

# Poincaré inequalities and dimension free concentration of measure

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**Abstract.** In this paper, we consider Poincaré inequalities for non-Euclidean metrics on  $\mathbb{R}^d$ . These inequalities enable us to derive precise dimension free concentration inequalities for product measures. This technique is appropriate for a large scope of concentration rate: between exponential and Gaussian and beyond. We give equivalent functional forms of these Poincaré type inequalities in terms of transportation-cost inequalities and inf-convolution inequalities. Workable sufficient conditions are given and a comparison is made with super Poincaré inequalities.

**Résumé.** Dans cet article, nous introduisons des inégalités de Poincaré pour des métriques non-euclidiennes sur  $\mathbb{R}^d$  et nous montrons qu'elles entraînent des inégalités de concentrations adimensionnelles pour les mesures produits. Cette technique nous permet d'atteindre un spectre très large de taux de concentration, aussi bien sous et sur-gaussiens. Par ailleurs, nous montrons que ces inégalités de Poincaré admettent des formes fonctionnelles équivalentes en termes d'inégalités de transport et d'inf-convolution. Enfin, nous donnons des conditions suffisantes pour ces inégalités de Poincaré et nous les comparons aux inégalités super-Poincaré.

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## 1. Introduction

### 1.1. Poincaré inequality and concentration of measure

One says that a probability measure on a metric space  $(\mathcal{X}, d)$  satisfies a Poincaré inequality also called spectral gap inequality with the constant  $C$ , if for all locally Lipschitz function  $f$ , one has

$$\mathrm{Var}_\mu(f) \leq C \int |\nabla f|^2 d\mu, \quad (1.1)$$

where the length of the gradient is defined by

$$|\nabla f|(x) := \limsup_{y \rightarrow x} \frac{|f(x) - f(y)|}{d(x, y)} \quad (1.2)$$

(when  $x$  is not an accumulation point of  $\mathcal{X}$ , one defines  $|\nabla f|(x) = 0$ ).

It is well known since the works [1,2,12,23] that the inequality (1.1) implies dimension free concentration inequalities for the product measures  $\mu^n$ ,  $n \geq 1$ .

For example, in [12], Ledoux and Bobkov proved that if  $\mu$  verifies (1.1), then there exists a constant  $L$  depending only on  $C$  such that for all subset  $A$  of  $\mathcal{X}^n$  with  $\mu^n(A) \geq 1/2$ ,

$$\forall h \geq 0 \quad \mu^n(A^h) \geq 1 - e^{-Lh}, \tag{1.3}$$

where the set  $A^h$  is the enlargement of  $A$  defined by

$$A^h = \left\{ y \in \mathcal{X}^n : \inf_{x \in A} \sum_{i=1}^n \alpha(d(x_i, y_i)) \leq h \right\},$$

where  $\alpha(u) = \min(|u|, u^2)$  for all  $u \in \mathbb{R}$  (see [12], Corollary 3.2, and Section 2 of the present paper).

Inequalities such as (1.3) were first obtained by Talagrand in different articles using completely different techniques (see e.g. [36]).

In this paper, one will say that a probability measure  $\mu$  satisfies the *classical Poincaré inequality with constant*  $C > 0$  on  $\mathbb{R}^d$ , if  $\mu$  satisfies (1.1) on  $\mathbb{R}^d$  equipped with its standard Euclidean norm  $|\cdot|_2$ . In that case, one will write that  $\mu$  satisfies the inequality  $\mathbb{S}\mathbb{G}(C)$ , where  $\mathbb{S}\mathbb{G}$  stands for spectral gap. In all the sequel,  $B_p$  will denote the  $\ell^p$  unit ball of  $\mathbb{R}^m$ :  $B_p = \{x \in \mathbb{R}^m : |x_1|^p + \dots + |x_m|^p \leq 1\}$ .

If  $\mu$  satisfies the inequality  $\mathbb{S}\mathbb{G}(C)$  on  $\mathbb{R}^d$  then (1.3) can be rewritten in a more pleasant way: for all subset  $A$  of  $(\mathbb{R}^d)^n$  with  $\mu^n(A) \geq 1/2$ ,

$$\forall h \geq 0 \quad \mu^n(A + \sqrt{h}B_2 + hB_1) \geq 1 - e^{-hL} \tag{1.4}$$

with a constant  $L$  depending on  $C$  and the dimension  $d$ . The archetypic example of a measure satisfying the classical Poincaré inequality is the exponential measure on  $\mathbb{R}^d$   $\nu_1^d$ , where  $d\nu_1(x) = \frac{1}{2}e^{-|x|} dx$ . For this probability, (1.4) cannot be improved (a version of (1.4) with sharp constants has been established by Talagrand in [34] see also Maurey [30], Corollary 1). Thus (1.4) expresses that the probability measures  $\mu^n$  concentrate at least as fast as the exponential measure on  $(\mathbb{R}^d)^n$ .

Some probability measures concentrate faster than the exponential measure. For example, the standard Gaussian measure  $\gamma^m$  on  $\mathbb{R}^m$  verifies for all  $A \subset \mathbb{R}^m$  with  $\gamma^m(A) \geq 1/2$ ,

$$\forall h \geq 0 \quad \gamma^m(A + hB_2) \geq 1 - e^{-h^2/2}. \tag{1.5}$$

One cannot derive such a bound from the classical Poincaré inequality. The inequality (1.5) requires stronger tools. For example, it is now well known that (1.5) follows from the Logarithmic-Sobolev inequality, introduced by Gross in [24], which is strictly stronger than the classical Poincaré inequality (see [27], Chapter 5). Let us recall, that a probability measure  $\mu$  on  $\mathbb{R}^d$  is said to satisfy the Logarithmic-Sobolev inequality with a constant  $C > 0$ , if

$$\text{Ent}_\mu(f^2) \leq C \int |\nabla f|_2^2 d\mu \tag{1.6}$$

holds for all locally Lipschitz function  $f$  on  $\mathbb{R}^d$ , where the entropy functional is defined by

$$\forall f \geq 0 \quad \text{Ent}_\mu(f) = \int f \log(f) d\mu - \left( \int f d\mu \right) \cdot \log \left( \int f d\mu \right).$$

### 1.2. Changing the metric improves the concentration

The aim of this paper is to show that considering Poincaré inequality on  $\mathbb{R}^d$  equipped with other metrics than the Euclidean distance makes possible to reach a large scope of concentration properties including Gaussian or even stronger behaviors. The metrics we are going to equip  $\mathbb{R}^d$  with are of the form:

$$\forall x, y \in \mathbb{R}^d \quad d_\omega(x, y) = \left[ \sum_{i=1}^d |\omega(x_i) - \omega(y_i)|^2 \right]^{1/2}, \tag{1.7}$$

where, in all the paper, we will assume that  $\omega : \mathbb{R} \rightarrow \mathbb{R}$  is increasing and verifies:

- $\omega$  is such that  $x \mapsto \omega(x)/x$  is nondecreasing on  $(0, +\infty)$ ,
- $\omega$  is nonnegative on  $\mathbb{R}^+$ ,
- $\omega$  is such that  $\omega(-x) = -\omega(x)$ , for all  $x \in \mathbb{R}$ .

Note that the first assumption is verified as soon as  $\omega$  is convex on  $\mathbb{R}^+$  with  $\omega(0) = 0$ .

**Definition 1.1.** One says that a probability measure  $\mu$  on  $\mathbb{R}^d$  satisfies the inequality  $\mathbb{S}\mathbb{G}(\omega, C)$  if  $\mu$  satisfies the Poincaré inequality (1.1) for the distance  $d_\omega(\cdot, \cdot)$  defined by (1.7) with the constant  $C > 0$ .

The following proposition gives examples of the variety of concentration rates enabled by our approach:

**Proposition 1.2.** Let  $\omega_p(x) = \max(x, x^p)$  on  $\mathbb{R}^+$  with  $\omega_p(-x) = -\omega_p(x)$  for all  $x \in \mathbb{R}$ .

Suppose that  $\mu$  satisfies the inequality  $\mathbb{S}\mathbb{G}(\omega_p, C)$  on  $\mathbb{R}^d$  for some  $C > 0$ .

If  $p \in [1, 2]$ , then for all  $n \geq 1$  and all  $A \subset (\mathbb{R}^d)^n$  with  $\mu^n(A) \geq 1/2$ ,

$$\forall h \geq 0 \quad \mu^n(A + 2\sqrt{h}B_2 + 2h^{1/p}B_p) \geq 1 - e^{-Lh}. \tag{1.8}$$

If  $p \geq 2$ , then for all  $n \geq 1$  and all  $A \subset (\mathbb{R}^d)^n$  with  $\mu^n(A) \geq 1/2$ ,

$$\forall h \geq 0 \quad \mu^n(A + 2\sqrt{h}B_2 \cap 2h^{1/p}B_p) \geq 1 - e^{-Lh}, \tag{1.9}$$

where  $L$  is a constant depending only on  $C$  and the dimension  $d$ ; one can take  $L = \alpha(\frac{1}{\sqrt{C\kappa}})/(16d)$ , where  $\kappa = \sqrt{18e\sqrt{5}}$ .

This result will be easily deduced from (1.3) and from an elementary comparison between the metric  $d_{\omega_p}(\cdot, \cdot)$  and the norms  $|\cdot|_p$ .

This paper will provide a lot of sufficient conditions for the inequalities  $\mathbb{S}\mathbb{G}(\omega, C)$ . Let us just say for the moment that, in particular, for all  $p \in [1, +\infty)$ , the probability measure  $d\nu_p(x) = \frac{1}{Z_p} e^{-|x|^p} dx$  verifies  $\mathbb{S}\mathbb{G}(\omega_p, C)$  for some  $C$  on  $\mathbb{R}$ . For these  $\nu_p$  one thus formally recovers a famous result by Talagrand ([35], Theorem 2.4). Let us emphasize here that the above proposition only gives an example of the concentration results we can obtain with this approach. It is for instance possible to derive adapted concentration results for fast decreasing probabilities such as  $d\mu(x) = \frac{1}{Z} \exp(-\exp(x^2)) dx$ .

Before presenting in details our results, let us outline some of the positive features of the inequalities  $\mathbb{S}\mathbb{G}(\omega, \cdot)$ :

- They enjoy the classical properties of Poincaré inequalities: *tensorization* and *stability under bounded perturbation*.
- A lot of workable *sufficient conditions* are available. In dimension one, one proves a *necessary and sufficient condition*.
- A large variety of Talagrand’s like concentration inequalities can be obtained. Moreover it is interesting to note that the *same family* of functional inequalities yields as well *sub-Gaussian and super-Gaussian estimates*.
- These inequalities are *weak*. For example, we are going to show that for all  $p \in (1, 2]$  the Poincaré inequality  $\mathbb{S}\mathbb{G}(\omega_p, \cdot)$  is *strictly weaker* than the Latała–Oleszkiewicz inequality  $\mathbb{L}\mathbb{O}(p, \cdot)$  defined below and gives the same kind of concentration.
- Finally, inequalities  $\mathbb{S}\mathbb{G}(\omega, \cdot)$  are *equivalent* to certain *transportation-cost inequalities* and *inf-convolution inequalities*. As a byproduct, our paper furnishes new results for these inequalities.

### 1.3. About the literature

In recent years, several authors developed many different tools in order to obtain dimension free concentration estimates such as (1.8) and (1.9) for  $1 < p \leq 2$  (see e.g. [5,7,8,19,25,26,37]) and  $p > 2$  [13,14,17,18,21,39]. It will be a difficult task to give a complete summary of these various attempts. We will focus on four important functional approaches to the concentration of measure phenomenon: the Latała–Oleszkiewicz inequalities, the modified Logarithmic-Sobolev inequalities, the super Poincaré inequalities and the transportation-cost inequalities.

*The Latała–Oleszkiewicz inequalities.* We have already indicate how the concentration inequalities (1.8) for  $p = 1$  and  $p = 2$  can be derived from the classical Poincaré inequality and the Logarithmic-Sobolev inequality (1.6) respectively. In [25], Latała and Oleszkiewicz proposed a family of inequalities interpolating between Poincaré and Log-Sobolev. These inequalities are defined as follows. Let  $p \in [1, 2]$ , one will say that a probability measure  $\mu$  on  $\mathbb{R}^d$  satisfies the inequality  $\mathbb{L}\mathbb{O}(p, C)$  if

$$\sup_{a \in (1,2)} \frac{\int f^2 d\mu - (\int |f|^a d\mu)^{2/a}}{(2-a)^{2(1-1/p)}} \leq C \int |\nabla f|_2^2 d\mu, \tag{1.10}$$

holds for all  $f$  smooth enough. For  $p = 1$ , the inequality (1.10) is Poincaré inequality  $\mathbb{S}\mathbb{G}(C)$  and for  $p = 2$  it is equivalent to the Logarithmic-Sobolev inequality (see [25], Corollary 1). The  $\mathbb{L}\mathbb{O}(p, C)$  inequalities on  $\mathbb{R}$  were completely characterized by Barthe and Roberto in [7]. Several extensions of this inequality were considered (see e.g. [41] or [5]). According to [25], Theorem 1, if  $\mu$  is a probability measure on  $\mathbb{R}^d$  satisfying  $\mathbb{L}\mathbb{O}(p, C)$ , then there is a constant  $L > 0$  such that  $\mu^n$  verifies the concentration inequality (1.8). So, roughly speaking, if  $\mu$  verifies  $\mathbb{L}\mathbb{O}(p, C)$  it concentrates independently of the dimension like  $dv_p(x) = \frac{1}{Z_p} e^{-|x|^p} dx$ ,  $p \in [1, 2]$ .

*Modified Logarithmic-Sobolev inequalities.* These inequalities first appear in a paper of Bobkov and Ledoux [11]. Let  $H : \mathbb{R} \rightarrow \mathbb{R}^+$  be a convex function; one says that a probability  $\mu$  on  $\mathbb{R}^d$  verifies the modified Logarithmic-Sobolev inequality  $\mathbb{L}\mathbb{S}(H, C)$ , if

$$\text{Ent}_\mu(f^2) \leq C \int \sum_{i=1}^d H\left(\frac{\partial_i f}{f}\right) f^2 d\mu, \tag{1.11}$$

holds for all positive and locally Lipschitz function  $f$ . When  $H(x) = x^2$ , the preceding inequality is simply the Logarithmic-Sobolev inequality, and if  $H(x) = x^2$  for  $|x| \leq 1$  and  $+\infty$  otherwise, the resulting inequality was shown to be equivalent to the Poincaré inequality (see [12], Theorem 3.1).

- Let  $p \geq 2$  and consider  $H_q(x) = |x|^q$  with  $1/p + 1/q = 1$ ; the inequality  $\mathbb{L}\mathbb{S}(H_q, C)$  was studied by Bobkov and Ledoux in [13] and by Bobkov and Zegarlinski in [14], where a complete characterization on  $\mathbb{R}$  was achieved (see [14], Theorem 5.3). This inequality is associated to super-Gaussian concentration. More precisely, if  $\mu$  verifies  $\mathbb{L}\mathbb{S}(H_q, C)$  then for all subset  $A$  of  $(\mathbb{R}^d)^n$  with  $\mu^n(A) \geq 1/2$ ,

$$\forall t \geq 0 \quad \mu^n(A + t^{1/p} B_p) \geq 1 - e^{-Lt},$$

where  $L$  is independent of  $n$ . For  $p \geq 2$ , the measure  $dv_p(x) = \frac{1}{Z_p} e^{-|x|^p} dx$  verifies  $\mathbb{L}\mathbb{S}(H_q, C)$  for some  $C$  and  $1/p + 1/q = 1$ .

- Let  $p \in [1, 2]$  and consider  $H_q(x) = \max(x^2, |x|^q)$  with  $1/p + 1/q = 1$ . The family  $\mathbb{L}\mathbb{S}(H_q, C)$  was first studied by Gentil, Guillin and Miclo in [19] where it was shown that  $\mathbb{L}\mathbb{S}(H_q, C)$  was fulfilled by  $dv_p(x) = \frac{1}{Z_p} e^{-|x|^p} dx$  for  $p \in [1, 2]$  and  $1/p + 1/q = 1$ . It was recently completely characterized on the real line by Barthe and Roberto (see [5], Theorem 10). As shown in [19] or [5], Example 31, if  $\mu$  verifies  $\mathbb{L}\mathbb{S}(H_q, C)$  for some  $C$  then it verifies the concentration inequality (1.8) for some  $L > 0$ . Other choices of  $H$  were considered in [5] and a general concentration inequality established (see [5], Theorem 29, and the remarks after). These results are available under the assumption that  $H(x)/x^2$  is increasing. The resulting concentrations inequalities are thus always *sub-Gaussian*.

