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Quantitative recurrence in two-dimensional extended processes

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Abstract. Under some mild condition, a random walk in the plane is recurrent. In particular each trajectory is dense, and a natural question is how much time one needs to approach a given small neighbourhood of the origin. We address this question in the case of some extended dynamical systems similar to planar random walks, including $\mathbb{Z}^2$-extension of mixing subshifts of finite type. We define a pointwise recurrence rate and relate it to the dimension of the process, and establish a result of convergence in distribution of the rescaled return time near the origin.

Résumé. Sous certaines conditions, une marche aléatoire dans le plan est récurrente. En particulier, chaque trajectoire est dense, et il est naturel d'estimer le temps nécessaire pour revenir dans un petit voisinage de l'origine. Nous nous intéressons à cette question dans le cas de systèmes dynamiques étendus similaires à des marches aléatoires planaires, notamment celui des $\mathbb{Z}^2$-extension de sous-shifts de type fini mélangeants. Nous déterminons une vitesse de convergence ponctuelle que nous relions à la dimension du processus et nous établissons un résultat de convergence en loi du temps de retour à l'origine, correctement normalisé.

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1. Introduction

1.1. *Motivation*

Let us consider a recurrent random walk or a recurrent dynamical system (with finite or σ-finite invariant measure). Given an initial condition, say x, we thus know that the process will return back ε close to its starting point x. A basic question is *when*? For finite measure preserving dynamical systems this question has some deep relations to the Hausdorff dimension of the invariant measure. Namely, if $\tau_\varepsilon(x)$ represents this time, in many situations

$$\tau_\varepsilon(x) \approx \frac{1}{\varepsilon^{\dim}}$$

for typical points x, where dim is the Hausdorff dimension of the underlying invariant measure. This has been proved for interval maps [15] and rapidly mixing systems [1,14]. Another type of result is the exponential distribution of rescaled return times and the lognormal fluctuations of the return times [4,9].

In this paper we are dealing with systems where the underlying natural measure is indeed infinite. This causes the return time to be non-integrable, in contrast with the finite measure case. However, the systems we are thinking about have in common the property that, in some sense, the behaviours at small scale and at large scale are independent. The large scale dynamics being some kind of recurrent random walk, and the small scale dynamics a finite measure

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Table 1
Recurrence for $\mathbb{Z}^k$-extensions. B_ε denotes the ball of radius ε, ν is a Gibbs measure on the base and d is the Hausdorff dimension of ν

Dimension	$\mathbb{Z}^0$-extension	$\mathbb{Z}^1$-extension	$\mathbb{Z}^2$-extension
Scale	$\lim_{\varepsilon\to 0} \frac{\log \tau_\varepsilon}{-\log\varepsilon} = d$	$\lim_{\varepsilon\to 0} \frac{\log \sqrt{\tau_\varepsilon}}{-\log\varepsilon} = d$	$\lim_{\varepsilon\to 0} \frac{\log \mathbf{log}\tau_\varepsilon}{-\log\varepsilon} = d$
Local law	$\nu(\nu(B_\varepsilon)\tau_\varepsilon > t) \to \mathrm{e}^{-t}$	$\overline{\lim}_{\varepsilon\to 0}\, \nu(\nu(B_\varepsilon)\sqrt{\tau_\varepsilon} > t) \le \frac{1}{1+\beta t}$	$\nu(\nu(B_\varepsilon)\mathbf{log}\tau_\varepsilon > t) \to \frac{1}{1+\beta t}$
Lognormal fluctuations	$\varepsilon^d \tau_\varepsilon$	$\varepsilon^d \sqrt{\tau_\varepsilon}$	$\varepsilon^d \mathbf{log}\tau_\varepsilon$

preserving system. We provide results in three different cases: two probabilistic models (with some hypothesis of independence) and an infinite measure dynamical system (a $\mathbb{Z}^2$-extension of a finite measure dynamical system). The first case treated in Section 2 is a toy model designed to give the hint of the general case. Then, in Section 3 we consider the case of planar random walks. Finally, in Section 4 we give a complete analysis of the quantitative behaviour of return times in the case of $\mathbb{Z}^2$-extensions of subshifts of finite type. The recurrence for $\mathbb{Z}^2$-extensions of subshifts of finite type essentially comes from Guivarc'h and Hardy's local limit theorem [8]. It was proven afterwards by Conze [6] and Schmidt [16] that it is also a consequence of the central limit theorem.

1.2. *Description of the main result*: $\mathbb{Z}^2$*-extensions of subshifts of finite type*

We emphasize that this dimension 2 is at the threshold between recurrent and non-recurrent processes, since in higher dimension these processes are not recurrent (except if degenerate). It makes sense to show how our results behave with respect to the dimension. For completeness, we call the non-extended system itself a $\mathbb{Z}^0$-extension. In this non-extended case, the type of results we have in mind (see Table 1) have already been established respectively by Ornstein and Weiss [13], Hirata [9] and Collet, Galves and Schmitt [4]. The case of $\mathbb{Z}^2$-extension is completely done in Section 4. The case of $\mathbb{Z}^1$-extension can be partly derived following the technique used in the present paper.[2] The essential difference is that the local limit theorem has the one-dimensional scaling in $\frac{1}{\sqrt{n}}$, instead of $\frac{1}{n}$ in the two-dimensional case. The following table summarizes the different results as the dimension changes. The first line of results corresponds to Theorem 8, the second to Theorem 9 and the third to Corollary 10. We refer to Section 4 for precise statements.

