(\Sigma_{\gamma,\delta}^l - \Sigma_{\alpha,\beta}^k \wedge \Sigma_{\gamma,\delta}^l\big)\big) \wedge \big(\Gamma_{\gamma,\delta}^l - \Gamma_{\gamma,\delta}^l \wedge \big(\Sigma_{\alpha,\beta}^k - \Sigma_{\alpha,\beta}^k \wedge \Sigma_{\gamma,\delta}^l\big)\big), \tag{2.16}$$

$$G_{\alpha,\beta;\gamma,\delta}^{0;l} = \Gamma_{\gamma,\delta}^l - \Gamma_{\gamma,\delta}^l \wedge \big(\Sigma_{\alpha,\beta}^0 - \Sigma_{\alpha,\beta}^0 \wedge \Sigma_{\gamma,\delta}^l\big), \tag{2.17}$$

$$G_{\alpha,\beta;\gamma,\delta}^{k;0} = \Gamma_{\alpha,\beta}^k - \Gamma_{\alpha,\beta}^k \wedge \big(\Sigma_{\gamma,\delta}^0 - \Sigma_{\alpha,\beta}^k \wedge \Sigma_{\gamma,\delta}^0\big), \tag{2.18}$$

$$G_{\alpha,\beta;\gamma,\delta}^{0;0} = 0, \tag{2.19}$$

where the $\Sigma_{\alpha,\beta}^k$'s denote partial sums of rates for $k \geq 0$,

$$\Sigma_{\alpha,\beta}^k = \sum_{k'>k} \Gamma_{\alpha,\beta}^{k'}. \tag{2.20}$$

In other words, at time t, configurations ξ_{t-}, ζ_{t-} are changed into $\xi_t = S_{x,y}^k\xi_{t-}, \zeta_t = S_{x,y}^l\zeta_{t-}$ with rate $G_{\xi_{t-}(x),\xi_{t-}(y);\zeta_{t-}(x),\zeta_{t-}(y)}^{k;l}(y-x)$. Rates with superscript '$k;0$' (resp. '$0;l$') correspond to uncoupled changes in the $\xi_.$ (resp. $\zeta_.$) marginal process.

Since the coupling rates are constructed for a fixed pair of sites (x,y), we dropped from the notation the explicit dependence of the rates on (x,y) and wrote $\Gamma_{\alpha,\beta}^k$ for $\Gamma_{\alpha,\beta}^k(y-x)$ and $G_{\alpha,\beta;\gamma,\delta}^{k;l}$ for $G_{\alpha,\beta;\gamma,\delta}^{k;l}(y-x)$.

Remark 2.12. *The partial sums $\Sigma_{\alpha,\beta}^k$'s in (2.20) are well defined by (2.11). As a direct consequence of expressions (2.16)–(2.19) for the coupling rates, the coupled process is well defined on $\Omega_0 \times \Omega_0$ (cf. (2.1)).*

Remark 2.13. *Formulas (2.16)–(2.19) are symmetric under exchange of marginals: for all $(\alpha,\beta,\gamma,\delta)$, k and l*

$$G_{\alpha,\beta;\gamma,\delta}^{k;l} = G_{\gamma,\delta;\alpha,\beta}^{l;k}.$$

Remark 2.14. *Like in the basic coupling construction, the diagonal coupling rates are equal to the original rates of the marginal processes: for $(\alpha,\beta,k) = (\gamma,\delta,l)$, (2.16) becomes*

$$G_{\alpha,\beta;\alpha,\beta}^{k;k} = \big(\Gamma_{\alpha,\beta}^k - \Gamma_{\alpha,\beta}^k \wedge \big(\Sigma_{\alpha,\beta}^k - \Sigma_{\alpha,\beta}^k \wedge \Sigma_{\alpha,\beta}^k\big)\big) = \Gamma_{\alpha,\beta}^k.$$

In the next two propositions, we give some hints on the construction and structure of the coupling rates; for any fixed $(\alpha,\beta,\gamma,\delta)\in X^4$, non-zero coupling rates are located on a "staircase-shaped" path in the (k,l) quadrant: most of the coupling rates are zero and for $n>0$ there is at most one pair (k,l) such that $k+l=n$ and $G_{\alpha,\beta;\gamma,\delta}^{k;l} > 0$. Alternatively, the coupling rates can be defined by induction.

Proposition 2.15. *For any* $(\alpha,\beta,\gamma,\delta)\in X^4$, *all* $z\in S$, *let* $(k_0,l_0)=(0,0)$ *and* $\mathcal{P}_{\alpha,\beta;\gamma,\delta}=\{(k_i,l_i)\}_{i>0}$ *be the sequence of points in* $\mathbb{N}\times\mathbb{N}$ *defined for all* $i>0$ *as*

$$\begin{cases} k_i=k_{i-1}+1, & l_i=l_{i-1} \quad \text{if } \Sigma^{k_{i-1}}_{\alpha,\beta}\geq \Sigma^{l_{i-1}}_{\gamma,\delta},\\ k_i=k_{i-1}, & l_i=l_{i-1}+1 \quad \text{otherwise}.\end{cases} \tag{2.21}$$

The coupling rates defined by Eqs (2.16)–(2.19) *satisfy*:

(i) *For all* $(k_i,l_i)\in\mathcal{P}_{\alpha,\beta;\gamma,\delta}$,

$$G^{k_i;l_i}_{\alpha,\beta;\gamma,\delta}=\Sigma^{k_{i-1}}_{\alpha,\beta}\vee\Sigma^{l_{i-1}}_{\gamma,\delta}-\Sigma^{k_i}_{\alpha,\beta}\vee\Sigma^{l_i}_{\gamma,\delta}. \tag{2.22}$$

(ii) *For all* $(k,l)\notin\mathcal{P}_{\alpha,\beta;\gamma,\delta}$,

$$G^{k;l}_{\alpha,\beta;\gamma,\delta}=0. \tag{2.23}$$

Proposition 2.16. *The set of coupling rates defined by Eqs* (2.16)–(2.19) *is the unique solution of the recursion relations*:

For all $(\alpha,\beta,\gamma,\delta)\in X^4$, *all* $z\in S$ *and all positive* k,l,

$$G^{k;l}_{\alpha,\beta;\gamma,\delta}=\Bigl(\Gamma^k_{\alpha,\beta}-\sum_{l'>l}G^{k;l'}_{\alpha,\beta;\gamma,\delta}\Bigr)\wedge\Bigl(\Gamma^l_{\gamma,\delta}-\sum_{k'>k}G^{k';l}_{\alpha,\beta;\gamma,\delta}\Bigr), \tag{2.24}$$

$$G^{0;l}_{\alpha,\beta;\gamma,\delta}=\Gamma^l_{\gamma,\delta}-\sum_{k'>0}G^{k';l}_{\alpha,\beta;\gamma,\delta}, \tag{2.25}$$

$$G^{k;0}_{\alpha,\beta;\gamma,\delta}=\Gamma^k_{\alpha,\beta}-\sum_{l'>0}G^{k;l'}_{\alpha,\beta;\gamma,\delta}, \tag{2.26}$$

$$G^{0;0}_{\alpha,\beta;\gamma,\delta}=0. \tag{2.27}$$

We end this subsection with a proposition giving the last step to Theorem 2.9.

Proposition 2.17. *Under conditions* (2.13)–(2.14), *the Markovian coupling defined in Proposition* 2.11 *is increasing. More precisely, if* (α,β), (γ,δ) *are ordered, then the coupling rate* $G^{k;l}_{\alpha,\beta;\gamma,\delta}$ (*with non-negative* k,l) *is nonzero only if the increments satisfy*

$$\begin{cases} l-k\in\{-(\delta-\beta),\dots,\gamma-\alpha\} & \text{if } (\alpha,\beta)\leq(\gamma,\delta),\\ l-k\in\{-(\alpha-\gamma),\dots,\beta-\delta\} & \text{if } (\alpha,\beta)\geq(\gamma,\delta).\end{cases} \tag{2.28}$$

2.3. *Evolution of discrepancies*

Attractiveness expresses that two ordered configurations remain such (stochastically) under the dynamics. Many applications require also a control over unordered pairs of configurations. This can either be a consequence of attractiveness or require some additional hypotheses, that we investigate in this subsection.

Definition 2.18. *In the coupled process* $(\xi_t,\zeta_t)_{t\geq 0}$, *there is a* discrepancy *at site* $z\in S$ *at time* t *if* $\xi_t(z)\neq\zeta_t(z)$. *This discrepancy is* positive (*resp*. negative) of width $a>0$ *if* $\xi_t(z)-\zeta_t(z)=a$ (*resp*. $\xi_t(z)-\zeta_t(z)=-a$).

We first show that under the Markovian increasing coupling constructed in Proposition 2.11, the sum of the widths of the discrepancies (involved in a transition) between the two coupled processes $(\xi_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ is non-increasing (later on, we will sometimes forget the word ‘width,’ and speak of non-increasing or decreasing discrepancies).

We fix x, y two sites of S. Denote by $\overline{\mathcal{L}}_{x,y}$ the generator of jumps between them,

$$\overline{\mathcal{L}}_{x,y} f(\xi,\zeta) = \sum_{\alpha,\beta\in X}\sum_{\gamma,\delta\in X} \chi_{x,y}^{\alpha,\beta}(\xi)\chi_{x,y}^{\gamma,\delta}(\zeta)\bigg(\sum_{k,l} G_{\alpha,\beta;\gamma,\delta}^{k;l}(y-x)\big(f\big(S_{x,y}^k\xi, S_{x,y}^l\zeta\big) - f(\xi,\zeta)\big) + \sum_{k,l} G_{\beta,\alpha;\delta,\gamma}^{l;k}(x-y)\big(f\big(S_{y,x}^l\xi, S_{y,x}^k\zeta\big) - f(\xi,\zeta)\big)\bigg) \tag{2.29}$$

for a local function f, and two configurations $\xi, \zeta \in \Omega_0$.

Let $f_{x,y}^{\pm} = f_{y,x}^{\pm}$ be the functions which measure the width of positive or negative discrepancies between ξ and ζ on sites x and y

$$\begin{aligned} &\psi_x^{\pm}(\xi,\zeta) := \big[\xi(x) - \zeta(x)\big]^{\pm}, \\ &f_{x,y}^{\pm}(\xi,\zeta) := \psi_x^{\pm}(\xi,\zeta) + \psi_y^{\pm}(\xi,\zeta). \end{aligned} \tag{2.30}$$

The quantities

$$\Delta^{\pm}\big(\xi(x),\xi(y),\zeta(x),\zeta(y),k,l\big) := f_{x,y}^{\pm}\big(S_{x,y}^k\xi, S_{x,y}^l\zeta\big) - f_{x,y}^{\pm}(\xi,\zeta) \tag{2.31}$$

measure how, in a coupled transition on sites x, y, discrepancies evolve. Since by the conservation rule,

$$f_{x,y}^{+}(\xi,\zeta) - f_{x,y}^{-}(\xi,\zeta) = \big(\xi(x)+\xi(y)\big) - \big(\zeta(x)+\zeta(y)\big) = f_{x,y}^{+}\big(S_{x,y}^k\xi, S_{x,y}^l\zeta\big) - f_{x,y}^{-}\big(S_{x,y}^k\xi, S_{x,y}^l\zeta\big)$$

the two quantities $\Delta^{\pm}$ in (2.31) are equal and we set

$$\Delta\big(\xi(x),\xi(y),\zeta(x),\zeta(y),k,l\big) = \Delta^{\pm}\big(\xi(x),\xi(y),\zeta(x),\zeta(y),k,l\big) \tag{2.32}$$

with the meaning that positive and negative discrepancies disappear (or get reduced) by pair annihilation. Notice the symmetry

$$\Delta\big(\xi(y),\xi(x),\zeta(y),\zeta(x),l,k\big) = \Delta\big(\xi(x),\xi(y),\zeta(x),\zeta(y),k,l\big), \tag{2.33}$$

because by exchanging respectively $\xi(x)$ and $\xi(y)$, $\zeta(x)$ and $\zeta(y)$, k and l,

$$\begin{aligned} &\big[\big(\xi(y)-l\big) - \big(\zeta(y)-k\big)\big]^+ = \big[\big(\xi(y)+k\big) - \big(\zeta(y)+l\big)\big]^+, \\ &\big[\big(\xi(x)+l\big) - \big(\zeta(x)+k\big)\big]^+ = \big[\big(\xi(x)-k\big) - \big(\zeta(x)-l\big)\big]^+. \end{aligned}$$

Next theorem details how discrepancies decrease.

Theorem 2.19. *For all* $(\xi,\zeta)\in\Omega_0\times\Omega_0$, $(x,y)\in S^2$,

$$\overline{\mathcal{L}}_{x,y} f_{x,y}^{+}(\xi,\zeta) = \overline{\mathcal{L}}_{x,y} f_{x,y}^{-}(\xi,\zeta) \le 0. \tag{2.34}$$

More precisely, setting $(\alpha,\beta,\gamma,\delta) = (\xi(x),\xi(y),\zeta(x),\zeta(y))$, *the coupling rate* $G_{\alpha,\beta;\gamma,\delta}^{k;l}(y-x)$ *may be positive only for non-negative increments* k, l *satisfying*:

$$-[\alpha-\gamma]^+ - [\delta-\beta]^+ \le l-k \le [\gamma-\alpha]^+ + [\beta-\delta]^+ \tag{2.35}$$

and in that case

$$\begin{cases} \Delta(\alpha,\beta,\gamma,\delta,k,l) = 0 & \text{for } (\alpha,\beta),(\gamma,\delta) \text{ ordered}, \\ \Delta(\alpha,\beta,\gamma,\delta,k,l) = 0 & \text{for } (\alpha,\beta),(\gamma,\delta) \text{ not ordered and } k-l\in\big\{0,(\alpha-\gamma)+(\delta-\beta)\big\}, \\ \Delta(\alpha,\beta,\gamma,\delta,k,l) < 0 & \text{otherwise}. \end{cases} \tag{2.36}$$

In (2.36), the first line expresses attractiveness. The second line deals with discrepancies of opposite signs and contains two cases: when $k - l = 0$, the same number of particles moves together in both processes so that discrepancies do not change (as in basic coupling, for which $k = l = 1$); when $k - l = (\alpha - \gamma) + (\delta - \beta)$, the positions of discrepancies at sites x and y are exchanged. The third line expresses a decrease of discrepancies, it will be the key tool in the determination of $(\mathcal{I} \cap \mathcal{S})_e$ (see Section 5). Notice that for basic coupling, this case corresponds only to uncoupled jumps, while here it may involve non-zero values for both k and l; the jump gets the two marginals closer by merging together two discrepancies of opposite signs. The resulting pairs of values need not be ordered since discrepancies can reduce their widths without disappearing and possibly exchange their signs (as in the second line). This new phenomenon of (partial) *exchange of discrepancies* does exist *beyond* basic coupling (it requires $|k - l| \geq 2$) and induces some changes in the behavior of the coupled process. More precisely:

Definition 2.20. *The increasing coupling defined by* (2.15) allows exchanges of discrepancies *whenever there exist* $(x, y) \in S^2$, $(\alpha, \beta, \gamma, \delta) \in X^4$ *with* $(\alpha - \gamma)(\beta - \delta) < 0$, *non-negative* k, l *with* $|k - l| > |\alpha - \gamma| \vee |\beta - \delta|$ *such that* $G^{k;l}_{\alpha,\beta;\gamma,\delta}(y - x) > 0$. *The exchange of discrepancies is* total *if* $k - l = (\alpha - \gamma) + (\delta - \beta)$ *and* partial *otherwise.*

Lemma 2.21. *The increasing coupling defined by* (2.15) *does not allow exchanges of discrepancies if and only if for any* $(x, y) \in S^2$, $(\alpha, \beta, \gamma, \delta) \in X^4$, *non-negative* k, l *such that* $G^{k;l}_{\alpha,\beta;\gamma,\delta}(y - x) > 0$, *we have*

$$-\big([\alpha - \gamma]^+ \vee [\delta - \beta]^+\big) \leq l - k \leq [\gamma - \alpha]^+ \vee [\beta - \delta]^+. \tag{2.37}$$

Remark 2.22. *When* $(\alpha, \beta), (\gamma, \delta)$ *are ordered*, (2.37) *reduces to inequalities* (2.28) *for attractiveness. Otherwise*, (2.37) *is stronger than* (2.35) (*which is a consequence of attractiveness*).

A first application of Theorem 2.19 is to determine extremal invariant and translation invariant probability measures for the model (see Theorem 5.13) under irreducibility conditions on the rates, that cope with exchanges of discrepancies and other phenomena which appear beyond the case $k \leq 1$.

A second application is *macroscopic stability*, an essential step to derive the hydrodynamic limit of one-dimensional models (see Section 7). This property, introduced in [7], Proposition 3.1, for one-dimensional finite-range simple exclusion processes, gives a control over the time evolution of discrepancies in the coupled process. However, its proof does not extend to models with jumps of size $k \geq 1$, in particular because the latter induce exchanges of discrepancies. So we prove the following proposition.

Proposition 2.23. *We assume that the particle system defined by* (2.8) *is one-dimensional* ($S = \mathbb{Z}$) *with only nearest neighbor transitions, and that the increasing coupling defined by* (2.15) *does not allow exchanges of discrepancies. Then, if the coupled process* $(\xi_s, \zeta_s)_{s\geq 0}$ *is such that* $\sum_{x\in\mathbb{Z}}[\xi_0(x) + \zeta_0(x)] < +\infty$, *we have for every* $t > 0$,

$$S(\xi_t, \zeta_t) \leq S(\xi_0, \zeta_0), \tag{2.38}$$

where, for $x \in \mathbb{Z}$,

$$s_x(\xi_s, \zeta_s) := \sum_{y\leq x}\big[\xi_s(y) - \zeta_s(y)\big], \qquad S(\xi_s, \zeta_s) := \sup_{x\in\mathbb{Z}}\big|s_x(\xi_s, \zeta_s)\big|. \tag{2.39}$$

Remark 2.24. *Both restriction to nearest neighbor interactions and condition* (2.37) *are necessary to get the macroscopic stability property* (2.38). *See Section* 3 *for counter-examples.*

Proposition 2.25. *The following inequalities imply* (2.37): *For all* $(\alpha, \beta, \gamma, \delta) \in X^4$, *non-negative* k, l,

$$\Sigma^l_{\gamma,\delta} \geq \Sigma^L_{\alpha,\beta}, \quad L = l + [\alpha - \gamma]^+ \vee [\delta - \beta]^+, \tag{2.40}$$

$$\Sigma^k_{\alpha,\beta} \geq \Sigma^K_{\gamma,\delta}, \quad K = k + [\gamma - \alpha]^+ \vee [\beta - \delta]^+. \tag{2.41}$$

Remark 2.26. *When* $(\alpha, \beta), (\gamma, \delta)$ *are ordered*, (2.40)–(2.41) *coincide with* (2.13)–(2.14) *and thus reduce to attractiveness conditions.*

3. Proofs

Proof of Proposition 2.10. We proceed in two steps. The first one yields an inequality for the generator of an attractive system, when applied on indicators of increasing sets. Such inequalities were first derived in [19] for Markov jump processes on a countable state space E, with an infinitesimal generator bounded in $l_1(E)$. The second step specializes the previous inequality to our model on which it reads (2.13)–(2.14).

Step 1. Let $(\xi,\zeta)\in\Omega_0\times\Omega_0$ be two configurations such that $\xi\le\zeta$. Let $V\subset\Omega$ be an increasing cylinder set. If $\xi\in V$ or $\zeta\notin V$,

$$\mathbf{1}_V(\xi)=\mathbf{1}_V(\zeta). \tag{3.1}$$

By attractiveness, for all $t\ge 0$,

$$\big(T(t)\mathbf{1}_V\big)(\xi)\le\big(T(t)\mathbf{1}_V\big)(\zeta)$$

(use Remark 2.4 and Theorem 2.5(a)). Combining this with (3.1),

$$t^{-1}\big[\big(T(t)\mathbf{1}_V\big)(\xi)-\mathbf{1}_V(\xi)\big]\le t^{-1}\big[\big(T(t)\mathbf{1}_V\big)(\zeta)-\mathbf{1}_V(\zeta)\big],$$

which gives, by Assumption (2.1),

$$(\mathcal{L}\mathbf{1}_V)(\xi)\le(\mathcal{L}\mathbf{1}_V)(\zeta). \tag{3.2}$$

We have

$$\begin{aligned}(\mathcal{L}\mathbf{1}_V)(\xi)&=\sum_{x,y\in S}\sum_{\alpha,\beta\in X}\chi_{x,y}^{\alpha,\beta}(\xi)\sum_k\Gamma_{\alpha,\beta}^k(y-x)\big(\mathbf{1}_V\big(S_{x,y}^k\xi\big)-\mathbf{1}_V(\xi)\big)\\&=-\mathbf{1}_V(\xi)\sum_{x,y\in S}\sum_{\alpha,\beta\in X}\chi_{x,y}^{\alpha,\beta}(\xi)\sum_k\mathbf{1}_{\Omega_0\setminus V}\big(S_{x,y}^k\xi\big)\Gamma_{\alpha,\beta}^k(y-x)\\&\quad+\mathbf{1}_{\Omega_0\setminus V}(\xi)\sum_{x,y\in S}\sum_{\alpha,\beta\in X}\chi_{x,y}^{\alpha,\beta}(\xi)\sum_k\mathbf{1}_V\big(S_{x,y}^k\xi\big)\Gamma_{\alpha,\beta}^k(y-x).\end{aligned} \tag{3.3}$$

Step 2. We fix $(x,y)\in S^2$, $x\ne y$, $(\alpha,\beta,\gamma,\delta)\in X^4$, with $(\alpha,\beta)\le(\gamma,\delta)$, and two configurations (ξ,ζ) in $\Omega_0\times\Omega_0$ such that $\xi(x)=\alpha$, $\xi(y)=\beta$, $\zeta(x)=\gamma$, $\zeta(y)=\delta$, and $\xi(z)=\zeta(z)$ for all $z\ne x,y$. Thus $\xi\le\zeta$.

We fix numbers $(p_z\in X, z\in S)$, which satisfy $p_x<\xi(x)=\alpha$, $p_y>\zeta(y)=\delta$, and $p_z=\xi(z)=\zeta(z)$ for all $z\in S\setminus\{x,y\}$. We denote by $\|\cdot\|$ the L_1-norm on S. For $n\ge\|y-x\|$, we define

$$I_y(n)=\big\{\eta\in\Omega:\ \eta(z)\ge p_z,\ \text{for all } z\ne x \text{ such that } \|y-z\|\le n\big\}.$$

By Example 2.2, $I_y(n)$ is an increasing cylinder set. Since neither ζ nor ξ belong to $I_y(n)$, we have using (3.3)

$$\begin{aligned}(\mathcal{L}\mathbf{1}_{I_y(n)})(\xi)&=\sum_{x',y'\in S}\sum_{\alpha',\beta'\in X}\chi_{x',y'}^{\alpha',\beta'}(\xi)\sum_{k'}\mathbf{1}_{I_y(n)}\big(S_{x',y'}^{k'}\xi\big)\Gamma_{\alpha',\beta'}^{k'}\big(y'-x'\big)\\&=\sum_{k'\in\mathbb{N}}\mathbf{1}_{\{\beta+k'\ge p_y\}}\Gamma_{\alpha,\beta}^{k'}(y-x)+\sum_{z:\|y-z\|>n}\sum_{k'\in\mathbb{N}}\mathbf{1}_{\{\beta+k'\ge p_y\}}\Gamma_{\xi(z),\beta}^{k'}(y-z)\end{aligned}$$

and similarly

$$(\mathcal{L}\mathbf{1}_{I_y(n)})(\zeta)=\sum_{l'\in\mathbb{N}}\mathbf{1}_{\{\delta+l'\ge p_y\}}\Gamma_{\gamma,\delta}^{l'}(y-x)+\sum_{z:\|y-z\|>n}\sum_{l'\in\mathbb{N}}\mathbf{1}_{\{\delta+l'\ge p_y\}}\Gamma_{\zeta(z),\delta}^{l'}(y-z)$$

so that (3.2) writes

$$\sum_{k'\in\mathbb{N}:\beta+k'\geq p_y}\Gamma^{k'}_{\alpha,\beta}(y-x)+\sum_{z:\|y-z\|>n}\sum_{k'\in\mathbb{N}}\mathbf{1}_{\{\beta+k'\geq p_y\}}\Gamma^{k'}_{\xi(z),\beta}(y-z)$$
$$\leq\sum_{l'\in\mathbb{N}:\delta+l'\geq p_y}\Gamma^{l'}_{\gamma,\delta}(y-x)+\sum_{z:\|y-z\|>n}\sum_{l'\in\mathbb{N}}\mathbf{1}_{\{\delta+l'\geq p_y\}}\Gamma^{l'}_{\zeta(z),\delta}(y-z).$$

Taking the monotone limit $n\to\infty$ gives

$$\sum_{k'\in\mathbb{N}:\beta+k'\geq p_y}\Gamma^{k'}_{\alpha,\beta}(y-x)\leq\sum_{l'\in\mathbb{N}:\delta+l'\geq p_y}\Gamma^{l'}_{\gamma,\delta}(y-x).\tag{3.4}$$

A similar argument can be used with the decreasing set

$$D_x(n)=\bigl\{\eta\in\Omega:\ \eta(z)\leq p_z,\ \text{for all } z\neq y \text{ such that } \|x-z\|\leq n\bigr\}.$$

Configurations ξ and ζ belong to its complement $\Omega_0\setminus D_x(n)$ (which is increasing by Remark 2.3) and inequality (3.2) for this set leads to

$$\sum_{k'\in\mathbb{N}:\alpha-k'\leq p_x}\Gamma^{k'}_{\alpha,\beta}(y-x)\geq\sum_{l'\in\mathbb{N}:\gamma-l'\leq p_x}\Gamma^{l'}_{\gamma,\delta}(y-x).\tag{3.5}$$

Finally, taking $p_y=\delta+l+1$ in (3.4) and $p_x=\alpha-k-1$ in (3.5) gives (2.13)–(2.14). □

In order to prove Proposition 2.11, we give equivalent expressions for the coupling rates.