*The super Poincaré inequality.* Let  $\beta : [1, +\infty) \rightarrow \mathbb{R}^+$  be a nonincreasing function; one says that a probability  $\mu$  on  $\mathbb{R}^d$  verifies the super Poincaré inequality with the function  $\beta$  if

$$\forall s \geq 1 \quad \int f^2 d\mu \leq \beta(s) \int |\nabla f|_2^2 d\mu + s \left( \int |f| d\mu \right)^2, \tag{1.12}$$

holds true for all locally Lipschitz function  $f$ . If  $\mu$  verifies (1.12), one will write for short that  $\mu$  satisfies the inequality  $\mathbb{S}\mathbb{P}(\beta)$ . Super Poincaré inequalities were introduced by Wang in [39]. They are of great interest in spectral theory or for isoperimetric problems (see [6]). Another nice feature of this family is that several other functional inequalities

are encoded among it, i.e. correspond to specific choices of  $\beta$ . For example, defining for all  $p \geq 1$ ,  $\beta_p(s) = (\log(e + s))^{-2(1-1/p)}$ , then the Latała–Oleszkiewicz inequality  $\mathbb{L}\mathbb{O}(p, C)$ ,  $p \in [1, 2]$  is equivalent to  $\mathbb{S}\mathbb{P}(\tilde{C}\beta_p)$  for some  $\tilde{C}$  as shown in [41], Corollary 1.2. The same is true for  $F$ -Sobolev inequalities (see [39], Theorems 3.1 and 3.2) or weak Logarithmic-Sobolev inequalities (see [15]). For a general  $\beta$  only quite rough concentration estimates can be deduced from  $\mathbb{S}\mathbb{P}(\beta)$ . For example, if  $\mu$  verifies the inequality  $\mathbb{S}\mathbb{P}(C\beta_p)$  for some  $C$  with the function  $\beta_p$  defined above, then  $\int e^{a|x|^p} d\mu(x) < +\infty$  for some  $a > 0$ . The general case is more intricate (see [39], Theorem 6.2, or Proposition 5.2 of the present paper). Moreover, unlike the functional inequalities presented above, the super Poincaré inequality does not tensorize properly and thus the concentration bounds may be affected by the dimension.

*The transportation-cost inequalities.* Transportation-cost inequalities were first introduced by Marton and Talagrand in [28,29,37]. In these inequalities one tries to bound an optimal transportation-cost in the sense of Kantorovich by the relative entropy functional. More precisely, if  $c : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+$  is a measurable map on some metric space  $\mathcal{X}$ , the optimal transportation-cost between  $\nu$  and  $\mu \in \mathcal{P}(\mathcal{X})$  (the set of probability measures on  $\mathcal{X}$ ) is defined by

$$\mathcal{T}_c(\nu, \mu) = \inf_{\pi \in P(\nu, \mu)} \int c(x, y) d\pi, \quad (1.13)$$

where  $P(\nu, \mu)$  is the set of probability measures  $\pi$  on  $\mathcal{X} \times \mathcal{X}$  such that  $\pi(dx, \mathcal{Y}) = \nu(dx)$  and  $\pi(\mathcal{X}, dy) = \mu(dy)$ . One says that  $\mu$  satisfies the transportation-cost inequality with the cost function  $c(x, y)$  if

$$\forall \nu \in \mathcal{P}(\mathcal{X}) \quad \mathcal{T}_c(\nu, \mu) \leq H(\nu|\mu), \quad (1.14)$$

where  $H(\nu|\mu)$  denotes the relative entropy of  $\nu$  with respect to  $\mu$  and is defined by  $H(\nu|\mu) = \int \log(\frac{d\nu}{d\mu}) d\nu$  if  $\nu$  is absolutely continuous with respect to  $\mu$  and  $H(\nu|\mu) = +\infty$  otherwise. Transportation-cost inequalities are known to have good tensorization properties and to yield concentration results independent of the dimension (all these facts are recalled in Section 4 – see also [22]).

One will say that  $\mu$  satisfies the inequality  $\mathbb{T}\mathbb{C}_p(C)$ ,  $p \in [1, 2]$  if it satisfies the transportation-cost inequality with the cost function  $c(x, y) = \frac{1}{C} \min(|x - y|_2^2, |x - y|_2^p)$ . It is now classical that the inequality  $\mathbb{T}\mathbb{C}_p(C)$  implies a concentration inequality similar to (1.8). When  $p = 2$ , the inequality  $\mathbb{T}\mathbb{C}_2(\cdot)$  is usually denoted by  $\mathbb{T}_2$ . In [37], Talagrand proved that the inequality  $\mathbb{T}\mathbb{C}_2(\cdot)$  is satisfied by Gaussian measures.

In dimension one, an almost complete characterization of transportation-cost inequalities was proposed by the author in [21]. It covers in particular the case of the  $\mathbb{T}\mathbb{C}_p(\cdot)$  inequalities for all  $p \in [1, 2]$ . In higher dimensions, one only knows that  $\mathbb{T}\mathbb{C}_p(\cdot)$  inequalities and modified logarithmic Sobolev inequalities are related:

- For  $p = 2$ , a celebrated result by Otto and Villani shows that the Logarithmic-Sobolev inequality implies  $\mathbb{T}\mathbb{C}_2(\cdot)$  (see [33]). It was shown by Cattiaux and Guillin in [16] that the implication is strict: there exist probability measures satisfying  $\mathbb{T}\mathbb{C}_2(\cdot)$  and not the Logarithmic-Sobolev inequality. Wang provides extensions of Otto and Villani's result to Riemannian manifolds and path spaces in [40,42].
- The case  $p = 1$  is very interesting. Bobkov, Gentil and Ledoux have shown in [9] that the inequality  $\mathbb{T}\mathbb{C}_1(\cdot)$  is equivalent to the Poincaré inequality  $\mathbb{S}\mathbb{G}(\cdot)$  (see Theorem 4.7 for a precise statement).
- For  $p \in (1, 2)$ , it was shown by Gentil, Guillin and Miclo in [19] that the modified Logarithmic-Sobolev inequality  $\mathbb{L}\mathbb{S}(H_q, \cdot)$  with  $1/p + 1/q = 1$  implies the transportation-cost inequality  $\mathbb{T}\mathbb{C}_p(\cdot)$ .
- The case  $p > 2$  is much less known. Examples of probability measures satisfying the transportation-cost inequality with a cost function of the form  $|x - y|_p^p$  appear in [13] or [17].

Another very efficient functional approach to the concentration of measure phenomenon was proposed by Maurey in [30]: the so-called  $(\tau)$  property also called inf-convolution inequality. As we will see in Section 4, inf-convolution inequalities are in fact equivalent to transportation-cost inequalities (see Proposition 4.13).

#### 1.4. Presentation of the results

The map  $\omega$  is defined on  $\mathbb{R}$  but we will also denote by  $\omega$  the map defined on  $\mathbb{R}^m$  (for every  $m \geq 1$ ) by  $(x_1, \dots, x_m) \mapsto (\omega(x_1), \dots, \omega(x_m))$ . The image of a probability measure  $\mu$  on a space  $\mathcal{X}$  under a measurable map  $T : \mathcal{X} \rightarrow \mathcal{Y}$  will be denoted by  $T\sharp\mu$ . We recall that it is defined by

$$\forall A \subset \mathcal{Y} \quad T\sharp\mu(B) = \mu(T^{-1}(A)).$$

Our paper is organized as follows:

- In Section 2, we first recall some well-known facts about Poincaré inequalities. One explains then how to derive general Talagrand’s concentration results from the inequalities  $\mathbb{S}\mathbb{G}(\omega, \cdot)$ . The main result of this section is Proposition 2.4, where we show that if  $\mu$  verifies  $\mathbb{S}\mathbb{G}(\omega, C)$  for some  $C > 0$ , then  $\mu^n$  concentrates independently of the dimension in the following way: for all  $n \geq 1$  and all  $A \subset (\mathbb{R}^d)^n$ , one has

$$\forall h \geq 0 \quad \mu^n(A + B_\omega(h)) \geq 1 - e^{-Lh},$$

where  $L$  is a constant depending only on  $C$  and  $B_\omega(h)$  is the Orlicz ball defined by

$$B_\omega(h) = \left\{ (x_1, \dots, x_n) \in (\mathbb{R}^d)^n : \sum_{i=1}^n \sum_{j=1}^d \alpha \circ \omega \left( \frac{|x_{i,j}|}{2} \right) \leq h \right\}.$$

(For all  $1 \leq i \leq n$ ,  $x_{i,j}$ ,  $1 \leq j \leq d$ , are the coordinates of the vector  $x_i \in \mathbb{R}^d$ .) Proposition 2.4 easily implies Proposition 1.2 for the special case of the functions  $\omega_p$ .

- In Section 3 we address the problem of finding workable sufficient conditions for the Poincaré inequalities  $\mathbb{S}\mathbb{G}(\omega, \cdot)$ . To do so, we relate the inequality  $\mathbb{S}\mathbb{G}(\omega, \cdot)$  to the classical Poincaré inequality  $\mathbb{S}\mathbb{G}(\cdot)$ . One shows in Proposition 3.1, that

$$\mu \text{ verifies } \mathbb{S}\mathbb{G}(\omega, C) \iff \omega^\sharp \mu \text{ verifies } \mathbb{S}\mathbb{G}(C). \tag{1.15}$$

So, according to (1.15), to prove that a probability measure  $\mu$  verifies  $\mathbb{S}\mathbb{G}(\omega, \cdot)$ , all we have to do is to apply to the measure  $\omega^\sharp \mu$  one of the known criteria for the classical Poincaré inequality  $\mathbb{S}\mathbb{G}(\cdot)$ . In dimension one, one thus easily derive from the celebrated Muckenhoupt theorem a necessary and sufficient condition for the inequality  $\mathbb{S}\mathbb{G}(\omega, \cdot)$  (see Theorem 3.2). Using this criteria, one can give a large collection of examples. Under mild regularity conditions, one proves in Proposition 3.3 that a symmetric probability  $d\mu(x) = e^{-V(x)} dx$  on  $\mathbb{R}$  satisfies the inequality  $\mathbb{S}\mathbb{G}(\omega, C)$  for some  $C$  if and only if

$$\liminf_{x \rightarrow +\infty} \frac{V'(x)}{\omega'(x)} > 0. \tag{1.16}$$

The same strategy can be applied in dimension  $d$ . It is well known that a probability  $d\mu(x) = e^{-V(x)} dx$  on  $\mathbb{R}^d$  satisfies the Poincaré inequality as soon as  $\liminf_{|x| \rightarrow +\infty} \frac{1}{2} |\nabla V|_2(x)^2 - \Delta V(x) > 0$ . Combined with (1.15), this criteria yields a sufficient condition for the inequality  $\mathbb{S}\mathbb{G}(\omega, \cdot)$  (see Proposition 3.5).

- In Section 4, we show the equivalence between the Poincaré inequalities for the metrics  $d_\omega$  and certain transportation-cost inequalities.

**Definition 1.3.** *Let us say that  $\mu \in \mathcal{P}(\mathbb{R}^d)$  satisfies the inequality  $\mathbb{T}\mathbb{C}(\omega, a)$  if it satisfies the transportation-cost inequality (1.14) with the cost function  $(x, y) \mapsto \alpha(ad_\omega(x, y))$ , where  $d_\omega(x, y)$  is defined in (1.7).*

In Theorem 4.6, which is one of the main results of this paper, one proves that  $\mu$  satisfies the inequality  $\mathbb{S}\mathbb{G}(\omega, C)$  for some  $C$  if and only if it satisfies the inequality  $\mathbb{T}\mathbb{C}(\omega, a)$  for some  $a$ . The link between  $a$  and  $C$  is made precise in Theorem 4.6. This theorem is an extension of a result by Bobkov, Gentil and Ledoux concerning the equivalence of the classical Poincaré inequality and the inequality  $\mathbb{T}\mathbb{C}_1(\cdot)$  (see [9], Corollary 5.1). This extension is performed using a very simple contraction principle for transportation-cost inequalities. This technique was previously used by the author in [21] to characterize a large class of transportation-cost inequalities on the real line. Since the inequality  $\mathbb{T}\mathbb{C}(\omega_p, \cdot)$  is easily shown to be stronger than  $\mathbb{T}\mathbb{C}_p(\cdot)$ , Theorem 4.6 offers new sufficient conditions for the transportation-cost inequalities  $\mathbb{T}\mathbb{C}_p(\cdot)$  (see Corollary 4.11). Up to now, Corollary 4.11 gives the weakest known sufficient condition for  $\mathbb{T}\mathbb{C}_p$  inequalities.

- In Section 5, we compare the inequalities  $\mathbb{S}\mathbb{G}(\omega, \cdot)$  to other functional inequalities.

The main result of this paper, Theorem 5.4, states that under not very restrictive conditions on the function  $\beta$ , the super Poincaré inequality  $\mathbb{SP}(\beta)$  implies an inequality  $\mathbb{SG}(\omega_\beta, \cdot)$  where  $\omega_\beta$  depends only on the function  $\beta$ . Since a lot of functional inequalities are encoded as super Poincaré inequalities, this result is extremely general.

As a consequence, one deduces in particular the following relationships.

For  $p \in [1, 2]$ ,

$$\mu \text{ verifies } \mathbb{LO}(p, \cdot) \implies \mu \text{ verifies } \mathbb{SG}(\omega_p, \cdot).$$

Moreover, a counter example of Cattiaux and Guillin shows that the Logarithmic-Sobolev inequality (which corresponds to  $p = 2$ ) is strictly stronger than the inequality  $\mathbb{SG}(\omega_2, \cdot)$  (see Remark 5.19).

For  $p \geq 2$  and  $1/p + 1/q = 1$ ,

$$\mu \text{ verifies } \mathbb{LS}(H_q, \cdot) \implies \mu \text{ verifies } \mathbb{SG}(\omega_p, \cdot).$$

Let us emphasize another interesting fact about Theorem 5.4. One knows that super Poincaré inequalities do not tensorize properly. If  $\mu$  verifies a super Poincaré inequality, then  $\mu^n$  will satisfy a super Poincaré inequality with  $\beta(s)$  replaced by  $\beta(s/n)$ . Thus the inequalities deteriorate when the dimension increases. On the other hand, the inequality  $\mathbb{SG}(\omega_\beta, \cdot)$  implied by the super Poincaré inequality has a *good tensorization property* and implies *concentration independent of the dimension*. From this follows that super Poincaré inequalities (almost) always imply dimension free concentration estimates.

## 2. Poincaré inequalities and concentration of measure

### 2.1. A reminder on Poincaré inequalities

Let us recall the two classical structural properties of Poincaré inequalities: tensorization property and stability under bounded perturbations.

**Proposition 2.1.** *Let  $\mu$  be a probability on  $\mathbb{R}^d$  satisfying the Poincaré inequality  $\mathbb{SG}(\omega, C)$  for some constant  $C > 0$ .*

- For all  $n \geq 1$ , the probability measure  $\mu^n$  verifies  $\mathbb{SG}(\omega, C)$  on  $(\mathbb{R}^d)^n$ .
- If  $\bar{\mu}$  is a probability measure on  $\mathbb{R}^d$  absolutely continuous with respect of  $\mu$  with a density of the form  $d\bar{\mu}(x) = e^{h(x)} d\mu(x)$  with  $h$  bounded, then  $\bar{\mu}$  verifies the Poincaré inequality  $\mathbb{SG}(\omega, e^{\text{Osc}(h)} C)$ , where  $\text{Osc}(h) = \sup(h) - \inf(h)$ .

The reader will find a proof (in the general case) in e.g. [27], Corollary 5.7.

### 2.2. Poincaré inequalities and concentration – the abstract case

Now let us recall how concentration estimates can be derived from the Poincaré inequality. We follow the work by Bobkov and Ledoux [12].

**Theorem 2.2 (Bobkov–Ledoux).** *If  $\mu$  satisfies (1.1), then for every bounded function  $f$  on  $\mathcal{X}^n$  such that  $\sum_{i=1}^n |\nabla_i f|^2 \leq a^2$  and  $\max_{i=1, \dots, n} |\nabla_i f| \leq b$ ,  $\mu^n$  a.e. (where  $|\nabla_i f|$  denotes the length of the gradient with respect to the  $i$ th coordinate) one has*

$$\forall t \geq 0 \quad \mu^n \left( f \geq \int f d\mu^n + t \right) \leq \exp \left( - \min \left( \frac{t^2}{C\kappa^2 a^2}, \frac{t}{\sqrt{C}\kappa b} \right) \right), \quad \text{with } \kappa = \sqrt{18e\sqrt{5}}. \quad (2.1)$$

The preceding deviation inequality expresses that Lipschitz functions are almost constant on  $\mathcal{X}^n$ .

Another way to express the concentration of the product measure  $\mu^n$  is given in the following corollary which can be easily deduced from the preceding theorem:

**Corollary 2.3 (Bobkov–Ledoux).** *Let  $\mu$  be a probability measure on  $\mathcal{X}$  satisfying the Poincaré inequality (1.1) on  $(\mathcal{X}, d)$  with the constant  $C > 0$ . There is a constant  $L$  depending only on  $C$  such that for all subset  $A$  of  $\mathcal{X}^n$  with  $\mu^n(A) \geq 1/2$ ,*

$$\forall h \geq 0 \quad \mu^n(A^h) \geq 1 - e^{-Lh}, \tag{2.2}$$

where the set  $A^h$  is the enlargement of  $A$  defined by

$$A^h = \left\{ y \in \mathcal{X}^n : \inf_{x \in A} \sum_{i=1}^n \alpha(d(x_i, y_i)) \leq h \right\},$$

where  $\alpha(u) = \min(|u|, u^2)$  for all  $u \in \mathbb{R}$ . One can take  $L = \alpha(\frac{1}{\sqrt{C\kappa}})/16$ , where as before  $\kappa = \sqrt{18e\sqrt{5}}$ .

For the sake of completeness, the reader will find a proof of these two results in the [Annex](#).