2. A toy model in dimension two

We present a toy model designed to posses a lot of independence. It has the advantage of giving the right formulas with elementary proofs.

2.1. *Description of the model and statement of the results*

Let us consider two sequences of independent identically distributed random variables $(X_n)_{n\ge 1}$ and $(Y_n)_{n\ge 0}$ independent one from the other such that:

- the random variable X_1 is uniformly distributed on $\{(1,0), (-1,0), (0,1), (0,-1)\}$;
- the random variable Y_0 is uniformly distributed on $[0,1]^2$.

Let us notice that $S_n := \sum_{k=1}^n X_k$ (with the convention $S_0 := 0$) is the symmetric random walk on $\mathbb{Z}^2$. We study a kind of random walk M_n on $\mathbb{R}^2$ given by $M_n = S_n + Y_n$.

[2] For $\mathbb{Z}^1$-extensions, the method easily gives the three results presented in the table. However, the result on the local law is not sharp. The determination of the precise law will be the object of a forthcoming paper.

Another representation of our model could be the following. Let $S=\mathbb{R}^d$ and consider the system $\mathbb{Z}^2\times S$. Attached to each site $i\in\mathbb{Z}^2$ of the lattice, there is a local system which lives on S and σ_n is a i.i.d. sequence of S-valued random variables with some density ρ, independent of the X_n's. Then we look at the random walk (S_n,σ_n), thinking at σ_n as a spin.

We want to study the asymptotic behaviour, as ε goes to zero, of the return time in the open ball $B(M_0,\varepsilon)$ of radius ε centered at M_0 (for the euclidean metric). Let

$$\tau_\varepsilon:=\min\bigl\{m\geq 1:\ |M_m-M_0|<\varepsilon\bigr\}.$$

We will prove the following:

Theorem 1. *Almost surely, $\frac{\log\log\tau_\varepsilon}{-\log\varepsilon}$ converges to the dimension 2 of the Lebesgue measure on $\mathbb{R}^2$ as ε goes to zero.*

Theorem 2. *For all $t\geq 0$ we have*:

$$\lim_{\varepsilon\to 0}\mathbb{P}\bigl(\lambda\bigl(B(M_0,\varepsilon)\bigr)\log\tau_\varepsilon\leq t\bigr)=\frac{1}{1+\pi/t}.$$

2.2. *Proof of the pointwise convergence of the recurrence rate to the dimension*

First, let us define $R_1:=\min\{m\geq 1:\ S_m=0\}$. According to [7], we know that we have:

$$\mathbb{P}(R_1>s)\sim\frac{\pi}{\log s}\quad\text{as } s \text{ goes to infinity.}\tag{1}$$

We then define for any $p\geq 0$ the pth return time R_p in $[0,1)^2$ by induction:

$$R_{p+1}:=\inf\{m>R_p:\ S_m=0\}.$$

Observe that R_p is the pth return time at the origin of the random walk S_n on the lattice, thus the delays between successive return times R_p-R_{p-1}, setting $R_0=0$, are independent and identically distributed. Consequently:

$$\mathbb{P}(R_p-R_{p-1}>s)=\mathbb{P}(R_1>s).\tag{2}$$

The proof of Theorem 1 follows from these two lemmas.

Lemma 3. *Almost surely, $\frac{\log\log R_n}{\log n}\to 1$ as $n\to\infty$.*

Proof. It suffices to prove that for any $0<\alpha<1$, almost surely, $\mathrm{e}^{n^{1-\alpha}}\leq R_n\leq 2n\mathrm{e}^{n^{1+\alpha}}$ provided n is sufficiently large. By independence and Eq. (2) we have

$$\mathbb{P}\bigl(\log R_n\leq n^{1-\alpha}\bigr)\leq\mathbb{P}\bigl(\forall p\leq n,\log(R_p-R_{p-1})\leq n^{1-\alpha}\bigr)=\mathbb{P}\bigl(\log R_1\leq n^{1-\alpha}\bigr)^n.$$

According to the asymptotic formula (1), for n sufficiently large

$$\mathbb{P}\bigl(\log R_1\leq n^{1-\alpha}\bigr)^n\leq\left(1-\frac{\pi}{2n^{1-\alpha}}\right)^n\leq\mathrm{e}^{-\pi n^\alpha/2}.$$

The first inequality follows then from the Borel–Cantelli lemma.