Lemma 3.1. *The following expressions are equivalent to* (2.16)–(2.18) *for all* $(\alpha,\beta,\gamma,\delta)\in X^4$ *and positive* k,l:

$$G^{k;l}_{\alpha,\beta;\gamma,\delta}=\Gamma^k_{\alpha,\beta}\wedge\bigl(\Sigma^{l-1}_{\gamma,\delta}-\Sigma^k_{\alpha,\beta}\wedge\Sigma^{l-1}_{\gamma,\delta}\bigr)-\Gamma^k_{\alpha,\beta}\wedge\bigl(\Sigma^l_{\gamma,\delta}-\Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\bigr),\tag{3.6}$$
$$G^{k;l}_{\alpha,\beta;\gamma,\delta}=\bigl(\Sigma^{k-1}_{\alpha,\beta}-\Sigma^{k-1}_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\bigr)\wedge\Gamma^l_{\gamma,\delta}-\bigl(\Sigma^k_{\alpha,\beta}-\Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\bigr)\wedge\Gamma^l_{\gamma,\delta},\tag{3.7}$$
$$G^{0;l}_{\alpha,\beta;\gamma,\delta}=\bigl(\Sigma^{l-1}_{\gamma,\delta}-\Sigma^0_{\alpha,\beta}\wedge\Sigma^{l-1}_{\gamma,\delta}\bigr)-\bigl(\Sigma^l_{\gamma,\delta}-\Sigma^0_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\bigr),\tag{3.8}$$
$$G^{k;0}_{\alpha,\beta;\gamma,\delta}=\bigl(\Sigma^{k-1}_{\alpha,\beta}-\Sigma^{k-1}_{\alpha,\beta}\wedge\Sigma^0_{\gamma,\delta}\bigr)-\bigl(\Sigma^k_{\alpha,\beta}-\Sigma^k_{\alpha,\beta}\wedge\Sigma^0_{\gamma,\delta}\bigr).\tag{3.9}$$

Proof. We prove only (3.7), (3.9); the two other equations follow by symmetry from Remark 2.13. Let $(\alpha,\beta,\gamma,\delta)\in X^4$, $k>0$ and $l\geq 0$. Using the elementary identity

$$\forall a,b,c\geq 0\quad a\wedge(c-b\wedge c)=(a+b)\wedge c-b\wedge c\tag{3.10}$$

with $a=\Gamma^k_{\alpha,\beta}$, $b=\Sigma^k_{\alpha,\beta}$, $c=\Sigma^l_{\gamma,\delta}$ leads to (we have $\Gamma^k_{\alpha,\beta}=\Sigma^{k-1}_{\alpha,\beta}-\Sigma^k_{\alpha,\beta}$)

$$\begin{aligned}\Gamma^k_{\alpha,\beta}-\Gamma^k_{\alpha,\beta}\wedge\bigl(\Sigma^l_{\gamma,\delta}-\Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\bigr)&=\bigl[(a+b)-(a+b)\wedge c\bigr]-[b-b\wedge c]\\&=\bigl(\Sigma^{k-1}_{\alpha,\beta}-\Sigma^{k-1}_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\bigr)-\bigl(\Sigma^k_{\alpha,\beta}-\Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\bigr).\end{aligned}\tag{3.11}$$

Recalling (2.18) and setting $l=0$ in (3.11) gives (3.9). Inserting (3.11) in (2.16), and using identity (3.10) again with $a=(\Sigma^{k-1}_{\alpha,\beta}-\Sigma^{k-1}_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta})-(\Sigma^k_{\alpha,\beta}-\Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta})$, $b=\Sigma^k_{\alpha,\beta}-\Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}$ and $c=\Gamma^l_{\gamma,\delta}$ gives

$$\begin{aligned}G^{k;l}_{\alpha,\beta;\gamma,\delta}&=\bigl[\bigl(\Sigma^{k-1}_{\alpha,\beta}-\Sigma^{k-1}_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\bigr)-\bigl(\Sigma^k_{\alpha,\beta}-\Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\bigr)\bigr]\wedge\bigl[\Gamma^l_{\gamma,\delta}-\Gamma^l_{\gamma,\delta}\wedge\bigl(\Sigma^k_{\alpha,\beta}-\Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\bigr)\bigr]\\&=\bigl(\Sigma^{k-1}_{\alpha,\beta}-\Sigma^{k-1}_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\bigr)\wedge\Gamma^l_{\gamma,\delta}-\bigl(\Sigma^k_{\alpha,\beta}-\Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\bigr)\wedge\Gamma^l_{\gamma,\delta}\end{aligned}$$

which is (3.7). □

Proof of Proposition 2.11. Let $(\alpha,\beta,\gamma,\delta)\in X^4$ and positive k,l. Using (2.18), (3.6) and a telescopic argument, partial sums of coupling rates read

$$\sum_{l'=0}^{l} G^{k;l'}_{\alpha,\beta;\gamma,\delta} = G^{k;0}_{\alpha,\beta;\gamma,\delta} + \sum_{l'=1}^{l} G^{k;l'}_{\alpha,\beta;\gamma,\delta} = \Gamma^k_{\alpha,\beta} - \Gamma^k_{\alpha,\beta}\wedge\big(\Sigma^l_{\gamma,\delta} - \Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\big).$$

In the limit $l\to\infty$, $\Sigma^l_{\gamma,\delta}\to 0$ by assumption (2.11), thus

$$\lim_{l\to\infty} \Gamma^k_{\alpha,\beta}\wedge\big(\Sigma^l_{\gamma,\delta} - \Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\big) = 0. \tag{3.12}$$

This gives the correct left marginal for all $k>0$,

$$\sum_{l'\ge 0} G^{k;l'}_{\alpha,\beta;\gamma,\delta} = G^{k;0}_{\alpha,\beta;\gamma,\delta} + \sum_{l'>0} G^{k;l'}_{\alpha,\beta;\gamma,\delta} = \Gamma^k_{\alpha,\beta}. \tag{3.13}$$

The right marginal can be treated in a similar way using (2.17) and (3.7). We get

$$\lim_{k\to\infty} \big(\Sigma^k_{\alpha,\beta} - \Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\big)\wedge\Gamma^l_{\gamma,\delta} = 0 \tag{3.14}$$

thus, for all $l>0$,

$$\sum_{k'\ge 0} G^{k';l}_{\alpha,\beta;\gamma,\delta} = G^{0;l}_{\alpha,\beta;\gamma,\delta} + \sum_{k'>0} G^{k';l}_{\alpha,\beta;\gamma,\delta} = \Gamma^l_{\gamma,\delta}. \tag{3.15}$$

□

Proofs of Propositions 2.15 and 2.16 rely on the following lemmas.

Lemma 3.2. *The coupling rates defined by* (2.16)–(2.19) *satisfy the recursion relations* (2.24)–(2.27).

Proof. Equations (2.25)–(2.26) are verified by (2.16)–(2.18) as a direct consequence of (3.13)–(3.15) (and (2.19) is identical to (2.27)). Equation (2.24) requires a partial resummation of the coupling rates: using (3.6), (3.12) on the one hand and (3.7), (3.14) on the other hand, gives

$$\sum_{l'>l} G^{k;l'}_{\alpha,\beta;\gamma,\delta} = \Gamma^k_{\alpha,\beta}\wedge\big(\Sigma^l_{\gamma,\delta} - \Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\big) \tag{3.16}$$

for $k>0$ and $l\ge 0$, and

$$\sum_{k'>k} G^{k';l}_{\alpha,\beta;\gamma,\delta} = \big(\Sigma^k_{\alpha,\beta} - \Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\big)\wedge\Gamma^l_{\gamma,\delta} \tag{3.17}$$

for $k\ge 0$ and $l>0$. Both right-hand sides above appear in (2.16); substituting them by the respective left-hand side of (3.16) and (3.17) in (2.16) gives (2.24). □

Lemma 3.3. *For all* $(\alpha,\beta,\gamma,\delta)\in X^4$, *any solution of* (2.24)–(2.27) *satisfies for all* $(k,l)\in\mathbb{N}\times\mathbb{N}$,

$$\Sigma^k_{\alpha,\beta}\ge\Sigma^l_{\gamma,\delta} \quad\Longrightarrow\quad \forall k'\le k,\forall l'>l \quad G^{k';l'}_{\alpha,\beta;\gamma,\delta}=0, \tag{3.18}$$

$$\Sigma^k_{\alpha,\beta}<\Sigma^l_{\gamma,\delta} \quad\Longrightarrow\quad \forall k'>k,\forall l'\le l \quad G^{k';l'}_{\alpha,\beta;\gamma,\delta}=0. \tag{3.19}$$

Proof. Inserting (2.25)–(2.26) into (2.24) writes

$$\Big(\sum_{0\le l'\le l} G^{k;l'}_{\alpha,\beta;\gamma,\delta}\Big)\wedge\Big(\sum_{0\le k'\le k} G^{k';l}_{\alpha,\beta;\gamma,\delta}\Big)=G^{k;l}_{\alpha,\beta;\gamma,\delta}.$$

Subtracting $G^{k;l}_{\alpha,\beta;\gamma,\delta}$ on both sides gives

$$\Big(\sum_{0\le k'<k} G^{k';l}_{\alpha,\beta;\gamma,\delta}\Big)\wedge\Big(\sum_{0\le l'<l} G^{k;l'}_{\alpha,\beta;\gamma,\delta}\Big)=0. \tag{3.20}$$

On the other hand, we have

$$\Sigma^k_{\alpha,\beta}-\Sigma^l_{\gamma,\delta}=\sum_{k'>k;l'\ge 0} G^{k';l'}_{\alpha,\beta;\gamma,\delta}-\sum_{k'\ge 0;l'>l} G^{k';l'}_{\alpha,\beta;\gamma,\delta}=\sum_{k'>k;l'\le l} G^{k';l'}_{\alpha,\beta;\gamma,\delta}-\sum_{k'\le k;l'>l} G^{k';l'}_{\alpha,\beta;\gamma,\delta}. \tag{3.21}$$

Let $\Sigma^k_{\alpha,\beta}\ge\Sigma^l_{\gamma,\delta}$. Suppose that the first sum in (3.21) is zero; by non-negativity of the coupling rates, all terms in the second sum in (3.21) are zero, which gives (3.18). Now suppose that the first sum in (3.21) is not zero. Then there are $k^*>k$, $l^*\le l$ such that $G^{k^*;l^*}_{\alpha,\beta;\gamma,\delta}>0$. Thus, for all $l''>l$, we have

$$\sum_{0\le l'<l''} G^{k^*;l'}_{\alpha,\beta;\gamma,\delta}>0$$

and by (3.20) applied to each l''

$$\sum_{0\le k'<k^*} G^{k';l''}_{\alpha,\beta;\gamma,\delta}=0.$$

By non-negativity of the coupling rates, all the elements in the above sums are zero; in particular, since $k<k^*$, (3.18) holds.

We skip a similar derivation for (3.19). □

Lemma 3.4. *The rates defined by* (2.22)–(2.23) *are the unique solution of the recursion relations* (2.24)–(2.27).

Proof. It consists in an explicit resolution of (2.24)–(2.27). We first prove that any solution is zero outside $\mathcal{P}_{\alpha,\beta;\gamma,\delta}$, that is (2.23). Let $(k,l)\notin\mathcal{P}_{\alpha,\beta;\gamma,\delta}$ and define $i=k+l$ and (k_i,l_i) as in (2.21). Notice that by construction $k_i+l_i=i$ (hence $k\ne k_i$, $l\ne l_i$), $\{k_0,k_1,\dots,k_i\}=\{0,1,\dots,k_i\}$ and $\{l_0,l_1,\dots,l_i\}=\{0,1,\dots,l_i\}$.

Suppose first that $k<k_i$. Let i' denote the greatest element of $\{j\ge 0\colon k_j=k\}$. Hence $k_{i'+1}=k_{i'}+1$ and by (2.21), we have $\Sigma^{k_{i'}}_{\alpha,\beta}\ge\Sigma^{l_{i'}}_{\gamma,\delta}$. Since $k=k_{i'}$ and $l>l_i\ge l_{i'}$, $G^{k;l}_{\alpha,\beta;\gamma,\delta}=0$ by (3.18).

Suppose now $k>k_i$. Then $l<l_i$ and we denote by i'' the greatest element of $\{j\ge 0: l_j=l\}$. Hence $l_{i''+1}=l_{i''}+1$ and by (2.21), we have $\Sigma^{k_{i''}}_{\alpha,\beta}<\Sigma^{l_{i''}}_{\gamma,\delta}$. Since $k>k_i\ge k_{i''}$ and $l=l_{i''}$, $G^{k;l}_{\alpha,\beta;\gamma,\delta}=0$ by (3.19).

We are now able to compute explicitely the values of the rates on $\mathcal{P}_{\alpha,\beta;\gamma,\delta}$.

Take $i\ge 0$ and (k_i,l_i) as in (2.21). Suppose first that $\Sigma^{k_i}_{\alpha,\beta}\ge\Sigma^{l_i}_{\gamma,\delta}$. Then $k_j>k_i$ for all $j>i$ and thus, using (2.23),

$$\sum_{j>i} G^{k_j;l_j}_{\alpha,\beta;\gamma,\delta}=\sum_{k>k_i}\sum_{l\ge 0} G^{k;l}_{\alpha,\beta;\gamma,\delta}=\sum_{k>k_i}\Gamma^k_{\alpha,\beta}=\Sigma^{k_i}_{\alpha,\beta}.$$

Suppose now that $\Sigma^{k_i}_{\alpha,\beta}<\Sigma^{l_i}_{\gamma,\delta}$. Then $l_j>l_i$ for all $j>i$ and we have

$$\sum_{j>i} G^{k_j;l_j}_{\alpha,\beta;\gamma,\delta}=\sum_{l>l_i}\sum_{k\ge 0} G^{k;l}_{\alpha,\beta;\gamma,\delta}=\sum_{l>l_i}\Gamma^k_{\alpha,\beta}=\Sigma^{l_i}_{\gamma,\delta}.$$

Putting both cases together gives for all $i \geq 0$

$$\sum_{j>i} G^{k_j;l_j}_{\alpha,\beta;\gamma,\delta} = \Sigma^{k_i}_{\alpha,\beta} \vee \Sigma^{l_i}_{\gamma,\delta}.$$

These relations lead to the expression (2.22): for all $i > 0$,

$$G^{k_i;l_i}_{\alpha,\beta;\gamma,\delta} = \sum_{j>i-1} G^{k_j;l_j}_{\alpha,\beta;\gamma,\delta} - \sum_{j>i} G^{k_j;l_j}_{\alpha,\beta;\gamma,\delta} = \left(\Sigma^{k_{i-1}}_{\alpha,\beta} \vee \Sigma^{l_{i-1}}_{\gamma,\delta}\right) - \left(\Sigma^{k_i}_{\alpha,\beta} \vee \Sigma^{l_i}_{\gamma,\delta}\right).$$

□

Proofs of Propositions 2.15 and 2.16. From Lemma 3.2 the coupling rates defined by (2.16)–(2.19) verify the recursion relations (2.24)–(2.27), which, by Lemma 3.4, have a unique solution given by (2.22)–(2.23). Therefore expressions (2.16)–(2.19) and (2.22)–(2.23) are identical. □

Proof of Proposition 2.17. Using conditions (2.13)–(2.14), we now prove that the process is attractive. We need to show that $G^{k;l}_{\alpha,\beta;\gamma,\delta} = 0$ whenever $(\alpha, \beta) \leq (\gamma, \delta)$ and $(\alpha - k, \beta + k) \not\leq (\gamma - l, \delta + l)$. This last condition splits into two possible cases

$$l > \gamma - \alpha + k, \tag{3.22}$$

$$k > \delta - \beta + l. \tag{3.23}$$

In the first case, using (3.22), non-negativity of the Γ's and (2.14), we write

$$\Sigma^{l-1}_{\gamma,\delta} = \sum_{l'\geq l} \Gamma^{l'}_{\gamma,\delta} \leq \sum_{l'>\gamma-\alpha+k} \Gamma^{l'}_{\gamma,\delta} \leq \sum_{k'>k} \Gamma^{k'}_{\alpha,\beta} = \Sigma^{k}_{\alpha,\beta}.$$

Since $\Sigma^{l}_{\gamma,\delta} \leq \Sigma^{l-1}_{\gamma,\delta}$, one has both $\Sigma^{l}_{\gamma,\delta} \leq \Sigma^{k}_{\alpha,\beta}$ and $\Sigma^{l-1}_{\gamma,\delta} \leq \Sigma^{k}_{\alpha,\beta}$ and the two terms appearing between parentheses in expressions (3.6) for $k \neq 0$ or (3.8) for $k = 0$ are equal to zero. Hence $G^{k;l}_{\alpha,\beta;\gamma,\delta} = 0$.

In the second case (3.23), one writes in a similar way, using (2.13)

$$\Sigma^{k}_{\alpha,\beta} \leq \Sigma^{k-1}_{\alpha,\beta} = \sum_{k'\geq k} \Gamma^{k'}_{\alpha,\beta} \leq \sum_{k'>\delta-\beta+l} \Gamma^{k'}_{\alpha,\beta} \leq \sum_{l'>l} \Gamma^{l'}_{\gamma,\delta} = \Sigma^{l}_{\gamma,\delta}.$$

Inserting these inequalities into the expression for the coupling rate, that is (3.7) for $l > 0$ or (3.9) for $l = 0$, one gets again $G^{k;l}_{\alpha,\beta;\gamma,\delta} = 0$.

In conclusion, when $(\alpha, \beta) \leq (\gamma, \delta)$, we have

$$G^{k;l}_{\alpha,\beta;\gamma,\delta} = 0 \quad \text{if } l - k \notin \left\{-(\delta - \beta), \ldots, \gamma - \alpha\right\}.$$

The case $(\alpha, \beta) \geq (\gamma, \delta)$ is treated by symmetry using Remark 2.13, which yields (2.28). □

Proof of Theorem 2.19. We first prove that for all $(\alpha, \beta, \gamma, \delta) \in X^4$, the coupling rate $G^{k;l}_{\alpha,\beta;\gamma,\delta}$ is zero if $k - l$ is outside some range depending only on $\alpha - \gamma$ and $\beta - \delta$.

From Proposition 2.17, we already know that if (α, β), (γ, δ) are ordered, then (2.28) is satisfied. We now show similar restrictions for non-ordered pairs, for which Eqs (2.13)–(2.14) lead also, indirectly, to inequalities between partial sums of rates. Suppose for instance that $\alpha > \gamma$, $\beta < \delta$. We insert an intermediate pair of values and write

$$(\alpha, \beta) \geq (\gamma, \beta) \quad \text{and} \quad (\gamma, \delta) \geq (\gamma, \beta).$$

We apply Eqs (2.13)–(2.14) on both pairs of values and get two sets of inequalities valid for all $k \geq 0$,

$$\sum_{k'>k} \Gamma^{k'}_{\gamma,\beta} \leq \sum_{l'>k} \Gamma^{l'}_{\alpha,\beta}; \qquad \sum_{l'>\alpha-\gamma+k} \Gamma^{l'}_{\alpha,\beta} \leq \sum_{k'>k} \Gamma^{k'}_{\gamma,\beta},$$

$$\sum_{k'>\delta-\beta+k} \Gamma^{k'}_{\gamma,\beta} \le \sum_{l'>k} \Gamma^{l'}_{\gamma,\delta}; \qquad \sum_{l'>k} \Gamma^{l'}_{\gamma,\delta} \le \sum_{k'>k} \Gamma^{k'}_{\gamma,\beta}.$$

Combining them by pairs in order to eliminate the intermediate values (γ, β) gives

$$\sum_{l'>k} \Gamma^{l'}_{\gamma,\delta} \le \sum_{k'>k} \Gamma^{k'}_{\alpha,\beta},$$

$$\sum_{k'>\delta-\beta+\alpha-\gamma+k} \Gamma^{k'}_{\alpha,\beta} \le \sum_{l'>k} \Gamma^{l'}_{\gamma,\delta},$$

which implies

$$\Sigma^{l}_{\gamma,\delta} \le \Sigma^{l-1}_{\gamma,\delta} \le \Sigma^{k}_{\alpha,\beta} \quad \text{for all } l > k, \tag{3.24}$$

$$\Sigma^{k}_{\alpha,\beta} \le \Sigma^{k-1}_{\alpha,\beta} \le \Sigma^{l}_{\gamma,\delta} \quad \text{for all } k > \delta - \beta + \alpha - \gamma + l. \tag{3.25}$$

Inserting (3.24) in (3.6) (or (3.8) if $k = 0$) and (3.25) in (3.7) (or (3.9) if $l = 0$), one gets

$$G^{k;l}_{\alpha,\beta;\gamma,\delta} = 0 \quad \text{if } \alpha > \gamma, \beta < \delta \text{ and } k - l \notin \{0, \dots, \alpha - \gamma + \delta - \beta\}. \tag{3.26}$$

We have also by symmetry through Remark 2.13

$$G^{k;l}_{\alpha,\beta;\gamma,\delta} = 0 \quad \text{if } \alpha < \gamma, \beta > \delta \text{ and } l - k \notin \{0, \dots, \gamma - \alpha + \beta - \delta\}. \tag{3.27}$$

Thus, putting together (2.28), (3.26), (3.27), the coupling rates $G^{k;l}_{\alpha,\beta;\gamma,\delta}$ are non-zero only if the increments satisfy

$$\begin{cases} l - k \in \bigl\{-(\delta - \beta), \dots, (\gamma - \alpha)\bigr\} & \text{if } \alpha \le \gamma \text{ and } \beta \le \delta, \\ l - k \in \bigl\{-(\alpha - \gamma), \dots, (\beta - \delta)\bigr\} & \text{if } \alpha \ge \gamma \text{ and } \beta \ge \delta, \\ l - k \in \bigl\{-(\alpha - \gamma) - (\delta - \beta), \dots, 0\bigr\} & \text{if } \alpha > \gamma \text{ and } \beta < \delta, \\ l - k \in \bigl\{0, \dots, (\gamma - \alpha) + (\beta - \delta)\bigr\} & \text{if } \alpha < \gamma \text{ and } \beta > \delta. \end{cases} \tag{3.28}$$

All four cases can be written together as the single statement (2.35) or equivalently

$$\left| l - k + \frac{1}{2}(\alpha - \gamma) - \frac{1}{2}(\beta - \delta) \right| \le \frac{1}{2}|\alpha - \gamma| + \frac{1}{2}|\beta - \delta|. \tag{3.29}$$

These restrictions induce in turn a control on the sign of $\Delta(\alpha, \beta, \gamma, \delta, k, l)$ through the inequality: For all a, b, x in $\mathbb{R}$ such that $|x| \le |a| + |b|$,

$$[a + b + x]^+ + [a + b - x]^+ \le 2[a]^+ + 2[b]^+. \tag{3.30}$$

Inequality (3.30) is trivial if its left-hand side is equal to 0. When exactly one of the two quantities on its left-hand side is non-zero, (3.30) follows directly from the bound on x; otherwise, the value of the left-hand side is independent of x and trivially bounded by the right-hand side.