### 2.3. The $\mathbb{S}\mathbb{G}(\omega, \cdot)$ inequality and concentration

**Proposition 2.4.** *Suppose that  $\mu$  satisfies  $\mathbb{S}\mathbb{G}(\omega, C)$  on  $\mathbb{R}^d$  for some  $C > 0$ . Then for all  $n \geq 1$  and all  $A \subset (\mathbb{R}^d)^n$  with  $\mu^n(A) \geq 1/2$ , one has*

$$\forall h \geq 0 \quad \mu^n(A + B_\omega(h)) \geq 1 - e^{-Lh},$$

where  $L = \alpha(\frac{1}{\sqrt{C\kappa}})/(16d)$  and  $B_\omega(h)$  is defined by

$$B_\omega(h) = \left\{ (x_1, \dots, x_n) \in (\mathbb{R}^d)^n : \sum_{i=1}^n \sum_{j=1}^d \alpha \circ \omega \left( \frac{|x_{i,j}|}{2} \right) \leq h \right\}. \tag{2.3}$$

(For all  $1 \leq i \leq n$ ,  $x_{i,j}$ ,  $1 \leq j \leq d$  are the coordinates of the vector  $x_i \in \mathbb{R}^d$ .)

**Remark 2.5.** *The fact that the dimension  $d$  appears in the preceding result is not important. The important thing is that the constants do not depend on the dimension  $n$ .*

We need the following elementary facts:

#### Lemma 2.6.

- (1) For all  $x, y \in \mathbb{R}$ ,  $|\omega(x) - \omega(y)| \geq \omega(\frac{|x-y|}{2})$ .
- (2) The function  $\alpha(u) = \min(|u|, u^2)$  is such that  $\alpha(au) \geq \alpha(a)\alpha(u)$ , for all  $a, u \geq 0$ .

**Proof.** Let us prove the first point. The function  $x \mapsto \omega(x)/x$  is nondecreasing on  $\mathbb{R}^+$ . It follows that  $\omega$  is super additive on  $\mathbb{R}^+$ . Indeed, if  $0 < x \leq y$  then  $\omega(x+y) = \omega(y(1+x/y)) \geq (1+x/y)\omega(y) = \omega(y) + x\omega(y)/y \geq \omega(y) + x\omega(x)/x = \omega(y) + \omega(x)$ .

Let  $x \geq y$ . If  $x \geq y \geq 0$ , then using the super additivity of  $\omega$ , one gets  $\omega(x) = \omega((x-y) + y) \geq \omega(x-y) + \omega(y)$ , so  $\omega(x) - \omega(y) \geq \omega(x-y) \geq \omega((x-y)/2)$ . The case  $0 \geq x \geq y$  is similar. Now, if  $x \geq 0 \geq y$ , then  $\omega(x) - \omega(y) = \omega(x) + \omega(-y) \geq \omega(\max(x, -y)) \geq \omega((x-y)/2)$ .

Now let us prove the second point. If  $0 < a \leq 1$ , then  $\alpha(au)/\alpha(a) = u^2$  if  $u \leq 1/a$  and  $\alpha(au)/\alpha(a) = u/a$  if  $u \geq 1/a$ . If  $u \leq 1$ , one has  $\alpha(au)/\alpha(a) = \alpha(u)$ . If  $u \in [1, 1/a]$ , then  $u^2 \geq u$  and so  $\alpha(au)/\alpha(a) \geq \alpha(u)$ . If  $u \geq 1/a$ , then  $u/a \geq u$  and so  $\alpha(au)/\alpha(a) \geq \alpha(u)$ . The case  $a \geq 1$  can be handled in a similar way.  $\square$

**Proof of Proposition 2.4.** First,  $d_\omega(u, v) \geq \frac{1}{\sqrt{d}} \sum_{i=1}^d |\omega(u_i) - \omega(v_i)|$ , for all  $u, v \in \mathbb{R}^d$ .

Now,

$$\begin{aligned} \alpha(d_\omega(u, v)) &\geq \alpha\left(\sum_{i=1}^d \frac{1}{\sqrt{d}} |\omega(u_i) - \omega(v_i)|\right) \stackrel{(i)}{\geq} \sum_{i=1}^d \alpha\left(\frac{1}{\sqrt{d}} |\omega(u_i) - \omega(v_i)|\right) \\ &\stackrel{(ii)}{\geq} \sum_{i=1}^d \alpha\left(\frac{1}{\sqrt{d}} \omega\left(\frac{|u_i - v_i|}{2}\right)\right) \stackrel{(iii)}{\geq} \frac{1}{d} \sum_{i=1}^d \alpha \circ \omega\left(\frac{|u_i - v_i|}{2}\right), \end{aligned}$$

where (i) comes from the super additivity of the function  $\alpha$ , (ii) from Lemma 2.6(1) and (iii) from Lemma 2.6(2).

Consequently, for all  $x \in (\mathbb{R}^d)^n$  and  $A \subset (\mathbb{R}^d)^n$ ,

$$\inf_{a \in A} \sum_{i=1}^n \alpha(d_\omega(x_i, a_i)) \geq \frac{1}{d} \inf_{a \in A} \sum_{i=1}^n \sum_{j=1}^d \alpha \circ \omega\left(\frac{|x_{i,j} - a_{i,j}|}{2}\right).$$

Applying (2.2) yields immediately the desired result. □

**Proof of Proposition 1.2.** Suppose  $p \in [1, 2]$ ; in view of Proposition 2.4, it is enough to prove that

$$\sum_{k=1}^{nd} \alpha \circ \omega_p(u_k) \leq h \implies u = (u_1, \dots, u_{nd}) \in \sqrt{h} B_2 + h^{1/p} B_p.$$

Let  $s = (s_1, \dots, s_{nd})$  and  $t = (t_1, \dots, t_{nd})$  be defined by  $s_k = u_k$  if  $u_k \in [-1, 1]$  and  $s_k = 0$  if  $|u_k| > 1$  and  $t = u - s$ . Then,

$$\sum_{k=1}^{nd} \alpha \circ \omega_p(u_k) = |s|_2^2 + |t|_p^p \leq h.$$

So,  $|s|_2 \leq \sqrt{h}$  and  $|t|_p \leq h^{1/p}$ . Since  $u = s + t$ , one concludes that  $u \in \sqrt{h} B_2 + h^{1/p} B_p$ .

Now, if  $p \geq 2$ , then  $\forall x \geq 0, \alpha \circ \omega_p(x) = \max(x^2, x^p)$ . This observation together with Proposition 2.4 easily implies the result. □

Let us conclude this section with a remark concerning centering. If  $\mu$  is a probability measure on  $\mathbb{R}^d$  and  $z \in \mathbb{R}^d$ , let us denote by  $\mu_z$  the translate of  $\mu$  defined by:

$$\mu_z(A) = \mu(A + z) \tag{2.4}$$

for all measurable set  $A$ .

The following corollary is immediate.

**Corollary 2.7.** *Suppose that there is some  $z \in \mathbb{R}^d$  such that  $\mu_z$  verifies the inequality  $\mathbb{S}\mathbb{G}(\omega, C)$  for some  $C > 0$ , then for all  $n \geq 1$  and all  $A \subset (\mathbb{R}^d)^n$  with  $\mu^n(A) \geq 1/2$ , one has*

$$\forall h \geq 0 \quad \mu^n(A + B_\omega(h)) \geq 1 - e^{-Lh},$$

where  $L = \alpha(\frac{1}{\sqrt{C\kappa}})/(16d)$  and  $B_\omega(h)$  is defined by (2.3).

**Definition 2.8.** *One will say that  $\mu$  verifies the centered Poincaré inequality  $\mathbb{S}\mathbb{G}(\omega, C)$  if  $\mu_{\int x \, d\mu}$  verifies the inequality  $\mathbb{S}\mathbb{G}(\omega, C)$ .*

This definition will play an important role in Section 5.

### 3. Workable sufficient conditions for $\mathbb{S}\mathbb{G}(\omega, \cdot)$

#### 3.1. Links with the classical Poincaré inequality

In order to obtain sufficient conditions for the inequalities  $\mathbb{S}\mathbb{G}(\omega, \cdot)$ , one relates them to (weighted) forms of the classical Poincaré inequality, which is quite well known.

**Proposition 3.1.** *Let  $\mu$  be a probability measure on  $\mathbb{R}^d$  and  $C$  a positive number. The following properties are equivalent:*

- (i) *The probability measure  $\mu$  verifies  $\mathbb{S}\mathbb{G}(\omega, C)$ .*
- (ii) *The probability measure  $\omega^\sharp \mu$  verifies  $\mathbb{S}\mathbb{G}(C)$ .*
- (iii) *The probability measure  $\mu$  satisfies the following weighted Poincaré inequality:*

$$\text{Var}_\mu(f) \leq C \int \sum_{i=1}^d \frac{1}{\omega'(x_i)^2} \left( \frac{\partial f}{\partial x_i}(x) \right)^2 d\mu(x) \quad (3.1)$$

for all  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $f \circ \omega^{-1}$  is of class  $C^1$ .

**Proof.** Let us denote  $|\nabla f|_\omega$  (resp.  $|\nabla f|_2$ ) the length of the gradient computed with respect to the metric  $d_\omega(\cdot, \cdot)$  (see (1.2)). If  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  is locally Lipschitz for the Euclidean metric, then according to Rademacher theorem, one has

$$\limsup_{y \rightarrow x} \frac{|f(x) - f(y)|}{|x - y|_2} = \left[ \sum_{i=1}^d \left( \frac{\partial f}{\partial x_i} \right)^2(x) \right]^{1/2} = |\nabla f|_2(x)$$

for  $\mu$  a.e.  $x \in \mathbb{R}^d$ , and so the length of the gradient equals the norm of the vector  $\nabla f$   $\mu$  a.e.

Locally Lipschitz function for  $d_\omega(\cdot, \cdot)$  and  $|\cdot|_2$  are related in the following way. A function  $g: \mathbb{R}^d \rightarrow \mathbb{R}$  is locally Lipschitz for  $d_\omega(\cdot, \cdot)$  if and only if  $g \circ \omega^{-1}$  is locally Lipschitz for  $|\cdot|_2$ .

(i)  $\Rightarrow$  (ii) Define  $\tilde{\mu} = \omega^\sharp \mu$ . Let  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  be locally Lipschitz for  $|\cdot|_2$ , then  $f \circ \omega$  is locally Lipschitz for  $d_\omega(\cdot, \cdot)$ , and

$$\text{Var}_{\tilde{\mu}}(f) = \text{Var}_\mu(f \circ \omega) \leq \int |\nabla(f \circ \omega)|_\omega^2 d\mu \stackrel{(*)}{=} \int |\nabla f|_2^2 \circ \omega d\mu = \int |\nabla f|_2^2 d\tilde{\mu},$$

where (\*) follows from the easy to check identity:  $|\nabla(f \circ \omega)|_\omega = |\nabla f|_2 \circ \omega$ :

(ii)  $\Rightarrow$  (i) The proof is the same.

(ii)  $\Rightarrow$  (iii) Take  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $f \circ \omega^{-1}$  is of class  $C^1$ . Then

$$\text{Var}_\mu(f) = \text{Var}_{\tilde{\mu}}(f \circ \omega^{-1}) \leq \int |\nabla(f \circ \omega^{-1})|_2^2 \circ \omega d\mu = \int \sum_{i=1}^d \frac{1}{\omega'(x_i)^2} \left( \frac{\partial f}{\partial x_i}(x) \right)^2 d\mu(x).$$

(iii)  $\Rightarrow$  (ii) Apply the weighted Poincaré inequality to the function  $f \circ \omega$  with  $f$  of class  $C^1$ . □

#### 3.2. Dimension one

In the following proposition, a necessary and sufficient condition is given for  $\mathbb{S}\mathbb{G}(\omega, \cdot)$  inequalities.

**Proposition 3.2.** *A probability measure  $\mu$  on  $\mathbb{R}$  absolutely continuous with density  $h > 0$  satisfies the inequality  $\mathbb{S}\mathbb{G}(\omega, C)$  for some  $C > 0$  if and only if*

$$D_\omega^- = \sup_{x \leq m} \mu(-\infty, x] \int_x^m \frac{\omega'(u)^2}{h(u)} du < +\infty \quad \text{and} \quad D_\omega^+ = \sup_{x \geq m} \mu[x, +\infty) \int_m^x \frac{\omega'(u)^2}{h(u)} du < +\infty, \quad (3.2)$$

where  $m$  denotes the median of  $\mu$ . Moreover, the optimal constant  $C$  in (1.1) denoted by  $C_{\text{opt}}$  verifies

$$\max(D_{\omega}^-, D_{\omega}^+) \leq C_{\text{opt}} \leq 4 \max(D_{\omega}^-, D_{\omega}^+).$$

This proposition follows at once from the celebrated Muckenhoupt condition for the classical Poincaré inequality (see [32]).

**Proof of Proposition 3.2.** According to Muckenhoupt condition, a probability measure  $\nu = h \, dx$  having a positive continuous density with respect to Lebesgue measure, satisfies the classical Poincaré inequality if and only if

$$D^- = \sup_{x \leq m} \nu(-\infty, x] \int_x^m \frac{1}{h(u)} \, du < +\infty \quad \text{and} \quad D^+ = \sup_{x \geq m} \nu[x, +\infty) \int_m^x \frac{1}{h(u)} \, du < +\infty,$$

and the optimal constant  $C_{\text{opt}}$  verifies  $\max(D^-, D^+) \leq C_{\text{opt}} \leq 4 \max(D^-, D^+)$ . Now, according to Proposition 3.1,  $\mu$  satisfies  $\mathbb{S}\mathbb{G}(\omega, C)$  if and only if  $\tilde{\mu} = \omega^{\sharp} \mu$  satisfies  $\mathbb{S}\mathbb{G}(C)$ . The density of  $\tilde{\mu}$  is  $\tilde{h} = \frac{h \circ \omega^{-1}}{\omega' \circ \omega^{-1}}$ . Plugging  $\tilde{h}$  in Muckenhoupt conditions gives immediately the announced result.  $\square$

The following result completes the picture giving a large class of examples:

**Proposition 3.3.** Let  $\mu$  be an absolutely continuous probability measure on  $\mathbb{R}$  with density  $d\mu(x) = e^{-V(x)} \, dx$ . Assume that the potential  $V$  is of class  $C^1$  and that  $\omega$  verifies the following regularity condition:

$$\frac{\omega''(x)}{\omega'^2(x)} \xrightarrow{x \rightarrow +\infty} 0.$$

If  $V$  is such that

$$\liminf_{x \rightarrow \pm\infty} \frac{\text{sgn}(x)V'(x)}{\omega'(x)} > 0, \tag{3.3}$$

then the probability measure  $\mu$  verifies the Poincaré inequality  $\mathbb{S}\mathbb{G}(\omega, C)$  for some  $C > 0$ .

**Proof.** Let  $\tilde{\mu} = \omega^{\sharp} \mu$  and let  $\nu$  be the symmetric exponential probability measure on  $\mathbb{R}$ , that is the probability measure with density  $d\nu(x) = \frac{1}{2} e^{-|x|} \, dx$ . It is well known that it verifies the following Poincaré inequality:

$$\text{Var}_{\nu}(g) \leq 4 \int g^2(x) \, d\nu(x) \tag{3.4}$$

for all smooth  $g$  (see, for example, [12], Lemma 2.1). Let  $T: \mathbb{R} \rightarrow \mathbb{R}$  be the map defined by  $T(x) = F_{\tilde{\mu}}^{-1} \circ F_{\nu}(x)$ , with  $F_{\nu}(x) = \nu(-\infty, x]$  and  $F_{\tilde{\mu}}(x) = \tilde{\mu}(-\infty, x]$ . It is well known that  $T$  is increasing and transports  $\nu$  on  $\tilde{\mu}$  which means that  $T^{\sharp} \nu = \tilde{\mu}$ . Let us apply inequality (3.4) to a function  $g = f \circ T$ . It yields immediately:

$$\text{Var}_{\tilde{\mu}}(f) \leq 4 \int f'^2(T' \circ T^{-1})^2 \, d\tilde{\mu} \leq 4 \left( \sup_{x \in \mathbb{R}} T'(x) \right)^2 \int f'^2 \, d\tilde{\mu}.$$

As a conclusion, if the map  $T$  is  $L$  Lipschitz then  $\tilde{\mu}$  verifies Poincaré inequality  $\mathbb{S}\mathbb{G}(4L^2)$ . The probability  $\tilde{\mu}$  has density  $d\tilde{\mu}(x) = e^{-\tilde{V}(x)} \, dx$ , with  $\tilde{V}(x) = V(\omega^{-1}(x)) + \log \omega' \circ \omega^{-1}(x)$ . It is proved in [21] (see Proposition 34) that a sufficient condition for  $T$  to be Lipschitz is that  $\liminf_{x \rightarrow \pm\infty} \text{sgn}(x) \tilde{V}'(x) > 0$ . But  $\tilde{V}'(\omega(x)) = \frac{V'(x)}{\omega'(x)} + \frac{\omega''(x)}{\omega'^2(x)}$  and by assumption  $\frac{\omega''(x)}{\omega'^2(x)} \rightarrow 0$  when  $x$  goes to  $\infty$ . Thus  $\liminf_{x \rightarrow \pm\infty} \text{sgn}(x) \tilde{V}'(x) = \liminf_{x \rightarrow \pm\infty} \frac{\text{sgn}(x)V'(x)}{\omega'(x)}$ , which completes the proof.  $\square$

**Remark 3.4.** The condition  $\liminf_{x \rightarrow \pm\infty} \frac{\text{sgn}(x)V'(x)}{\omega'(x)} > 0$  can also be derived from Proposition 3.2 using the same techniques as in e.g. [3], Theorem 6.4.3. But this method has the disadvantage of introducing useless technical assumptions such as  $\lim_{\pm\infty} V''/(V'^2) = 0$ .