Moreover, according to formulas (2) and (1), we have $\sum_{n\geq 1}\mathbb{P}(\log(R_n-R_{n-1})>n^{1+\alpha}))<+\infty$. Hence, by the Borel–Cantelli lemma, we know that almost surely, for all n sufficiently large, we have $R_n-R_{n-1}\leq\mathrm{e}^{n^{1+\alpha}}$. From this we get the second inequality. □

Let $T_\varepsilon := \min\{\ell \geq 1 : |Y_{R_\ell} - Y_0| < \varepsilon\}$.

Lemma 4. *Almost surely,* $\frac{\log T_\varepsilon}{-\log \lambda(B(Y_0,\varepsilon))} \to 1$ *as* $\varepsilon \to 0$.

Proof. Since (Y_ℓ) is an i.i.d. sequence independent of the X_k's, the random variable T_ε has a geometric distribution with parameter $\lambda_\varepsilon := \lambda(B(Y_0,\varepsilon)) = \pi\varepsilon^2$. For any $\alpha > 0$ we have the simple decomposition

$$\mathbb{P}\left(\left|\frac{\log T_\varepsilon}{-\log \lambda_\varepsilon} - 1\right| > \alpha\right) = \mathbb{P}\left(T_\varepsilon > \lambda_\varepsilon^{-1-\alpha}\right) + \mathbb{P}\left(T_\varepsilon < \lambda_\varepsilon^{-1+\alpha}\right).$$

The first term is directly handled by Markov inequality:

$$\mathbb{P}\left(T_\varepsilon > \lambda_\varepsilon^{-1-\alpha}\right) \leq \lambda_\varepsilon^\alpha,$$

while the second term may be computed using the geometric distribution:

$$\begin{aligned}
\mathbb{P}\left(T_\varepsilon < \lambda_\varepsilon^{-1+\alpha}\right) &= 1 - (1-\lambda_\varepsilon)^{\lceil \lambda_\varepsilon^{-1+\alpha}\rceil} \\
&= 1 - \exp\left[\left\lceil \lambda_\varepsilon^{-1+\alpha}\right\rceil \log(1-\lambda_\varepsilon)\right] \\
&\leq -\left\lceil \lambda_\varepsilon^{-1+\alpha}\right\rceil \log(1-\lambda_\varepsilon) \\
&= \mathrm{O}\left(\lambda_\varepsilon^\alpha\right).
\end{aligned}$$

Let us define $\varepsilon_n := n^{-1/\alpha}$. According to the Borel–Cantelli lemma, $\frac{\log T_{\varepsilon_n}}{-\log \lambda_{\varepsilon_n}}$ converges almost surely to the constant 1. The conclusion follows from the facts that $(\varepsilon_n)_{n\geq 1}$ is a decreasing sequence of real numbers satisfying $\lim_{n\to+\infty} \varepsilon_n = 0$ and $\lim_{n\to+\infty} \frac{\varepsilon_n}{\varepsilon_{n+1}} = 1$, and T_ε is monotone in ε. □

Proof of Theorem 1. Observe that whenever $B(M_0,\varepsilon)$ is contained in $(0,1)^2$, we have $\tau_\varepsilon = R_{T_\varepsilon}$. The theorem follows from Lemmas 3 and 4 since

$$\frac{\log\log \tau_\varepsilon}{-\log \varepsilon} = \frac{\log\log R_{T_\varepsilon}}{\log T_\varepsilon} \frac{\log T_\varepsilon}{-\log \lambda_\varepsilon} \frac{\log \lambda_\varepsilon}{\log \varepsilon} \to 1 \times 1 \times 2$$

almost surely as $\varepsilon \to 0$. □

2.3. *Proof of the convergence in distribution of the rescaled return time*

Proof of Theorem 2. Observe that $\frac{\log \tau_\varepsilon}{\log R_{T_\varepsilon}}$ converges almost surely to 1. Let $t > 0$. By independence of T_ε and the R_n we have

$$F_\varepsilon(t) := \mathbb{P}\left(\lambda\left(B(Y_0,\varepsilon)\right) \log R_{T_\varepsilon} \leq t\right) = \sum_{n\geq 1} \mathbb{P}(T_\varepsilon = n)\mathbb{P}\left(\log R_n \leq \frac{t}{\lambda_\varepsilon}\right).$$

Since T_ε has a geometric law with parameter λ_ε, $F_\varepsilon(t)$ is equal to $G_{\lambda_\varepsilon}(t)$ with:

$$G_\delta(t) := \sum_{n\geq 1} \delta(1-\delta)^{n-1}\mathbb{P}\left(\log R_n \leq \frac{t}{\delta}\right).$$

First, we notice that the independence of the times between the successive returns gives for any $u > 0$ that

$$\mathbb{P}(R_n \leq u) \leq \mathbb{P}\left(\max_{k=1,\dots,n} R_k - R_{k-1} \leq u\right) = \mathbb{P}(R_1 \leq u)^n.$$

Let $\alpha < 1$. Using the inequality above and the equivalence relation (1) we get that for any $\delta > 0$ sufficiently small,

$$G_\delta(t) \le \sum_{n\ge 1} \delta(1-\delta)^{n-1}\left(1-\alpha\frac{\pi\delta}{t}\right)^n = \frac{1}{1+\alpha\pi/t} + \mathrm{O}(\delta).$$

This implies that $\limsup_{\varepsilon\to 0} F_\varepsilon(t) \le \frac{1}{1+\pi/t}$.