We now apply (3.30), choosing (in view of (3.29))

$$a = \frac{1}{2}(\alpha - \gamma), \qquad b = \frac{1}{2}(\beta - \delta), \qquad x = l - k + \frac{1}{2}(\alpha - \gamma) - \frac{1}{2}(\beta - \delta) \tag{3.31}$$

we get

$$\bigl[(\alpha - k) - (\gamma - l)\bigr]^+ + \bigl[(\beta + k) - (\delta + l)\bigr]^+ \le [\alpha - \gamma]^+ + [\beta - \delta]^+. \tag{3.32}$$

Hence, recalling definition (2.31) of $\Delta(\alpha, \beta, \gamma, \delta, k, l)$, and (2.33),

$$G^{k;l}_{\alpha,\beta;\gamma,\delta} \ne 0 \quad \Longrightarrow \quad \Delta(\alpha, \beta, \gamma, \delta, k, l) = \Delta(\beta, \alpha, \delta, \gamma, l, k) \le 0. \tag{3.33}$$

To go from (3.33) to (2.34), notice first that for $k=l$, $\Delta(\xi(x),\xi(y),\zeta(x),\zeta(y),k,l)=0$. We have, since $f^+_{x,y}(\xi,\zeta)=f^+_{y,x}(\xi,\zeta)$, and by (2.33),

$$\begin{aligned}\overline{\mathcal{L}}_{x,y}f^+_{x,y}(\xi,\zeta) &= \sum_{\alpha,\beta\in X}\sum_{\gamma,\delta\in X}\chi^{\alpha,\beta}_{x,y}(\xi)\chi^{\gamma,\delta}_{x,y}(\zeta)\\ &\quad\times\sum_{k\neq l}\bigl(G^{k;l}_{\alpha,\beta;\gamma,\delta}(y-x)\Delta(\alpha,\beta,\gamma,\delta,k,l)+G^{l;k}_{\beta,\alpha;\delta,\gamma}(x-y)\Delta(\beta,\alpha,\delta,\gamma,l,k)\bigr) \qquad (3.34)\\ &= \sum_{\alpha,\beta\in X}\sum_{\gamma,\delta\in X}\chi^{\alpha,\beta}_{x,y}(\xi)\chi^{\gamma,\delta}_{x,y}(\zeta)\\ &\quad\times\sum_{k\neq l}\bigl(G^{k;l}_{\alpha,\beta;\gamma,\delta}(y-x)+G^{l;k}_{\beta,\alpha;\delta,\gamma}(x-y)\bigr)\Delta(\alpha,\beta,\gamma,\delta,k,l). \qquad (3.35)\end{aligned}$$

By (3.33), each term of the sum in the right-hand side of (3.34) is non-negative. Since by (2.32), $\overline{\mathcal{L}}_{x,y}f^+_{x,y}(\xi,\zeta)=\overline{\mathcal{L}}_{x,y}f^-_{x,y}(\xi,\zeta)$, (2.34) is satisfied.

To derive (2.36), we first check that equality holds in (3.30) either for $a+b=\pm(|a|+|b|)$ or for $x=\pm(|a|+|b|)$. Recalling (3.31), the first case corresponds to ordered pairs of values, $(\alpha,\beta)\le(\gamma,\delta)$ or $(\alpha,\beta)\ge(\gamma,\delta)$, for all values of k and l. The other one gives two new possibilities for non-ordered pairs of values: either $l-k=0$ or $l-k=-(\alpha-\gamma)+(\beta-\delta)$. Notice that in this last case, the values of the discrepancies at sites x and y are exchanged. □

Proof of Lemma 2.21. Let $(x,y)\in S^2$, $(\alpha,\beta,\gamma,\delta)\in X^4$, non-negative k,l such that $G^{k;l}_{\alpha,\beta;\gamma,\delta}(y-x)>0$. Either (α,β), (γ,δ) are ordered and (2.37) reduces to the expression (2.28) for attractiveness; or (α,β), (γ,δ) are not ordered and (2.37) is Definition 2.20 of absence of exchanges of discrepancies. □

Proof of Proposition 2.23. We fix $(\xi_0,\zeta_0)\in\mathbf{X}^2$ with $\sum_{y\in\mathbb{Z}}[\xi_0(y)+\zeta_0(y)]<+\infty$. We prove (2.38) by checking that

$$\overline{\mathcal{L}}S(\xi_t,\zeta_t)\le 0 \quad \text{for all } t\ge 0. \qquad (3.36)$$

We have

$$\begin{aligned}\overline{\mathcal{L}}S(\xi_t,\zeta_t) &= \sum_{x,y\in\mathbb{Z},|x-y|=1}\sum_{\alpha,\beta\in X}\sum_{\gamma,\delta\in X}\chi^{\alpha,\beta}_{x,y}(\xi_t)\chi^{\gamma,\delta}_{x,y}(\zeta_t)\sum_{k,l}G^{k;l}_{\alpha,\beta;\gamma,\delta}(y-x)\\ &\quad\times\Bigl(\sup_{z\in\mathbb{Z}}\bigl|s_z\bigl(S^k_{x,y}\xi_t,S^l_{x,y}\zeta_t\bigr)\bigr|-\sup_{z\in\mathbb{Z}}\bigl|s_z(\xi_t,\zeta_t)\bigr|\Bigr). \qquad (3.37)\end{aligned}$$

We note that

$$\forall z\notin[x\wedge y,x\vee y)\quad s_z\bigl(S^k_{x,y}\xi_t,S^l_{x,y}\zeta_t\bigr)=s_z(\xi_t,\zeta_t), \qquad (3.38)$$

because, for $z<x\wedge y$, jumps involve only sites x,y and because, for $z\ge x\vee y$, the dynamics is conservative (cf. (2.32)). Due to the restriction to nearest neighbor interactions, $|y-x|=1$, so that s_z may change only for $z=x\wedge y$,

$$\begin{cases} s_z\bigl(S^k_{x,y}\xi_t,S^l_{x,y}\zeta_t\bigr)=s_z(\xi_t,\zeta_t)+l-k & \text{if } z=x=y-1,\\ s_z\bigl(S^k_{x,y}\xi_t,S^l_{x,y}\zeta_t\bigr)=s_z(\xi_t,\zeta_t)+k-l & \text{if } z=y=x-1\end{cases} \qquad (3.39)$$

and we have

$$\begin{aligned}\overline{\mathcal{L}}S(\xi_t,\zeta_t) &= \sum_{x\in\mathbb{Z}}\sum_{\alpha,\beta\in X}\sum_{\gamma,\delta\in X}\chi^{\alpha,\beta}_{x,x+1}(\xi_t)\chi^{\gamma,\delta}_{x,x+1}(\zeta_t)\sum_{k,l}\bigl(G^{k;l}_{\alpha,\beta;\gamma,\delta}(1)+G^{l;k}_{\beta,\alpha;\delta,\gamma}(-1)\bigr)\\ &\quad\times\Bigl(\bigl|s_x(\xi_t,\zeta_t)+l-k\bigr|\vee\sup_{z\neq x}\bigl|s_z(\xi_t,\zeta_t)\bigr|-S(\xi_t,\zeta_t)\Bigr)\end{aligned}$$

$$\leq \sum_{x\in\mathbb{Z}}\sum_{\alpha,\beta\in X}\sum_{\gamma,\delta\in X}\chi_{x,x+1}^{\alpha,\beta}(\xi_t)\chi_{x,x+1}^{\gamma,\delta}(\zeta_t)\sum_{k,l}\bigl(G_{\alpha,\beta;\gamma,\delta}^{k;l}(1)+G_{\beta,\alpha;\delta,\gamma}^{l;k}(-1)\bigr)$$
$$\times\bigl[\bigl|s_x(\xi_t,\zeta_t)+l-k\bigr|-S(\xi_t,\zeta_t)\bigr]^+. \tag{3.40}$$

Now we prove that if the coupling is increasing and does not allow exchanges of discrepancies, all terms in the last line of (3.40) are zero. Let (ξ_t,ζ_t) be such that $\xi_t(x)=\alpha$, $\xi_t(x+1)=\beta$, $\zeta_t(x)=\gamma$, $\zeta_t(x+1)=\delta$. For all transition rates with $G_{\alpha,\beta;\gamma,\delta}^{k;l}(1)+G_{\beta,\alpha;\delta,\gamma}^{l;k}(-1)>0$, we apply Lemma 2.21: Indeed, either $G_{\alpha,\beta;\gamma,\delta}^{k;l}(1)>0$ hence we have (2.37), or $G_{\beta,\alpha;\delta,\gamma}^{l;k}(-1)>0$, which implies inequalities (2.37) again (by symmetry the latter stay the same when exchanging α with β, γ with δ, k with l). They have the equivalent formulation

$$0\wedge(\gamma-\alpha)\wedge(\beta-\delta)\leq l-k\leq 0\vee(\gamma-\alpha)\vee(\beta-\delta). \tag{3.41}$$

Recalling (2.39) we have

$$s_{x-1}(\xi_t,\zeta_t)=s_x(\xi_t,\zeta_t)+\gamma-\alpha,\qquad s_{x+1}(\xi_t,\zeta_t)=s_x(\xi_t,\zeta_t)+\beta-\delta. \tag{3.42}$$

Thus, by adding $s_x(\xi_t,\zeta_t)$ to each term of (3.41) we get

$$s_x(\xi_t,\zeta_t)\wedge s_{x-1}(\xi_t,\zeta_t)\wedge s_{x+1}(\xi_t,\zeta_t)\leq s_x(\xi_t,\zeta_t)+l-k\leq s_x(\xi_t,\zeta_t)\vee s_{x-1}(\xi_t,\zeta_t)\vee s_{x+1}(\xi_t,\zeta_t)$$

which implies

$$\bigl|s_x(\xi_t,\zeta_t)+l-k\bigr|\leq\bigl|s_x(\xi_t,\zeta_t)\bigr|\vee\bigl|s_{x-1}(\xi_t,\zeta_t)\bigr|\vee\bigl|s_{x+1}(\xi_t,\zeta_t)\bigr|\leq S(\xi_t,\zeta_t) \tag{3.43}$$

and finally, inserting (3.43) in (3.40), we get (3.36). □

Proof of Remark 2.24. We exhibit two counter-examples for which (2.38) is wrong: in (i), interactions are not only nearest neighbor; in (ii), (2.37) is not fulfilled. For both cases, let (ξ_0,ζ_0) be such that for $x<y$, $\xi_0(x)=\alpha$, $\xi_0(y)=\beta$, $\zeta_0(x)=\gamma$, $\zeta_0(y)=\delta$ with (α,β), (γ,δ) not ordered, and $\xi_0(z)=\zeta_0(z)=0$ for $z\notin[x-1,y]$. Let the first coupled transition occur at time $t>0$ between sites x, y at rate $G_{\alpha,\beta;\gamma,\delta}^{k;l}(y-x)$.

(i) Assume that $y=x+2$, $\xi_0(x+1)=\widetilde{\alpha}$, $\zeta_0(x+1)=\widetilde{\gamma}$, $\xi_0(x-1)=\zeta_0(x-1)=0$, with $\gamma>\alpha$, $\beta>\delta$, $\gamma-\alpha=\beta-\delta$, $\widetilde{\alpha}-\widetilde{\gamma}=\gamma-\alpha+1$, and that $l-k+1=\gamma-\alpha$ in $G_{\alpha,\beta;\gamma,\delta}^{k;l}(2)>0$. Therefore property (3.38) is still valid, and (3.41) is satisfied. An analogous computation to (3.37)–(3.40) yields

$$\sup_{z\in\mathbb{Z}}\bigl|s_z(\xi_{t^-},\zeta_{t^-})\bigr|=\sup(\gamma-\alpha,1,\gamma-\alpha+1),$$

$$\sup_{z\in\mathbb{Z}}\bigl|s_z\bigl(S_{x,x+2}^k\xi_{t^-},S_{x,x+2}^l\zeta_{t^-}\bigr)\bigr|=\sup(1,\gamma-\alpha+2,\gamma-\alpha+1),$$

$$\overline{\mathcal{L}}S(\xi_t,\zeta_t)\geq G_{\alpha,\beta;\gamma,\delta}^{k;l}(2)\Bigl(\sup_{z\in\mathbb{Z}}\bigl|s_z\bigl(S_{x,x+2}^k\xi_{t^-},S_{x,x+2}^l\zeta_{t^-}\bigr)\bigr|-\sup_{z\in\mathbb{Z}}\bigl|s_z(\xi_{t^-},\zeta_{t^-})\bigr|\Bigr)>0$$

thus (3.36) is wrong.

(ii) Assume that $y=x+1$, $\xi_0(x-1)=\gamma$, $\zeta_0(x-1)=\alpha$, and that in $G_{\alpha,\beta;\gamma,\delta}^{k;l}(1)$,

$$0<\beta-\delta\leq\gamma-\alpha<l-k\leq(\gamma-\alpha)+(\beta-\delta), \tag{3.44}$$

so that (2.37) is not satisfied while (2.35) is. We have by (3.38), (3.39), (3.42)

$$S(\xi_{t^-},\zeta_{t^-})=s_{x-1}(\xi_t,\zeta_t)=s_{x-1}(\xi_{t^-},\zeta_{t^-})=\gamma-\alpha,$$

$$s_{x+1}(\xi_t,\zeta_t)=s_{x+1}(\xi_{t^-},\zeta_{t^-})=\beta-\delta,$$

$$s_x(\xi_{t^-},\zeta_{t^-})=0;\qquad s_x(\xi_t,\zeta_t)=l-k=S(\xi_t,\zeta_t),$$

so that (2.38) fails at time t because of (3.44). □

Proof of Proposition 2.25. Let $(\alpha,\beta,\gamma,\delta)\in X^4$ and k,l non-negative. Assuming (2.40)–(2.41), we show that if k,l do not satisfy (2.37), then $G^{k;l}_{\alpha,\beta;\gamma,\delta}=0$.

If $l>K$, (2.41) implies that $\Sigma^k_{\alpha,\beta}\ge\Sigma^{l-1}_{\gamma,\delta}$ and thus $\Sigma^k_{\alpha,\beta}-\Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\ge\Gamma^l_{\gamma,\delta}$. Therefore $G^{k;l}_{\alpha,\beta;\gamma,\delta}=0$ by (2.16) (if $k>0$) or by (2.17) (if $k=0$).

If $k>L$, (2.40) implies that $\Sigma^l_{\gamma,\delta}\ge\Sigma^{k-1}_{\alpha,\beta}$ and thus $\Sigma^l_{\gamma,\delta}-\Sigma^k_{\alpha,\beta}\wedge\Sigma^l_{\gamma,\delta}\ge\Gamma^k_{\alpha,\beta}$. Therefore $G^{k;l}_{\alpha,\beta;\gamma,\delta}=0$ by (2.16) (if $l>0$) or by (2.18) (if $l=0$). □

4. Coupling and attractiveness on examples

In this section, we apply results of Section 2 to examples, first the classical ones, SEP–ZRP–MP, then a stick process (StP), and finally a two species exclusion model (S_2EP).

4.1. *SEP–ZRP–MP*

According to notations in Sections 1 and 2, let $p(x,y)=p(y-x)$ be an irreducible translation invariant probability transition on $\mathbb{Z}^d$; set the transition rates $\Gamma^k_{\alpha,\beta}(y-x)$ equal to 0 unless $k=1$, and write $\Gamma^1_{\alpha,\beta}(y-x)=p(y-x)b(\alpha,\beta)$. We specify below (cf. (2.1)) the necessary conditions on the rates and the proper reduction of the state space (when needed) which ensure that (2.8) is the generator of a well-defined Markov process (see [1,8,18]) with a one parameter family of invariant and translation invariant product probability measures, denoted by $\{\mu_\rho\}_\rho$. The parameter ρ is the average particles' density per site. We assume that the transition rates satisfy condition (A) of Section 1.

(a) *MP: The misanthropes process* [8]. There, $X=\mathbb{N}$, $S=\mathbb{Z}^d$. We assume that the transition rates $b(\alpha,\beta)$, $\alpha,\beta\in\mathbb{N}$ are Lipschitz functions w.r.t. the first variable, with a Lipschitz constant uniform in the second variable; the state space is reduced to

$$\Omega_0=\left\{\eta\in X^S;\sum_{x\in S}a(x)\eta(x)<+\infty\right\},\tag{4.1}$$

where a is a function which satisfies $\sum_{y\in S}p(x,y)a(y)\le Ma(x)$ for all $x\in S$, for some constant M (see [1], whose construction for ZRP extends for MP to these conditions, weaker than in [8], formula (1.7)).

We quote [8], formulas (2.3) and (2.4), which imply the existence of product probability measures $\{\mu_\rho\}_\rho$, with $\rho\in[0,+\infty)$:

(M1) $b(0,\cdot)=0$ and $b(\alpha,\beta)>0$ for $\alpha>0$ and $\beta\ge0$ (irreducibility on X^S),

$$\text{(M2)}\quad\begin{cases}\big(\forall z\in\mathbb{Z}^d,\ p(z)=p(-z)\big)\quad\text{or}\quad b(\alpha,\beta)-b(\beta,\alpha)=b(\alpha,0)-b(\beta,0),\\ \frac{b(\alpha,\beta)}{b(\beta+1,\alpha-1)}=\frac{b(\alpha,0)b(1,\beta)}{b(\beta+1,0)b(1,\alpha-1)}.\end{cases}$$

(b) *ZRP: The zero-range process* [1] is a misanthropes process for which the rates depend on the first coordinate only, that is, $b(\alpha,\beta)=g(\alpha)$ (for more general existence conditions in the totally asymmetric case, we refer to [6]). Condition (M1) reduces to $g(0)=0$ and

$$g(\alpha)>0\quad\text{for all }\alpha>0.\tag{4.2}$$

Condition (M2) is verified in all cases and $\rho\in[0,\lim_\alpha g(\alpha))$.

(c) *SEP: The simple exclusion process* [17,18] has state space $\{0,1\}^{\mathbb{Z}^d}$. The only non-zero jump rate is $\Gamma^1_{1,0}(y-x)=p(y-x)b(1,0)=p(y-x)$. Irreducibility for $p(x,y)$ is relaxed to

$$\forall x,y\in\mathbb{Z}^d,\forall t>0\quad p_t(x,y)+p_t(y,x)>0,\tag{4.3}$$

where $p_t(x,y)$ are the transition probabilities for the continuous time Markov chain associated to $p(x,y)$. The invariant product probability measures are Bernoulli, with $\rho=\mu_\rho(\eta(x)=1)\in[0,1]$.

In those three examples, (2.13)–(2.14) reduce to condition (A), which thus insures attractiveness. For ZRP, (A) reads: $g(\alpha)$ is a non-decreasing function of α. For SEP, (A) is empty and does not impose anything.

The coupling rates' formulas (2.16)–(2.17) reduce to those of basic coupling, since the only non-zero possibility for k and l is 1:

$$\begin{cases} G^{1;1}_{\alpha,\beta;\gamma,\delta} = \Gamma^1_{\alpha,\beta} \wedge \Gamma^1_{\gamma,\delta}, \\ G^{0;1}_{\alpha,\beta;\gamma,\delta} = \Gamma^1_{\gamma,\delta} - \Gamma^1_{\alpha,\beta} \wedge \Gamma^1_{\gamma,\delta}, \\ G^{1;0}_{\alpha,\beta;\gamma,\delta} = \Gamma^1_{\alpha,\beta} - \Gamma^1_{\alpha,\beta} \wedge \Gamma^1_{\gamma,\delta}. \end{cases} \tag{4.4}$$

In Section 6 we revisit the conditions previously stated to determine extremal invariant and translation invariant probability measures for SEP–ZRP–MP.

4.2. *A stick process* (*StP*)

This example is motivated by its connections with a generalization of the discrete Hammersley–Aldous–Diaconis (HAD) process, a continuous-time Markov process taking values in a subset of $\mathcal{X} = \{0,1\}^{\mathbb{Z}}$ (see [10]). Let $p(x,y) = p(y-x)$ be an irreducible nearest-neighbor translation invariant probability transition on $\mathbb{Z}$, with $p(1) + p(-1) = 1$. For each empty site j, the closest particle to its left (resp. right) jumps to site j with rate $p(1)$ (resp. $p(-1)$). The formal generator of the generalized HAD process is

$$L_H f(\widetilde{\eta}) = \sum_{j\in\mathbb{Z}} \sum_{z=\pm 1} p(z)\bigl[f\bigl(S^1_{x,x_z}\widetilde{\eta}\bigr) - f(\widetilde{\eta})\bigr], \tag{4.5}$$

where, for all $\widetilde{\eta} \in \mathcal{X}$, $x \in \mathbb{Z}$ and $z = \pm 1$,

$$x_z = x_z(\widetilde{\eta}, x) = z \max\bigl\{y < zx \colon \widetilde{\eta}(zy) = 1\bigr\}. \tag{4.6}$$

The existence of this dynamics is proved through a graphical construction, and to exclude the possibility of particles escaping immediately to infinity, the state space is restricted to

$$\widetilde{\mathcal{X}} = \left\{\widetilde{\eta} \in \mathcal{X} \colon \lim_{m\to+\infty} m^{-1/2} \sum_{j=-m}^{-1} \widetilde{\eta}(j) = \lim_{m\to+\infty} m^{-1/2} \sum_{j=1}^{m} \widetilde{\eta}(j) = \infty\right\}.$$

Notice that particles cannot jump over each other, so they keep their relative order. We denote by $(z_i(t), i \in \mathbb{Z}, t \ge 0)$ their positions, with $z_0(0)$ the position of the first $\widetilde{\eta}$-particle initially to the right of (or at) the origin.

We study here a *generalized stick process* (StP) which is a discrete version with a generalization to nearest neighbor jumps of the *stick process* defined in [20] (for a totally asymmetric case). It is defined on the state space

$$\Omega_0 = \left\{\eta \in \mathbb{N}^{\mathbb{Z}}; \lim_{n\to-\infty} n^{-2} \sum_{x=n}^{-1} \eta(x) = \lim_{n\to+\infty} n^{-2} \sum_{x=1}^{n} \eta(x) = 0\right\}$$

so that $\Omega_0 \subset X^S$ with $X = \mathbb{N}$, $S = \mathbb{Z}$; it describes translation invariant nearest-neighbor jumps. Its transition rates are

$$\begin{cases} \Gamma^k_{\alpha,\beta}(z) = p(z)\mathbf{1}_{\{k\le\alpha\}} & \text{for } z = \pm 1, \\ \Gamma^k_{\alpha,\beta}(z) = 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

They satisfy (2.11).

Similarly to the original models (see [20]), the generalized stick process is in bijection with the generalized discrete HAD process seen from a tagged particle: If we assume that at time 0, $\widetilde{\eta}_0(0) = 1$ so that $z_0(0) = 0$, we have for all $t \ge 0$, $i \in \mathbb{Z}$,

$$\eta_t(i) = z_{i+1}(t) - z_i(t). \tag{4.8}$$

In other words, holes (resp. positions of particles) of the HAD are turned into particles (resp. sites) for the stick model.

The geometric product probability measures $\{\mu_\rho, \rho \in [0, +\infty)\}$ with parameter $\rho(1+\rho)^{-1}$ are invariant for the generalized stick process (ρ is the average particles' density per site).

Proposition 4.1. *The stick process is attractive. The rates of the coupling generator $\overline{\mathcal{L}}$ are, for all $(\alpha, \beta, \gamma, \delta) \in \mathbb{N}^4$, positive k, l, $z = \pm 1$,*

$$G^{k;l}_{\alpha,\beta;\gamma,\delta}(z) = p(z)\mathbf{1}_{\{k\le\alpha, l\le\gamma, \alpha-\gamma=k-l\}}, \tag{4.9}$$

$$G^{0;l}_{\alpha,\beta;\gamma,\delta}(z) = p(z)\mathbf{1}_{\{l\le\gamma-\alpha\}}, \quad \text{where } \gamma > \alpha, \tag{4.10}$$

$$G^{k;0}_{\alpha,\beta;\gamma,\delta}(z) = p(z)\mathbf{1}_{\{k\le\alpha-\gamma\}}, \quad \text{where } \alpha > \gamma. \tag{4.11}$$

This increasing coupling does not allow exchanges of discrepancies.