### 3.3. Dimension $d$

In dimension  $d$ , one gets:

**Proposition 3.5.** *Let  $\mu$  be a probability measure on  $\mathbb{R}^d$  absolutely continuous with respect to the Lebesgue measure, with  $d\mu(x) = e^{-V(x)} dx$  with  $V$  a function of class  $C^2$ . Suppose that  $\omega$  is of class  $C^3$  on  $\mathbb{R}$  and such that  $\omega'(0) > 0$  and*

$$\forall x \in \mathbb{R} \quad \left| \frac{\omega^{(3)}(x)}{(\omega')^3} \right| \leq M$$

for some  $M \geq 0$ . If there is some constant  $u > 0$  such that

$$\liminf_{|x| \rightarrow +\infty} \frac{1}{u^2} \sum_{i=1}^d \left[ \frac{1}{10} \left( \frac{\partial V}{\partial x_i} \right)^2 \left( \frac{x}{u} \right) - \frac{\partial^2 V}{\partial x_i^2} \left( \frac{x}{u} \right) \right] \frac{1}{\omega'(x_i)^2} > dM,$$

then the probability measure  $\mu$  satisfies  $\mathbb{S}\mathbb{G}(\tilde{\omega}, C)$  for some  $C$ , where  $\tilde{\omega}(x) = \omega(ux)$ , for all  $x \in \mathbb{R}$ .

**Proof.** It is well known that a probability  $d\nu(x) = e^{-W(x)} dx$  on  $\mathbb{R}^d$  satisfies the classical Poincaré inequality if  $W$  verifies the following condition:

$$\liminf_{|x| \rightarrow +\infty} \frac{1}{2} |\nabla W|_2^2(x) - \Delta W(x) > 0. \tag{3.5}$$

This condition is rather classical; a nice elementary proof can be found in [4].

Suppose that  $\mu$  is an absolutely continuous probability measure on  $\mathbb{R}^d$  with density  $d\mu(x) = e^{-V(x)} dx$  with  $V$  of class  $C^2$ . Then  $\tilde{\mu} = \omega^\# \mu$  has density  $d\tilde{\mu}(x) = e^{-\tilde{V}(x)} dx$ , with

$$\forall x \in \mathbb{R}^d \quad \tilde{V}(x) = V(\omega^{-1}(x)) + \sum_{i=1}^d \log \omega' \circ \omega^{-1}(x_i).$$

According to Proposition 3.1, to show that  $\mu$  satisfies the inequality  $\mathbb{S}\mathbb{G}(\omega, C)$  for some  $C > 0$  it is enough to show that  $\tilde{\mu}$  satisfies the inequality  $\mathbb{S}\mathbb{G}(C)$  and a sufficient condition for this is that  $\tilde{V}$  fulfills condition (3.5).

Elementary computations yield

$$\begin{aligned} \frac{\partial \tilde{V}}{\partial x_i}(\omega(x)) &= \frac{1}{\omega'(x_i)} \frac{\partial V}{\partial x_i}(x) + \frac{\omega''(x_i)}{\omega'^2(x_i)}, \\ \frac{\partial^2 \tilde{V}}{\partial x_i^2}(\omega(x)) &= -\frac{\omega''(x_i)}{\omega'^3(x_i)} \frac{\partial V}{\partial x_i}(x) + \frac{1}{\omega'^2(x_i)} \frac{\partial^2 V}{\partial x_i^2}(x) + \frac{\omega^{(3)}(x_i)}{\omega'^3(x_i)} - 2 \frac{\omega''^2(x_i)}{\omega'^4(x_i)}. \end{aligned}$$

Let  $I(x) = \frac{1}{2} |\nabla \tilde{V}|_2^2(\omega(x)) - \Delta \tilde{V}(\omega(x))$ ; one has:

$$I(x) = \sum_{i=1}^d \frac{1}{\omega'^2(x_i)} \left[ \frac{1}{2} \left( \frac{\partial V}{\partial x_i} \right)^2(x) - \frac{\partial^2 V}{\partial x_i^2}(x) \right] + 2 \sum_{i=1}^d \frac{\omega''(x_i)}{\omega'^3(x_i)} \frac{\partial V}{\partial x_i}(x) + \frac{5}{2} \sum_{i=1}^d \frac{\omega''^2(x_i)}{\omega'^4(x_i)} - \sum_{i=1}^d \frac{\omega^{(3)}(x_i)}{\omega'^3(x_i)}.$$

Using the inequality  $uv \geq -\frac{5}{4}u^2 - \frac{1}{5}v^2$ , one has

$$\begin{aligned} 2 \sum_{i=1}^d \frac{\omega''(x_i)}{\omega'^3(x_i)} \frac{\partial V}{\partial x_i}(x) &= 2 \sum_{i=1}^d \left( \frac{\omega''(x_i)}{\omega'^2(x_i)} \right) \cdot \left( \frac{1}{\omega'(x_i)} \frac{\partial V}{\partial x_i}(x) \right) \\ &\geq -\frac{5}{2} \sum_{i=1}^d \frac{\omega''^2(x_i)}{\omega'^4(x_i)} - \frac{2}{5} \sum_{i=1}^d \frac{1}{\omega'^2(x_i)} \left( \frac{\partial V}{\partial x_i} \right)^2(x), \end{aligned}$$

and so

$$I(x) \geq \sum_{i=1}^d \frac{1}{\omega'^2(x_i)} \left[ \frac{1}{10} \left( \frac{\partial V}{\partial x_i} \right)^2(x) - \frac{\partial^2 V}{\partial x_i^2}(x) \right] - \sum_{i=1}^d \frac{\omega^{(3)}(x_i)}{\omega'^3(x_i)}.$$

Since,  $\liminf_{|x| \rightarrow +\infty} I(x) = \liminf_{|y| \rightarrow +\infty} \frac{1}{2} |\nabla \tilde{V}|_2^2(y) - \Delta \tilde{V}(y)$  and  $\sum_{i=1}^d \frac{\omega^{(3)}(x_i)}{\omega'^3(x_i)} \leq dM$ , one concludes that  $\tilde{V}$  satisfies (3.5) as soon as

$$\liminf_{|x| \rightarrow +\infty} \sum_{i=1}^d \frac{1}{\omega'^2(x_i)} \left[ \frac{1}{10} \left( \frac{\partial V}{\partial x_i} \right)^2(x) - \frac{\partial^2 V}{\partial x_i^2}(x) \right] > dM.$$

Applying this latter condition to the probability measure  $\mu_u = (u \text{Id})^\# \mu$  (where  $\text{Id}$  is the identity function) which has density  $d\mu_u(x) = \frac{1}{u^d} e^{-V(x/u)} dx$  gives the condition of Proposition 3.5.  $\square$

#### 4. Transportation-cost inequalities

Let us recall the notation relative to this family of inequalities. A probability measure  $\mu$  satisfies the transportation-cost inequality with the cost function  $c(x, y)$  on  $\mathbb{R}^d$  if for all probability measure  $\nu$  on  $\mathbb{R}^d$ , the following holds:

$$\inf_{\pi \in P(\nu, \mu)} \int c(x, y) d\pi(x, y) \leq H(\nu|\mu), \quad (4.1)$$

where  $P(\nu, \mu)$  is the set of all probability measures on  $\mathbb{R}^d \times \mathbb{R}^d$  such that  $\pi(dx \times \mathbb{R}^d) = \nu(dx)$  and  $\pi(\mathbb{R}^d \times dy) = \mu(dy)$  and  $H(\nu|\mu)$  is the relative entropy of  $\nu$  with respect to  $\mu$ .

One writes for short that  $\mu$  satisfies the inequality  $\text{TC}(\omega, a)$  if there is some  $a > 0$  such that

$$\forall \nu \quad \inf_{\pi \in P(\nu, \mu)} \int \alpha(ad_\omega(x, y)) d\pi(x, y) \leq H(\nu|\mu),$$

with  $\alpha(u) = \min(|u|, u^2)$  and  $d_\omega(\cdot, \cdot)$  the distance defined by (1.7). The purpose of this section is to show that the inequalities  $\text{SG}(\omega, \cdot)$  are equivalent to transportation-cost inequalities  $\text{TC}(\omega, \cdot)$ .

Transportation-cost inequalities of the form  $\text{TC}(\omega, \cdot)$  are quite unusual. Let us define another family of transportation-cost inequalities appearing often in the literature (see [9,19,37]).

Let  $p \geq 1$ ; one says that  $\mu$  verifies the inequality  $\text{TC}_p(C)$  if

$$\text{when } p \in [1, 2], \forall \nu, \quad \inf_{\pi \in P(\nu, \mu)} \int \min(|x - y|_2^2, |x - y|_2^p) d\pi(x, y) \leq CH(\nu|\mu),$$

$$\text{when } p \in [2, +\infty), \forall \nu, \quad \inf_{\pi \in P(\nu, \mu)} \int \max(|x - y|_2^2, |x - y|_2^p) d\pi(x, y) \leq CH(\nu|\mu).$$

As we will see, the inequality  $\text{TC}_p(\cdot)$  is slightly weaker than the inequality  $\text{TC}(\omega_p, \cdot)$  (see the proof of Corollary 4.11). So in this case, our characterization of inequalities  $\text{TC}(\omega_p, \cdot)$  in terms of Poincaré inequalities brings new information and criteria for the study of the  $\text{TC}_p(\cdot)$ .

##### 4.1. Basic properties

Like Poincaré inequalities, transportation-cost inequalities enjoy a tensorization property and are related to Talagrand's concentration inequalities.

**Proposition 4.1 (Tensorization).** *Suppose that a probability measure  $\mu$  on a space  $\mathcal{X}$  satisfies the transportation-cost inequality (4.1) with the cost function  $c(x, y)$ , then  $\mu^n$  satisfies the transportation-cost inequality on  $\mathcal{X}^n$  with the cost function  $c^{\oplus n}(x, y) = \sum_{i=1}^n c(x_i, y_i)$ . In other words,*

$$\forall \nu \in \mathcal{P}(\mathcal{X}^n) \quad \inf_{\pi \in P(\nu, \mu^n)} \int \sum_{i=1}^n c(x_i, y_i) \, d\pi \leq H(\nu | \mu^n),$$

where  $P(\nu, \mu^n)$  is the set of probability measures on  $\mathcal{X}^n \times \mathcal{X}^n$  such that  $\pi(dx, \mathcal{X}^n) = \nu(dx)$  and  $\pi(\mathcal{X}^n, dy) = \mu^n(dy)$ .

This result goes back to the first works of Marton on the subject (see [28,29]). A proof can be found in [22].

Let us explain how to derive concentration inequalities from the inequality  $\mathbb{T}\mathbb{C}(\omega, a)$ .

**Proposition 4.2.** *If  $\mu$  satisfies the transportation-cost inequality  $\mathbb{T}\mathbb{C}(\omega, a)$ , then for all  $n \geq 1$  and all  $A \subset (\mathbb{R}^d)^n$ ,*

$$\forall h \geq 0 \quad \mu^n(A + B_\omega(h)) \geq 1 - \frac{1}{\mu^n(A)} e^{-ha(a/\sqrt{d})/2},$$

where  $B_\omega(h)$  is defined as in Proposition 2.4.

**Remark 4.3.** *According to Theorem 4.6 below, if  $\mu$  satisfies the inequality  $\mathbb{S}\mathbb{G}(\omega, C)$  then it satisfies  $\mathbb{T}\mathbb{C}(\omega, a)$  with  $a = \frac{1}{\sqrt{C\kappa}}$ . With this value of  $a$  the concentration inequality given by Proposition 4.2 is almost the same as the one derived in Proposition 2.4.*

We will need the following lemma:

**Lemma 4.4.** *The function  $\alpha(u) = \min(|u|, u^2)$  is such that  $\alpha(x + y) \leq 2(\alpha(x) + \alpha(y))$ , for all  $x, y \geq 0$ .*

**Proof.** If  $x + y \leq 1$ , then  $\alpha(x + y) = (x + y)^2 \leq 2(x^2 + y^2) = 2(\alpha(x) + \alpha(y))$ .

Now, suppose that  $x + y \geq 1$ .

If  $x \leq 1$  and  $y \leq 1$ , then  $\alpha(x + y) = x + y \leq (x + y)^2 \leq 2(x^2 + y^2) = 2(\alpha(x) + \alpha(y))$ .

If  $x \leq 1$  and  $y \geq 1$ , then  $x \leq y \Rightarrow x - 2x^2 \leq y \Rightarrow x + y \leq 2(x^2 + y) \Rightarrow \alpha(x + y) \leq 2(\alpha(x) + \alpha(y))$ .

If  $x \geq 1$  and  $y \geq 1$ , then  $\alpha(x + y) = x + y = \alpha(x) + \alpha(y) \leq 2(\alpha(x) + \alpha(y))$ . □

**Proof of Proposition 4.2.** If  $\mu$  satisfies  $\mathbb{T}\mathbb{C}(\omega, a)$  on  $\mathbb{R}^d$  then according to Proposition 4.1,  $\mu^n$  satisfies the transportation-cost inequality on  $(\mathbb{R}^d)^n$  with the cost function  $c$  defined by

$$c : ((x_1, \dots, x_n), (y_1, \dots, y_n)) \in (\mathbb{R}^d)^n \times (\mathbb{R}^d)^n \mapsto \sum_{i=1}^n \alpha(ad_\omega(x_i, y_i)).$$

Using the triangle inequality for the metric  $d_\omega(\cdot, \cdot)$  and Lemma 4.4, one has

$$\forall x, y, z \in (\mathbb{R}^d)^n \quad c(x, z) \leq 2c(x, y) + 2c(y, z).$$

Now, let  $\nu_1$  and  $\nu_2$  be two probability measures on  $(\mathbb{R}^d)^n$ . Take  $\pi_1 \in P(\nu_1, \mu^n)$  and  $\pi_2 \in P(\mu^n, \nu_2)$ , then one can construct three random variables  $X, Y, Z$  such that  $\mathcal{L}(X, Y) = \pi_1$  and  $\mathcal{L}(Y, Z) = \pi_2$  (see, for instance, the Gluing lemma of [38], p. 208). Then, one has

$$\begin{aligned} \mathcal{I}_c(\nu_1, \nu_2) &\leq \mathbb{E}[c(X, Z)] \leq 2\mathbb{E}[c(X, Y)] + 2\mathbb{E}[c(Y, Z)] \\ &= 2 \int c(x, y) \, d\pi_1(x, y) + 2 \int c(y, z) \, d\pi_2(y, z). \end{aligned}$$

Optimizing on  $\pi_1$  and  $\pi_2$  gives

$$\mathcal{I}_c(\nu_1, \nu_2) \leq 2\mathcal{I}_c(\nu_1, \mu^n) + 2\mathcal{I}_c(\nu_2, \mu^n).$$

Consequently,  $\mu^n$  satisfies the following symmetrized transportation-cost inequality: for all  $\nu_1, \nu_2$  probability measures on  $(\mathbb{R}^d)^n$ ,

$$\mathcal{T}_c(\nu_1, \nu_2) \leq 2\mathsf{H}(\nu_1|\mu^n) + 2\mathsf{H}(\nu_2|\mu^n).$$

Take  $d\nu_1 = \mathbb{1}_A d\mu^n/\mu^n(A)$  and  $d\nu_2 = \mathbb{1}_{\tilde{A}} d\mu^n/\mu^n(\tilde{A})$ , for some  $A, \tilde{A} \subset (\mathbb{R}^d)^n$ , then

$$\begin{aligned} \inf_{x \in A, y \in \tilde{A}} c(x, y) &\leq \mathcal{T}_c(\nu_1, \nu_2) \leq 2\mathsf{H}(\nu_1|\mu^n) + 2\mathsf{H}(\nu_2|\mu^n) \\ &= 2\log(1/\mu^n(A)) + 2\log(1/\mu^n(\tilde{A})). \end{aligned}$$

Letting  $c(A, \tilde{A}) = \inf_{x \in A, y \in \tilde{A}} c(x, y)$ , one gets

$$\mu^n(A)\mu^n(\tilde{A}) \leq e^{-c(A, \tilde{A})/2}.$$

Defining

$$\tilde{A} = \left\{ y: \inf_{x \in A} c(x, y) > \alpha(a/\sqrt{d})h \right\}$$

one gets  $\mu^n(\tilde{A}) \leq \frac{1}{\mu^n(A)} e^{-\alpha(a/\sqrt{d})h/2}$ . To obtain the announced inequality it is thus enough to compare  $A + B_\omega(h)$  and  $\tilde{A}$ . Take  $x = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$  and  $y = (y_1, \dots, y_n) \in (\mathbb{R}^d)^n$ ; then for all  $i \in 1, \dots, n$ , one has

$$\begin{aligned} \alpha(ad_\omega(x_i, y_i)) &\stackrel{(a)}{\geq} \alpha\left(\frac{a}{\sqrt{d}} \sum_{j=1}^d |\omega(x_{i,j}) - \omega(y_{i,j})|\right) \stackrel{(b)}{\geq} \sum_{j=1}^d \alpha\left(\frac{a}{\sqrt{d}} |\omega(x_{i,j}) - \omega(y_{i,j})|\right) \\ &\stackrel{(c)}{\geq} \sum_{j=1}^d \alpha\left(\frac{a}{\sqrt{d}} \omega\left(\frac{x_{i,j} - y_{i,j}}{2}\right)\right) \stackrel{(d)}{\geq} \alpha(a/\sqrt{d}) \sum_{j=1}^d \alpha \circ \omega\left(\frac{x_{i,j} - y_{i,j}}{2}\right), \end{aligned}$$

where (a) follows from the comparison between the norms  $|\cdot|_2$  and  $|\cdot|_1$  in  $\mathbb{R}^d$ , (b) from the super additivity of  $\alpha$ , (c) from Lemma 2.6(1) and (d) from Lemma 2.6(2).