Fix $A > 0$ and keeping the same notations observe that we have $F_\varepsilon(t) \ge H_{\lambda_\varepsilon}(t)$ with:

$$H_\delta(t) := \sum_{1\le n\le A/\delta} \delta(1-\delta)^{n-1}\mathbb{P}\left(\log R_n \le \frac{t}{\delta}\right).$$

Note that the independence gives in addition that for any $u > 0$

$$\mathbb{P}(R_n \le u) \ge \mathbb{P}\Big(\max_{k=1,\dots,n} R_k - R_{k-1} \le u/n\Big) = \mathbb{P}(R_1 \le u/n)^n.$$

Let $\alpha > 1$. Using the inequality above and the equivalence relation (1) we get that for sufficiently small $\delta > 0$

$$H_\delta(t) \ge \sum_{1\le n\le A/\delta} \delta(1-\delta)^{n-1}\left(1-\alpha\frac{\pi}{t/\delta - \log n}\right)^n \ge \sum_{1\le n\le A/\delta} \delta(1-\delta)^{n-1}\left(1-\alpha^2\frac{\pi\delta}{t}\right)^n.$$

Evaluating the limit when $\delta \to 0$ of the geometric sum and then letting $A \to \infty$ we end up with $\liminf_{\delta\to 0} H_\delta(t) \ge \frac{1}{1+\pi/t}$, which gives the result. □

3. Random walk on the plane

3.1. *Almost sure convergence*

We now consider a true random walk on $\mathbb{R}^2$, $S_n = X_1 + \cdots + X_n$ where the X_i's are i.i.d. random variables distributed with a law μ of zero mean, with (invertible) covariance matrix Σ^2 and characteristic function $\hat{\mu}(t) = \int \mathrm{e}^{\mathrm{i}t\cdot x}\,\mathrm{d}\mu(x)$. Let τ_ε be the first return time of the walk in the ε-neighbourhood of the origin:

$$\tau_\varepsilon := \min\{n \ge 1\colon\ |S_n| < \varepsilon\}.$$

Let $\Omega^* = \{\forall n \ge 1, S_n \ne 0\}$. We notice that outside Ω^* the return time τ_ε is obviously bounded by the first time $n \ge 1$ for which $S_n = 0$. Therefore, we will only consider the asymptotic behaviour of τ_ε on Ω^*.

Theorem 5. *Assume additionally that the distribution μ satisfies the Cramer condition*

$$\limsup_{|t|\to\infty} \big|\hat{\mu}(t)\big| < 1.$$

Then on Ω^* we have $\lim_{\varepsilon\to 0} \frac{\log\log \tau_\varepsilon}{-\log\varepsilon} = 2$ almost surely.

We remark that a kind of Cramer's condition on the law is necessary, since there exist some planar recurrent random walks for which the statement of the theorem is false (the return time being even larger than expected). We discovered after the completion of the proof of this theorem that its statement is contained in Theorem 2 of Cheliotis's recent paper [5]. For completeness we describe the strategy of our original proof. A key point is a uniform version of the local limit theorem. Indeed we need an estimation of the type $\mathbb{P}(|S_n| < \varepsilon) \sim \frac{c\varepsilon^2}{n}$, with some uniformity in ε (for some $c > 0$). We will follow the classical proof of the local limit theorem (see Theorem 10.17 of [3]) to get the following (see Section 3.3 for its proof):

Lemma 6. *There exists $a > 0$, $\varepsilon_0 > 0$, an integer N, a sequence $\kappa_n \to 0$ such that, for any $n > N$, any $\varepsilon \in (0, \varepsilon_0)$ and for any $A \in \mathbb{R}^2$ we have*

$$\left|\mathbb{P}\big(S_n \in A + [-\varepsilon, \varepsilon]^2\big) - \frac{\gamma\varepsilon^2}{n}\exp\left(-\frac{\Sigma^{-2}A \cdot A}{2n}\right)\right| \le \frac{\varepsilon^2\kappa_n}{n} + \frac{\exp(-a\sqrt{n})}{\varepsilon^2},$$

with $\gamma = \frac{2}{\pi\sqrt{\det \Sigma^2}}$.

Then, this information on the probability of return is strong enough to estimate the first return time to the ε-neighbourhood of the origin.

Proof of Theorem 5. For any $\alpha > \frac{1}{2}$, using $\varepsilon_n = 1/\log^\alpha n$, we get that $\mathbb{P}(|S_n| < \varepsilon_n)$ is summable. By the Borel–Cantelli lemma, for almost every $x \in \Omega^*$, for all n large enough, we have $\tau_{\varepsilon_n}(x) > n$ and thus:

$$\liminf_{n\to\infty} \frac{\log\log \tau_{\varepsilon_n}(x)}{-\log \varepsilon_n} \ge \liminf_{n\to\infty} \frac{\log\log n}{\log\log^\alpha n} = \frac{1}{\alpha},$$

which implies by monotonicity and the fact that α is arbitrary that $\liminf_{\varepsilon\to 0} \frac{\log\log\tau_\varepsilon(x)}{-\log\varepsilon} \ge 2$.