Proof. Inequalities (2.13)–(2.14) are satisfied for the rates (4.7) of StP, which is thus attractive. The coupling rates formulas (2.16)–(2.18) write here (4.9)–(4.11). Finally, recalling Theorem 2.19, notice that the case $\Delta(\alpha, \beta, \gamma, \delta, k, l) = 0$, (α, β), (γ, δ) not ordered (line 2 of (2.36)) is not possible; more generally (cf. Definition 2.20) exchanges of discrepancies are impossible since $|k-l| \le |\alpha-\gamma|$ and thus $\alpha-\gamma+l-k$ cannot change sign. □

Remark 4.2. *The rates (4.9)–(4.11) are very different from the basic coupling rates. For k, l positive, $k = l$ in $G^{k;l}_{\alpha,\beta;\gamma,\delta}(z)$ is possible if and only if $\alpha = \gamma$, and*

$$G^{k;k}_{\alpha,\beta;\alpha,\delta}(z) = p(z)\mathbf{1}_{\{k\le\alpha\}} = \Gamma^k_{\alpha,\beta}(z) = \Gamma^k_{\alpha,\delta}(z).$$

4.3. *The two species exclusion model (S_2EP)*

This one-dimensional model with charge conservation was introduced and studied in [9] (see also [22,23] for a full account on conservation laws in this model). Its state space is $\{-1, 0, 1\}^{\mathbb{Z}}$, with the interpretation that a site on $\mathbb{Z}$ is either empty or occupied by at most one charge, positive or negative. A transition consists in a simultaneous change of the charges on two nearest-neighbor sites, such that the total charge is conserved: in a transition, the value at a site can change by $+1, -1, +2$ or -2, therefore the admissible values for k are 1 and 2. Since in [9] the model is interpreted as a solid-on-solid interface, conventions there are different from the present ones, and the rates $\Gamma^k_{\alpha,\beta}(1)$ (resp. $\Gamma^k_{\alpha,\beta}(-1)$) here are equal to their $\Gamma^{-k}_{\alpha,\beta}$ (resp. $\Gamma^k_{\beta,\alpha}$). There are possibly ten different transition rates, namely: $\Gamma^1_{0,-1}(z)$, $\Gamma^2_{1,-1}(z)$, $\Gamma^1_{1,-1}(z)$, $\Gamma^1_{0,0}(z)$, $\Gamma^1_{1,0}(z)$, $z = \pm 1$. We do not consider the case $\Gamma^1_{0,0}(z) = \Gamma^1_{1,-1}(z) = 0$, where there is a second conserved quantity (the number of particles involved in each transition). We assume, in order to avoid degeneracies, that

$$\begin{cases} \Gamma^1_{1,-1}(1) + \Gamma^1_{1,-1}(-1) > 0, \\ \Gamma^1_{0,0}(1) + \Gamma^1_{0,0}(-1) > 0. \end{cases} \tag{4.12}$$

Under the condition

$$\frac{\Gamma^1_{0,0}(-1)\Gamma^1_{1,-1}(1) - \Gamma^1_{0,0}(1)\Gamma^1_{1,-1}(-1)}{\Gamma^1_{0,0}(1) + \Gamma^1_{0,0}(-1)} = \sum_{z=\pm 1} z\big(\Gamma^2_{1,-1}(z) + \Gamma^1_{1,-1}(z) - \Gamma^1_{1,0}(z) - \Gamma^1_{0,-1}(z)\big) \tag{4.13}$$

the process has a one-parameter family of stationary product probability measures $\{\mu_\rho, \rho \in [-1, 1]\}$, where $\mu_1 = \delta_{\underline{1}}$ and $\mu_{-1} = \delta_{\underline{-1}}$ are the Dirac measures concentrating respectively on $\eta \equiv 1$ and $\eta \equiv -1$. See Section 7 below for more details.

Proposition 4.3. (1) *The two species exclusion model is attractive if and only if the rates satisfy, for* $z = \pm 1$,

$$\Gamma^2_{1,-1}(z) \vee \Gamma^1_{0,0}(z) \le \Gamma^1_{0,-1}(z) \wedge \Gamma^1_{1,0}(z) \le \Gamma^1_{0,-1}(z) \vee \Gamma^1_{1,0}(z) \le \Gamma^1_{1,-1}(z) + \Gamma^2_{1,-1}(z). \tag{4.14}$$

(2) *The increasing coupling for the two-species model does not allow exchanges of discrepancies if and only if, for* $z = \pm 1$,

$$\Gamma^2_{1,-1}(z) \le \Gamma^1_{0,0}(z). \tag{4.15}$$

Proof. (1) Inequalities (4.14) are just a compact form of (2.13)–(2.14) for this model.

(2) By Definition 2.20, we have exchanges of discrepancies for $(x, y) \in S^2$, $(\alpha, \beta, \gamma, \delta) \in X^4$ with $(\alpha - \gamma)(\beta - \delta) < 0$, non-negative k, l with $|k-l| > |\alpha-\gamma| \vee |\beta-\delta|$. Since $|\alpha-\gamma| \vee |\beta-\delta| \ge 1$, the only possibility is $|k-l| = 2$ hence $(k, l) \in \{(2, 0), (0, 2)\}$. Also $y - x = z$ has to belong to $\{-1, 1\}$. For $\alpha < \gamma$, $\beta > \delta$ the only possible positive coupling rate is

$$G^{0;2}_{0,0;1,-1}(z) = \Gamma^2_{1,-1}(z) - \Gamma^1_{0,0}(z)$$

and for $\alpha > \gamma$, $\beta < \delta$ it is

$$G^{2;0}_{1,-1;0,0}(z) = \Gamma^2_{1,-1}(z) - \Gamma^1_{0,0}(z). \qquad \square$$

The rates of the coupling process are here rather different from those of basic coupling. Their explicit form is derived in the Appendix.

Remark 4.4. *Hydrodynamic limits of non attractive two species exclusion models have been studied in* [12] (*where the dynamics has two conserved quantities*), *and in* [11], *Section* 6, *where the rates are* $\Gamma^2_{1,-1}(1) = 2 + \sigma$, $\Gamma^2_{1,-1}(-1) = \sigma$, $\Gamma^1_{0,-1}(1) = \Gamma^1_{1,0}(1) = 1 + \sigma$, $\Gamma^1_{0,-1}(-1) = \Gamma^1_{1,0}(-1) = \sigma$, $\Gamma^1_{0,0}(1) = \beta + \sigma$, $\Gamma^1_{0,0}(-1) = \sigma$, $\Gamma^1_{1,-1}(1) = \sigma$, $\Gamma^1_{1,-1}(-1) = \beta + \sigma$ (*for suitable positive parameters* β, σ), *which do not satisfy* (4.14).

We determine in Section 6 $(\mathcal{I} \cap \mathcal{S})_e$ and in Section 7 the hydrodynamic limit for StP and attractive S_2EP under assumption (4.15).

5. Invariant measures: Determination of $(\mathcal{I} \cap \mathcal{S})_e$

The main result in this section, Theorem 5.13, is the determination of the extremal invariant and translation invariant probability measures of the process $(\eta_t)_{t\ge 0}$ with generator (2.8), assuming it is attractive. This result, which does not depend on the existence of product invariant probability measures, was already known in some cases (for instance, SEP). The proof's skeleton is classical (and goes back to [17]), but the derivation of its main step ((5.2) below) in a rather general setting is new; it sheds some light on the relevant properties of the model, and will enable (see Section 6) to relax the assumptions on the rates on the examples ZRP–MP. This proof is linked to Section 2.3.

5.1. *Preliminaries and statement of Theorem* 5.13

We first give *irreducibility conditions for the coupled process* $(\xi_t, \zeta_t)_{t\ge 0}$ of Proposition 2.11, which are required assumptions for Theorem 5.13. Their goal is to ascertain that if a coupled configuration contains a pair of discrepancies of opposite signs, there is a positive probability (for the coupled evolution) that it evolves into a locally ordered coupled configuration. Whenever basic coupling can be used ($k \le 1$), such event can be described as a motion of discrepancies towards neighboring sites where they eventually merge; as a consequence, irreducibility conditions are more intuitive and can be stated directly in terms of the marginal rates. Here more involved events may occur, such as (partial or total) exchanges of discrepancies (see Definition 2.20) and irreducibility conditions need to be expressed in terms of the coupling rates.

Definition 5.1. *A pair of sites $(x,y)\in S^2$ is* an edge for Γ, *and we write $(x,y)\in E_\Gamma$, if*

$$\sum_k\sum_{\alpha,\beta\in X}\left[\Gamma^k_{\alpha,\beta}(y-x)+\Gamma^k_{\beta,\alpha}(x-y)\right]>0. \tag{5.1}$$

Remark 5.2. *Because the left-hand side of* (5.1) *is symmetric and the rates are translation invariant, we have*

$$(x,y)\in E_\Gamma \quad\Longleftrightarrow\quad (y,x)\in E_\Gamma \quad\Longleftrightarrow\quad (0,y-x)\in E_\Gamma.$$

Remark 5.3. *Due to the definitions* (2.16)–(2.18) *of the coupling rates, the set of edges E_Γ for Γ coincides with the set E_G of edges for G, defined as*

$$E_G=\left\{(x,y)\in S^2\colon \sum_{k,l}\sum_{\alpha,\beta\in X}\sum_{\gamma,\delta\in X}\left[G^{k;l}_{\alpha,\beta;\gamma,\delta}(y-x)+G^{l;k}_{\beta,\alpha;\delta,\gamma}(x-y)\right]>0\right\}.$$

Definition 5.4. *For any pair of sites $(x,y)\in S^2$, the distance $d(x,y)$ is the number of edges in the shortest* path of sites *p between x and y,*

$$d(x,y)=\inf\left\{n\colon \exists p\in\mathcal{P}^n(x,y)\right\},$$
$$\mathcal{P}^n(x,y)=\left\{p=(x_0,\dots,x_n)\in S^{n+1}\colon x_0=x, x_n=y, \forall 1\le i\le n, (x_{i-1},x_i)\in E_\Gamma\right\}.$$

Definition 5.5. *For any edge $(x,y)\in E_\Gamma$, a* finite path of coupled transitions on (x,y) *starting from $(\alpha',\beta',\gamma',\delta')\in X^4$ and ending at $(\alpha'',\beta'',\gamma'',\delta'')\in X^4$ is a finite sequence $\{(\alpha_r,\beta_r,\gamma_r,\delta_r)\}_{0\le r\le u}\subset(X^4)^{u+1}$, for $u\in\mathbb{N}\setminus\{0\}$, such that $(\alpha_0,\beta_0,\gamma_0,\delta_0)=(\alpha',\beta',\gamma',\delta')$, $(\alpha_u,\beta_u,\gamma_u,\delta_u)=(\alpha'',\beta'',\gamma'',\delta'')$, and for all $0\le r\le u-1$,*

$$\alpha_r-\alpha_{r+1}=\beta_{r+1}-\beta_r=:k_r,\qquad \gamma_r-\gamma_{r+1}=\delta_{r+1}-\delta_r=:l_r$$

and either

$$\left(k_r\ge 0, l_r\ge 0, G^{k_r;l_r}_{\alpha_r,\beta_r;\gamma_r,\delta_r}(y-x)>0\right)\quad\text{or}\quad\left(k_r\le 0, l_r\le 0, G^{-k_r;-l_r}_{\beta_r,\alpha_r;\delta_r,\gamma_r}(x-y)>0\right).$$

Remark 5.6. *This notion is the natural 'building block' to construct an event in which the width of discrepancies decreases. It plays the same role as a single coupled jump in the basic coupling construction.*

We also define the set

$$X^4_D=\left\{(\alpha,\beta,\gamma,\delta)\in X^4\colon (\alpha<\gamma,\beta>\delta)\text{ or }(\alpha>\gamma,\beta<\delta)\right\}$$

of values corresponding to couples of discrepancies of opposite signs (cf. Definition 2.18) located on two sites x,y of S, that is $\xi(x)=\alpha,\xi(y)=\beta,\zeta(x)=\gamma,\zeta(y)=\delta$.

In order to cope with 'strongly' asymmetric processes (in the sense that $\sum_k\Gamma^k_{\alpha,\beta}(y-x)=0$ for some $(x,y)\in E_\Gamma$ and $(\alpha,\beta)\in X^2$), we introduce the following notions.

Definition 5.7. *An* oriented wedge *is an ordered triple $(x_0,x_1,x_2)\in X^3$ such that $(x_0,x_1)\in E_\Gamma$, $(x_1,x_2)\in E_\Gamma$ and $x_0\ne x_2$.*

Definition 5.8. *Let W_Γ be a set of oriented wedges. For any pair of sites $(x,y)\in S^2$, a W_Γ-*path *from x to y is a set of oriented wedges $p_{W_\Gamma}=(w_1,\dots,w_{n-1})$, $n\ge 2$, such that $w_i=(x^i_0,x^i_1,x^i_2)$ for all $1\le i\le n-1$, $x^i_1=x^{i+1}_0$, $x^i_2=x^{i+1}_1$ for all $1\le i\le n-2$ and $x^1_0=x$, $x^{n-1}_2=y$. We denote by $\mathcal{P}^n_{W_\Gamma}(x,y)$ the set of all W_Γ-pathes with x,y as endpoints and $n-1$ wedges.*

Definition 5.9. *The set of sites S is W_Γ-*connected *if (S,E_Γ) is a connected graph and for all $(x,y)\in S^2$ such that $d(x,y)\ge 2$, there exists a W_Γ-path from x to y or from y to x.*

Definition 5.10. *If S is W_Γ-connected, for all $(x, y) \in S^2$, we define the quantity $d_{W_\Gamma}(x, y)$ as*

$$d_{W_\Gamma}(x, y) = \begin{cases} d(x, y) & \text{if } d(x, y) \le 1, \\ \inf\{n\colon \exists p \in \mathcal{P}^n_{W_\Gamma}(x, y)\} & \text{otherwise.} \end{cases}$$

Remark 5.11. *d_{W_Γ} is a pseudo-distance since triangular inequality may not hold. If W_Γ is taken as the set of all oriented wedges, $d(\cdot,\cdot)$ and $d_{W_\Gamma}(\cdot,\cdot)$ coincide. In general, we have only $d_{W_\Gamma}(x, y) \ge d(x, y)$ since the existence of a W_Γ-path with $n-1$ wedges, $p_{W_\Gamma} = (w_1, \dots, w_{n-1})$, $w_i = (x_0^i, x_1^i, x_2^i)$ for all $1 \le i \le n-1$, implies the existence of a path of sites $p = (x_0^1, x_1^1, \dots, x_1^{n-1}, x_2^{n-1})$ with $n+1$ elements and the same endpoints. See in Examples 6.2 cases where $d_{W_\Gamma}(x, y) > d(x, y)$.*

Definition 5.12. *The* irreducibility condition *(IC) for the coupled process is*:
(o) *(S, E_Γ) forms a connected lattice.*

For all $(x, y) \in E_\Gamma$,

(a) *for any couple of discrepancies $(\alpha, \beta, \gamma, \delta) \in X_D^4$, there is a finite path of coupled transitions on (x, y) starting from $(\alpha, \beta, \gamma, \delta)$ along which $f^+_{x,y}$ decreases*;
(b) *let $X_R(y-x) = X_L(x-y)$ be the set of values $\varepsilon \in X$ such that for any discrepancy with values (α, γ) located at x, there is a finite path of coupled transitions on (x, y) starting from $(\alpha, \varepsilon, \gamma, \varepsilon)$ which ends with a discrepancy on y; then either $X_R(y-x) = X$ or $X_L(y-x) = X$.*

When (b) is not satisfied, we replace (b) by (b′):
There exists a set of oriented wedges W_Γ such that:

(b'_0) *S is W_Γ-connected*;
(b'_1) *for all $w = (x_0, x_1, x_2) \in W_\Gamma$, $X_R(x_1 - x_0) \cup X_R(x_1 - x_2) = X$*;
(b'_2) *for all $w = (x_0, x_1, x_2) \in W_\Gamma$, all $\varepsilon_1 \notin X_R(x_1 - x_0)$ and all $\varepsilon_2 \in X_R(x_2 - x_1)$, there is a finite path of coupled transitions on (x_1, x_2) from $(\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2)$ to $(\varepsilon_3, \varepsilon_4, \varepsilon_3, \varepsilon_4)$ such that $\varepsilon_3 \in X_R(x_1 - x_0)$.*

Theorem 5.13. *If the process $(\eta_t)_{t\ge 0}$ with generator (2.8) is attractive and satisfies assumption* (IC), *then*:

(1) *if the state space of $(\eta_t)_{t\ge 0}$ is compact, that is $X = \{a, \dots, b\}$ for $(a, b) \in \mathbb{Z}^2$, we have $(\mathcal{I} \cap \mathcal{S})_e = \{\mu_\rho, \rho \in \mathcal{R}\}$, where $\mathcal{R}$ is a closed subset of $[a, b]$ containing $\{a, b\}$, and μ_ρ is a translation invariant probability measure on Ω with $\mu_\rho[\eta(0)] = \rho$; the measures μ_ρ are stochastically ordered, that is, $\mu_\rho \le \mu_{\rho'}$ if $\rho \le \rho'$*;
(2) *if $(\eta_t)_{t\ge 0}$ possesses a one parameter family $\{\mu_\rho\}_\rho$ of product invariant and translation invariant probability measures, we have $(\mathcal{I} \cap \mathcal{S})_e = \{\mu_\rho\}_\rho$.*

5.2. *Proof of Theorem 5.13*

We consider case (2). For (1), we refer to [3], Proposition 3.1, which deals with a misanthrope type model. The only difference to cope with here is (5.2) below, the difficult part common to both cases.

• *The method.*

Let $\mu \in (\mathcal{I} \cap \mathcal{S})_e$, and $\overline{\nu}$ be an extremal invariant and translation invariant probability measure for the coupled process $(\xi_t, \zeta_t)_{t\ge 0}$ with marginals μ and μ_ρ, for some value ρ of the parameter (see [1], Lemma 4.4 or [18], Proposition VIII.2.14). We claim that

$$\forall (x, y) \in S^2 \quad \overline{\nu}\{(\xi, \zeta)\colon \xi(x) > \zeta(x) \text{ and } \xi(y) < \zeta(y)\} = 0. \tag{5.2}$$

Then $\overline{\nu}\{(\xi, \zeta)\colon \xi \le \zeta \text{ or } \xi \ge \zeta\} = 1$ (see [1], Corollary 4.8; [18], Lemma VIII.3.2(b)), so that $\mu = \mu_{\rho_0}$, where $\rho_0 = \sup\{\rho, \mu \ge \mu_\rho\}$ (see [1], Proposition 5.5; [18], Proposition VIII.2.13 and Theorem VIII.3.9), which concludes the proof.

We are left to prove (5.2), that is, there cannot be discrepancies of opposite signs under the coupled measure $\overline{\nu}$, which we can also write

$$\forall (x, y) \in S^2, \forall (\alpha, \beta, \gamma, \delta) \in X_D^4 \text{ with } \alpha > \gamma, \beta < \delta$$
$$\overline{\nu}\{(\xi, \zeta)\colon \xi(x) = \alpha, \zeta(x) = \gamma, \xi(y) = \beta, \zeta(y) = \delta\} = 0. \tag{5.3}$$

For this, we show that in a coupled evolution where the initial configuration has at least on a pair of sites $(x,y)\in S^2$ a couple of discrepancies of opposite signs $(\alpha,\beta,\gamma,\delta)\in X_D^4$, then with a positive probability, the width of the positive discrepancy decreases to 0 in a (succession of) coupled transition(s). The latter exists by assumption (IC).

We prove (5.3) by induction on the distance $d(x,y)=n$ (resp. $d_{W_\Gamma}(x,y)=n$ under assumptions (IC, b′)) between the discrepancies of opposite signs.

To clarify the presentation, we turn some claims into lemmas, whose proofs are postponed to next subsection.

• *The case* $d(x,y)=1$.

By Theorem 2.19, the function $f^+_{x,y}=\psi^+_x+\psi^+_y$ with $\psi^+_z(\xi,\zeta)=[\xi(z)-\zeta(z)]^+$ defined in (2.30) satisfies (2.34), that is $\overline{\mathcal{L}}_{x,y}f^+_{x,y}(\xi,\zeta)\le 0$. Since $\overline{\nu}$ is invariant and translation invariant for the coupled process,

$$\begin{aligned}
0&=\int\overline{\mathcal{L}}\psi_0^+\,\mathrm{d}\overline{\nu}=\int\sum_{u\in S}\overline{\mathcal{L}}_{u,0}\psi_0^+\,\mathrm{d}\overline{\nu}+\int\sum_{v\in S}\overline{\mathcal{L}}_{0,v}\psi_0^+\,\mathrm{d}\overline{\nu}\\
&=\int\sum_{v\in S}\big(\overline{\mathcal{L}}_{-v,0}\psi_0^++\overline{\mathcal{L}}_{0,v}\psi_0^+\big)\,\mathrm{d}\overline{\nu}=\int\sum_{v\in S}\overline{\mathcal{L}}_{0,v}\big(\psi_v^++\psi_0^+\big)\,\mathrm{d}\overline{\nu}\\
&=\sum_{v\in S}\int\overline{\mathcal{L}}_{0,v}f^+_{0,v}\,\mathrm{d}\overline{\nu}.
\end{aligned}$$

Since each term is non-positive by (2.36), $\int\overline{\mathcal{L}}_{0,v}f^+_{0,v}\,\mathrm{d}\overline{\nu}=0$ for any site v, and by translation invariance we also get $\int\overline{\mathcal{L}}_{x,y}f^+_{x,y}\,\mathrm{d}\overline{\nu}=0$. Using (3.35), we obtain

$$\begin{aligned}
0=\;&\overline{\nu}\big\{\xi(x)=\alpha,\zeta(x)=\gamma,\xi(y)=\beta,\zeta(y)=\delta\big\}\\
&\times\sum_{k-l\notin\{0,(\alpha-\gamma)+(\delta-\beta)\}}\big(G^{k;l}_{\alpha,\beta;\gamma,\delta}(y-x)+G^{l;k}_{\beta,\alpha;\delta,\gamma}(x-y)\big)\Delta(\alpha,\beta,\gamma,\delta,k,l).
\end{aligned}\tag{5.4}$$

We conclude with the following lemma.

Lemma 5.14. *Under assumption* (IC, a), *Eq.* (5.4) *implies* (5.3) *for* $d(x,y)=1$.

• *The first steps to* $n\ge 2$.

We suppose (5.3) satisfied for $d(x,y)\le n-1$ (resp. for $d_{W_\Gamma}(x,y)\le n-1$ under assumptions (IC, b′)). Consider any path of sites $p=(x_0,\dots,x_n)\in\mathcal{P}^n(x,y)$ (resp. W_Γ-path $p_{W_\Gamma}=((x_0,x_1,x_2),(x_1,x_2,x_3),\dots,(x_{n-2},x_{n-1},x_n))\in\mathcal{P}^n_{W_\Gamma}(x,y)$). For all $1\le i\le n-1$, by the induction hypothesis,

$$\begin{aligned}
&\overline{\nu}\big\{(\xi,\zeta)\colon\ \xi(x_0)>\zeta(x_0),\xi(x_n)<\zeta(x_n),\xi(x_i)>\zeta(x_i)\big\}\\
&\quad+\overline{\nu}\big\{(\xi,\zeta)\colon\ \xi(x_0)>\zeta(x_0),\xi(x_n)<\zeta(x_n),\xi(x_i)<\zeta(x_i)\big\}=0.
\end{aligned}\tag{5.5}$$

Therefore if for $(\varepsilon_1,\dots,\varepsilon_{n-1})\in X^{n-1}$, we define the event

$$\begin{aligned}
\mathcal{E}_p(\varepsilon_1,\dots,\varepsilon_{n-1})=\big\{&(\xi,\zeta)\colon\ \xi(x_0)=\alpha>\zeta(x_0)=\gamma,\xi(x_n)=\beta<\zeta(x_n)=\delta;\\
&\big(\forall 1\le i\le n-1,\xi(x_i)=\zeta(x_i)=\varepsilon_i\big)\big\}
\end{aligned}\tag{5.6}$$

we now have

$$\overline{\nu}\big\{(\xi,\zeta)\colon\ \xi(x_0)=\alpha>\zeta(x_0)=\gamma,\xi(x_n)=\beta<\zeta(x_n)=\delta\big\}=\sum_{(\varepsilon_1,\dots,\varepsilon_{n-1})\in X^{n-1}}\overline{\nu}\big(\mathcal{E}_p(\varepsilon_1,\dots,\varepsilon_{n-1})\big).$$

We show that the left-hand side is zero by proving the existence of a path of sites $p\in\mathcal{P}^n(x,y)$ such that for all $(\varepsilon_1,\dots,\varepsilon_{n-1})\in X^{n-1}$,

$$\overline{\nu}\big(\mathcal{E}_p(\varepsilon_1,\dots,\varepsilon_{n-1})\big)=0.\tag{5.7}$$

We fix arbitrary $(\varepsilon_1, \dots, \varepsilon_{n-1}) \in X^{n-1}$ and $p = (x_0, \dots, x_n) \in \mathcal{P}^n(x, y)$ (resp. $p_{W_\Gamma} = ((x_0, x_1, x_2), (x_1, x_2, x_3), \dots, (x_{n-2}, x_{n-1}, x_n)) \in \mathcal{P}^n_{W_\Gamma}(x, y)$).