Consequently, if  $y \notin A + B_\omega(h)$ , then  $\inf_{x \in A} \sum_{i=1}^n \sum_{j=1}^d \alpha \circ \omega\left(\frac{x_{i,j} - y_{i,j}}{2}\right) \geq h$ , and so  $y$  belongs to  $\tilde{A}$ . From this follows that  $\mu^n(A + B_\omega(h)) \geq \mu^n(\tilde{A}^c) \geq 1 - \frac{1}{\mu^n(A)} e^{-\alpha(a/\sqrt{d})h/2}$ , which completes the proof.  $\square$

**Remark 4.5.** The idea of deriving concentration estimates from transportation-cost inequalities goes back to Marton's seminal work [28]. The above proof is essentially due to Talagrand (see the proof of [37], Corollary 1.3).

#### 4.2. Links with Poincaré inequalities

**Theorem 4.6.** Let  $\mu$  be a probability measure on  $\mathbb{R}^d$  absolutely continuous with respect to Lebesgue measure. Then  $\mu$  satisfies the Poincaré inequality  $\mathbb{S}\mathbb{G}(\omega, C)$  for some  $C > 0$  if and only if it satisfies the transportation-cost inequality  $\mathbb{T}\mathbb{C}(\omega, a)$  for some  $a > 0$ .

More precisely:

- If  $\mu$  satisfies  $\mathbb{S}\mathbb{G}(\omega, C)$  then it satisfies  $\mathbb{T}\mathbb{C}(\omega, \frac{1}{\sqrt{C\kappa}})$ , with  $\kappa = \sqrt{18e\sqrt{5}}$ .
- If  $\mu$  satisfies the inequality  $\mathbb{T}\mathbb{C}(\omega, a)$ , then  $\mu$  satisfies the inequality  $\mathbb{S}\mathbb{G}(\omega, \frac{1}{2a^2})$ .

The proof of Theorem 4.6 relies on two ingredients. The first one is the following result by Bobkov, Gentil and Ledoux ([9], Corollary 5.1):

**Theorem 4.7 (Bobkov, Gentil and Ledoux).** *If an absolutely continuous probability measure  $\mu$  satisfies the inequality  $\mathbb{S}\mathbb{G}(C)$  on  $\mathbb{R}^d$  then it satisfies the transportation-cost inequality for the cost function  $(x, y) \mapsto \alpha_s(|x - y|_2)$  for all  $s < \frac{2}{\sqrt{C}}$ , where*

$$\alpha_s(t) = \begin{cases} \frac{t^2}{4L(s)} & \text{if } |t| \leq 2L(s)s, \\ s|t| - L(s)s^2 & \text{otherwise,} \end{cases} \quad \text{with } L(s) = \frac{C}{2} \left( \frac{2 + \sqrt{Cs}}{2 - \sqrt{Cs}} \right)^2 e^{s\sqrt{5C}}.$$

In particular, if one takes  $s = \frac{1}{\sqrt{C}}$ , then it is easy to check that  $\alpha_s(t) \geq \alpha(\frac{t}{\sqrt{C\kappa}})$ , where  $\alpha(u) = \min(|u|, u^2)$  and  $\kappa = \sqrt{18e\sqrt{5}}$ . Thus if  $\mu$  satisfies  $\mathbb{S}\mathbb{G}(C)$  it satisfies the transportation-cost inequality with the cost function  $(x, y) \mapsto \alpha(\frac{|x-y|_2}{\sqrt{C\kappa}})$ . In other words, with the definition of the transportation-cost inequality  $\mathbb{T}\mathbb{C}(\omega, a)$ , the preceding result can be restated as follows:

**Corollary 4.8.** *If  $\mu$  is an absolutely continuous probability measure on  $\mathbb{R}^d$  satisfying the classical Poincaré inequality  $\mathbb{S}\mathbb{G}(C)$  for some  $C > 0$ , then it satisfies the transportation-cost inequality  $\mathbb{T}\mathbb{C}(\text{Id}, \frac{1}{\sqrt{C\kappa}})$ . (Where  $\text{Id}: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto x$  is the identity function.)*

The converse is also true:

**Proposition 4.9.** *If  $\mu$  satisfies  $\mathbb{T}\mathbb{C}(\text{Id}, a)$ , for some  $a > 0$ , then  $\mu$  satisfies the inequality  $\mathbb{S}\mathbb{G}(\frac{1}{2a^2})$ .*

The proof of Proposition 4.9 is classical and can be found in various places (see e.g. the proofs of [9], Corollary 5.1, or [30], Corollary 3).

The second argument is a very simple contraction principle:

**Proposition 4.10.** *Let  $\mu$  be a probability measure on a metric space  $\mathcal{X}$ ; if  $\mu$  satisfies the transportation-cost inequality with the cost function  $c: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+$ , and if  $T: \mathcal{X} \rightarrow \mathcal{Y}$  is a measurable bijection then,  $T^\# \mu$  satisfies the transportation-cost inequality with the cost function  $(x, y) \mapsto c(T^{-1}(x), T^{-1}(y))$ .*

This contraction principle goes back to Maurey's work on infimum convolution inequalities (see [30]). A proof can also be found in [21], where this simple property was intensively used to derive necessary and sufficient conditions for transportation-cost inequalities on the real line.

Now let us apply the contraction principle together with Theorem 4.7 to prove that Poincaré inequalities  $\mathbb{S}\mathbb{G}(\omega, \cdot)$  and transportation-cost inequalities  $\mathbb{T}\mathbb{C}(\omega, \cdot)$  are qualitatively equivalent.

**Proof of Theorem 4.6.** If  $\mu$  satisfies  $\mathbb{S}\mathbb{G}(\omega, C)$ , then according to Proposition 3.1,  $\omega^\# \mu$  satisfies the classical Poincaré inequality  $\mathbb{S}\mathbb{G}(C)$ , and according to Corollary 4.8, this implies that  $\omega^\# \mu$  satisfies  $\mathbb{T}\mathbb{C}(\text{Id}, a)$ , with  $a = \frac{1}{\sqrt{C\kappa}}$ . According to the contraction principle,  $\mu$  (which is the image of  $\omega^\# \mu$  under the map  $\omega^{-1}$ ) satisfies the transportation-cost inequality with the cost function  $(x, y) \mapsto \alpha(a|\omega(x) - \omega(y)|_2) = \alpha(ad_\omega(x, y))$  by definition of the metric  $d_\omega(\cdot, \cdot)$  (see (1.7)).

Now suppose that  $\mu$  satisfies  $\mathbb{T}\mathbb{C}(\omega, a)$  for some  $a > 0$ . According to the contraction principle,  $\omega^\# \mu$  satisfies  $\mathbb{T}\mathbb{C}(\text{Id}, a)$ , and according to Proposition 4.9, this implies that  $\omega^\# \mu$  satisfies  $\mathbb{S}\mathbb{G}(\frac{1}{2a^2})$ . Using Proposition 3.1, one concludes that  $\mu$  satisfies  $\mathbb{S}\mathbb{G}(\omega, \frac{1}{2a^2})$ . This concludes the proof.  $\square$

**Corollary 4.11.** *If an absolutely continuous probability measure  $\mu$  verifies the inequality  $\mathbb{S}\mathbb{G}(\omega_p, C)$  on  $\mathbb{R}^d$ , for some  $C$  and  $p \geq 1$ , then:*

- if  $p \in [1, 2]$  it satisfies the transportation-cost inequality

$$\forall \nu \quad \inf_{\pi \in P(\nu, \mu)} \int \min(|x - y|_2^2, |x - y|_2^p) d\pi(x, y) \leq \frac{4}{\alpha(1/(\sqrt{C}d\kappa))} H(\nu|\mu);$$

- if  $p \geq 2$  it satisfies the transportation-cost inequality

$$\forall \nu \quad \inf_{\pi \in P(\nu, \mu)} \int \max(|x - y|_2^2, |x - y|_p^p) d\pi(x, y) \leq \frac{2^p}{\alpha(1/(\sqrt{Cd}\kappa))} H(\nu|\mu).$$

**Proof.** Let  $c_p(x, y) = \sum_{i=1}^d \alpha \circ \omega_p(\frac{x_i - y_i}{2})$ . During the proof of Proposition 4.2, we have shown that

$$\alpha(a/\sqrt{d})c_p(x, y) \leq \alpha(ad_{\omega_p}(x, y)).$$

So, if  $\mu$  satisfies the inequality  $\mathbb{TC}(\omega_p, a)$ , it satisfies the transportation-cost inequality with the cost function  $\alpha(a/\sqrt{d})c_p(x, y)$ .

For  $p \in [1, 2]$ , the function  $\alpha \circ \omega_p(\sqrt{\cdot})$  is concave, so

$$c_p(x, y) \geq \alpha \circ \omega_p(|x - y|_2/2) \geq 1/4 \min(|x - y|_2^2, |x - y|_p^p).$$

For  $p \geq 2$ ,

$$c_p(x, y) \geq \max(1/4|x - y|_2^2, 1/2^p|x - y|_p^p) \geq 1/2^p \max(|x - y|_2^2, |x - y|_p^p).$$

The result follows from Theorem 4.6. □

**Remark 4.12.** In particular, the inequality  $\mathbb{SG}(\omega_2, \cdot)$  implies  $\mathbb{TC}(\omega_2, \cdot)$  which is stronger than Talagrand's  $\mathbb{T}_2$  inequality, that is to say the transportation-cost inequality with a cost function of the form  $(x, y) \mapsto a|x - y|_2^2$  for some  $a > 0$ . The transportation-cost inequalities  $\mathbb{T}_2$  and  $\mathbb{TC}(\omega_2, \cdot)$  seem to be very close; we do not know if they are equivalent.

### 4.3. Links with inf-convolution inequalities

Transportation-cost inequalities are closely related to another type of inequalities introduced by Maurey in [30], the so called inf-convolution inequalities.

Let us say that a probability measure  $\mu$  on a metric space  $\mathcal{X}$  satisfies the inf-convolution inequality with the cost function  $c: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+$ , if the following holds for all measurable nonnegative functions  $f: \mathcal{X} \rightarrow \mathbb{R}^+$ :

$$\int e^{Q_c f} d\mu \cdot \int e^{-f} d\mu \leq 1, \tag{4.2}$$

where the inf-convolution operator  $Q_c$  is defined by

$$Q_c f(x) = \inf_{y \in \mathcal{X}} \{f(y) + c(x, y)\}. \tag{4.3}$$

One will say that a probability measure  $\mu$  on  $\mathbb{R}^d$  satisfies the inf-convolution inequality  $\mathbb{IC}(\omega, a)$  if it satisfies the inf-convolution inequality (4.2) with the cost function  $c(x, y) = \alpha(ad_{\omega}(x, y))$ .

The inequalities  $\mathbb{TC}(\omega, \cdot)$  and  $\mathbb{IC}(\omega, \cdot)$  are qualitatively equivalent, as shown by the following proposition:

**Proposition 4.13.** *If  $\mu$  verifies the inequality  $\mathbb{IC}(\omega, a)$  then it verifies the inequality  $\mathbb{TC}(\omega, a)$ . Conversely, if  $\mu$  verifies the inequality  $\mathbb{TC}(\omega, a)$  then it verifies the inf-convolution inequality with the cost function  $2\alpha(\frac{a}{2}d_{\omega}(x, y))$ ; in particular, it satisfies the inequality  $\mathbb{IC}(\omega, \frac{a}{2})$ .*

**Proof.** Let  $Q^a f(x) = \inf_{y \in X} \{f(y) + \alpha(ad_{\omega}(x, y))\}$ . If  $\mu$  verifies the inequality  $\mathbb{IC}(\omega, a)$  then, applying Jensen inequality, it holds:

$$\int e^{Q^a f} d\mu \leq e^{\int f d\mu} \tag{4.4}$$

for all bounded measurable  $f: \mathbb{R}^d \rightarrow \mathbb{R}$ . According to [22], Corollary 1, this latter inequality is equivalent to the transportation-cost inequality  $\mathbb{TC}(\omega, a)$ .

Conversely, suppose that  $\mu$  verifies the transportation-cost inequality  $\mathbb{TC}(\omega, a)$ . According to [22], Corollary 1, the inequality (4.4) holds. Applying (4.4) to  $Q^a f$  instead of  $f$ , one gets

$$\int e^{Q^a(Q^a f)} d\mu \cdot e^{-\int Q^a f d\mu} \leq 1$$

and applying again (4.4) with  $-Q^a f$  instead of  $f$ , one gets

$$\int e^{Q^a(-Q^a(f))} d\mu \cdot e^{\int Q^a f d\mu} \leq 1.$$

Multiplying these two inequalities yields to

$$\int e^{Q^a(Q^a f)} d\mu \cdot \int e^{Q^a(-Q^a(f))} d\mu \leq 1.$$

Now, for all  $x, y \in \mathbb{R}^d$ , one has:  $-f(x) + Q^a f(y) \leq \alpha(ad_\omega(x, y))$ , and consequently,  $-f(x) \leq Q^a(-Q^a(f))(x)$ . Plugging this into the last inequality gives

$$\int e^{Q^a(Q^a f)} d\mu \cdot \int e^{-f} d\mu \leq 1.$$

An easy computation gives:

$$Q^a(Q^a f)(x) = \inf_{y \in \mathbb{R}^d} \left\{ f(y) + 2\alpha\left(\frac{a}{2}d_\omega(x, y)\right) \right\}.$$

This completes the proof. □

The following corollary is an immediate consequence of Theorem 4.6:

**Corollary 4.14.** *Let  $\mu$  be a probability measure on  $\mathbb{R}^d$  absolutely continuous with respect to Lebesgue measure with a positive density. Then  $\mu$  satisfies the Poincaré inequality  $\mathbb{SG}(\omega, C)$  for some  $C > 0$  if and only if it satisfies the inequality  $\mathbb{IC}(\omega, a)$  for some  $a > 0$ .*

More precisely:

- If  $\mu$  satisfies  $\mathbb{SG}(\omega, C)$  then it satisfies  $\mathbb{IC}(\omega, \frac{1}{2\sqrt{C\kappa}})$ , with  $\kappa = \sqrt{18e\sqrt{5}}$ .
- If  $\mu$  satisfies the inequality  $\mathbb{IC}(\omega, a)$ , then  $\mu$  satisfies the inequality  $\mathbb{SG}(\omega, \frac{1}{2a^2})$ .

### 5. Comparison with other functional inequalities

In this section, one shows that the Poincaré inequalities  $\mathbb{SG}(\omega, \cdot)$  are weaker than super Poincaré inequalities.

Let us recall that  $\mu$  verifies the super Poincaré inequality  $\mathbb{SP}(\beta)$  if for every locally Lipschitz  $f$  on  $\mathbb{R}^d$ , one has

$$\forall s \geq 1 \quad \int f^2 d\mu \leq \beta(s) \int |\nabla f|_2^2 d\mu + s \left( \int |f| d\mu \right)^2, \tag{5.1}$$

where  $\beta: [1, +\infty) \rightarrow \mathbb{R}^+$  is nonincreasing.

**Remark 5.1.** *Of course super Poincaré inequalities are stronger than the classical Poincaré inequality. Namely, if  $\mu$  satisfies  $\mathbb{SP}(\beta)$ , then  $\mu$  verifies  $\mathbb{SG}(2\beta(1))$ . Indeed, taking  $s = 1$  in (5.1) and applying it to  $(f - m)_+$ , where  $m$*

denotes the median of the function  $f$ , gives:

$$\begin{aligned} \int_{f \geq m} (f - m)^2 d\mu &\leq \beta(1) \int_{f \geq m} |\nabla f|_2^2 d\mu + \left( \int_{f \geq m} f - m d\mu \right)^2 \\ &\leq \beta(1) \int_{f \geq m} |\nabla f|_2^2 d\mu + \frac{1}{2} \int_{f \geq m} (f - m)^2 d\mu. \end{aligned}$$

Thus,  $\int_{f \geq m} (f - m)^2 d\mu \leq 2\beta(1) \int_{f \geq m} |\nabla f|_2^2 d\mu$ . Doing the same with  $(f - m)_-$  yields,  $\int_{f \leq m} (f - m)^2 d\mu \leq 2\beta(1) \int_{f \leq m} |\nabla f|_2^2 d\mu$ . Adding these inequalities gives  $\int (f - m)^2 d\mu \leq 2\beta(1) \int |\nabla f|_2^2 d\mu$ . Since  $\text{Var}_\mu(f) \leq \int (f - m)^2 d\mu$ , this concludes the proof.

### 5.1. Concentration involved by super Poincaré

As noted by Wang in [39], Theorems 6.1 and 6.2, super Poincaré inequalities imply concentration results. This is recalled in the following proposition.