Let $\alpha < \frac{1}{2}$. To control the lim sup, we will take $n = n_\varepsilon = \lceil \exp(\varepsilon^{-1/\alpha})\rceil$. Let $\gamma' < \gamma$ and $m = (\log \varepsilon)^4$. We use a similar decomposition to that of Dvoretsky and Erdös in [7]. Let $A_k = \{|S_k| < \varepsilon \text{ and } \forall p = k+1, \dots, n, |S_p - S_k| > 2\varepsilon\}$. The A_k's are disjoint, hence by independence and invariance, and according to Lemma 6, for ε small enough, we have:

$$1 \ge \sum_{k=m}^{n} \mathbb{P}(A_k) \ge \sum_{k=m}^{n} \mathbb{P}\big(|S_k| < \varepsilon\big)\mathbb{P}(\tau_{2\varepsilon} > n-k) \ge \sum_{k=m}^{n} \mathbb{P}\big(|S_k| < \varepsilon\big)\mathbb{P}(\tau_{2\varepsilon} > n) \ge \sum_{k=m}^{n} \frac{\gamma'\varepsilon^2}{k}\mathbb{P}(\tau_{2\varepsilon} > n).$$

Hence we have $\mathbb{P}(\tau_\varepsilon > n) \le \frac{1}{c\varepsilon^2 \log n} \le c\varepsilon^{1/\alpha - 2}$, if n is large enough. Let $\varepsilon_p = p^{-2/(\alpha-2)}$. By the Borel–Cantelli lemma we have $\log \tau_{\varepsilon_p} \le \varepsilon_p^{-1/\alpha}$ almost surely, hence $\limsup_{p\to\infty} \frac{\log\log\tau_{\varepsilon_p}}{-\log\varepsilon_p} \le \frac{1}{\alpha}$. By monotonicity and the fact that α is arbitrary we get the result. □

3.2. *Limit distribution of return times in neighbourhoods*

We introduce a slight modification of the model. Let $\varepsilon > 0$. Let M_0^ε be uniformly distributed on the ball $B(0, \varepsilon)$ and independent[3] of the sequence (S_n) introduced in the Section 3.1.

We define the random walk $M_n^\varepsilon = M_0^\varepsilon + S_n$. We are interested in the limiting distribution of the first return time in the ball $B(0, \varepsilon)$:

$$\tilde{\tau}_\varepsilon = \inf\big\{n \ge 1\colon \left|M_n^\varepsilon\right| < \varepsilon\big\}.$$

Theorem 7. *Under the hypotheses of Theorem 5, the random variable $\varepsilon^2 \log \tilde{\tau}_\varepsilon$ converges in distribution to a random variable Y with distribution* $\mathbb{P}(Y > t) = \frac{\mathbb{P}(\Omega^*)}{1 + 2\sqrt{\det(\Sigma^{-2})}\mathbb{P}(\Omega^*)t}$ $(t > 0)$.

Proof. To simplify notations, we omit the indices ε.

Upper bound: Fix $R > 0$ and an integer $K > 0$. Let

$$\Gamma = \big\{\forall i = 1, \dots, K,\ M_i \ne M_0 \text{ and } |M_i| \le R\big\}.$$

[3] Using if necessary a larger probability space, let U be a random variable uniformly distributed in the ball $B(0, 1)$ and independent of the sequence (S_n). Then one can take $M_0^\varepsilon = \varepsilon U$.

Using the same decomposition as in the proof of Theorem 5, we have

$$\mathbb{P}(\Gamma)=\sum_{k=0}^{n}\mathbb{P}\big(\Gamma\cap\big\{|M_k|<\varepsilon-2\nu \text{ and } \forall\ell=k+1,\dots,n,|M_\ell|\geq\varepsilon-2\nu\big\}\big),$$

where n is equal to $\lfloor\exp(\frac{t}{\varepsilon^2})\rfloor$ for some $t>0$. Let $\nu=\varepsilon^2$ and $\mathcal{A}=\frac{1}{2\sqrt{2}\nu}\mathbb{Z}^2\cap B(0,\varepsilon-3\nu)$. Notice that the sets $Q_a:=(a-\nu/\sqrt{2},a+\nu/\sqrt{2})^2$ with $a\in\mathcal{A}$ are pairwise disjoints and contained in $B(0,\varepsilon-2\nu)$. Let $m=(\log\varepsilon)^4$. Hence we have