Lemma 5.15. *Equation* (5.7) *is satisfied for* $(\varepsilon_1, \dots, \varepsilon_{n-1}) \in X^{n-1}$ *if either* (a) $\varepsilon_1 \in X_R(x_1 - x_0)$, *or* (b) $\varepsilon_{n-1} \in X_R(x_{n-1} - x_n)$.

If, moreover, either $X_R(x_1 - x_0) = X$ or $X_R(x_{n-1} - x_n) = X$ Lemma 5.15(a) or 5.15(b) implies that (5.7) holds for any $(\varepsilon_1, \dots, \varepsilon_{n-1}) \in X^{n-1}$, and we are done. Otherwise, we have both $X_R(x_1 - x_0) \neq X$ and $X_R(x_{n-1} - x_n) \neq X$. Some more work is required, depending on which assumption we work with:

• *Assumption* (IC, b) *is satisfied.*

Thus $X_R(x_0 - x_1) = X_R(x_n - x_{n-1}) = X$, and we construct a new path of sites $p' = (x'_0, \dots, x'_n)$ where

$$\begin{cases} x'_0 = x_0 = x, x'_n = x_n = y, \\ x'_1 = x_0 + y - x_{n-1}, \\ x'_{n-1} = y + x_0 - x_1, \\ x'_{j+1} - x'_j = x_{j+1} - x_j \quad \text{for all } 1 \le j < n - 1. \end{cases} \tag{5.8}$$

By translation invariance (cf. Remark 5.2), $p' \in \mathcal{P}^n(x, y)$. Since $X_R(x'_1 - x) = X_R(y - x_{n-1}) = X$ and $X_R(x'_{n-1} - y) = X_R(x - x_1) = X$, both cases of Lemma 5.15 apply for the path p', and $\overline{\nu}(\mathcal{E}_{p'}(\varepsilon_1, \dots, \varepsilon_{n-1})) = 0$ for all $(\varepsilon_1, \dots, \varepsilon_{n-1}) \in X^{n-1}$.

• *Assumptions* (IC, b′) *are satisfied.*

We consider the path of wedges $p_{W_\Gamma} = ((x_0, x_1, x_2), (x_1, x_2, x_3), \dots, (x_{n-2}, x_{n-1}, x_n)) \in \mathcal{P}^n_{W_\Gamma}(x, y)$ and the induced path of sites $p = (x_0, x_1, \dots, x_n) \in \mathcal{P}_n(x, y)$. Using (IC, b'_1), we have

$$X_R(x_1 - x_0) \cup X_R(x_1 - x_2) = X. \tag{5.9}$$

When $n = 2$, we conclude by Lemma 5.15.

When $n \ge 3$, we assume that

$$\varepsilon_1 \notin X_R(x_1 - x_0), \qquad \varepsilon_{n-1} \notin X_R(x_{n-1} - x_n) \tag{5.10}$$

(otherwise Lemma 5.15 implies (5.7)). Then by (5.9),

$$\varepsilon_1 \notin X_R(x_1 - x_0), \qquad \varepsilon_{n-1} \in X_R(x_{n-1} - x_{n-2}). \tag{5.11}$$

Lemma 5.16. *Under assumption* (IC, b'_2), *Eq.* (5.7) *is satisfied* $\forall (\varepsilon_1, \dots, \varepsilon_{n-1}) \in X^{n-1}$ *such that* $\varepsilon_1 \notin X_R(x_1 - x_0)$ *and* $\varepsilon_{n-1} \in X_R(x_{n-1} - x_{n-2})$.

By (5.11), Lemma 5.16 exhausts the remaining cases for $n \ge 3$ and $\overline{\nu}(\mathcal{E}_p(\varepsilon_1, \dots, \varepsilon_{n-1})) = 0$ for all $(\varepsilon_1, \dots, \varepsilon_{n-1}) \in X^{n-1}$.

Theorem 5.13 is proven.

5.3. *Proofs of lemmas*

All three proofs share a common part that we state as another lemma.

Lemma 5.17. *Let* $(\widetilde{x}, \widetilde{y}) \in E_\Gamma$ *and a finite path of coupled transitions* $\{(\alpha_r, \beta_r, \gamma_r, \delta_r)\}_{0 \le r \le u}$ *on* $(\widetilde{x}, \widetilde{y})$. *Define a family of cylinder indicator functions* $\{\phi_r\}_{0 \le r \le u}$ *on coupled configurations such that for all* $(\xi, \zeta) \in \Omega^2$,

$$\begin{cases} \forall 0 \le r \le u & \phi_r(\xi, \zeta) \le \chi^{\alpha_r, \beta_r}_{\widetilde{x}, \widetilde{y}}(\xi) \chi^{\gamma_r, \delta_r}_{\widetilde{x}, \widetilde{y}}(\zeta), \\ \forall 0 \le r \le u - 1 & \phi_r(\xi, \zeta) = \begin{cases} \phi_{r+1}\big(S^{k_r}_{\widetilde{x}, \widetilde{y}} \xi, S^{l_r}_{\widetilde{x}, \widetilde{y}} \zeta\big) & \text{if } k_r + l_r > 0, \\ \phi_{r+1}\big(S^{-k_r}_{\widetilde{y}, \widetilde{x}} \xi, S^{-l_r}_{\widetilde{y}, \widetilde{x}} \zeta\big) & \text{if } k_r + l_r < 0. \end{cases} \end{cases} \tag{5.12}$$

Then $\int \phi_v \, d\overline{\nu} = 0$ *for* $v = u - 1$ *or for* $v = u$ *implies* $\int \phi_0 \, d\overline{\nu} = 0$.

Proof. The key idea is: if there is a succession of coupled transitions from an event of $\overline{\nu}$-probability 0 to another event, then the latter must also be of $\overline{\nu}$-probability 0.

We make a downwards induction on $0 \le r \le v$. We start the induction with the hypothesis $\int \phi_v \, d\overline{\nu} = 0$. Then we suppose that for $0 \le r \le v - 1$ we have $\int \phi_{r+1} \, d\overline{\nu} = 0$ and write

$$\begin{aligned} 0 &= \int \overline{\mathcal{L}} \phi_{r+1} \, d\overline{\nu} \\ &= \sum_{x',y'} \sum_{\alpha',\beta'} \sum_{\gamma',\delta'} \int \chi_{x',y'}^{\alpha',\beta'}(\xi) \chi_{x',y'}^{\gamma',\delta'}(\zeta) \sum_{k,l} \left(G_{\alpha',\beta';\gamma',\delta'}^{k;l}\left(y'-x'\right) \left[\phi_{r+1}\left(S_{x',y'}^{k}\xi, S_{x',y'}^{l}\zeta\right) - \phi_{r+1}(\xi,\zeta) \right] \right) d\overline{\nu}(\xi,\zeta) \\ &= \sum_{x',y'} \sum_{\alpha',\beta'} \sum_{\gamma',\delta'} \int \chi_{x',y'}^{\alpha',\beta'}(\xi) \chi_{x',y'}^{\gamma',\delta'}(\zeta) \sum_{k,l} G_{\alpha',\beta';\gamma',\delta'}^{k;l}\left(y'-x'\right) \phi_{r+1}\left(S_{x',y'}^{k}\xi, S_{x',y'}^{l}\zeta\right) d\overline{\nu}(\xi,\zeta). \end{aligned}$$

The integral over negative terms in the second line is zero as a function integrated on an event of $\overline{\nu}$-probability 0 (by the induction hypothesis). The third line is a sum of non-negative terms, so each of them is separately zero, in particular the term corresponding to k_r, l_r given in Definition 5.5, that is either

$$\begin{aligned} &k_r \ge 0, l_r \ge 0, x' = \widetilde{x}, y' = \widetilde{y}, k = k_r, l = l_r, \alpha' = \alpha_r, \beta' = \beta_r, \gamma' = \gamma_r, \delta' = \delta_r, \\ &\quad G_{\alpha_r,\beta_r;\gamma_r,\delta_r}^{k_r;l_r}(\widetilde{y} - \widetilde{x}) > 0; \qquad \chi_{\widetilde{x},\widetilde{y}}^{\alpha_r,\beta_r}(\xi) \chi_{\widetilde{x},\widetilde{y}}^{\gamma_r,\delta_r}(\zeta) \phi_{r+1}\left(S_{\widetilde{x},\widetilde{y}}^{k_r}\xi, S_{\widetilde{x},\widetilde{y}}^{l_r}\zeta\right) = \phi_r(\xi,\zeta) \end{aligned}$$

or

$$\begin{aligned} &k_r \le 0, l_r \le 0, x' = \widetilde{y}, y' = \widetilde{x}, k = -k_r, l = -l_r, \alpha' = \beta_r, \beta' = \alpha_r, \gamma' = \delta_r, \delta' = \gamma_r, \\ &\quad G_{\beta_r,\alpha_r;\delta_r,\gamma_r}^{-k_r;-l_r}(\widetilde{x} - \widetilde{y}) > 0; \qquad \chi_{\widetilde{x},\widetilde{y}}^{\alpha_r,\beta_r}(\xi) \chi_{\widetilde{x},\widetilde{y}}^{\gamma_r,\delta_r}(\zeta) \phi_{r+1}\left(S_{\widetilde{y},\widetilde{x}}^{-k_r}\xi, S_{\widetilde{y},\widetilde{x}}^{-l_r}\zeta\right) = \phi_r(\xi,\zeta). \end{aligned}$$

In both cases, this yields

$$\int \phi_r \, d\overline{\nu} = 0.$$

The induction is proven and Lemma 5.17 follows. □

Proof of Lemma 5.14. We have fixed an arbitrary element $(\alpha,\beta,\gamma,\delta)$ of X_D^4. By assumption (IC, a), for any coupled configuration (ξ,ζ) such that $\xi(x) = \alpha$, $\xi(y) = \beta$, $\zeta(x) = \gamma$, $\zeta(y) = \delta$, there is a finite path of coupled transitions $\{(\alpha_r,\beta_r,\gamma_r,\delta_r)\}_{0 \le r \le u}$ on (x,y) starting from $(\alpha_0,\beta_0,\gamma_0,\delta_0) = (\alpha,\beta,\gamma,\delta)$ such that

$$f_{x,y}^{+}\left(S_{x,y}^{K_{u-1}}\xi, S_{x,y}^{L_{u-1}}\zeta\right) < f_{x,y}^{+}(\xi,\zeta), \tag{5.13}$$

where $K_s = \sum_{r=0}^{s} k_r$, $L_s = \sum_{r=0}^{s} l_r$ for $0 \le s \le u-1$. Without loss of generality, we assume u to be the first step of the path of coupled transitions in which $f_{x,y}^{+}$ decreases, and, by Theorem 2.19, $f_{x,y}^{+}(S_{x,y}^{K_r}\xi, S_{x,y}^{L_r}\zeta) = f_{x,y}^{+}(\xi,\zeta)$ for all $0 \le r < u-1$, so that $(\alpha_r,\beta_r,\gamma_r,\delta_r) \in X_D^4$, and equality (5.4) is valid for $(\alpha_r,\beta_r,\gamma_r,\delta_r)$. For $0 \le r \le u-1$, we use Lemma 5.17 for

$$\varphi_r(\xi,\zeta) = \chi_{x,y}^{\alpha_r,\beta_r}(\xi) \chi_{x,y}^{\gamma_r,\delta_r}(\zeta).$$

By our assumption (5.13) on u and Definition 5.5, for $r = u-1$, we have either

$$\begin{aligned} &k_r \ge 0, l_r \ge 0 \text{ and } G_{\alpha_r,\beta_r;\gamma_r,\delta_r}^{k_r;l_r}(y-x) > 0, \\ &\quad \Delta(\alpha_r,\beta_r,\gamma_r,\delta_r,k_r,l_r) = f_{x,y}^{+}\left(S_{x,y}^{K_r}\xi, S_{x,y}^{L_r}\zeta\right) - f_{x,y}^{+}\left(S_{x,y}^{K_{r-1}}\xi, S_{x,y}^{L_{r-1}}\zeta\right) < 0 \end{aligned}$$

or

$$k_r \le 0, l_r \le 0 \text{ and } G^{-k_r;-l_r}_{\beta_r,\alpha_r;\delta_r,\gamma_r}(x-y) > 0,$$
$$\Delta(\beta_r,\alpha_r,\delta_r,\gamma_r,-k_r,-l_r) = f^+_{x,y}\big(S^{K_r}_{x,y}\xi, S^{L_r}_{x,y}\zeta\big) - f^+_{x,y}\big(S^{K_{r-1}}_{x,y}\xi, S^{L_{r-1}}_{x,y}\zeta\big) < 0.$$

In both cases, the sum in (5.4) applied to $(\alpha_r,\beta_r,\gamma_r,\delta_r)$ contains a negative term, thus is negative by Theorem 2.19. Hence, (5.4) applied to $(\alpha_r,\beta_r,\gamma_r,\delta_r)$ gives

$$\int \varphi_{u-1}\,\mathrm{d}\overline{\nu} = \overline{\nu}\big\{(\xi,\zeta)\colon\ \xi(x)=\alpha_{u-1}, \zeta(x)=\gamma_{u-1}, \xi(y)=\beta_{u-1}, \zeta(y)=\delta_{u-1}\big\} = 0.$$

For $u>1$, by Lemma 5.17 with $v=u-1$ we get

$$\int \varphi_0\,\mathrm{d}\overline{\nu} = \overline{\nu}\big\{(\xi,\zeta)\colon\ \xi(x)=\alpha, \zeta(x)=\gamma, \xi(y)=\beta, \zeta(y)=\delta\big\} = 0. \qquad\square$$

Proof of Lemma 5.15. We only derive Lemma 5.15(a), supposing that $\varepsilon_1 \in X_R(x_1-x)$. (The proof of Lemma 5.15(b), when $\varepsilon_{n-1} \in X_R(x_{n-1}-y)$, is similar and omitted.) Then, there is a finite path of coupled transitions on (x,x_1), $\{(\widetilde{\alpha}_r,\widetilde{\beta}_r,\widetilde{\gamma}_r,\widetilde{\delta}_r)\}_{0\le r\le u}$ such that $(\widetilde{\alpha}_0,\widetilde{\beta}_0,\widetilde{\gamma}_0,\widetilde{\delta}_0)=(\alpha,\varepsilon_1,\gamma,\varepsilon_1)$, $\widetilde{\beta}_r=\widetilde{\delta}_r$ for all $1\le r<u$ and $\widetilde{\beta}_u>\widetilde{\delta}_u$ (by attractiveness $(\alpha,\varepsilon_1)\ge(\gamma,\varepsilon_1)$ implies that $(\widetilde{\alpha}_r,\widetilde{\beta}_r)\ge(\widetilde{\gamma}_r,\widetilde{\delta}_r)$).

For $0\le r\le u$, we define the cylinder indicator functions

$$\varphi^{(n)}_r(\xi,\zeta) = \mathbf{1}\big\{(\xi,\zeta)\colon\ \xi(x)=\widetilde{\alpha}_r, \zeta(x)=\widetilde{\gamma}_r, \xi(x_1)=\widetilde{\beta}_r, \zeta(x_1)=\widetilde{\delta}_r;$$
$$\big(\xi(x_i)=\zeta(x_i)=\varepsilon_i, \forall 2\le i\le n-1\big); \xi(y)=\beta, \zeta(y)=\delta\big\}.$$

These functions fulfill conditions (5.12) of Lemma 5.17 for the finite path of coupled transitions $\{(\widetilde{\alpha}_r,\widetilde{\beta}_r,\widetilde{\gamma}_r,\widetilde{\delta}_r)\}_{0\le r\le u}$ on (x,x_1). In addition, we have

$$\varphi^{(n)}_u(\xi,\zeta) \le \mathbf{1}\big\{(\xi,\zeta)\colon\ \xi(x_1)<\zeta(x_1), \xi(y)>\zeta(y)\big\}.$$

Since $d(x_1,y)=n-1$ (resp. $d_{W_\Gamma}(x_1,y)=n-1$),

$$\overline{\nu}\big\{(\xi,\zeta)\colon\ \xi(x_1)<\zeta(x_1), \xi(y)>\zeta(y)\big\} = 0$$

by the induction hypothesis (we assumed (5.3) satisfied for $d(x,y)\le n-1$ under assumption (IC, b), resp. for $d_{W_\Gamma}(x,y)\le n-1$ under assumptions (IC, b′)). Hence $\int\varphi^{(n)}_u\,\mathrm{d}\overline{\nu}=0$. Applying Lemma 5.17 with $v=u$ implies then

$$0 = \int \varphi^{(n)}_0\,\mathrm{d}\overline{\nu} = \overline{\nu}\big(\mathcal{E}_p(\varepsilon_1,\dots,\varepsilon_{n-1})\big). \qquad\square$$

Proof of Lemma 5.16. We define

$$j_0 = \inf\big\{j\colon\ 1\le j\le n-1, \varepsilon_j\in X_R(x_j-x_{j-1})\big\}.$$

By (5.11), j_0 exists and $1<j_0$. Using (IC, b'_2), we make a downwards induction on $1\le m\le j_0-1$ to construct a sequence of finite pathes of coupled transitions on (x_m,x_{m+1}), $\{(\widetilde{\varepsilon}^{(m,r)}_m, \widetilde{\varepsilon}^{(m,r)}_{m+1}, \widetilde{\varepsilon}^{(m,r)}_m, \widetilde{\varepsilon}^{(m,r)}_{m+1})\}_{0\le r\le u_m}$ such that

$$\begin{cases} \widetilde{\varepsilon}^{(j_0-1,0)}_{j_0-1} = \varepsilon_{j_0-1} \notin X_R(x_{j_0-1}-x_{j_0-2}), \\ \widetilde{\varepsilon}^{(j_0-1,0)}_{j_0} = \varepsilon_{j_0} \in X_R(x_{j_0}-x_{j_0-1}), \\ \widetilde{\varepsilon}^{(j_0-1,u_{j_0-1})}_{j_0-1} \in X_R(x_{j_0-1}-x_{j_0-2}) \end{cases}$$

starts the induction for $m = j_0 - 1$ (thanks to the definition of j_0), and for each step $1 \le m < j_0 - 1$,

$$\begin{cases} \widetilde{\varepsilon}_m^{(m,0)} = \varepsilon_m \notin X_R(x_m - x_{m-1}), \\ \widetilde{\varepsilon}_{m+1}^{(m,0)} = \widetilde{\varepsilon}_{m+1}^{(m+1,u_{m+1})} \in X_R(x_{m+1} - x_m), \\ \widetilde{\varepsilon}_m^{(m,u_m)} \in X_R(x_m - x_{m-1}). \end{cases} \tag{5.14}$$

For all $1 \le m \le j_0 - 1$ and all $0 \le r \le u_m$, we define the cylinder indicator function

$$\begin{aligned} \widetilde{\varphi}_r^m(\xi,\zeta) = \mathbf{1}\big\{&(\xi,\zeta)\colon \xi(x) = \alpha, \zeta(x) = \gamma, \big(\xi(x_i) = \zeta(x_i) = \varepsilon_i, \forall 1 \le i \le m-1\big); \\ &\xi(x_m) = \zeta(x_m) = \widetilde{\varepsilon}_m^{(m,r)}, \xi(x_{m+1}) = \zeta(x_{m+1}) = \widetilde{\varepsilon}_{m+1}^{(m,r)}, \\ &\big(\xi(x_i) = \zeta(x_i) = \widetilde{\varepsilon}_i^{(i-1,u_{i-1})}, \forall m+2 \le i \le j_0\big); \\ &\big(\xi(x_i) = \zeta(x_i) = \varepsilon_i, \forall j_0 + 1 \le i \le n-1\big); \xi(y) = \beta, \zeta(y) = \delta\big\} \end{aligned}$$

(with the expressions between parentheses possibly empty according to the values of m and j_0) which verify (5.12) and in addition

$$\widetilde{\varphi}_0^m(\xi,\zeta) = \widetilde{\varphi}_{u_{m+1}}^{m+1}(\xi,\zeta) \tag{5.15}$$

for all $1 \le m < j_0 - 1$. By (5.14) for $m = 1$, we can now use Lemma 5.15(a) to get $\int \widetilde{\varphi}_{u_1}^1 \, \mathrm{d}\overline{\nu} = 0$, then we apply Lemma 5.17 for $v = u_1$ which gives $\int \widetilde{\varphi}_0^1 \, \mathrm{d}\overline{\nu} = 0$. This is the first step ($m = 1$) of another induction to prove that $\int \widetilde{\varphi}_0^m \, \mathrm{d}\overline{\nu} = 0$ for all $1 \le m \le j_0 - 1$. To go from m to $m+1$, we use (5.15) then another application of Lemma 5.17 to $\{\widetilde{\varphi}_r^m\}_{0 \le r \le u_m}$ for $v = u_m$. For $m = j_0 - 1$, we get

$$0 = \int \widetilde{\varphi}_0^{j_0-1} \, \mathrm{d}\overline{\nu} = \overline{\nu}\big(\mathcal{E}_p(\varepsilon_1, \dots, \varepsilon_{n-1})\big).$$

The lemma is proven. □

6. Irreducibility condition and invariant measures on examples

In this section, we determine what assumption (IC) imposes on the rates of the various examples of Section 4, so that Theorem 5.13 applies: For ZRP–MP, assumption (IC) enables to extend the previously known results to probability transitions $p(\cdot,\cdot)$ such that

$$q(x,y) = \frac{p(x,y) + p(y,x)}{2} \quad \text{satisfies (4.3).} \tag{6.1}$$

For SEP, assumption (IC) is proven equivalent to (4.3). Results on stick process (StP) and on the two-species exclusion process (S_2EP) are new.

6.1. *SEP*

Theorem 5.13 for SEP is equivalent to [18], Theorem 3.9(a). Indeed

Proposition 6.1. *For SEP, condition* (4.3) *for* $p(\cdot,\cdot)$ *is equivalent to assumption* (*IC*).

Proof. We assume condition (4.3), and check whether assumption (IC) is satisfied. The lattice being connected by (4.3), (IC, o) holds;

There exists two pairs of discrepancies of opposite signs

$$X_D^4 = \big\{(0,1,1,0), (1,0,0,1)\big\}.$$

By symmetry and Remark 2.13, we can restrict the analysis to the first element in X_D^4. By definition, for any edge $(x,y) \in E_\Gamma$, we have

$$0 < G_{0,1,1,0}^{0;1}(y-x) + G_{1,0,0,1}^{1;0}(x-y) = p(y-x) + p(x-y)$$

thus at least one of the two terms in the last sum is non-zero.

- (IC, a). A path of coupled transitions on (x,y) starting from $(0,1,1,0)$ along which $f_{x,y}^+$ decreases is given by

$$\begin{cases} \{(0,1,1,0),(0,1,0,1)\} & \text{if } p(y-x) > 0, \\ \{(0,1,1,0),(1,0,1,0)\} & \text{if } p(x-y) > 0, \end{cases}$$

so that assumption (IC, a) is always satisfied.

- (IC, b), (IC, b′). The value of $X_R(y-x)$ can be read from (cf. (4.4))

$$G_{0,0,1,0}^{0;1}(y-x) = G_{1,0,0,0}^{1;0}(y-x) = p(y-x),$$
$$G_{1,1,1,0}^{0;1}(x-y) = G_{1,1,1,0}^{1;0}(x-y) = p(x-y),$$

which gives

$$\begin{cases} 0 \in X_R(y-x) & \text{if and only if } \ p(y-x) > 0, \\ 1 \in X_R(y-x) & \text{if and only if } \ p(x-y) > 0. \end{cases} \tag{6.2}$$

Thus assumption (IC, b) requires

$$\forall (x,y) \in E_\Gamma \quad p(y-x) > 0 \quad \text{and} \quad p(x-y) > 0. \tag{6.3}$$

If this fails, we take

$$W_\Gamma = \big\{(x_0,x_1,x_2) \in S^3\colon\ p(x_1-x_0)p(x_2-x_1) > 0$$
$$\text{and either } p(x_1-x_2) = 0 \text{ or } p(x_0-x_1)p(x_1-x_2) > 0\big\}. \tag{6.4}$$

Thus (4.3) implies that for any pair $(x,y) \in S^2$, there is a W_Γ-path from either x to y, or from y to x and (IC, b_0') holds.