**Proposition 5.2.** *Suppose that  $\mu$  verifies (5.1) with a continuous decreasing function  $\beta$  such that  $\beta(s) \rightarrow 0$  when  $s$  goes to  $+\infty$  and define  $a = 1/\sqrt{2\beta(1)}$ , then for all 1-Lipschitz function  $f$  on  $\mathbb{R}^d$  such that  $\int f d\mu = 0$ , one has:*

$$\forall \lambda \geq 0 \quad \int e^{\lambda f} d\mu \leq \exp\left(\lambda \int_0^\lambda \phi(t \vee a) dt\right),$$

where the function  $\phi$  is defined by

$$\forall t > 0 \quad \phi(t) = \frac{1}{t^2} \log\left(2\beta^{-1}\left(\frac{1}{2t^2}\right)\right).$$

As a consequence, defining for all  $\lambda \geq 0$ ,  $\Lambda_\beta(\lambda) = \lambda \int_0^\lambda \phi(t \vee a) dt$  and for all  $t \geq 0$ ,  $\Lambda_\beta^*(t) = \sup_{\lambda \geq 0} \{\lambda t - \Lambda_\beta(\lambda)\}$ , one has

$$\forall t \geq 0 \quad \mu(f \geq t) \leq e^{-\Lambda_\beta^*(t)}.$$

Moreover, the inverse function of  $\Lambda_\beta^*$  can be expressed as follows

$$\forall t \geq 0 \quad \Lambda_\beta^{*-1}(t) = \int_0^t \psi(u) du,$$

where  $\psi : (0, +\infty] \rightarrow \mathbb{R}^+$  is defined by:

$$\psi(t) = \begin{cases} \sqrt{\frac{2\log(2)\beta(1)}{t}} & \text{if } t \leq \log(2), \\ \sqrt{2\beta\left(\frac{e^t}{2}\right)} & \text{if } t \geq \log(2). \end{cases}$$

The observation concerning the inverse of  $\Lambda_\beta^*$  seems to be new and will be very useful in the sequel. The proof below is simpler than the one proposed by Wang in [39].

**Proof of Proposition 5.2.** Let  $f$  be a 1-Lipschitz function with  $\int f d\mu = 0$ ; define  $Z(\lambda) = \int e^{\lambda f} d\mu$  and  $\Lambda(\lambda) = \log Z(\lambda)$ . Applying (5.1) to the function  $e^{\lambda f}$  yields:

$$Z(2\lambda) \leq \lambda^2 \beta(s) Z(2\lambda) + s Z(\lambda)^2.$$

So, if  $s > \beta^{-1}(1/\lambda^2)$ , one easily gets

$$\Lambda(2\lambda) \leq \log\left(\frac{s}{1 - \lambda^2\beta(s)}\right) + 2\Lambda(\lambda).$$

Since the function  $\Lambda$  is convex, one has  $\Lambda(2\lambda) \geq \Lambda(\lambda) + \lambda\Lambda'(\lambda)$ , and so

$$\left[\frac{\Lambda(\lambda)}{\lambda}\right]' = \frac{\lambda\Lambda'(\lambda) - \Lambda(\lambda)}{\lambda^2} \leq \frac{1}{\lambda^2} \log\left(\frac{s}{1 - \lambda^2\beta(s)}\right). \tag{5.2}$$

If  $\lambda < 1/\sqrt{2\beta(1)} = a$ , then taking  $s = 1$  in (5.2) yields

$$\left[\frac{\Lambda(\lambda)}{\lambda}\right]' \leq -\frac{1}{\lambda^2} \log(1 - \lambda^2\beta(1)) \leq 2\log(2)\beta(1) = \phi(a).$$

If  $\lambda \geq 1/\sqrt{2\beta(1)} = a$ , then taking  $s = \beta^{-1}(\frac{1}{2\lambda^2})$  in (5.2) gives

$$\left[\frac{\Lambda(\lambda)}{\lambda}\right]' \leq \phi(\lambda).$$

So, for all  $\lambda > 0$ ,  $[\frac{\Lambda(\lambda)}{\lambda}]' \leq \phi(\lambda \vee a)$ ; since  $\Lambda(\lambda)/\lambda \rightarrow_{\lambda \rightarrow 0} 0$  one gets the result.

The inequality  $\mu(f \geq t) \leq e^{-\Lambda_\beta^*(t)}$  follows at once from the preceding using routine arguments.

Now, let us prove the claim concerning the inverse of  $\Lambda_\beta^*$ . It is easy to check that

$$\int_0^\lambda \phi(u \vee a) \, du = \int_{1/\lambda}^{+\infty} \frac{1}{u^2} \phi\left(\frac{1}{u} \vee a\right) \, du = \int_{1/\lambda}^{+\infty} \psi^{-1}(u) \, du = - \int_0^{\psi^{-1}(1/\lambda)} v\psi'(v) \, dv.$$

Now integrating by part yields

$$\int_0^{\psi^{-1}(1/\lambda)} v\psi'(v) \, dv = \frac{\psi^{-1}(1/\lambda)}{\lambda} - \int_0^{\psi^{-1}(1/\lambda)} \psi(u) \, du.$$

Let  $h(\lambda) = \lambda \int_0^t \psi(u) \, du - \Lambda_\beta(\lambda)$ , then

$$h(\lambda) = \lambda \int_{\psi^{-1}(1/\lambda)}^t \psi(u) \, du + \psi^{-1}(1/\lambda) = \lambda \int_{\psi^{-1}(1/\lambda)}^t (\psi(u) - 1/\lambda) \, du + t.$$

Observing that  $\psi$  is decreasing and  $\lambda \mapsto \psi^{-1}(1/\lambda)$  is increasing, it is easy to check that the integral term above is always nonpositive and vanishes when  $\lambda = 1/\psi(t)$ . One concludes that

$$\sup_{\lambda \geq 0} h(\lambda) = \Lambda_\beta^*\left(\int_0^t \psi(u) \, du\right) = t,$$

which concludes the proof. □

**Lemma 5.3.** *Suppose that  $\beta : [1, +\infty) \rightarrow \mathbb{R}^+$  is a continuous decreasing function such that  $s \mapsto s\beta(s)$  is nondecreasing on  $[1, +\infty)$  and define  $\omega_\beta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  as follows:*

$$\forall t \geq 0 \quad \omega_\beta^{-1}(t) = 4 \int_0^t \sqrt{\beta(e^u)} \, du. \tag{5.3}$$

Then one has

$$\forall t \geq 0 \quad \alpha \circ \omega_\beta(t) \leq \Lambda_\beta^*(t) \leq \alpha \circ \omega_\beta(5t), \tag{5.4}$$

where  $\alpha(t) = \min(t^2, t)$  for all  $t \geq 0$ .

**Proof.** Let us prove the lower bound in (5.4). According to Proposition 5.2, this inequality is equivalent to the following one

$$\forall t \geq 0 \quad \Lambda_\beta^{*-1}(\alpha(t)) = \int_0^{t^2 \wedge t} \psi(u) \, du \leq 4 \int_0^t \sqrt{\beta(e^u)} \, du, \quad (5.5)$$

where the function  $\psi$  is defined in Proposition 5.2. In fact a slightly better inequality holds true:

$$\forall t \geq 0 \quad \Lambda_\beta^{*-1}(\alpha(t)) = \int_0^{t^2 \wedge t} \psi(u) \, du \leq 2\sqrt{2} \int_0^t \sqrt{\beta(e^u/2)} \, du, \quad (5.6)$$

with the convention  $\beta(s) = \beta(1)$ , when  $s \in [0, 1]$ . Since the function  $s\beta(s)$  is nondecreasing on  $[0, +\infty)$  it is easy to check that  $\beta(e^u/2) \leq 2\beta(e^u)$ , and so (5.6) implies (5.5). To prove (5.6), let us distinguish the following cases:

- If  $t \leq \log(2)$ , then  $\Lambda_\beta^{*-1}(\alpha(t)) = \Lambda_\beta^{*-1}(t^2) = 2\sqrt{2\log(2)\beta(1)t} \leq 2\sqrt{2\beta(1)t} = 2\sqrt{2} \int_0^t \sqrt{\beta(e^u/2)} \, du$ .
- If  $\log(2) \leq t$ , then

$$\begin{aligned} \Lambda_\beta^{*-1}(\alpha(t)) &\leq \Lambda_\beta^{*-1}(t) = 2\sqrt{2\beta(1)\log(2)} + \int_{\log(2)}^t \sqrt{2\beta(e^u/2)} \, du \\ &\leq 2 \int_0^{\log(2)} \sqrt{2\beta(e^u/2)} \, du + \int_{\log(2)}^t \sqrt{2\beta(e^u/2)} \, du \\ &\leq 2\sqrt{2} \int_0^t \sqrt{\beta(e^u/2)} \, du. \end{aligned}$$

The proof of the upper bound in (5.4) is similar and left to the reader. □

**Examples.** Let  $p \geq 1$ , and define  $\beta_p(s) = \log(e+s)^{2(1/p-1)}$  (which verifies the condition  $s\beta_p(s)$  increasing according to Lemma 5.14). Then, one can show that

$$\forall t \geq 0 \quad \omega_p\left(\frac{t}{4p(2^{1/p}-1)}\right) \geq \omega_{\beta_p}(t) \geq \omega_p\left(\frac{t}{4p}\right), \quad (5.7)$$

where  $\omega_p(u) = u \vee u^p$  for all  $u \geq 0$ . In particular, if  $\mu$  verifies inequality  $\mathbb{SP}(C\beta_p)$  for some  $C > 0$ , then one has

$$\forall t \geq 0 \quad \mu\left(|x|_2 \geq t + \int |x|_2 \, d\mu\right) \leq e^{-\alpha \omega_p(t/(4\sqrt{C}p))},$$

and this implies that  $\int e^{\varepsilon|x|_2^p} \, d\mu < +\infty$  for some  $\varepsilon > 0$ . Since the probability measure  $d\nu_p(x) = \frac{1}{Z_p} e^{-|x|^p} \, dx$  verifies  $\mathbb{SP}(C\beta_p)$ , for some  $C > 0$ , one concludes that the function  $\omega_{\beta_p}$  gives the right order of concentration. We think that more generally the function  $\omega_\beta$  is of the right order.

Now we can state our main result:

**Theorem 5.4.** Let  $\beta : [1, +\infty) \rightarrow \mathbb{R}^+$  be a continuous decreasing function such that  $s \mapsto s\beta(s)$  is increasing and such that there is some  $\lambda \geq 4$  for which the following holds

$$\forall s \geq 1 \quad \lambda\beta(\lambda s) \geq 4\beta(s). \quad (5.8)$$

If a probability measure  $\mu$  on  $\mathbb{R}^d$  verifies the super Poincaré inequality  $\mathbb{SP}(\beta)$ , then there is some  $a > 0$  such that  $\mu$  verifies  $\mathbb{SG}(\omega_\beta(\cdot/a), 4\lambda^2)$ , where  $\omega_\beta$  is defined by (5.3) for  $t \geq 0$  and extended to  $\mathbb{R}^-$  by  $\omega_\beta(t) = -\omega_\beta(-t)$ , for  $t \neq 0$ . One can take

$$a = \max(\lambda, \Lambda_\beta^*(m)),$$

where  $m = \int |x|_2 d\mu$ .

Moreover, under the same assumptions, the probability measure  $\mu$  verifies the centered Poincaré inequality  $\mathbb{S}\mathbb{G}(\omega_\beta(\cdot/\bar{a}), 4\lambda^2)$  (see Definition 2.8) with

$$\bar{a} = \max(\lambda, \Lambda_\beta^*(\sqrt{2\beta(1/d)}).$$

The constant  $\bar{a}$  above depends only on  $\beta$  and enjoys the following invariant property: if  $\beta$  is replaced by  $t\beta$  with  $t > 0$ , then  $\bar{a}$  is unchanged.

Finally, under the same assumptions, the probability measure  $\mu$  verifies the following transportation-cost inequality

$$\inf_{\pi \in P(v, \mu)} \int \sum_{i=1}^d \alpha \circ \omega_\beta \left( \frac{|x_i - y_i|}{2\bar{a}} \right) d\pi(x, y) \leq \frac{1}{\alpha(1/(2\lambda\kappa\sqrt{d}))} H(v|\mu)$$

for all probability measure  $v$  on  $\mathbb{R}^d$ .

**Remark 5.5.** The assumptions concerning  $\beta$  are not very restrictive. They are in particular fulfilled by the functions  $\beta_p(s) = \log(e + s)^{2(1/p-1)}$  (see Lemma 5.14). The last part of Theorem 5.4 can be seen as a generalization of Otto and Villani result concerning Talagrand’s  $\mathbb{T}_2$  inequality.

### 5.2. A capacity measure criterion for super Poincaré inequality

Our approach to compare the inequalities  $\mathbb{S}\mathbb{G}(\omega, \cdot)$  to the super Poincaré inequalities relies on the capacity-measure results of Barthe, Cattiaux and Roberto [5,6].

Let us recall the definition of a capacity-measure inequality (a good reference for this type of inequalities is the book of Mazja [31]).

**Definition 5.6.** Let  $\mu$  be a probability measure on  $\mathbb{R}^d$ . Let  $A \subset \Omega$  be Borel sets. One defines

$$\text{Cap}_\mu(A, \Omega) = \inf \left\{ \int |\nabla f|_2^2 d\mu : \mathbb{1}_A \leq f \leq \mathbb{1}_\Omega \right\}.$$

The capacity of a set  $A$  with  $\mu(A) \leq 1/2$  is defined by

$$\begin{aligned} \text{Cap}_\mu(A) &= \inf \{ \text{Cap}_\mu(A, \Omega) : A \subset \Omega \text{ and } \mu(\Omega) \leq 1/2 \} \\ &= \inf \left\{ \int |\nabla f|_2^2 d\mu : f : \mathbb{R}^d \rightarrow [0, 1], f|_A = 1 \text{ and } \mu(f = 0) \geq 1/2 \right\}. \end{aligned}$$

One says that  $\mu$  satisfies a capacity-measure inequality if there is a function  $\Psi : [0, 1] \rightarrow \mathbb{R}^+$  such that for all  $A$  with  $\mu(A) \leq 1/2$ ,

$$\Psi(\mu(A)) \leq \text{Cap}_\mu(A).$$

Many functional inequalities admit a transcription in terms of capacity measure. The simplest example is the classical Poincaré inequality on  $\mathbb{R}^d$ .

**Theorem 5.7.** A probability measure  $\mu$  on  $\mathbb{R}^d$  verifies the inequality  $\mathbb{S}\mathbb{G}(C)$  for some  $C > 0$  if and only if there is some  $D > 0$  such that for all  $A \subset \mathbb{R}^d$  with  $\mu(A) \leq 1/2$ ,

$$\mu(A) \leq D \text{Cap}_\mu(A).$$

Moreover, optimal constants verify  $D_{\text{opt}}/2 \leq C_{\text{opt}} \leq 4D_{\text{opt}}$ .

A proof of Theorem 5.7 can be found in [5], Proposition 13.

Under some assumptions on the function  $\beta$  the same holds true for super Poincaré inequalities. The following theorem due to Barthe, Cattiaux and Roberto shows how to deduce a super Poincaré inequality from a capacity measure inequality (see [6], Theorem 1 and Corollary 6).

**Theorem 5.8 (Barthe–Cattiaux–Roberto).** *Let  $\beta : [1, +\infty) \rightarrow \mathbb{R}^+$  be a nonincreasing function such that  $s \mapsto s\beta(s)$  is nondecreasing. Suppose that for all  $A \subset \mathbb{R}^d$ , with  $\mu(A) \leq 1/2$ ,*

$$\frac{\mu(A)}{\beta(1/\mu(A))} \leq \text{Cap}_\mu(A),$$

then  $\mu$  verifies the super Poincaré inequality  $\mathbb{SP}(8\beta)$ .

In fact, for our purpose one is only interested in the converse proposition:

**Proposition 5.9.** *Let  $\beta : [1, +\infty) \rightarrow \mathbb{R}^+$  be a nonincreasing function such that  $s \mapsto s\beta(s)$  is nondecreasing. Suppose also that there exists  $\lambda \geq 4$  such that*

$$\forall s \geq 1 \quad \lambda\beta(\lambda s) \geq 4\beta(s).$$

Under the preceding assumption, if  $\mu$  verifies the super Poincaré inequality  $\mathbb{SP}(\beta)$ , then for all  $A \subset \mathbb{R}^d$ , with  $\mu(A) \leq 1/2$  one has

$$\frac{\mu(A)}{\beta(1/\mu(A))} \leq 4\lambda \text{Cap}_\mu(A).$$

**Proof.** The following proof is a straightforward adaptation of the proof of [5], Theorem 22, and we will only sketch it. Let  $A \subset \mathbb{R}^d$  with  $\mu(A) \leq 1/2$  and  $f : \mathbb{R}^d \rightarrow [0, 1]$  a function which is 1 on  $A$  and vanishes with probability more than  $1/2$ . For all  $k \in \mathbb{Z}$ , define  $f_k = (f - 2^k)_+ \wedge 2^k$  and  $\Omega_k = \{f \geq 2^k\}$ . Applying the super Poincaré inequality (5.1) to the function  $f_k$  one obtains:

$$\begin{aligned} \int f_k^2 d\mu &\leq \beta(s) \int |\nabla f_k|_2^2 d\mu + s \left( \int |f_k| d\mu \right)^2 \\ &\leq \beta(s) \int |\nabla f|_2^2 d\mu + s\mu(\Omega_k) \int f_k^2 d\mu. \end{aligned}$$

Taking  $s = \frac{1}{2\mu(\Omega_k)} \geq 1$  and noticing that  $f_k^2 \geq 2^{2k}$  on  $\Omega_{k+1}$  gives

$$\mu(\Omega_{k+1})2^{2k} \leq \int f_k^2 d\mu \leq 2\beta\left(\frac{1}{2\mu(\Omega_k)}\right) \int |\nabla f|_2^2 d\mu.$$

Defining  $F(x) = \frac{1}{2\beta(x/2)}$  for  $x \geq 2$ ,  $a_k = \mu(\Omega_k)$  and  $C = \int |\nabla f|_2^2 d\mu$  one gets  $2^{2k}a_{k+1}F(1/a_k) \leq C$ , as soon as  $a_k > 0$ . Applying [5], Lemma 23, one concludes that  $2^{2k}a_kF(1/a_k) \leq \lambda C$  as soon as  $a_k > 0$ . If one takes  $k = 0$ , one has  $A \subset \Omega_0$  so  $a_0 \geq \mu(A)$  and since  $s\beta(s)$  is nondecreasing,  $a_0F(1/a_0) \geq \mu(A)F(1/\mu(A))$ . Consequently,

$$\frac{\mu(A)}{4\beta(1/\mu(A))} \leq \frac{\mu(A)}{2\beta(1/(2\mu(A)))} \leq \lambda \int |\nabla f|_2^2 d\mu.$$

Optimizing over  $f$  gives the result. □

5.3. Proof of Theorem 5.4

In all what follows, we will adopt the following convention: for  $s \leq 1$ , one defines  $\beta(s) = \beta(1)$ .