$$\begin{aligned}\mathbb{P}(\Gamma)&\geq\mathbb{P}\big(\Gamma\cap\big\{\forall\ell=1,\dots,n\ |M_\ell|\geq\varepsilon\big\}\big)\\&\qquad+\sum_{k=m}^{n}\sum_{a\in\mathcal{A}}\mathbb{P}\big(\Gamma\cap\big\{M_k\in Q_a \text{ and } \forall\ell=k+1,\dots,n,\ M_\ell\notin B(0,\varepsilon-2\nu)\big\}\big)\\&\geq\mathbb{P}\big(\Gamma\cap\big\{\forall\ell=1,\dots,n,\ |M_\ell|\geq\varepsilon\big\}\big)\\&\qquad+\sum_{k=m}^{n}\sum_{a\in\mathcal{A}}\mathbb{P}\big(\Gamma\cap\big\{M_k\in Q_a \text{ and } \forall\ell=k+1,\dots,n,\ S_\ell-S_k\notin B(-a,\varepsilon-\nu)\big\}\big)\\&\geq\mathbb{P}\big(\Gamma\cap\{\tilde\tau_\varepsilon>n\}\big)+\sum_{k=m}^{n}\sum_{a\in\mathcal{A}}\mathbb{P}\big(\Gamma\cap\{M_k\in Q_a\}\big)\mathbb{P}\big(\forall\ell=1,\dots,n,\ S_\ell\notin B(-a,\varepsilon-\nu)\big)\end{aligned}$$

by independence (since $m>K$ whenever ε is sufficiently small). Note that

$$\mathbb{P}\big(\Gamma\cap\{M_k\in Q_a\}\big)=\int_{\{\forall i,x_i\neq x_0,|x_i|\leq R\}}\mathbb{P}(S_{k-K}\in -x_K+Q_a)\,\mathrm{d}\mathbb{P}_{(M_0,\dots,M_K)}(x_0,\dots,x_K).$$

According to Lemma 6 we get

$$\forall k\geq m\quad \mathbb{P}\big(\Gamma\cap\{M_k\in Q_a\}\big)\geq\mathbb{P}(\Gamma)\frac{\gamma'\nu^2}{2k}$$

for some fixed $\gamma'<\gamma$, provided ε is sufficiently small. Hence we have

$$\begin{aligned}&\sum_{k=m}^{n}\sum_{a\in\mathcal{A}}\mathbb{P}\big(\Gamma\cap\{M_k\in Q_a\}\big)\mathbb{P}\big(\forall\ell=1,\dots,n,S_\ell\notin B(-a,\varepsilon-\nu)\big)\\&\quad\geq\sum_{k=m}^{n}\sum_{a\in\mathcal{A}}\mathbb{P}(\Gamma)\frac{\gamma'\nu^2}{2k}\mathbb{P}\big(\forall\ell=1,\dots,n,M_\ell\notin B(0,\varepsilon)|M_0\in Q_a\big)\\&\quad\geq\sum_{k=m}^{n}\sum_{a\in\mathcal{A}}\mathbb{P}(\Gamma)\frac{\gamma'\pi\varepsilon^2}{k}\mathbb{P}(M_0\in Q_a)\mathbb{P}\big(\forall\ell=1,\dots,n,M_\ell\notin B(0,\varepsilon)|M_0\in Q_a\big)\\&\quad\geq\sum_{k=m}^{n}\mathbb{P}(\Gamma)\frac{\gamma'\pi\varepsilon^2}{k}\big(\mathbb{P}\big(\forall\ell=1,\dots,n,M_\ell\notin B(0,\varepsilon)\big)-\mathbb{P}\big(\varepsilon-4\nu\leq|M_0|\leq\varepsilon\big)\big)\\&\quad\geq\mathbb{P}(\Gamma)\gamma'\pi\varepsilon^2\log(n)\big(1+\mathrm{o}(1)\big)\mathbb{P}(\tilde\tau_\varepsilon>n)-\mathrm{o}(1).\end{aligned}$$

Therefore since $\varepsilon^2\log n=t+\mathrm{o}(1)$ we get

$$\begin{aligned}\mathbb{P}(\Gamma)&\geq\mathbb{P}\big(\Gamma\cap\{\tilde\tau_\varepsilon>n\}\big)+\gamma'\pi t\mathbb{P}(\Gamma)\mathbb{P}(\tilde\tau_\varepsilon>n)+\mathrm{o}(1)\\&\geq\mathbb{P}\big(\{\tilde\tau_\varepsilon>n\}\big)-\mathbb{P}\big(\exists 1\leq i\leq K,|M_i|>R\big)+\gamma'\pi t\mathbb{P}(\Gamma)\mathbb{P}(\tilde\tau_\varepsilon>n)+\mathrm{o}(1).\end{aligned}$$

Hence we have

$$\limsup_{\varepsilon\to 0}\mathbb{P}\left(\tilde\tau_\varepsilon > \exp\left(\frac{t}{\varepsilon^2}\right)\right) \le \frac{\mathbb{P}(\Gamma)+\mathbb{P}(\exists 1\le i\le K, |M_i|>R)}{1+\gamma'\pi t\mathbb{P}(\Gamma)}.$$

Taking $R\to\infty$ first then $K\to\infty$ we obtain

$$\limsup_{\varepsilon\to 0}\mathbb{P}\left(\tilde\tau_\varepsilon > \exp\left(\frac{t}{\varepsilon^2}\right)\right) \le \frac{\mathbb{P}(\Omega^*)}{1+\gamma'\pi t\mathbb{P}(\Omega^*)}.$$

This holds for all $\gamma'<\gamma$, which gives the upper bound.