Let $(x_0,x_1,x_2) \in W_\Gamma$. By (6.2), $0 \in X_R(x_1-x_0)$, $1 \in X_R(x_1-x_2)$ and (IC, b_1') holds. Now suppose $X_R(x_1-x_0) \neq X$; then $p(x_0-x_1) = 0$, $p(x_1-x_2) = 0$ and $X_R(x_1-x_0) = X_R(x_2-x_1) = \{0\}$. Since $p(x_2-x_1) > 0$, there is a finite path of coupled transitions from $(1,0,1,0)$ to $(0,1,0,1)$ on (x_1,x_2) and (IC, b_2') holds. Assumption (IC) is thus satisfied.

For the converse, we assume (IC). Assumption (IC, o) implies condition (6.1). By (6.3), (IC, b) implies then (4.3). If assumptions (IC, b′) hold for some W_Γ, let $(x_0,x_1,x_2) \in W_\Gamma$ be such that $1 \notin X_R(x_1-x_0)$. Assumption (IC, b_1') implies $1 \in X_R(x_1-x_2)$, hence $p(x_2-x_1) > 0$ thus $0 \in X_R(x_2-x_1)$ by (6.2); then assumption (IC, b_2') implies that $1 \notin X_R(x_2-x_1)$ (hence $p(x_1-x_2) = 0$ by (6.2)), since there is no possible coupled transition starting from $(1,1,1,1)$. Similarly, if $0 \notin X_R(x_1-x_0)$ (hence $p(x_1-x_0) = 0$ by (6.2)), by (IC, b_1') we have $0 \in X_R(x_1-x_2)$ and $p(x_1-x_2) > 0$ by (6.2); then by (IC, b_2'), $0 \notin X_R(x_2-x_1)$ (hence $p(x_2-x_1) = 0$ by (6.2), thus $1 \notin X_R(x_1-x_2)$ and by (IC, b_1') we have $1 \in X_R(x_1-x_0)$ thus $p(x_1-x_0) > 0$ by (6.2)), since there is no possible transition starting from $(0,0,0,0)$. Therefore the only possible choice for W_Γ is (6.4).

This implies (4.3). □

Example 6.2. *Let $S = \mathbb{Z}^2$ be endowed with its canonical basis (e_1,e_2). We denote by $\underline{0}$ the origin.*

(1) *Let $p(e_1) = p_1$, $p(-e_1) = q_1$, $p(e_2) = p_2$, $p(-e_2) = q_2$, with p_1,q_1,p_2,q_2 all distinct, $p_1+q_1+p_2+q_2 = 1$. There is an edge in E_Γ between $\underline{0}$ and its four neighboring sites hence assumption* (IC, o) *holds, and $p(y-x)p(x-y) > 0$ for all $(x,y) \in E_\Gamma$ so that assumption* (IC, b) *is satisfied.*

(2) *Let* $p(2e_1)=p_1$, $p(-e_1)=q_1$, $p(e_2)=p_2$, *with* $p_1+q_1+p_2=1$.

$$E_\Gamma=\big\{(\underline{0},2e_1),(\underline{0},-e_1),(\underline{0},e_2)\big\},$$
$$W_\Gamma=\big\{(\underline{0},-e_1,e_1),(\underline{0},2e_1,2e_1+e_2),(2e_1,e_1,\underline{0}),(e_1-e_2,e_1,\underline{0}),(e_1-e_2,-e_2,\underline{0})\big\}.$$

Notice first that the distances d *and* d_{W_Γ} *are distinct: for* $x=\underline{0}$, $y=4e_1-e_2$ *we have* $d(x,y)=3$, $d_{W_\Gamma}(x,y)=5$, *corresponding to the following path of sites and* W_Γ*-path*

$$p=(\underline{0},2e_1,2e_1-e_2,4e_1-e_2),$$
$$p_{W_\Gamma}=\big((\underline{0},e_1,2e_1),(e_1,2e_1,2e_1-e_2),(2e_1,2e_1-e_2,3e_1-e_2),(2e_1-e_2,3e_1-e_2,4e_1-e_2)\big).$$

There is an edge in E_Γ *between* $\underline{0}$ *and its four neighboring sites hence assumption* (IC, o) *holds. There exists a* W_Γ*-path between* $\underline{0}$ *and each of its* 4 *second neighbors*, e_1+e_2, $-e_1+e_2$, $-e_1-e_2$, e_1-e_2 *so that* (IC, b'_0) *holds. By* (6.2), (IC, b'_1) *is satisfied, and so is* (IC, b'_2) (*by inspection of all elements of* W_Γ).

6.2. *ZRP*

Applied to this case, Theorem 5.13 generalizes [1], Theorem 1.9, since

Proposition 6.3. *For ZRP, assumption* (IC) *is equivalent to* (6.1) *and* (4.2).

Proof. Because the transition rates have a product form and $g(\cdot)$ is non decreasing (by attractiveness), for all $(x,y)\in E_\Gamma$ necessarily $p(x,y)+p(y,x)>0$, and for any $\gamma>0$ the existence of a coupled transition on (x,y) from $(0,\gamma,\gamma,0)\in X_D^4$ reads

$$\big(p(x,y)+p(y,x)\big)g(\gamma)>0,$$

that is, assumption (IC, a) implies (4.2), the usual irreducibility condition on $g(\cdot)$. We now verify that (6.1) and (4.2) imply assumption (IC).

• (IC, a). Consider $(x,y)\in E_\Gamma$ and a pair of discrepancies $(\alpha,\beta,\gamma,\delta)\in X_D^4$ on (x,y) such that $\alpha<\gamma$, $\beta>\delta$ (the other case being treated by reversing the roles of the two marginals). A path of coupled transitions on (x,y) starting from $(\alpha,\beta,\gamma,\delta)$ along which $f^+_{x,y}$ decreases (by 1) is given by either

$$\big\{(\alpha,\beta,\gamma,\delta),\dots,(0,\beta+\alpha,\gamma-\alpha,\delta+\alpha),(0,\beta+\alpha,\gamma-\alpha-1,\delta+\alpha+1)\big\},\tag{6.5}$$

when $p(y-x)>0$ (with rates $p(y-x)$ times $g(\alpha),g(\alpha-1),\dots,g(1),g(\gamma-\alpha)$), or by

$$\big\{(\alpha,\beta,\gamma,\delta),\dots,(\alpha+\delta,\beta-\delta,\gamma+\delta,0),(\alpha+\delta+1,\beta-\delta-1,\gamma+\delta,0)\big\},\tag{6.6}$$

when $p(x-y)>0$ (with rates $p(x-y)$ times $g(\delta),g(\delta-1),\dots,g(1),g(\beta-\delta)$).

• (IC, b). Consider $\alpha<\gamma$ and $\varepsilon\ge 0$. Suppose $p(y-x)>0$; then a path of coupled transitions on (x,y) starting from $(\alpha,\varepsilon,\gamma,\varepsilon)$ which creates a discrepancy on y is given by

$$\big\{(\alpha,\varepsilon,\gamma,\varepsilon),\dots,(0,\varepsilon+\alpha,\gamma-\alpha,\varepsilon+\alpha),(0,\varepsilon+\alpha,\gamma-\alpha-1,\varepsilon+\alpha+1)\big\}\tag{6.7}$$

(with rates $p(y-x)$ times $g(\alpha),g(\alpha-1),\dots,g(1),g(\gamma-\alpha)$), which leads to $X_R(y-x)=X$. □

[1], Lemma 4.6 constructs the path of coupled transitions (6.5) (resp. (6.6)) to go from $(\alpha,\beta,\gamma,\delta)$ to $(0,\beta+\alpha,\gamma-\alpha,\delta+\alpha)$ when $p(y-x)>0$ (resp. to $(\alpha+\delta,\beta-\delta,\gamma+\delta,0)$ when $p(x-y)>0$). Lemma 5.15 corresponds to [1], Lemma 4.7.

6.3. *MP*

This is the third example with single particle jumps. Applied to this case, Theorem 5.13 generalizes [8], Theorem 2.13: we do not require product invariant probability measures, that is [8], formulas (2.3) and (2.4), and we extend [8], formula (1.7).

Proposition 6.4. *For MP, assumption (IC) is equivalent to either* (i) $b(\gamma,\alpha)>0$ *for all* $\gamma>0$, $\alpha\geq 0$, *and* $p(\cdot,\cdot)$ *satisfies* (6.1); *or* (ii) *for all* $(x,y)\in E_\Gamma$, $p(x,y)p(y,x)>0$ *and* $b(\cdot,\cdot)$ *may take some zero values along with*

$$b(\gamma,\alpha)>0 \quad \text{for all } \gamma>\alpha\geq 0. \tag{6.8}$$

In [8], (i) and (ii) have to be simultaneously valid.

Proof of Proposition 6.4. As for ZRP, necessarily $p(x,y)+p(y,x)>0$ for all $(x,y)\in E_\Gamma$. For any $\gamma>\alpha$ the existence of a coupled transition on (x,y) from $(\alpha,\gamma,\gamma,\alpha)\in X_D^4$ reads (we cannot reduce ourselves here to $\alpha=0$ since $b(\cdot,\cdot)$ depends on α,γ, and not only on α like $g(\cdot)$ for ZRP)

$$\begin{aligned}&\big(p(x,y)+p(y,x)\big)\big(b(\alpha,\gamma)\wedge b(\gamma,\alpha)+\big[b(\gamma,\alpha)-b(\alpha,\gamma)\wedge b(\gamma,\alpha)\big]^+\\&\quad+\big[b(\alpha,\gamma)-b(\alpha,\gamma)\wedge b(\gamma,\alpha)\big]^+\big)=\big(p(x,y)+p(y,x)\big)b(\gamma,\alpha)>0\end{aligned}$$

and thus (6.8) is valid. On the other hand, suppose that there is $(x,y)\in E_\Gamma$ such that $p(x-y)=0$ and that $b(\gamma_0,\delta_0)=0$ for some pair (γ_0,δ_0) such that $0<\gamma_0\leq\delta_0$. By attractiveness, $b(\alpha,\beta)=0$ for all (α,β) such that $\alpha<\gamma_0$ and $\beta>\delta_0$ and there is no path of coupled transitions on (x,y) starting from $(\alpha,\beta,\gamma_0,\delta_0)\in X_D^4$. Hence assumption (IC) excludes having simultaneously $p(x-y)=0$ for some $(x,y)\in E_\Gamma$ and non-trivial zeros in $b(\cdot,\cdot)$. We thus have the two (non-mutually exclusive) cases (i) and (ii) stated above.

Case (i) follows exactly the same lines as for ZRP. For case (ii), the construction of the correct pathes of coupled transitions on $(x,y)\in E_\Gamma$ depends on the values α_0,β_0 for which $b(\alpha_0,\beta_0)=0$. To explain how to proceed, we show that a Misanthrope process with rates $b(\cdot,\cdot)$ such that $b(\gamma,\delta)>0$ for all $\gamma>\delta$ and $b(\gamma,\delta)=0$ for all $\gamma\leq\delta$ satisfies assumption (IC).

• (IC, a). Let $(x,y)\in E_\Gamma$ and $(\alpha,\beta,\gamma,\delta)\in X_D^4$ on (x,y) such that $\alpha<\gamma$, $\beta>\delta$. Suppose first that $\gamma>\delta$ and let r be the smallest integer such that $\gamma-r\leq\delta+r$. Observe that $\beta+r>\delta+r\geq\gamma-r>\alpha-r$ and thus $\alpha-r+1\leq\beta+r-1$ while by definition of r, $\gamma-r+1>\delta+r-1$. Now choose $r'\leq r-1$ to be the smallest integer verifying both $\alpha-r'\leq\beta+r'$ and $\gamma-r'>\delta+r'$; a path of coupled transitions on (x,y) starting from $(\alpha,\beta,\gamma,\delta)$ along which $f^+_{x,y}$ decreases (by 1) is given by

$$\big\{(\alpha,\beta,\gamma,\delta),\dots,\big(\alpha-r',\beta+r',\gamma-r',\delta+r'\big),\big(\alpha-r',\beta+r',\gamma-r'-1,\delta+r'+1\big)\big\}$$

(with rates $p(x,y)$ times $b(\alpha,\beta),b(\alpha-1,\beta+1),\dots,b(\alpha-r'+1,\beta+r'-1),b(\gamma-r',\delta+r')$).

Suppose now that $\gamma\leq\delta$. Take r as the smallest integer such that $\gamma+r\geq\delta-r$. Hence $r<(\delta-\gamma+1)/2$ and since $\beta>\delta>\gamma>\alpha$, $(\beta-r)-(\alpha+r)\geq(\delta-r)-(\gamma+r)+2\geq 1$. Thus $\beta-r>\alpha+r$ and a path of coupled transitions on (x,y) starting from $(\alpha,\beta,\gamma,\delta)$ along which $f^+_{x,y}$ decreases (by 1) is given by

$$\big\{(\alpha,\beta,\gamma,\delta),\dots,(\alpha+r,\beta-r,\gamma+r,\delta-r),(\alpha+r+1,\beta-r-1,\gamma+r,\delta-r)\big\}$$

(with rates $p(y,x)$ times $b(\delta,\gamma),b(\delta-1,\gamma+1),\dots,b(\delta-r+1,\gamma+r-1),b(\beta-r,\alpha+r)$).

• (IC, b). Consider $\alpha<\gamma$ and $\varepsilon\geq 0$. There are five possibilities and for each one we give a path of coupled transitions on (x,y) starting from $(\alpha,\varepsilon,\gamma,\varepsilon)$ which creates a discrepancy on y.

(1) $\varepsilon\leq\alpha$, $r_0=\inf\{r>0\colon \varepsilon+2r\geq\alpha\}$ and $\gamma>\varepsilon+2r_0$:

$$\big\{(\alpha,\varepsilon,\gamma,\varepsilon),\dots,(\alpha-r_0,\varepsilon+r_0,\gamma-r_0,\varepsilon+r_0),(\alpha-r_0,\varepsilon+r_0,\gamma-r_0-1,\varepsilon+r_0+1)\big\}$$

(with rates $p(x,y)$ times $b(\alpha,\varepsilon),b(\alpha-1,\varepsilon+1),\dots,b(\alpha-r_0+1,\varepsilon+r_0-1),b(\gamma-r_0,\varepsilon+r_0)$).

(2) $\varepsilon \le \alpha$ and $\gamma = \varepsilon + 2r_0 = \alpha + 1$:

$$\{(\alpha, \varepsilon, \gamma, \varepsilon), \dots, (\alpha - r_0, \varepsilon + r_0, \gamma - r_0, \varepsilon + r_0), (\alpha - r_0 + 1, \varepsilon + r_0 - 1, \gamma - r_0, \varepsilon + r_0)\}$$

(with rates $p(x, y)$ times $b(\alpha, \varepsilon), b(\alpha - 1, \varepsilon + 1), \dots, b(\alpha - r_0 + 1, \varepsilon + r_0 - 1)$, then $p(y, x)$ times $b(\varepsilon + r_0, \alpha - r_0)$).

(3) $\alpha < \varepsilon < \gamma$:

$$\{(\alpha, \varepsilon, \gamma, \varepsilon), (\alpha, \varepsilon, \gamma - 1, \varepsilon + 1)\}$$

(with rate $p(x, y)$ times $b(\gamma, \varepsilon)$).

(4) $\gamma \le \varepsilon$, $r_1 = \inf\{r > 0\colon\ \varepsilon - 2r \le \gamma\}$ and $\alpha = \varepsilon - 2r_1 = \gamma - 1$:

$$\{(\alpha, \varepsilon, \gamma, \varepsilon), \dots, (\alpha + r_1, \varepsilon - r_1, \gamma + r_1, \varepsilon - r_1), (\alpha + r_1, \varepsilon - r_1, \gamma + r_1 - 1, \varepsilon - r_1 + 1)\}$$

(with rates $p(y, x)$ times $b(\varepsilon, \gamma), b(\varepsilon - 1, \gamma + 1), \dots, b(\varepsilon - r_1 + 1, \gamma + r_1 - 1)$, then $p(x, y)$ times $b(\gamma + r_1, \varepsilon - r_1)$).

(5) $\gamma \le \varepsilon$ and $\alpha < \varepsilon - 2r_1$:

$$\{(\alpha, \varepsilon, \gamma, \varepsilon), \dots, (\alpha + r_1, \varepsilon - r_1, \gamma + r_1, \varepsilon - r_1), (\alpha + r_1 + 1, \varepsilon - r_1 - 1, \gamma + r_1, \varepsilon - r_1)\}$$

(with rates $p(y, x)$ times $b(\varepsilon, \gamma), b(\varepsilon - 1, \gamma + 1), \dots, b(\varepsilon - r_1 + 1, \gamma + r_1 - 1), b(\varepsilon - r_1, \alpha + r_1)$).

This leads to $X_R(y - x) = X$.

Assumptions (IC, a, b) are thus satisfied. □

6.4. *StP*

For all $(x, y) \in E_\Gamma$ (so that $|y - x| = 1$), $(\alpha, \beta) \in X^2$ with $\alpha \ne 0, \beta \ne 0$, by (4.7) and the definition of E_Γ,

$$\sum_k \bigl(\Gamma^k_{\alpha,\beta}(1) + \Gamma^k_{\beta,\alpha}(-1)\bigr) = p(1)\alpha + p(-1)\beta > 0$$

therefore necessarily $p(1) + p(-1) > 0$ and assumption (IC, o) holds. We now check that assumptions (IC, a, b) are satisfied.

• (IC, a). A path of coupled transition on (x, y) starting from $(\alpha, \beta, \gamma, \delta) \in X_D^4$ with for instance $\alpha < \gamma, \beta > \delta$, along which $f^+_{x,y}$ decreases is given by (cf. (4.9)–(4.11))

$$\begin{cases} \{(\alpha, \beta, \gamma, \delta), (\alpha, \beta, \alpha, \delta + \gamma - \alpha)\} & \text{if } p(1) > 0 \ \bigl(\text{with rate } G^{0;\gamma-\alpha}_{\alpha,\beta;\gamma,\delta}(1) = p(1)\bigr), \\ \{(\alpha, \beta, \gamma, \delta), (\alpha + \beta - \delta, \delta, \gamma, \delta)\} & \text{if } p(-1) > 0 \ \bigl(\text{with rate } G^{\beta-\delta;0}_{\beta,\alpha;\delta,\gamma}(-1) = p(-1)\bigr). \end{cases}$$

• (IC, b). Consider $\alpha < \gamma$ and $\varepsilon \ge 0$. Suppose $p(y - x) = p(1) > 0$; then a path of coupled transitions on (x, y) starting from $(\alpha, \varepsilon, \gamma, \varepsilon)$ which creates a discrepancy on y is given by

$$\{(\alpha, \varepsilon, \gamma, \varepsilon), (\alpha, \varepsilon, \alpha, \varepsilon + \gamma - \alpha)\} \quad \bigl(\text{with rate } G^{0;\gamma-\alpha}_{\alpha,\varepsilon;\gamma,\varepsilon}(1) = p(1)\bigr)$$

which leads to $X_R(1) = X$. When $p(x - y) = p(-1) > 0$, a path of coupled transitions on (x, y) starting from $(\varepsilon, \alpha, \varepsilon, \gamma)$ which creates a discrepancy on x is given by

$$\{(\varepsilon, \alpha, \varepsilon, \gamma), (\varepsilon, \alpha, \alpha, \varepsilon + \gamma - \alpha)\} \quad \bigl(\text{with rate } G^{0;\gamma-\alpha}_{\alpha,\varepsilon;\gamma,\varepsilon}(-1) = p(-1)\bigr)$$

which leads to $X_L(1) = X$.

6.5. S_2EP

We first give the general conditions on the rates under which assumption (IC) is satisfied. They are obtained by direct inspection of all possible cases, we rely on Table 1. To illustrate this, we present three examples which fulfill assumption (IC) in different manners. Notice first that since the dynamics is nearest neighbor and non-degenerated by (4.12), (IC, o) is always satisfied.

• (IC, a)

$$\begin{aligned} X_D^4 = \big\{&(-1,0,0,-1),(-1,0,1,-1),(-1,1,0,0),(-1,1,0,-1),(-1,1,1,-1),(-1,1,1,0),\\ &(0,0,1,-1),(0,1,1,0),(0,1,1,-1),(0,-1,-1,0),(0,-1,-1,1),(0,0,-1,1),\\ &(1,-1,-1,0),(1,-1,-1,1),(1,-1,0,0),(1,-1,0,1),(1,0,-1,1),(1,0,0,1)\big\}. \end{aligned}$$

The following expressions need to be positive for $z=\pm 1$. They correspond to coupled transitions on $(0,z)$ from $(\alpha,\beta,\gamma,\delta)\in X_D^4$ with $\alpha<\gamma,\beta>\delta$ for which $f_{0,z}^+$ decreases

$$\begin{aligned} &G^{0;1}_{-1,0;0,-1}(z)+G^{1;0}_{0,-1;-1,0}(-z)=\Gamma^1_{0,-1}(z)+\Gamma^1_{0,-1}(-z),\\ &G^{0;1}_{-1,1;1,-1}(z)+G^{0;2}_{-1,1;1,-1}(z)+G^{1;0}_{1,-1;-1,1}(-z)+G^{2;0}_{1,-1;-1,1}(-z)\\ &\quad=\Gamma^1_{1,-1}(z)+\Gamma^2_{1,-1}(z)+\Gamma^1_{1,-1}(-z)+\Gamma^2_{1,-1}(-z),\\ &G^{0;1}_{0,1;1,0}(z)+G^{1;0}_{1,0;0,1}(-z)=\Gamma^1_{1,0}(z)+\Gamma^1_{1,0}(-z),\\ &G^{0;1}_{-1,0;1,-1}(z)+G^{0;2}_{-1,0;1,-1}(z)+G^{1;0}_{0,-1;-1,1}(-z)=\Gamma^1_{1,-1}(z)+\Gamma^2_{1,-1}(z)+\Gamma^1_{0,-1}(-z),\\ &G^{0;1}_{0,1;1,-1}(z)+G^{0;2}_{0,1;1,-1}(z)+G^{1;0}_{1,0;-1,1}(-z)=\Gamma^1_{1,-1}(z)+\Gamma^2_{1,-1}(z)+\Gamma^1_{1,0}(-z). \end{aligned}$$

We need also that either

$$\begin{aligned} &G^{0;1}_{0,0;1,-1}(z)+G^{1;2}_{0,0;1,-1}(z)+G^{1;0}_{0,0;-1,1}(-z)\\ &\quad=\Gamma^1_{1,-1}(z)-\big[\Gamma^1_{0,0}(z)-\Gamma^2_{1,-1}(z)\big]^++\Gamma^1_{0,0}(z)\wedge\Gamma^2_{1,-1}(z)+\Gamma^1_{0,0}(-z) \end{aligned} \tag{6.9}$$

is positive for $z=\pm 1$ (this corresponds to a coupled transition on $(0,z)$ from $(0,0,1,-1)$ for which $f_{0,z}^+$ decreases) or

$$G^{1;1}_{0,0;1,-1}(-z)+G^{0;2}_{0,0;1,-1}(-z)=\big|\Gamma^1_{0,0}(-z)-\Gamma^2_{1,-1}(-z)\big| \tag{6.10}$$

and (6.9) are simultaneously positive for $z=1$ or for $z=-1$. There, $f_{0,z}^+$ decreases along paths of coupled transitions on $(0,z)$ for which the first one induces an exchange of discrepancies (see Example 6.5(iii) below).