For all  $x > 0$ , let

$$\Theta(x) = \frac{x}{4\lambda\beta(1/x)}, \tag{5.9}$$

where  $\lambda$  is defined in (5.8).

**Lemma 5.10.** *If  $\beta : (0, +\infty] \rightarrow \mathbb{R}^+$  is a nonincreasing function such that  $s \mapsto s\beta(s)$  is nondecreasing then the function  $\Theta$  defined by (5.9) is nondecreasing and verifies  $\Theta(x + y) \leq \Theta(x) + \Theta(y)$  for all  $x, y \in \mathbb{R}^+$ .*

**Proof.** Since  $s\beta(s)$  is nondecreasing, it follows that  $\Theta$  is nondecreasing. Moreover, since  $\beta$  is nonincreasing, it follows that  $\Theta(x)/x$  is nonincreasing. Thus, if  $x \geq y > 0$ ; one gets

$$\Theta(x + y) = \Theta(x(1 + y/x)) \leq (1 + y/x)\Theta(x) = \Theta(x) + y\Theta(x)/x \leq \Theta(x) + \Theta(y).$$

This completes the proof. □

The following lemma explains how behave capacity-measure inequalities under push-forward:

**Lemma 5.11.** *Suppose that  $\mu$  satisfies the capacity-measure inequality*

$$\forall A \text{ with } \mu(A) \leq 1/2 \quad \Psi(\mu(A)) \leq D \text{Cap}_\mu(A).$$

Then  $\tilde{\mu} = \omega^\# \mu$  verifies the inequality

$$\forall A \text{ with } \tilde{\mu}(A) \leq 1/2 \quad \Psi(\tilde{\mu}(A)) \leq D \overline{\text{Cap}}_{\tilde{\mu}}(A),$$

where

$$\overline{\text{Cap}}_{\tilde{\mu}} = \inf \left\{ \int \sum_{i=1}^d (\omega' \circ \omega^{-1}(x_i))^2 \left( \frac{\partial f}{\partial x_i} \right)^2(x) d\tilde{\mu} : f : \mathbb{R}^d \rightarrow [0, 1], f|_A = 1 \text{ and } \tilde{\mu}(f = 0) \geq 1/2 \right\}.$$

**Proof.** Let  $A$  be such that  $\tilde{\mu}(A) \leq 1/2$ , and  $f$  be such that  $f = 1$  on  $A$  and  $\tilde{\mu}(f = 0) \geq 1/2$ . Define  $B = \omega^{-1}(A)$  and  $g = f \circ \omega$ . Then  $\mu(B) = \tilde{\mu}(A) \leq 1/2$ ,  $g \geq 1$  on  $B$  and  $\{g = 0\} = \omega^{-1}\{f = 0\}$  and so  $\mu(g = 0) = \tilde{\mu}(f = 0) \geq 1/2$ . Applying the capacity-measure inequality verified by  $\mu$  to  $B$  and  $g$  yields

$$\Psi(\tilde{\mu}(A)) = \Psi(\mu(B)) \leq D \int |\nabla g|_2^2 d\mu = D \int \sum_{i=1}^d (\omega' \circ \omega^{-1}(x_i))^2 \left( \frac{\partial f}{\partial x_i} \right)^2(x) d\tilde{\mu}.$$

Optimizing over such functions  $f$  gives the announced inequality for  $\tilde{\mu}$ . □

The next lemma compares the capacity  $\overline{\text{Cap}}_{\tilde{\mu}}$  to the usual capacity  $\text{Cap}_\mu$ :

**Lemma 5.12.** *Suppose that  $\omega$  is convex and let  $B_\infty(r) = \{x \in \mathbb{R}^d : \max_{1 \leq i \leq d} |x_i| \leq r\}$ , for all  $r \geq 0$ . If  $A \subset B_\infty(r)$  and  $\mu(A) \leq 1/2$ , then*

$$\overline{\text{Cap}}_{\tilde{\mu}}(A) \leq 2(\omega' \circ \omega^{-1}(r + 1))^2 [\text{Cap}_{\tilde{\mu}}(A) + \tilde{\mu}(B_\infty(r)^c)].$$

**Proof.** Let

$$\text{Cap}_{\tilde{\mu}}^r(A) = \inf \left\{ \int |\nabla f|_2^2 d\tilde{\mu} : \mathbb{1}_A \leq f \leq \mathbb{1}_{B_\infty(r+1)} \text{ and } \tilde{\mu}(f=0) \geq 1/2 \right\}.$$

Using the fact that the function  $\omega' \circ \omega^{-1}$  is nondecreasing on  $\mathbb{R}^+$ , one clearly has:

$$\overline{\text{Cap}}_{\tilde{\mu}}(A) \leq (\omega' \circ \omega^{-1}(r+1))^2 \text{Cap}_{\tilde{\mu}}^r(A).$$

Now let  $f: \mathbb{R}^d \rightarrow [0, 1]$  be such that  $f|_A = 1$  and  $\tilde{\mu}(f=0) \geq 1/2$ . Let  $h: \mathbb{R} \rightarrow \mathbb{R}^+$  defined by  $h(t) = (r+1-t)_+ \wedge 1$  and consider  $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}^+$  defined by  $\varphi(x) = h(|x|_\infty)$ . It is not difficult to check that  $|\nabla \varphi|_2 \leq \mathbb{1}_{B_\infty(r)^c}$ . Let  $g = f\varphi$ ; one has  $\mathbb{1}_A \leq g \leq \mathbb{1}_{B_\infty(r+1)}$ ,  $\tilde{\mu}(g=0) \geq \tilde{\mu}(f=0) \geq 1/2$  and

$$\begin{aligned} \text{Cap}_{\tilde{\mu}}^r(A) &\leq \int |\nabla g|_2^2 d\tilde{\mu} = \int |\nabla f\varphi + f\nabla\varphi|_2^2 d\tilde{\mu} \\ &\leq 2 \int |\nabla f|_2^2 \varphi^2 d\tilde{\mu} + 2 \int f^2 |\nabla\varphi|_2^2 d\tilde{\mu} \\ &\leq 2 \int |\nabla f|_2^2 d\tilde{\mu} + 2\tilde{\mu}(B_\infty(r)^c). \end{aligned}$$

Optimizing over  $f$  yields:

$$\text{Cap}_{\tilde{\mu}}^r(A) \leq 2\text{Cap}_{\tilde{\mu}}(A) + 2\tilde{\mu}(B_\infty(r)^c). \quad \square$$

**Proof of Theorem 5.4.** Define  $\tilde{\mu}$  as the image of  $\mu$  under the map  $x \mapsto \omega_\beta(x/a)$  with  $a = \max(\lambda, \Lambda_\beta^*(m))$ . One wants to prove that  $\tilde{\mu}$  verifies the classical Poincaré inequality. According to Proposition 5.9, the probability measure  $\mu$  satisfies the capacity-measure inequality

$$\forall A \text{ with } \mu(A) \leq 1/2 \quad \Theta(\mu(A)) \leq \text{Cap}_\mu(A). \quad (5.10)$$

According to Lemma 5.11,  $\tilde{\mu}$  satisfies the capacity-measure type inequality:

$$\forall A \text{ with } \tilde{\mu}(A) \leq 1/2 \quad \Theta(\tilde{\mu}(A)) \leq \overline{\text{Cap}}_{\tilde{\mu}}(A),$$

where  $\overline{\text{Cap}}_{\tilde{\mu}}$  is defined in the lemma.

Let  $B_\infty(r) = \{x \in \mathbb{R}^d : \max_{1 \leq i \leq d} |x_i| \leq r\}$ , for all  $r \geq 0$ . Let  $A \subset \mathbb{R}^d$  with  $\tilde{\mu}(A) \leq 1/2$ ; one has

$$\begin{aligned} \Theta(\tilde{\mu}(A)) &\stackrel{(i)}{\leq} \Theta(\tilde{\mu}(A \cap B_\infty(r))) + \Theta(\tilde{\mu}(B_\infty(r)^c)) \\ &\stackrel{(ii)}{\leq} \overline{\text{Cap}}_{\tilde{\mu}}(A \cap B_\infty(r)) + \Theta(\tilde{\mu}(B_\infty(r)^c)) \\ &\stackrel{(iii)}{\leq} 2/a^2 (\omega'_\beta \circ \omega_\beta^{-1}(r+1))^2 [\text{Cap}_{\tilde{\mu}}(A \cap B_\infty(r)) + \tilde{\mu}(B_\infty(r)^c)] + \Theta(\tilde{\mu}(B_\infty(r)^c)) \\ &\stackrel{(iv)}{\leq} 2/a^2 (\omega'_\beta \circ \omega_\beta^{-1}(r+1))^2 [\text{Cap}_{\tilde{\mu}}(A) + \tilde{\mu}(B_\infty(r)^c)] + \Theta(\tilde{\mu}(B_\infty(r)^c)) \\ &\stackrel{(v)}{\leq} \frac{e}{8a^2\beta(e^r)} [\text{Cap}_{\tilde{\mu}}(A) + \tilde{\mu}(B_\infty(r)^c)] + \Theta(\tilde{\mu}(B_\infty(r)^c)), \end{aligned}$$

where (i) follows from the sub-additivity and the monotonicity of  $\Theta$ , (ii) from Lemma 5.11, (iii) from Lemma 5.12 and the convexity of  $\omega_\beta$ , (iv) from the fact that the function  $A \mapsto \text{Cap}_{\tilde{\mu}}(A)$  is nondecreasing and (v) from the definition (5.3) of  $\omega_\beta$  and the inequality  $\beta(e^{r+1}) \geq 1/e\beta(e^r)$ . Thanks to Lemma 5.13 below, one has

$$\tilde{\mu}(B_\infty(r)^c) \leq (2e)e^{-r}.$$

Using the monotonicity and the sub-additivity of  $\Theta$ , one has  $\Theta(\tilde{\mu}(B_\infty(r)^c)) \leq \Theta((2e)e^{-r}) \leq (2e)\Theta(e^{-r})$ . So, letting  $t = \tilde{\mu}(A)$  and using the definition of  $\Theta$ , one has:

$$\forall r \geq 0 \quad \frac{t}{\beta(1/t)} \leq \left( \frac{e\lambda}{2a^2} \text{Cap}_{\tilde{\mu}}(A) + \left( \frac{e^2\lambda}{a^2} + 2e \right) e^{-r} \right) \frac{1}{\beta(e^r)}.$$

Since  $a \geq \lambda \geq 4$ , one has  $\frac{e\lambda}{2a^2} \leq 1/2$  and  $\frac{e^2\lambda}{a^2} + 2e \leq 8$  and so letting  $b = \frac{t}{\beta(1/t)}$ , one gets

$$\sup_{s \geq 1} \{ b\beta(s) - 8/s \} \leq \frac{1}{2} \text{Cap}_{\tilde{\mu}}(A).$$

Let  $g(s) = s\beta(s)$ ,  $s \geq 1$ ; by hypotheses  $g$  is increasing and goes to  $+\infty$  when  $s \rightarrow +\infty$ . Taking  $s = g^{-1}(16/b)$  (which is well defined) yields

$$b\beta(g^{-1}(16/b)) \leq \text{Cap}_{\tilde{\mu}}(A).$$

According to (5.8), one has  $g(\lambda x) \geq 4g(x)$  for all  $x \geq 1$ ; from this follows that  $g^{-1}(4x) \leq \lambda g^{-1}(x)$  for all  $x \geq \beta(1)$  and by iteration,  $g^{-1}(16x) \leq \lambda^2 g^{-1}(x)$ , for all  $x \geq \beta(1)$ . Consequently,

$$g^{-1}(16/b) \leq \lambda^2 g^{-1}(1/b) = \lambda^2 g^{-1}(g(1/t)) = \lambda^2/t.$$

As  $\beta$  is nonincreasing, one concludes that  $\beta(\lambda^2/t) \leq \beta(g^{-1}(16/b))$ . Since  $\beta(\lambda^2/t) \geq \beta(1/t)/\lambda^2$ , one gets  $t/\lambda^2 \leq b\beta(g^{-1}(16/b)) \leq \text{Cap}_{\tilde{\mu}}(A)$ .

In other word, for all  $A \subset \mathbb{R}^d$  with  $\tilde{\mu}(A) \leq 1/2$

$$\tilde{\mu}(A) \leq \lambda^2 \text{Cap}_{\tilde{\mu}}(A).$$

According to Theorem 5.7, one concludes that  $\tilde{\mu}$  verifies the classical Poincaré inequality  $\mathbb{S}\mathbb{G}(4\lambda^2)$ .

Let  $\bar{\mu} = \mu_{\int x d\mu}$ . If  $\mu$  verifies the super Poincaré inequality (5.1), then so does  $\bar{\mu}$ . So all the preceding results apply to  $\bar{\mu}$ . In particular,  $\bar{\mu}$  verifies the inequality  $\mathbb{S}\mathbb{G}(\omega_\beta(\cdot/a), 4\lambda^2)$ , with  $a = \max(\lambda, \Lambda_\beta^*(\bar{m}))$ , where  $\bar{m} = \int |x - \int y d\mu|_2 d\mu(x)$ . But,

$$\bar{m}^2 \leq \sum_{i=1}^d \int \left( x_i - \int y_i d\mu \right)^2 d\mu(x) \leq 2\beta(1)d,$$

where the first inequality follows from Cauchy–Schwarz inequality and the second from the fact that  $\mu$  verifies the Poincaré inequality  $\mathbb{S}\mathbb{G}(2\beta(1))$  (see Remark 5.1). This proves that  $\bar{\mu}$  verifies  $\mathbb{S}\mathbb{G}(\omega_\beta(\cdot/\bar{a}), 4\lambda^2)$  with  $\bar{a} = \max(\lambda, \Lambda_\beta^*(\sqrt{2\beta(1)d}))$ .

The invariance property of  $\bar{a}$  follows immediately from the definition of  $\Lambda_\beta^*$  given in Proposition 5.2.

Now, according to Theorem 4.6,  $\bar{\mu}$  verifies the inequality  $\mathbb{T}\mathbb{C}(\omega_\beta(\cdot/\bar{a}), \frac{1}{2\lambda\kappa})$ . Reasoning as in the proof of Corollary 4.11, one sees that this implies that  $\bar{\mu}$  satisfies the transportation-cost inequality with the cost function

$$c(x, y) = \alpha \left( \frac{1}{2\lambda\kappa\sqrt{d}} \right) \sum_{i=1}^d \alpha \circ \omega_\beta \left( \frac{|x_i - y_i|}{2\bar{a}} \right).$$

Since transportation-cost inequalities are translation invariant, this concludes the proof. □

During the proof of Theorem 5.4, one has used the following lemma.

**Lemma 5.13.** *The probability measure  $\tilde{\mu}$  which is the image of  $\mu$  under the map  $x \mapsto \omega_\beta(x/a)$  with  $a = \max(\lambda, \Lambda_\beta^*(m))$  and  $m = \int |x|_2 d\mu$  verifies*

$$\forall r \geq 0 \quad \tilde{\mu}(|x|_\infty \geq r) \leq (2e)e^{-r}.$$

**Proof.** According to Lemma 5.3 and e.g. [20], Lemma 2.3, one has

$$\forall \varepsilon \in [0, 1) \quad \int e^{\varepsilon \Lambda_\beta^*(\|x\|_2 - m)} d\mu \leq \frac{1 + \varepsilon}{1 - \varepsilon},$$

where  $\Lambda_\beta^*$  is defined in Proposition 5.2. Using the convexity of  $\Lambda_\beta^*$  and the fact that  $\Lambda_\beta^* \geq \alpha \circ \omega_\beta$  one gets since  $a \geq 2$ ,

$$\exp(\alpha \circ \omega_\beta(\|x\|_2/a)) \leq \exp(\Lambda_\beta^*(\|x\|_2/a)) \leq \exp\left(\frac{1}{a} \Lambda_\beta^*(\|x\|_2 - m)\right) \cdot \exp\left(\frac{1}{a} \Lambda_\beta^*(m)\right).$$

Since  $|\omega_\beta(x/a)|_\infty = \omega_\beta(\|x\|_\infty/a) \leq \omega_\beta(\|x\|_2/a)$ , integrating yields:

$$\int e^{\alpha(\|x\|_\infty)} d\tilde{\mu}(x) \leq \frac{1 + 1/a}{1 - 1/a} \cdot \exp\left(\frac{1}{a} \Lambda_\beta^*(m)\right) \leq 2e$$

which gives the result.  $\square$

#### 5.4. Consequences of Theorem 5.4

In this section we will draw consequences of Theorem 5.4. We will focus on the functions  $\beta_p(s) = \log(e + s)^{2(1/p-1)}$ , but more general results could be stated. First let us show that these functions verify the assumptions of Theorem 5.4.