Lower bound: We only provide a sketch proof since the arguments are very similar. Let $m=\lfloor(\log\varepsilon)^4\rfloor$, $n=\lfloor\exp(t/\varepsilon^2)\rfloor$, $\nu=\varepsilon^2$. We have:

$$\mathbb{P}(\Gamma)\le\mathbb{P}\big(\Gamma\cap\{\tilde\tau_\varepsilon>n\log n\}\big)+\sum_{k=1}^{m}p_k+\sum_{k=m}^{n\log n-n}p_k+\sum_{k=n\log n-n}^{n\log n}p_k,$$

where $p_k=\mathbb{P}(\Gamma\cap\{|M_k|<\varepsilon+2\nu \text{ and } \forall\ell=k+1,\dots,n, |M_\ell|\ge\varepsilon+2\nu\})$.

By Theorem 5 we have

$$\sum_{k=1}^{m}p_k\le\mathbb{P}\big(\Gamma\cap\{\tau_{3\varepsilon}\le m\}\big)\le\mathbb{P}\big(\Gamma\setminus\Omega^*\big)+\mathrm{o}(1)$$

since almost sure convergence implies convergence in probability.

Next, taking $\mathcal{A}'=\frac{1}{2\sqrt{2}\nu}\mathbb{Z}^2\cap B(0,\varepsilon+3\nu)$, $\bar Q_a:=[a-\nu/\sqrt{2},a+\nu/\sqrt{2}]^2$ and using the same arguments as for the upper bound we get

$$\begin{aligned}\sum_{k=m}^{n\log n-n}p_k &\le \sum_{k=m}^{n\log n-n}\sum_{a\in\mathcal{A}'}\mathbb{P}\big(\Gamma\cap\big\{M_k\in\bar Q_a \text{ and } \forall\ell=k+1,\dots,n,\ M_\ell\notin B(0,\varepsilon-2\nu)\big\}\big)\\ &\le\mathbb{P}(\Gamma)\gamma'\pi\varepsilon^2\log(n\log n-n)\mathbb{P}(\tilde\tau_\varepsilon>n)+\mathrm{o}(1),\end{aligned}$$

where now $\gamma'>\gamma$.

Finally,

$$\sum_{k=n\log n-n}^{n\log n}p_k\le\sum_{k=n\log n-n}^{n\log n}\mathbb{P}\big(\Gamma\cap\big\{|M_k|\le\varepsilon+2\nu\big\}\big)\le\sum_{k=n\log n-n}^{n\log n}\mathbb{P}(\Gamma)\frac{\gamma' 4\varepsilon^2}{k}=\mathrm{o}(1).$$

Putting these estimates together we end up with

$$\mathbb{P}(\Gamma)\le\mathbb{P}\big(\Gamma\cap\{\tilde\tau_\varepsilon>n\}\big)+\mathrm{o}(1)+\mathbb{P}\big(\Gamma\setminus\Omega^*\big)+\mathbb{P}(\Gamma)\gamma'\pi t\mathbb{P}(\tilde\tau_\varepsilon>n).$$

This gives

$$\liminf_{\varepsilon\to 0}\mathbb{P}\left(\tilde\tau_\varepsilon > \exp\left(\frac{t}{\varepsilon^2}\right)\right) \ge \frac{\mathbb{P}(\Gamma)-\mathbb{P}(\Gamma\setminus\Omega^*)+\mathrm{o}(1)}{1+\gamma'\pi t\mathbb{P}(\Gamma)}.$$

And the conclusion follows as for the upper bound. $\square$

3.3. *Proof of Lemma* 6

Proof. Fix $A \in \mathbb{R}^2$. Let $\varepsilon > 0$. Let $n \geq 2$. For any $0 < a < b$, let $h_{a,b}(t)$ be the piecewise affine function equal to 1 on $[-a, a]$, to 0 outside $[-b, b]$ and with slope $\frac{1}{b-a}$ in between. Let $H_{a,b}(x_1, x_2) = h_{a,b}(x_1)h_{a,b}(x_2)$. Let us notice that we have $H_{\varepsilon-\varepsilon/n,\varepsilon} \leq 1_{[-\varepsilon,\varepsilon]^2} \leq H_{\varepsilon,\varepsilon+\varepsilon/n}$ and thus

$$\mathbb{E}\big(H_{\varepsilon-\varepsilon/n,\varepsilon}(S_n - A)\big) \leq \mathbb{P}\big(|S_n - A|_\infty \leq \varepsilon\big) \leq \mathbb{E}\big(H_{\varepsilon,\varepsilon+\varepsilon/n}(S_n - A)\big).$$

For any $H \in L^1(\mathbb{R}^2)$ we define its Fourier transform $\hat{H}$ and its inverse Fourier transform $\check{H}$ by:

$$\forall u \in \mathbb{R}^2 \quad \hat{H}(u) = \frac{1}{2\pi} \int_{\mathbb{R}^2} H(x) \exp(-\mathrm{i}xu)\,\mathrm{d}x \quad \text{and} \quad \check{H}(u) = \hat{H}(-u).$$

With this definition, we immediately get that $\|\hat{H}_{a,b}\|_{L^\infty} = \frac{1}{2\pi}\|H_{a,b}\|_{L^1} = \frac{(a+b)^2}{2\pi}$ and after an easy computation $\|\hat{H}\|_{L^1} \leq (a + b + \frac{4}{b-a})^2$.