• (IC, b), (IC, b′)

Up to an exchange of marginals, there are three possible values (α,γ) with $\alpha<\gamma$ for a discrepancy on a given site: $(-1,0)$, $(-1,1)$, $(0,1)$. Therefore for each element $\varepsilon\in\{-1,0,1\}$ and each site $z\in\{-1,1\}$ neighbor of the origin, a set of 3 inequalities has to be verified by the coupling rates whenever ε belongs to $X_R(z)$,

$$-1\in X_R(z)\quad\Longleftrightarrow\quad\begin{cases} G^{0;1}_{-1,-1;0,-1}(z)>0,\\ G^{0;1}_{-1,-1;1,-1}(z)+G^{0;2}_{-1,-1;1,-1}(z)>0,\\ \begin{cases} G^{0;1}_{0,-1;1,-1}(z)+G^{0;2}_{0,-1;1,-1}(z)+G^{1;0}_{0,-1;1,-1}(z)+G^{1;2}_{0,-1;1,-1}(z)>0\\ \text{or}\\ \begin{cases} G^{1;1}_{0,-1;1,-1}(z)>0,\\ G^{0;1}_{-1,0;0,0}(z)+G^{0;1}_{0,-1;0,0}(-z)+G^{1;0}_{0,-1;0,0}(-z)>0. \end{cases} \end{cases} \end{cases}$$

Table 1
Non-zero coupling rates for the two species exclusion model

				$\zeta(x)\ \ \zeta(y)$								
$\xi(x)$	$\xi(y)$	k	l	$-1\ \ -1$	$-1\ \ 0$	$0\ \ -1$	$-1\ \ 1$	$0\ \ 0$	$1\ \ -1$	$0\ \ 1$	$1\ \ 0$	$1\ \ 1$
-1	-1	0	1			$\Gamma^1_{0,-1}$		$\Gamma^1_{0,0}$	$\Gamma^1_{1,-1}$		$\Gamma^1_{1,0}$	
-1	-1	0	2						$\Gamma^2_{1,-1}$			
-1	0	0	1			$\Gamma^1_{0,-1}$		$\Gamma^1_{0,0}$	$\Gamma^1_{1,-1}$		$\Gamma^1_{1,0}$	
-1	0	0	2						$\Gamma^2_{1,-1}$			
0	-1	0	1						$\Gamma^1_{1,-1}+\Gamma^2_{1,-1}-\Gamma^1_{0,-1}$		$[\Gamma^1_{1,0}-\Gamma^1_{0,-1}]^+$	
0	-1	1	0	$\Gamma^1_{0,-1}$	$\Gamma^1_{0,-1}$		$\Gamma^1_{0,-1}$	$\Gamma^1_{0,-1}-\Gamma^1_{0,0}$		$\Gamma^1_{0,-1}$	$[\Gamma^1_{0,-1}-\Gamma^1_{0,-1}]^+$	$\Gamma^1_{0,-1}$
0	-1	1	1			$\Gamma^1_{0,-1}$		$\Gamma^1_{0,0}$	$\Gamma^1_{0,-1}-\Gamma^2_{1,-1}$		$\Gamma^1_{1,0}\wedge\Gamma^1_{0,-1}$	
0	-1	1	2						$\Gamma^2_{1,-1}$			
-1	1	0	1			$\Gamma^1_{0,-1}$		$\Gamma^1_{0,0}$	$\Gamma^1_{1,-1}$		$\Gamma^1_{1,0}$	
-1	1	0	2						$\Gamma^2_{1,-1}$			
0	0	0	1			$\Gamma^1_{0,-1}-\Gamma^1_{0,0}$			$\Gamma^1_{1,-1}-[\Gamma^1_{0,0}-\Gamma^2_{1,-1}]^+$		$\Gamma^1_{1,0}-\Gamma^1_{0,0}$	
0	0	0	2						$[\Gamma^2_{1,-1}-\Gamma^1_{0,0}]^+$			
0	0	1	0	$\Gamma^1_{0,0}$	$\Gamma^1_{0,0}$		$\Gamma^1_{0,0}$			$\Gamma^1_{0,0}$		$\Gamma^1_{0,0}$
0	0	1	1			$\Gamma^1_{0,0}$		$\Gamma^1_{0,0}$	$[\Gamma^1_{0,0}-\Gamma^2_{1,-1}]^+$		$\Gamma^1_{0,0}$	
0	0	1	2						$\Gamma^1_{0,0}\wedge\Gamma^2_{1,-1}$			
1	-1	1	0	$\Gamma^1_{1,-1}$	$\Gamma^1_{1,-1}$	$\Gamma^1_{1,-1}+\Gamma^2_{1,-1}-\Gamma^1_{0,-1}$	$\Gamma^1_{1,-1}$	$\Gamma^1_{1,-1}-[\Gamma^1_{0,0}-\Gamma^2_{1,-1}]^+$		$\Gamma^1_{1,-1}$	$\Gamma^1_{1,-1}+\Gamma^2_{1,-1}-\Gamma^1_{1,0}$	$\Gamma^1_{1,-1}$
1	-1	1	1			$\Gamma^1_{0,-1}-\Gamma^2_{1,-1}$		$[\Gamma^1_{0,0}-\Gamma^2_{1,-1}]^+$	$\Gamma^1_{1,-1}$		$\Gamma^1_{1,0}-\Gamma^2_{1,-1}$	
1	-1	2	0	$\Gamma^2_{1,-1}$	$\Gamma^2_{1,-1}$		$\Gamma^2_{1,-1}$	$[\Gamma^2_{1,-1}-\Gamma^1_{0,0}]^+$		$\Gamma^2_{1,-1}$		$\Gamma^2_{1,-1}$
1	-1	2	1			$\Gamma^2_{1,-1}$		$\Gamma^2_{1,-1}\wedge\Gamma^1_{0,0}$			$\Gamma^2_{1,-1}$	
1	-1	2	2						$\Gamma^2_{1,-1}$			
0	1	0	1			$\Gamma^1_{0,-1}$		$\Gamma^1_{0,0}$	$\Gamma^1_{1,-1}$		$\Gamma^1_{1,0}$	
0	1	0	2						$\Gamma^2_{1,-1}$			
1	0	0	1			$[\Gamma^1_{0,-1}-\Gamma^1_{1,0}]^+$			$\Gamma^1_{1,-1}+\Gamma^2_{1,-1}-\Gamma^1_{1,0}$			
1	0	1	0	$\Gamma^1_{1,0}$	$\Gamma^1_{1,0}$	$[\Gamma^1_{1,0}-\Gamma^1_{0,-1}]^+$	$\Gamma^1_{1,0}$	$\Gamma^1_{1,0}-\Gamma^1_{0,0}$		$\Gamma^1_{1,0}$		$\Gamma^1_{1,0}$
1	0	1	1			$\Gamma^1_{1,0}\wedge\Gamma^1_{0,-1}$		$\Gamma^1_{0,0}$	$\Gamma^1_{1,0}-\Gamma^2_{1,-1}$		$\Gamma^1_{1,0}$	
1	0	1	2						$\Gamma^2_{1,-1}$			
1	1	0	1			$\Gamma^1_{0,-1}$		$\Gamma^1_{0,0}$	$\Gamma^1_{1,-1}$		$\Gamma^1_{1,0}$	
1	1	0	2						$\Gamma^2_{1,-1}$			

As previously, the 3 first lines correspond to coupled transitions which create a discrepancy on z, and the combination of the fourth and fifth lines corresponds to a finite path of coupled transitions on $(0,z)$, $\{(0,-1,1,-1),(-1,0,0,0),(-1,0,-1,1)\}$ which ends with a discrepancy on z,

$$0\in X_R(z)\quad\Longleftrightarrow\quad\begin{cases}\begin{cases}G^{0;1}_{-1,0;0,0}(z)+G^{0;1}_{0,-1;0,0}(-z)+G^{1;0}_{0,-1;0,0}(-z)>0\\ \text{or}\\ \begin{cases}G^{1;1}_{0,-1;0,0}(-z)>0,\\ G^{0;1}_{0,-1;1,-1}(z)+G^{0;2}_{0,-1;1,-1}(z)+G^{1;0}_{0,-1;1,-1}(z)+G^{1;2}_{0,-1;1,-1}(z)>0,\end{cases}\end{cases}\\ G^{0;1}_{-1,0;1,0}(z)+G^{1;0}_{0,-1;0,1}(-z)>0,\\ \begin{cases}G^{0;1}_{0,0;1,0}(z)+G^{1;0}_{0,0;1,0}(z)+G^{1;0}_{0,0;0,1}(-z)>0\\ \text{or}\\ \begin{cases}G^{1;1}_{0,0;1,0}(z)>0,\\ G^{1;0}_{1,-1;1,0}(-z)+G^{2;0}_{1,-1;1,0}(-z)+G^{0;1}_{1,-1;1,0}(-z)+G^{2;1}_{1,-1;1,0}(-z)>0,\end{cases}\end{cases}\end{cases}$$

$$1\in X_R(z)\quad\Longleftrightarrow\quad\begin{cases}\begin{cases}G^{1;0}_{1,-1;1,0}(-z)+G^{2;0}_{1,-1;1,0}(-z)+G^{0;1}_{1,-1;1,0}(-z)+G^{2;1}_{1,-1;1,0}(-z)>0\\ \text{or}\\ \begin{cases}G^{1;1}_{1,-1;1,0}(-z)>0,\\ G^{0;1}_{0,0;1,0}(z)+G^{1;0}_{0,0;1,0}(z)+G^{1;0}_{0,0;0,1}(-z)>0,\end{cases}\end{cases}\\ G^{1;0}_{1,-1;1,1}(-z)+G^{2;0}_{1,-1;1,1}(-z)>0,\\ G^{1;0}_{1,0;1,1}(-z)>0.\end{cases}$$

Using the coupling rates' expressions given in Table 1 as well as attractiveness conditions (4.14) leads to following conditions on the rates:

$$-1\in X_R(z)\quad\Longleftrightarrow\quad\begin{cases}\Gamma^1_{0,-1}(z)>0,\\ \Gamma^1_{1,-1}(z)+\Gamma^2_{1,-1}(z)>0,\\ \begin{cases}\big(\Gamma^1_{1,-1}(z)+\Gamma^2_{1,-1}(z)-\Gamma^1_{0,-1}(z)\big)+\Gamma^2_{1,-1}(z)>0\\ \text{or}\\ \Gamma^1_{0,-1}(z)-\Gamma^2_{1,-1}(z)>0\quad\text{and}\quad\Gamma^1_{0,0}(z)+\big(\Gamma^1_{0,-1}(-z)-\Gamma^1_{0,0}(-z)\big)>0,\end{cases}\end{cases}$$

$$0\in X_R(z)\quad\Longleftrightarrow\quad\begin{cases}\begin{cases}\Gamma^1_{0,0}(z)+\big(\Gamma^1_{0,-1}(-z)-\Gamma^1_{0,0}(-z)\big)>0\\ \text{or}\\ \Gamma^1_{0,0}(-z)>0\quad\text{and}\quad\big(\Gamma^1_{1,-1}(z)+\Gamma^2_{1,-1}(z)-\Gamma^1_{0,-1}(z)\big)+\Gamma^2_{1,-1}(z)>0,\end{cases}\\ \Gamma^1_{1,0}(z)+\Gamma^1_{0,-1}(-z)>0,\\ \begin{cases}\big(\Gamma^1_{1,0}(z)-\Gamma^1_{0,0}(z)\big)+\Gamma^1_{0,0}(-z)>0\\ \text{or}\\ \Gamma^1_{0,0}(z)>0\quad\text{and}\quad\big(\Gamma^1_{1,-1}(-z)+\Gamma^2_{1,-1}(-z)-\Gamma^1_{1,0}(-z)\big)+\Gamma^2_{1,-1}(-z)>0,\end{cases}\end{cases}$$

$$1\in X_R(z)\quad\Longleftrightarrow\quad\begin{cases}\begin{cases}\big(\Gamma^1_{1,-1}(-z)+\Gamma^2_{1,-1}(-z)-\Gamma^1_{1,0}(-z)\big)+\Gamma^2_{1,-1}(-z)>0\\ \text{or}\\ \Gamma^1_{1,0}(-z)-\Gamma^2_{1,-1}(-z)>0\quad\text{and}\quad\big(\Gamma^1_{1,0}(z)-\Gamma^1_{0,0}(z)\big)+\Gamma^1_{0,0}(-z)>0,\end{cases}\\ \Gamma^1_{1,-1}(-z)+\Gamma^2_{1,-1}(-z)>0,\\ \Gamma^1_{1,0}(-z)>0.\end{cases}$$

Example 6.5. *We give three examples of coupling rates which satisfy* (4.14) (*that is correspond to an attractive process*), *and assumption* (*IC*):

(i) *All the rates* $\Gamma^k_{\alpha,\beta}(z)$, $z=\pm 1$ *are positive. Hence* (IC, a) *holds and* $X_R(z)=X$ *for* $z=\pm 1$, *so that* (IC, b) *is satisfied.*

(ii) *All rates* $\Gamma^k_{\alpha,\beta}(-1)$ *are equal to* 0 *and all rates* $\Gamma^1_{\alpha,\beta}(1)$ *are positive* (*totally asymmetric case*). *Here* (IC, a) *holds only through* (6.9) *for* $z=\pm 1$ *if* $\Gamma^1_{0,0}(1)\le\Gamma^2_{1,-1}(1)$, *and through the combination of* (6.9), (6.10) *if* $0\le\Gamma^2_{1,-1}(1)<\Gamma^1_{0,0}(1)$. *We always have* $-1\in X_R(1), -1\notin X_R(-1)$ *and* $1\in X_R(-1), 1\notin X_R(1)$, *therefore* (IC, b) *never holds. Here* $W_\Gamma=\{(x,x+1,x+2)\in\mathbb{Z}^3\}$. *To have* (IC, b'_1), *we need in addition*

$$\begin{cases}\Gamma^1_{1,0}(1)>\Gamma^1_{0,0}(1) \quad\Longrightarrow\quad X_R(1)\supset\{0\}\\ \textit{or}\\ \Gamma^1_{0,-1}(1)>\Gamma^1_{0,0}(1) \quad\Longrightarrow\quad X_R(-1)\supset\{0\}.\end{cases}$$

In both cases, (IC, b'_2) *is satisfied.*

(iii) *We choose*:

$$\begin{cases}0=\Gamma^1_{1,-1}(1)=\Gamma^1_{0,0}(1), \qquad 0<\Gamma^2_{1,-1}(1)=\Gamma^1_{0,-1}(1)=\Gamma^1_{1,0}(1),\\ 0=\Gamma^2_{1,-1}(-1), \qquad 0<\Gamma^1_{0,0}(-1)=\Gamma^1_{0,-1}(-1)=\Gamma^1_{1,0}(-1)=\Gamma^1_{1,-1}(-1).\end{cases}$$

Here many coupling rates are equal to zero, *and so is* (6.9) *for* $z=-1$; *we have* (IC, a) *satisfied through the combination of* (6.9), (6.10) *for* $z=1$: *the pair of discrepancies* $(0,0,1,-1)$ *on* $(x,x+1)$ *can disappear only through paths of coupled transitions involving an exchange of discrepancies*:

$$\big\{(0,0,1,-1),(-1,1,0,0),(-1,1,-1,1)\big\} \quad \textit{or} \quad \big\{(0,0,1,-1),(0,0,-1,1),(-1,1,-1,1)\big\}.$$

By inspecting all possibilities, (IC, b) *is satisfied.*

For this model we can say more on invariant measures.

Proposition 6.6. *For the attractive two species exclusion model*, *under assumption* (IC), $\mu_\rho\in\mathcal{I}_e$ (*for* $\rho\in[-1,1]$ *under condition* (4.13), *or* $\rho\in\mathcal{R}$ *otherwise*).

Proof. It is similar to the one of [5], Proposition 1.1, but, as previously to derive (5.2), we have, moreover, to use assumption (IC). □

7. Hydrodynamic limits for the stick process and the two species exclusion model

In this section, we derive almost-sure hydrodynamics for these two nearest-neighbor one-dimensional models under Euler scaling following the constructive method of [4] (see also [2,3], where hydrodynamics is proved in the usual sense), which does not require product invariant probability measures for the dynamics. We quote the result from [4] (after explaining a few preliminaries), then indicate the scheme of its derivation. The fact that more than one particle can jump at a given time induces only one modification to the original proof.

7.1. *The hydrodynamic limit result*

A graphical construction of the models is needed; it is explained as follows. Since for both models the rates are bounded, the construction, restricted to compact state spaces in [4], can be extended here. For each $(x,z)\in\mathbb{Z}^2$, let $\{T^{x,z}_n, n\ge 1\}$ be the arrival times of mutually independent rate $\sup_{k\in\mathbb{N};\alpha,\beta\in X}\Gamma^k_{\alpha,\beta}(z)$ Poisson processes, let $\{U^{x,z}_n, n\ge 1\}$ be mutually independent (and independent of the Poisson processes) random variables, uniform on $[0,1]$.

We denote by $(\mathbf{X}, \mathcal{F}, \mathbb{P})$ the probability space corresponding to these families of variables. At time $t = T_n^{x,z}$, the configuration η_{t^-} becomes $S^k_{x,x+z}\eta_{t^-}$ if

$$U_n^{x,z} \le \frac{\Gamma^k_{\eta_{t^-}(x),\eta_{t^-}(x+z)}(z)}{\sup_{k\in\mathbb{N};\alpha,\beta\in X}\Gamma^k_{\alpha,\beta}(z)}$$

and stays unchanged otherwise. Let also $(\mathbf{X}_0, \mathcal{F}_0, \mathbb{P}_0)$ denote an 'initial' probability space large enough to construct random initial configurations $\eta_0 = \eta_0(\omega_0)$ for $\omega_0 \in \mathbf{X}_0$.

Let $N \in \mathbb{N}$ be the scaling parameter for the hydrodynamic limit. The empirical measure of a configuration η viewed on scale N is given by

$$\alpha^N(\eta)(\mathrm{d}x) = N^{-1}\sum_{y\in\mathbb{Z}}\eta(y)\delta_{y/N}(\mathrm{d}x).$$

It belongs to the set of positive, locally finite measures on $\mathbb{R}$, equipped with the topology of vague convergence.

Theorem 7.1 ([4], Theorem 2.1). *Let $(\eta_0^N, N \in \mathbb{N})$ be a sequence of Ω-valued random variables on $\mathbf{X}_0$. Assume there exists a measurable bounded profile $u_0(\cdot)$ on $\mathbb{R}$ such that*

$$\lim_{N\to\infty}\alpha^N\big(\eta_0^N\big)(\mathrm{d}x) = u_0(\cdot)\,\mathrm{d}x, \quad \mathbb{P}_0\text{-a.s.}, \tag{7.1}$$

that is,

$$\lim_{N\to\infty}\int_{\mathbb{R}}\psi(x)\alpha^N\big(\eta_0^N\big)(\mathrm{d}x) = \int\psi(x)u_0(x)\,\mathrm{d}x, \quad \mathbb{P}_0\text{-a.s.}$$

for every continuous function ψ on $\mathbb{R}$ with compact support. Let $(x,t)\mapsto u(x,t)$ denote the unique entropy solution to the scalar conservation law

$$\partial_t u + \partial_x\big[G(u)\big] = 0 \tag{7.2}$$

with initial condition u_0, where G is a Lipschitz-continuous flux function (*defined in* (7.5) *below*) *determined by the Γ's. Then, with $\mathbb{P}_0\otimes\mathbb{P}$-probability one, the convergence*

$$\lim_{N\to\infty}\alpha^N\big(\eta_{Nt}\big(\eta_0^N(\omega_0),\omega\big)\big)(\mathrm{d}x) = u(\cdot,t)\,\mathrm{d}x \tag{7.3}$$

holds uniformly on all bounded time intervals. That is, for every continuous function ψ on $\mathbb{R}$ with compact support, the convergence

$$\lim_{N\to\infty}\int_{\mathbb{R}}\psi(x)\alpha^N\big(\eta_{Nt}^N\big)(\mathrm{d}x) = \int\psi(x)u(\cdot,t)\,\mathrm{d}x$$

holds uniformly on all bounded time intervals.

For $x \in \mathbb{Z}$, the *microscopic current* between sites x and $x+1$ is defined by

$$J_{x,x+1}(\eta) = \sum_k k\big(\Gamma^k_{\eta(x),\eta(x+1)}(1) - \Gamma^k_{\eta(x+1),\eta(x)}(-1)\big). \tag{7.4}$$

The *macroscopic flux* G is defined by

$$G(\rho) = \mu_\rho\big[J_{0,1}(\eta)\big] \tag{7.5}$$

for $\rho \in \mathcal{R}$ (a set introduced in Theorem 5.13), and by linear interpolation on the complement of $\mathcal{R}$ (which is at most a countable union of disjoint open intervals).

The first step to prove Theorem 7.1 works here unchanged: it consists in first giving a variational formula for the solution of the hydrodynamic equation (7.2) under Riemann initial condition

$$u_0(x) = \lambda \mathbf{1}_{\{x<0\}} + \rho \mathbf{1}_{\{x\geq 0\}} \tag{7.6}$$

(where λ, ρ belong to $\mathcal{R}$), and then deriving hydrodynamics under this initial profile.

The second step consists in extending the hydrodynamic limit to any initial profile u_0 through an approximation scheme. Its key tools are a finite propagation property both at microscopic and macroscopic levels (which work unchanged: for particles, we refer to [2], Lemma 3.1, which does not rely on attractiveness) and the macroscopic stability property. The latter requires here a different proof, that we derived in Proposition 2.23.

In the following subsections, we compute the involved quantities, that is microscopic current, macroscopic flux, and check macroscopic stability for both examples. We saw that the stick process possesses a one-parameter family of product probability measures, and that for the two species exclusion it is the case under condition (4.13), otherwise the invariant measures are not explicit (cf. Theorem 5.13).

7.2. *Hydrodynamic limits for the stick process*

For $x \in \mathbb{Z}$, setting $p(1) = p$ and $p(-1) = q$, the microscopic current between x and $x+1$ is

$$\begin{aligned} J_{x,x+1}(\eta) &= \sum_k k\big(p\mathbf{1}_{\{k\leq \eta(x)\}} - q\mathbf{1}_{\{k\leq \eta(x+1)\}}\big) \\ &= \frac{p}{2}\eta(x)\big(\eta(x)+1\big) - \frac{q}{2}\eta(x+1)\big(\eta(x+1)+1\big) \end{aligned}$$

so that the macroscopic flux

$$G(\rho) = \frac{p-q}{2}\rho(1+\rho) \tag{7.7}$$

is a convex function for $p-q>0$.

Proposition 2.23 holds here by Proposition 4.1.

7.3. *Hydrodynamic limits for the two species exclusion model*

Under conditions (4.12)–(4.13), the model admits a one-parameter family of invariant product probability measures $\{\mu_\rho\}_{\rho\in(-1,1)}$ (cf. Section 4.3); the marginals of μ_ρ can be written as (see [9], Eqs (3.6)–(3.8))

$$\mu_\rho\big(\eta(x)=1\big) = \frac{cy}{1+cy+cy^{-1}} = \frac{\varphi(\rho)+\rho}{2}, \tag{7.8}$$

$$\mu_\rho\big(\eta(x)=0\big) = \frac{1}{1+cy+cy^{-1}} = 1-\varphi(\rho), \tag{7.9}$$

$$\mu_\rho\big(\eta(x)=-1\big) = \frac{cy^{-1}}{1+cy+cy^{-1}} = \frac{\varphi(\rho)-\rho}{2}, \tag{7.10}$$

where $y \in (0,+\infty)$ is an auxiliary parameter related both to the mean charge, $\rho = \int \eta(x)\,\mathrm{d}\mu_\rho(\eta)$ and to the mean squared charge, $\varphi(\rho) = \int \eta(x)^2\,\mathrm{d}\mu_\rho(\eta)$ as

$$\rho = \frac{c(y-y^{-1})}{1+cy+cy^{-1}}; \qquad \varphi(\rho) = \frac{c(y+y^{-1})}{1+cy+cy^{-1}}. \tag{7.11}$$

The constant c depends on the jump rates

$$c = \sqrt{\frac{\Gamma^1_{0,0}(1)+\Gamma^1_{0,0}(-1)}{\Gamma^1_{1,-1}(1)+\Gamma^1_{1,-1}(-1)}}. \tag{7.12}$$

After elimination of y in (7.11), $\varphi(\rho)$ appears as a function of $\rho \in (-1,1)$,

$$\varphi(\rho)=\begin{cases}\frac{\psi(\rho)-4c^2}{1-4c^2} & \text{with } \psi(\rho)=\sqrt{4c^2+\rho^2(1-4c^2)} \quad \text{if } 4c^2\neq 1,\\ \frac{\rho^2+1}{2} & \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } 4c^2=1.\end{cases} \tag{7.13}$$

Independently of (4.13), according to the values of the jump parameters, the scaling is either diffusive or Euler, and the model is studied either as a gradient model, or as an attractive one. In both cases, the microscopic current between x and $x+1$ reads

$$\begin{aligned}J_{x,x+1}(\eta)={}&\chi^{0,0}_{x,x+1}(\eta)\big(\Gamma^1_{0,0}(1)-\Gamma^1_{0,0}(-1)\big)\\&+\chi^{1,0}_{x,x+1}(\eta)\Gamma^1_{1,0}(1)-\chi^{0,1}_{x,x+1}(\eta)\Gamma^1_{1,0}(-1)\\&+\chi^{0,-1}_{x,x+1}(\eta)\Gamma^1_{0,-1}(1)-\chi^{-1,0}_{x,x+1}(\eta)\Gamma^1_{0,-1}(-1)\\&+\chi^{1,-1}_{x,x+1}(\eta)\big(\Gamma^1_{1,-1}(1)+2\Gamma^2_{1,-1}(1)\big)-\chi^{-1,1}_{x,x+1}(\eta)\big(\Gamma^1_{1,-1}(-1)+2\Gamma^2_{1,-1}(-1)\big).\end{aligned} \tag{7.14}$$

7.3.1. *Hydrodynamic limits in the diffusive case*

This case, for which we refer to [16,24], corresponds to the values of the jump rates under which the macroscopic flux computed under Euler scaling in (7.17) satisfies $G(\rho)\equiv 0$. We only derive the conditions under which the model is gradient.