**Lemma 5.14.** *For all  $p \geq 1$ , the function  $\beta_p(s) = \log(e + s)^{2(1/p-1)}$  is such that  $s \mapsto s\beta_p(s)$  is increasing on  $[0, +\infty)$ . Moreover, for all  $p \geq 1$ , there is some  $\lambda \geq 4$ , such that  $\lambda\beta_p(\lambda s) \geq 4\beta_p(s)$  for all  $s \geq 1$ . Let us denote by  $\lambda_p$  the smallest of these  $\lambda$ 's, then the map  $p \mapsto \lambda_p$  is increasing. Moreover, one always has  $\lambda_p \leq 205$  for all  $p \geq 1$  and for  $p \in [1, 2]$ , one has  $\lambda_p \leq 20$ .*

**Proof.** Let  $r = 2(1 - 1/p)$ ; then  $r \in [0, 2)$ . The map  $s \mapsto \log(e + s)^r$  is concave on  $[0, +\infty)$ . Consequently, the map  $s \mapsto (\log(e + s)^r - 1)/s$  decreases on  $(0, +\infty)$  and so does  $s \mapsto \log(e + s)^r/s$ . In other word  $s \mapsto s\beta_p(s)$  is increasing.

Next observe that  $\lambda\beta_p(\lambda s) \geq 4\beta_p(s) \Leftrightarrow \lambda \left[ \frac{\log(e + \lambda s)}{\log(e + \lambda s)} \right]^r \geq 4$ . This clearly implies that the map  $p \mapsto \lambda_p$  is nondecreasing.

Let  $f(s) = \frac{\log(e + s)}{\log(e + \lambda s)}$ ; then

$$\begin{aligned} f'(s) &= \frac{(e + \lambda s) \log(e + \lambda s) - \lambda(e + s) \log(e + s)}{\log(e + \lambda s)^2(e + s)(e + \lambda s)} \\ &= \frac{\varphi(\lambda s) - \lambda\varphi(s)}{\log(e + \lambda s)^2(e + s)(e + \lambda s)}, \end{aligned}$$

with  $\varphi(s) = (e + s) \log(e + s)$ . Then  $\frac{d}{ds} \frac{\varphi(s)}{s} = \frac{s - e \log(e + s)}{s^2}$ . If  $x \geq 6$ , then  $\frac{d}{ds} \frac{\varphi(s)}{s} \geq 0$  so  $s \mapsto \varphi(s)/s$  is nondecreasing and this implies that  $\varphi(\lambda s) \geq \lambda\varphi(s)$  for all  $s \geq 6$ . As a consequence,  $f'(s) \geq 0$  when  $s \geq 6$  and the function  $f$  is thus nondecreasing on  $[6, +\infty)$ . Consequently,  $f(s) \geq f(6)$  for  $s \geq 6$  and  $f(s) \geq \frac{1}{\log(e + 6\lambda)}$  for  $s \leq 6$ . Since  $f(6) \geq \frac{1}{\log(e + 6\lambda)}$ , one has  $f(s) \geq \frac{1}{\log(e + 6\lambda)}$  for all  $s \geq 1$ .

From what precedes one concludes it is enough to find  $\lambda \geq 4$  such that

$$\frac{\lambda}{\log(e + 6\lambda)^r} \geq 4.$$

For  $r = 2$ , one checks that  $\lambda = 205$  is convenient and for  $r = 1$ , one can take  $\lambda = 20$ . This the proof.  $\square$

##### 5.4.1. Comparison with Latała–Oleszkiewicz inequalities

Let us recall that  $\mu$  satisfies the Latała–Oleszkiewicz inequality  $\mathbb{L}\mathbb{O}(p, C)$  if

$$\forall f \quad \sup_{a \in (1, 2)} \frac{\int f^2 d\mu - (\int |f|^a d\mu)^{2/a}}{(2 - a)^{2(1-1/p)}} \leq C \int |\nabla f|_2^2 d\mu. \quad (5.11)$$

The following result is due to Wang (see [41], Theorem 1.1):

**Theorem 5.15.** *Let  $p \in [1, 2]$ ; a probability measure verifies the  $\mathbb{L}\mathbb{O}(p, C)$  for some  $C > 0$  if and only if it verifies the super Poincaré inequality  $\mathbb{S}\mathbb{P}(\tilde{C}\beta_p)$ .*

**Remark 5.16.** *If  $\mu$  verifies  $\mathbb{L}\mathbb{O}(p, C)$ , then it verifies  $\mathbb{S}\mathbb{P}(96C\beta_p)$ . This follows easily from [6], Corollary 8.*

**Corollary 5.17.** *If  $\mu$  verifies the inequality  $\mathbb{L}\mathbb{O}(p, C)$  on  $\mathbb{R}^d$ , with  $p \in [1, 2]$  then  $\mu$  verifies the centered inequality  $\mathbb{S}\mathbb{G}(\omega_p(\cdot/(a_1\sqrt{C})), a_2)$ , where  $a_1$  depends only on the dimension  $d$  and  $a_2$  is an absolute constant. One can take  $a_1 = 4\sqrt{6} \max(5d, 20)$  and  $a_2 = (320)^2$ .*

**Remark 5.18.** *The fact that the dimension  $d$  appears in the constant  $a_2$  above is not a problem, thanks to the tensorization property of the (centered) Poincaré inequality.*

**Proof of Corollary 5.17.** According to Theorem 5.15,  $\mu$  verifies  $\mathbb{S}\mathbb{P}(96C\beta_p)$  and according to Theorem 5.4,  $\mu$  verifies the centered Poincaré inequality  $\mathbb{S}\mathbb{G}(\omega, 4\lambda_p^2)$ , with  $\omega := \omega_{96C\beta_p}(\cdot/\bar{a}) = \omega_{\beta_p}(\cdot/(4\bar{a}\sqrt{6C}))$ . According to Lemma 5.14, one has  $\lambda_p \leq 20$ . Using the inequalities (5.4) and (5.7), one sees that  $\Lambda_{\beta_p}^*(\sqrt{2\beta_p(1)d}) \leq \Lambda_{\beta_2}^*(\sqrt{2d}) \leq 5d$ , so  $\bar{a} \leq \max(5d, 20)$ . It is easy to check that  $\omega'_{\beta_p} \geq \frac{1}{4p}\omega'_p \geq \frac{1}{8}\omega'_p$ . According to Proposition 3.1, one concludes that  $\mu$  verifies the centered Poincaré inequality  $\mathbb{S}\mathbb{G}(\omega_p(\cdot/(4 \max(5d, 20)\sqrt{6C})), (320)^2)$ .  $\square$

**Remark 5.19.** *According to Corollary 5.17, the Logarithmic-Sobolev inequality is stronger than the Poincaré inequality  $\mathbb{S}\mathbb{G}(\omega_2, \cdot)$ . In [16], Cattiaux and Guillin were able to construct a potential  $V$  on  $\mathbb{R}$  satisfying  $V(-x) = V(x)$  and  $\liminf_{x \rightarrow +\infty} V'(x)/x > 0$  but such that the probability measure  $d\mu(x) = e^{-V(x)} dx$  does not satisfy the Bobkov–Götze necessary and sufficient condition for the Logarithmic-Sobolev inequality (see [10]). According to Proposition 3.3, this shows that the Logarithmic-Sobolev inequality is strictly stronger than the inequality  $\mathbb{S}\mathbb{G}(\omega_2, \cdot)$ .*

#### 5.4.2. Comparison with modified Logarithmic-Sobolev inequalities

Let  $H : \mathbb{R} \rightarrow \mathbb{R}^+$ ; let us recall that  $\mu$  verifies the modified Log-Sobolev inequality  $\mathbb{L}\mathbb{S}(H, C)$  on  $\mathbb{R}^d$ , if for all locally Lipschitz positive function  $f$ ,

$$\text{Ent}_\mu(f^2) \leq C \sum_{i=1}^d \int H\left(\frac{\partial_i f}{f}\right) f^2 d\mu.$$

Let  $p \geq 2$  define  $q$  such that  $1/p + 1/q = 1$  and  $H_q(x) = |x|^q$ . The inequality  $\mathbb{L}\mathbb{S}(H_q, \cdot)$  is related to super Poincaré inequality  $\mathbb{S}\mathbb{P}(\beta_p)$  as explained in the following proposition.

**Proposition 5.20.** *Let  $p \geq 2$  and suppose that  $\mu$  verifies the inequality  $\mathbb{L}\mathbb{S}(H_q, C)$  on  $\mathbb{R}^d$  with  $1/p + 1/q = 1$ , then  $\mu$  verifies the super Poincaré inequality  $\mathbb{S}\mathbb{P}(C^{2(1-1/p)}k\beta_p)$ , where  $k$  is a constant depending only on the dimension  $d$  and  $p$ .*

**Proof.** Since the function  $x \mapsto x^{q/2}$  is concave, applying Jensen inequality yields:

$$\int H_q\left(\frac{\partial_i f}{f}\right) f^2 d\mu \leq \left(\int (\partial_i f)^2 d\mu\right)^{q/2} \left(\int f^2 d\mu\right)^{1-q/2}.$$

So, using concavity again,

$$\text{Ent}_\mu(f^2) \leq Cd^{1-q/2} \left(\int |\nabla f|_2^2 d\mu\right)^{q/2} \left(\int f^2 d\mu\right)^{1-q/2}.$$

Since,  $x^{q/2} = \inf_{s>0} \{sx + a_q s^{q/(q-2)}\}$ , with  $a_q = (\frac{2-q}{2})(\frac{2}{q})^{q/(q-2)}$ , one concludes that for all  $s > 0$ ,

$$\text{Ent}_\mu(f^2) \leq \tilde{C}s \int |\nabla f|_2^2 d\mu + \tilde{C}a_q s^{q/(q-2)} \int f^2 d\mu,$$

letting  $\tilde{C} = Cd^{1-q/2}$ . According to the proof of [39], Theorem 3.1, if a probability measure  $\mu$  verifies an inequality of the form:

$$\text{Ent}_\mu(f^2) \leq c_1 \int |\nabla f|_2^2 d\mu + c_2 \int f^2 d\mu,$$

then it verifies

$$\forall r > 0 \quad \int f^2 d\mu \leq r \int |\nabla f|_2^2 d\mu + \left(\frac{rc_2}{2c_1} + 1\right)^2 \exp\left(c_2 + \frac{2c_1}{r}\right) \left(\int |f| d\mu\right)^2.$$

From this follows, that

$$\int f^2 d\mu \leq r \int |\nabla f|_2^2 d\mu + \left(\frac{a_q}{2} r s^{2/(q-2)} + 1\right)^2 \exp(\tilde{C}a_q s^{q/(q-2)} + 2\tilde{C}s/r) \left(\int |f| d\mu\right)^2$$

holds for all  $s, r > 0$ . Choosing  $s = r^{(2-q)/2}$  yields:

$$\forall r > 0 \quad \int f^2 d\mu \leq r \int |\nabla f|_2^2 d\mu + \frac{1}{4}(a_q + 2)^2 \exp(\tilde{C}(a_q + 2)r^{-q/2}) \left(\int |f| d\mu\right)^2$$

or equivalently:

$$\forall s \geq \frac{1}{4}b_q \quad \int f^2 d\mu \leq \tilde{C}^{2/q} b_q^{1/q} \log\left(\frac{4s}{b_q}\right)^{-2/q} \int |\nabla f|_2^2 d\mu + s \left(\int |f| d\mu\right)^2,$$

where  $b_q = (a_q + 2)^2$ . According to [14], Proposition 2.3,  $\mu$  verifies the Poincaré inequality  $\mathbb{S}\mathbb{G}(c_q C^{2/q})$ , where  $c_q = 36 \cdot 6^{2/q}$ . Let  $\beta(s) = c_q \wedge d^{2/q-1} b_q^{1/q} \log(\frac{4s}{b_q})^{-2/q}$  for  $s \geq b_q/4$  and  $\beta(s) = c_q$  for  $s \in [1, b_q/4]$ , then  $\mu$  verifies the super Poincaré inequality  $\mathbb{S}\mathbb{P}(C^{2/q}\beta)$ . It is clear that one can find a constant  $k$  such that  $\beta \leq k\beta_p$ . This constant  $k$  depends only on  $d$  and  $q$ .  $\square$

Reasoning exactly as in Corollary 5.17, one proves the following result.

**Corollary 5.21.** *Let  $p \geq 2$  and suppose that  $\mu$  verifies the inequality  $\mathbb{L}\mathbb{S}(H_q, C)$  on  $\mathbb{R}^d$  with  $1/p + 1/q = 1$ , then  $\mu$  verifies  $\mathbb{S}\mathbb{G}(\omega_p(\cdot/aC^{1-1/p}), b)$ , where  $a$  and  $b$  are constants depending only on  $d$  and  $p$ .*

## Annex: Proof of Theorem 2.2 and its corollary

**Sketch of proof of Theorem 2.2.** *First step:* According to [12], Theorem 3.1 (which is the main result of [12]),  $\mu$  enjoys a modified Logarithmic-Sobolev inequality: for all  $0 < s < \frac{2}{\sqrt{C}}$  and for all locally Lipschitz  $f: \mathcal{X} \rightarrow \mathbb{R}$  such that  $|\nabla f| \leq s$   $\mu$  a.e. one has

$$\text{Ent}_\mu(e^f) \leq L(s) \int |\nabla f|^2 e^f d\mu, \tag{A.1}$$

where  $L(s) = \frac{C}{2} \left(\frac{2+\sqrt{Cs}}{2-\sqrt{Cs}}\right)^2 e^{s\sqrt{5C}}$ .

*Second step: Tensorization.* Thanks to the tensorization property of the entropy functional,

$$\text{Ent}_{\mu^n}(e^f) \leq \int \sum_{i=1}^n \text{Ent}_{\mu}(e^{f_i}) d\mu^n$$

for all  $f : \mathcal{X}^n \rightarrow \mathbb{R}$ .

Applying this inequality together with (A.1) yields

$$\text{Ent}_{\mu^n}(e^f) \leq L(s) \int \sum_{i=1}^n |\nabla_i f|^2 e^f d\mu \tag{A.2}$$

for all  $0 < s < \frac{2}{\sqrt{C}}$  and  $f : \mathcal{X}^n \rightarrow \mathbb{R}$  such that  $\max_{1 \leq i \leq n} |\nabla_i f| \leq s \mu^n$  a.e.

*Third step: Herbst argument.* Thanks to the homogeneity one can suppose that  $f : \mathcal{X}^n \rightarrow \mathbb{R}$  is such that  $\max_{1 \leq i \leq n} |\nabla_i f| \leq 1$  ( $b = 1$ ) and  $\sum_{i=1}^n |\nabla_i f|^2 \leq a^2$ . Define  $Z(\lambda) = \int e^{\lambda f} d\mu^n$ . Then, applying (A.2) to  $\lambda f$ , one easily obtains the following differential inequality

$$\forall 0 < \lambda \leq s < \frac{2}{\sqrt{C}} \quad \frac{d}{d\lambda} \left( \frac{\log(Z(\lambda))}{\lambda} \right) \leq L(s)a^2,$$

and since  $\frac{\log(Z(\lambda))}{\lambda} \rightarrow_{\lambda \rightarrow 0} \int f d\mu^n$ , one gets

$$\forall 0 < \lambda \leq s < \frac{2}{\sqrt{C}} \quad \int e^{\lambda f} d\mu^n \leq e^{\lambda^2 L(s)a^2 + \lambda \int f d\mu^n}.$$

*Fourth step: Tchebychev argument.* This latter inequality on the Laplace transform yields via Tchebychev argument:

$$\forall t \geq 0 \quad \mu^n \left( f \geq \int f d\mu^n + t \right) \leq e^{-h_s(t)},$$

where

$$h_s(t) = \sup_{\lambda \in [0, s]} \{ \lambda t - L(s)a^2 \lambda^2 \} = \begin{cases} \frac{t^2}{4L(s)a^2} & \text{if } 0 \leq t \leq 2L(s)a^2 s, \\ st - L(s)a^2 s^2 & \text{if } t \geq 2L(s)a^2 s. \end{cases}$$

Now it easy to see that,  $h_s(t) \geq \min(\frac{t^2}{4L(s)a^2}, \frac{st}{2})$ . For  $s = 1/\sqrt{C}$  one obtains after some computations,

$$h_s(t) \geq \min\left(\frac{t^2}{C\kappa^2 a^2}, \frac{t}{\sqrt{C}\kappa}\right) \quad \text{with } \kappa = \sqrt{18e\sqrt{5}}. \quad \square$$

**Sketch of proof of Corollary 2.3.** Take  $A \subset \mathcal{X}^n$ , such that  $\mu^n(A) \geq 1/2$  and define  $F(x) = \inf_{a \in A} \sum_{i=1}^n \alpha(d(x_i, a_i))$ , where  $\alpha(u) = \min(|u|, u^2)$ . Then for all  $r \geq 0$ , the function  $f = \min(F, r)$  verifies (see the details in [12]):  $\max_{1 \leq i \leq n} |\nabla_i f| \leq 2$  and  $\sum_{i=1}^n |\nabla_i f|^2 \leq 4r$ . Moreover, since  $\mu^n(A) \geq 1/2$ , one has  $\int f d\mu^n = \int f \mathbb{1}_{A^c} d\mu^n \leq r(1 - \mu^n(A)) \leq r/2$ . Consequently, applying (2.1) to  $f$  yields:

$$\mu^n(F \geq r) = \mu^n(f \geq r) \leq \mu^n \left( f \geq \int f d\mu^n + r/2 \right) \leq e^{-rK(C)},$$

with  $K(C) = \frac{1}{16} \min(\frac{1}{C\kappa^2}, \frac{1}{\sqrt{C}\kappa}) = \frac{1}{16} \alpha(\frac{1}{\sqrt{C}\kappa})$ . This concludes the proof of (2.2). □

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