Lemma 7.2. *The two species exclusion model is gradient if the rates satisfy*

$$\begin{cases}\Gamma^1_{0,0}(-1)=\Gamma^1_{0,0}(1); \qquad \Gamma^1_{1,0}(-1)=\Gamma^1_{1,0}(1); \qquad \Gamma^1_{0,-1}(-1)=\Gamma^1_{0,-1}(1),\\ \Gamma^1_{1,-1}(-1)+2\Gamma^2_{1,-1}(-1)=\Gamma^1_{1,-1}(1)+2\Gamma^2_{1,-1}(1)=\Gamma^1_{0,-1}(1)+\Gamma^1_{1,0}(1).\end{cases} \tag{7.15}$$

Proof. The model is gradient if there exists a function q such that

$$J_{x,x+1}(\eta)=(\tau_{x+1}-\tau_x)q(\eta).$$

Because the right-hand side of (7.4) depends only on $\eta(x)$, $\eta(x+1)$, $J_{x,x+1}(\eta)$ has to be of the form

$$g_1\big(\chi^{1,0}_{x,x+1}(\eta)-\chi^{0,1}_{x,x+1}(\eta)\big)+g_2\big(\chi^{0,-1}_{x,x+1}(\eta)-\chi^{-1,0}_{x,x+1}(\eta)\big)+g_3\big(\chi^{1,-1}_{x,x+1}(\eta)-\chi^{-1,1}_{x,x+1}(\eta)\big)$$

with $g_1+g_2=g_3$. This leads to (7.15), and

$$q(\eta)=-\big[\Gamma^1_{1,0}(1)+\Gamma^1_{0,-1}(1)\big]\frac{\eta(0)}{2}+\big[\Gamma^1_{0,-1}(1)-\Gamma^1_{1,0}(1)\big]\frac{\eta(0)^2}{2}. \tag{7.16}$$

We have used

$$\mathbf{1}_{\{\eta(x)=0\}}=1-\mathbf{1}_{\{\eta(x)=1\}}-\mathbf{1}_{\{\eta(x)=-1\}}; \qquad \mathbf{1}_{\{\eta(x)=1\}}=\eta(x)\frac{\eta(x)+1}{2}; \qquad \mathbf{1}_{\{\eta(x)=-1\}}=\eta(x)\frac{\eta(x)-1}{2}$$

to write in (7.14) the indicator functions as polynomials in $\eta(x)$, $\eta(x+1)$. □

7.3.2. *Hydrodynamic limit under Euler scaling in the attractive case*

Using expressions (7.14) for the microscopic current and (7.8)–(7.10) for the product measures, we obtain the macroscopic flux as a function of the density ρ,

$$\begin{aligned}G(\rho)={}&\big(\Gamma^1_{0,0}(1)-\Gamma^1_{0,0}(-1)\big)\big(1-\varphi(\rho)\big)^2\\&+\frac{1}{2}\big(\Gamma^1_{1,0}(1)-\Gamma^1_{1,0}(-1)\big)\big(1-\varphi(\rho)\big)\big(\varphi(\rho)+\rho\big)\end{aligned}$$

$$+\frac{1}{2}\big(\Gamma^1_{0,-1}(1)-\Gamma^1_{0,-1}(-1)\big)\big(1-\varphi(\rho)\big)\big(\varphi(\rho)-\rho\big)$$
$$+\frac{1}{4}\Big(\big(\Gamma^1_{1,-1}(1)-\Gamma^1_{1,-1}(-1)\big)+2\big(\Gamma^2_{1,-1}(1)-\Gamma^2_{1,-1}(-1)\big)\Big)\big(\varphi(\rho)^2-\rho^2\big). \tag{7.17}$$

It is a non-trivial function, a priori neither concave nor convex in the parameter ρ. Recalling (7.5) and (7.13), its second derivative reads

$$G''(\rho)=\frac{C_1-C_2}{1-4c^2}-\frac{2c^2(2C_1-(1+4c^2)C_2)}{(1-4c^2)\psi(\rho)^3}-\frac{C_3\rho}{\psi(\rho)}\left(\frac{1}{2}+\frac{c^2}{\psi(\rho)^2}\right), \tag{7.18}$$

where

$$C_1=2\big(\Gamma^1_{0,0}(1)-\Gamma^1_{0,0}(-1)\big)+2c^2\big(\Gamma^1_{1,-1}(1)-\Gamma^1_{1,-1}(-1)\big)+4c^2\big(\Gamma^2_{1,-1}(1)-\Gamma^2_{1,-1}(-1)\big),$$
$$C_2=\big(\Gamma^1_{1,0}(1)-\Gamma^1_{1,0}(-1)\big)+\big(\Gamma^1_{0,-1}(1)-\Gamma^1_{0,-1}(-1)\big),$$
$$C_3=\big(\Gamma^1_{1,0}(1)-\Gamma^1_{1,0}(-1)\big)-\big(\Gamma^1_{0,-1}(1)-\Gamma^1_{0,-1}(-1)\big).$$

According to the values of the jump rates, the flux function is either concave, or exhibits one or two inflexion points. This yields increasing or decreasing shocks, rarefaction fans as well as contact discontinuities for the unique entropy solution of the hydrodynamic equation (7.2) with initial condition (7.6).

Proposition 2.23 holds under condition (4.15).

Example. *We detail what happens for thermal bath dynamics. These can be compared to the simulations described in* [9], *though a discrete time Markov chain was used there. The model is used to describe the dynamics of a one-dimensional solid-on-solid interface, that is a collection of heights on* $\mathbb{Z}$ *and a configuration* η *describes only the height differences between neighboring sites. Jump rates are thus chosen according to the following rules*:

- *Detailed balance holds with respect to a formal Solid-on-Solid Hamiltonian.*

  $$\Gamma^k_{\eta(x),\eta(x+1)}(1)\exp\big\{H\big(S^k_{x,x+1}\eta\big)-H(\eta)\big\}=\Gamma^k_{\eta(x+1)+k,\eta(x)-k}(-1)$$

  for all η, *all* x *and all* $k>0$, *with*

  $$H(\eta)=J\sum_{x\in\mathbb{Z}}\big|\eta(x)\big|-E\sum_{x\in\mathbb{Z}}x\eta(x).$$

  There are two parameters: $J>0$ *weights the height differences between neighboring sites*, *and* $E>0$ *is an external field which drives the interface.*
- *Rates are chosen invariant under the symmetry* $x\to -x$, $\eta\to-\eta$.
- *Condition* (4.13) *for the existence of invariant product probability measures holds.*

The above three conditions allow to write all jump rates in terms of two physically significant parameters $a=\exp(-E/2)$ *and* $b=\exp(-J)$:

$$\begin{cases}\Gamma^1_{1,0}(1)=\Gamma^1_{0,-1}(1)=av_2,\\ \Gamma^1_{1,0}(-1)=\Gamma^1_{0,-1}(-1)=a^{-1}v_2,\\ \Gamma^1_{0,0}(1)=\Gamma^2_{1,-1}(1)=a^2v_0,\\ \Gamma^1_{0,0}(-1)=\Gamma^2_{1,-1}(-1)=a^{-2}v_0,\\ \Gamma^1_{1,-1}(1)=\Gamma^1_{1,-1}(-1)=b^{-2}v_0\end{cases} \tag{7.19}$$

with

$$v_2=\frac{a+a^{-1}}{2};\qquad v_0=\frac{a^2+a^{-2}}{a^2+a^{-2}+b^{-2}} \tag{7.20}$$

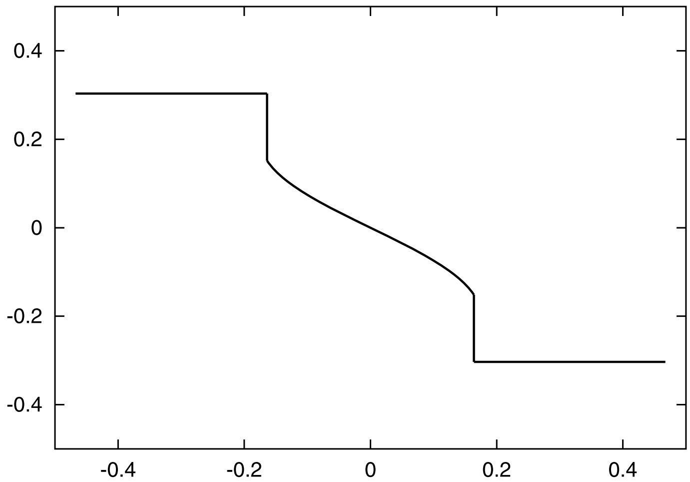

Fig. 1. Starting from an initial condition $u_0(x) = \lambda(\mathbf{1}_{\{x<0\}} - \mathbf{1}_{\{x\geq 0\}})$, the hydrodynamic limit may develop two shocks going in opposite directions at constant speeds and a rarefaction fan in-between. Here $c = 0.145$ and $\lambda = 0.303$.

so that

$$c^2 = \frac{b^2(a^{-2} + a^2)}{2}. \tag{7.21}$$

The macroscopic flux and its second derivative read

$$G(\rho) = \frac{(a^{-2} - a^2)}{2}\left(\rho^2 - \frac{\psi(\rho)}{1 - 4c^2}\right),$$

$$G''(\rho) = \left(a^{-2} - a^2\right)\left(1 - \frac{2c^2}{\psi(\rho)^3}\right),$$

which yields no inflexion point if $c \in [1/4, 1/\sqrt{2}]$. *When* $c > 1/\sqrt{2}$ *there are two inflexion points near* $\rho = \pm 1$; *when* $c < 1/4$ *the two inflexion points are near* $\rho = 0$. *The situation is thus rather similar to that of the discrete time model studied heuristically in* [9]. *We have derived rigorously all the cases they got using finite size scaling analysis. The situation in Fig.* 1 *was beyond the reach of their analysis*: *for* $c < 1/4$ *and initial condition* (7.6) *with* $\rho = -\lambda$ *and* λ *larger than the abscissa of the inflexion point, the system develops two shocks moving with opposite velocities and separated by a rarefaction fan.*

Appendix

In this part, we detail the derivation of coupling rates for the attractive two species exclusion model of Section 4.3. In order to illustrate the coupling construction of Proposition 2.11, we re-obtain the rates' values by direct inspection, without using Eqs (2.16)–(2.19) but relying on the following previously stated construction rules: (o) marginals have to be recovered; (i) in coupled jumps, the same departure and arrival sites are taken for both processes; (ii) partial order on departure and arrival sites has to be preserved; (iii) discrepancies should not increase. In some instances, these rules do not fully determine the coupling rates but give a family of solutions; a consistent choice can be done by the additional rule: (iv) the non-zero coupling rates are located on a "staircase-shaped path" in the (k, l)-quadrant (recall Proposition 2.15). Rules (o)–(iv) single out the solution given by (2.16)–(2.19).

By rule (i), we fix the same initial site x and final site y with $z = y - x = \pm 1$, consider only the local configuration $(\xi(x), \xi(y), \zeta(x), \zeta(y))$ as an element of X^4, with $X = \{-1, 0, 1\}$, and describe a coupled jump as a transition on this reduced state space. Furthermore, since in the initial process, jump rates from x to y depend only on values at these

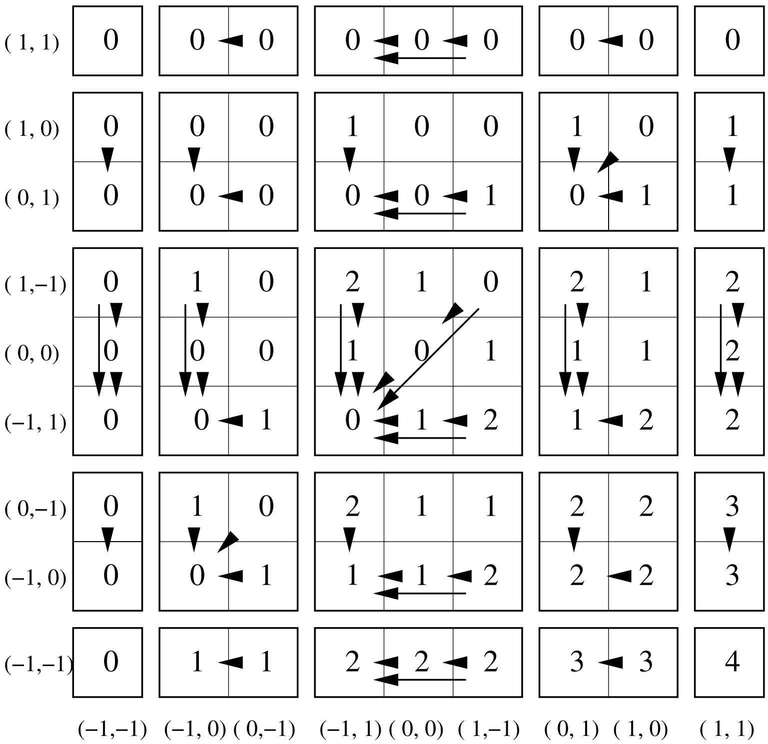

Fig. 2. The coupling rates between two sites for the two species model. Possible coupled jumps, represented by arrows, are subject to three restrictions: arrows connect two cells in the same sub-array, point leftwards and/or downwards, cannot point to a larger value than the initial one. All trivial arrows have been represented, leaving four non-trivial cases.

sites, one can look for a coupling process which has also this property so that the coupling construction can be given as if it were on X^4. In Fig. 2, we represent this reduced space as an $X^2 \times X^2$ square array with the values of $(\xi(x), \xi(y))$ (resp. $(\zeta(x), \zeta(y))$) as the first, horizontal (resp. second, vertical), coordinate from left to right on the horizontal axis, and from bottom to top on the vertical one. Before asking for the actual values of the coupling rates, note that three different factors limit the number of non-zero coupling rates:

(a) there is charge conservation on each marginal. Since both $\xi(x) + \xi(y)$ and $\zeta(x) + \zeta(y)$ take value in $\{-2, \dots, +2\}$, charge conservation splits the $X^2 \times X^2$ square array into 25 sub-arrays, and a transition takes place only between two elements of the same sub-array;

(b) only positive charges jump from x to y in both marginals so that in a jump, by rule (i), the values at x cannot increase;

(c) the coupling should be increasing, reflecting that the initial process is attractive by rule (ii) and that by rule (iii), the discrepancies cannot grow. Both properties are related to the function (cf. (2.30))

$$f^+_{x,y}(\xi, \zeta) = \big[\xi(x) - \zeta(x)\big]^+ + \big[\xi(y) - \zeta(y)\big]^+.$$

Once fixed the values of the local charges $\xi(x)+\xi(y)$ and $\zeta(x)+\zeta(y)$, $f^+_{x,y}(\xi, \zeta)$ takes its extremal values on ordered pairs, namely,

$$f^+_{x,y}(\xi, \zeta) = \begin{cases} 0 & \text{if } \big(\xi(x), \xi(y)\big) \le \big(\zeta(x), \zeta(y)\big), \\ \xi(x) + \xi(y) - \big(\zeta(x) + \zeta(y)\big) & \text{if } \big(\xi(x), \xi(y)\big) \ge \big(\zeta(x), \zeta(y)\big). \end{cases}$$

Rule (ii) requires that $f^+_{x,y}(\xi, \zeta)$ cannot escape its minimum by a jump between x and y, and rule (iii) extends this requirement to non ordered states by asking that $f^+_{x,y}(\xi, \zeta)$ is not increasing.

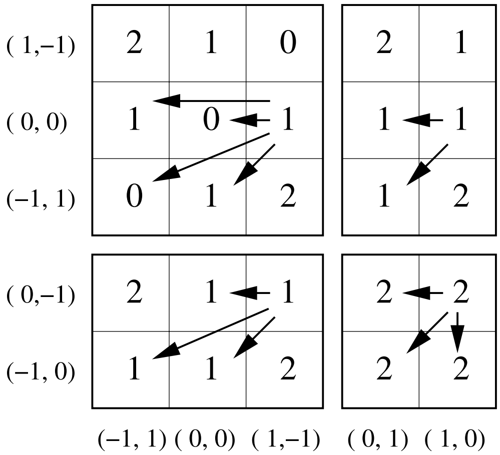

Fig. 3. Examples of non-trivial cases.

These three restrictions can be depicted on the graphical construction of Fig. 2, where we represent a possible coupled transition as an arrow from a 'cell' (initial values) to another one (final values). By (a), an arrow can be drawn only between two cells of a same subarray. By (b), since configurations are ordered by increasing charge and increasing value, arrows have to point left and/or downwards; In order to use (c), we have written in each cell the value of $f^{+}_{x,y}(\xi,\zeta)$: arrows cannot go from a given value to a larger one.

All this already fixes trivial rates for our coupling process, when only one transition from a cell is allowed, and we have indicated all of them on Fig. 2: these include non-coupled ξ-jumps (horizontal arrows), non-coupled ζ-jumps (vertical arrows) or fully coupled jumps between identical configurations (arrows on the main diagonal of each sub-array). We are thus left with only four types of coupled jumps, which we treat in more detail now, using the four examples represented in Fig. 3.

In the sub-array $A_1 = \{(0,1),(1,0)\} \times \{(-1,0),(0,-1)\}$, all four pairs are ordered and the function $f^{+}_{x,y}(\xi,\zeta)$ is constant. Starting from the state $((1,0),(0,-1))$, three jumps are possible and by rule (o), the coupling rates are related through

$$\begin{cases} G^{1;0}_{1,0;0,-1}(z) = \Gamma^{1}_{1,0}(z) - G^{1;1}_{1,0;0,-1}(z), \\ G^{0;1}_{1,0;0,-1}(z) = \Gamma^{1}_{0,-1}(z) - G^{1;1}_{1,0;0,-1}(z). \end{cases}$$

Any choice for $G^{1;1}_{1,0;0,-1}(z)$ which preserves non-negativity is valid, so that

$$0 \le G^{1;1}_{1,0;0,-1}(z) \le \Gamma^{1}_{1,0}(z) \wedge \Gamma^{1}_{0,-1}(z).$$

Here the only possible non-zero value for k and l is 1. Rule (iv) imposes that either $G^{1;0}_{1,0;0,-1}(z)$ or $G^{0;1}_{1,0;0,-1}(z)$ is zero, hence

$$G^{1;1}_{1,0;0,-1}(z) = \Gamma^{1}_{1,0}(z) \wedge \Gamma^{1}_{0,-1}(z)$$

and the non-zero coupling rates are either on the (k,l)-path $\mathcal{P}_{1,0;0,-1} = \{(1,0),(1,1)\}$ or $\mathcal{P}_{1,0;0,-1} = \{(0,1),(1,1)\}$, depending on the value of the marginals. Notice that the construction is here identical to the 'basic coupling' one.

In rectangle $A_2 = \{(0,1),(1,0)\} \times \{(-1,1),(0,0),(1,-1)\}$, we consider coupling rates from the initial state $((1,0),(0,0))$. Its coordinates being ordered (that is $(1,0) \ge (0,0)$), order preservation (rule (ii)) forbids a jump to $((1,0),(-1,1))$, thus

$$G^{0;1}_{1,0;0,0}(z) = 0.$$

The only choice to get the correct ζ-marginal (that is to satisfy rule (o)) is then

$$G^{1;1}_{1,0;0,0}(z) = \Gamma^1_{0,0}(z)$$

and the last rate follows

$$G^{1;0}_{1,0;0,0}(z) = \Gamma^1_{1,0}(z) - \Gamma^1_{0,0}(z).$$

It has to be non-negative, which was stated by (4.14). Notice that rule (iv) is automatically satisfied with $\mathcal{P}_{1,0;0,0} = \{(1,0),(1,1)\}$.

The situation in rectangle $A_3 = \{(-1,1),(0,0),(1,-1)\} \times \{(-1,0),(0,-1)\}$ is similar to the preceding one. We consider there the other type of non-trivial initial state, $((1,-1),(0,-1))$ whose coordinates are again ordered $((1,-1) \geq (0,-1))$, and by rule (ii)

$$G^{2;0}_{1,-1;0,-1}(z) = G^{0;-1}_{1,-1}(z) = 0.$$

Here the situation cannot be solved through basic coupling, since there is a marginal ξ-jump involving two positive charges. There is only one solution for the coupling rates,

$$\begin{aligned}
&G^{2;1}_{1,-1;0,-1}(z) = \Gamma^2_{1,-1}(z),\\
&G^{1;1}_{1,-1;0,-1}(z) = \Gamma^1_{0,-1}(z) - \Gamma^2_{1,-1}(z),\\
&G^{1;0}_{1,-1;0,-1}(z) = \Gamma^1_{1,-1}(z) + \Gamma^2_{1,-1}(z) - \Gamma^1_{0,-1}(z).
\end{aligned}$$

They are all non-negative by (4.14) and identical to the values given by (2.16)–(2.19). Rule (iv) is again satisfied and $\mathcal{P}_{1,-1;0,-1} = \{(1,0),(1,1),(2,1)\}$.

We find a last type of non-trivial case in the larger square

$$A_4 = \{(-1,1),(0,0),(1,-1)\} \times \{(-1,1),(0,0),(1,-1)\}$$

and we consider $((1,-1),(0,0))$ as initial state. Its coordinates are not ordered so that, rather than rule (ii), selection rule (iii) may apply. This does not fix completely the coupling rates and we have

$$\begin{aligned}
&G^{2;1}_{1,-1;0,0}(z) = \Gamma^2_{1,-1}(z) - G^{2;0}_{1,-1;0,0}(z),\\
&G^{1;1}_{1,-1;0,0}(z) = \Gamma^1_{0,0}(z) + G^{2;0}_{1,-1;0,0}(z) - \Gamma^2_{1,-1}(z),\\
&G^{1;0}_{1,-1;0,0}(z) = \Gamma^1_{1,-1}(z) + \Gamma^2_{1,-1}(z) - \Gamma^1_{0,0}(z) - G^{2;0}_{1,-1;0,0}(z),
\end{aligned}$$

where non-negativity is guaranteed by (4.14), and any choice for $G^{2;0}_{1,-1;0,0}(z)$ in the range

$$\left[\Gamma^2_{1,-1}(z) - \Gamma^1_{0,0}(z)\right]^+ \leq G^{2;0}_{1,-1;0,0}(z) \leq \Gamma^2_{1,-1}(z) \wedge \left(\Gamma^1_{1,-1}(z) + \Gamma^2_{1,-1}(z) - \Gamma^1_{0,0}(z)\right)$$

is valid. Rule (iv) imposes to set it to its minimal value,

$$G^{2;0}_{1,-1;0,0}(z) = \left[\Gamma^2_{1,-1}(z) - \Gamma^1_{0,0}(z)\right]^+$$

and gives a path $\mathcal{P}_{1,-1;0,0} = \{(1,0),(1,1),(2,1)\}$ or $\mathcal{P}_{1,-1;0,0} = \{(1,0),(2,0),(2,1)\}$ depending on the value of the marginals. The solution given by (2.16)–(2.19) is again recovered. Note that when $\Gamma^2_{1,-1}(z) \leq \Gamma^1_{0,0}(z)$, this choice is the only one which avoids an 'exchange of discrepancies' (cf. Proposition 4.3).

We conclude that (2.16)–(2.19) lead, as in the spirit of 'basic coupling', to changes as similar as possible in both marginals, which has in addition the property of minimizing exchanges of discrepancies.

For the sake of completeness, all the coupling rates for this model are reported in Table 1; they are also useful to derive conditions on the rates in Section 6.5.

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