Let us notice that we have for all $x \in \mathbb{R}^2$, $H_{a,b}(x) = \check{\hat{H}}_{a,b}(x)$. Hence with the use of Fubini's theorem we have

$$\begin{aligned}
\mathbb{E}\big[H_{a,b}(S_n - A)\big] &= \mathbb{E}\big[\check{\hat{H}}_{a,b}(S_n - A)\big] \\
&= \mathbb{E}\left[\frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{H}_{a,b}(u) \exp\big(\mathrm{i}u(S_n - A)\big)\,\mathrm{d}u\right] \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{H}_{a,b}(u)\mathbb{E}\big[\exp\big(\mathrm{i}u(S_n - A)\big)\big]\,\mathrm{d}u \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{H}_{a,b}(u)\big(\mathbb{E}\big[\exp(\mathrm{i}uX_1)\big]\big)^n \mathrm{e}^{-\mathrm{i}uA}\,\mathrm{d}u.
\end{aligned}$$

Denote the characteristic function of X_1 by $\phi(u) = \mathbb{E}[\exp(\mathrm{i}uX_1)]$. Because of the hypothesis on the distribution of X_1, we know that, if u is non-null, then $|\phi(u)| < 1$. Let us recall that we have:

$$\phi(u) = 1 - \frac{1}{2}\Sigma^2 u \cdot u + g(u)\Sigma^2 u \cdot u, \quad \text{with } \lim_{u \to 0} g(u) = 0.$$

Let us fix some $\beta > 0$ such that:

$$\forall u \in B(0, \beta) \quad \left|\phi(u) - \left(1 - \frac{1}{2}\Sigma^2 u \cdot u\right)\right| \leq \frac{1}{4}\Sigma^2 u \cdot u$$

and such that:

$$\sup_{\|u\|_2 > \beta} \big|\phi(u)\big| < 1 - \frac{1}{4}\alpha\beta^2,$$

where α is the smallest modulus of the eigenvalues of Σ^2. Now we take $(a, b) = (\varepsilon, \varepsilon + \frac{\varepsilon}{n})$ or $(a, b) = (\varepsilon - \frac{\varepsilon}{n}, \varepsilon)$. Hence we have:

$$\begin{aligned}
\left|\int_{\|u\|_2 > \beta n^{-1/6}} \hat{H}_{a,b}(u)\big(\phi(u)\big)^n \mathrm{e}^{-\mathrm{i}uA}\,\mathrm{d}u\right| &\leq \|\hat{H}_{a,b}\|_{L^1}\left(1 - \frac{\alpha\beta^2 n^{-1/3}}{4}\right)^n \\
&\leq \left(3\varepsilon + \frac{4n}{\varepsilon}\right)^2 \exp\left(-\frac{\alpha\beta^2 n^{2/3}}{4}\right).
\end{aligned} \tag{3}$$

Hence, it remains to estimate the following quantity:

$$\int_{B(0,\beta n^{-1/6})} \hat{H}_{a,b}(u)\big(\phi(u)\big)^n \mathrm{e}^{-\mathrm{i}uA}\,\mathrm{d}u = \frac{1}{n}\int_{B(0,\beta n^{1/3})} \hat{H}_{a,b}\big(v/\sqrt{n}\big)\big(\phi\big(v/\sqrt{n}\big)\big)^n \mathrm{e}^{-\mathrm{i}vA/\sqrt{n}}\,\mathrm{d}v.$$

We will compare this quantity to $\frac{1}{n}\int_{\mathbb{R}^2} \hat{H}_{a,b}(0)\exp(-\frac{1}{2}\Sigma^2 v\cdot v)\mathrm{e}^{-\mathrm{i}vA/\sqrt{n}}\,\mathrm{d}v$. First we have:

$$\begin{aligned}
&\left|\frac{1}{n}\int_{B(0,\beta n^{1/3})} \hat{H}_{a,b}\big(v/\sqrt{n}\big)\mathrm{e}^{-\mathrm{i}vA/\sqrt{n}}\left\{\phi\big(v/\sqrt{n}\big)^n - \exp\left(-\frac{1}{2}\Sigma^2 v\cdot v\right)\right\}\mathrm{d}v\right|\\
&\quad\le \frac{1}{n}\int_{B(0,\beta n^{1/3})} \big|\hat{H}_{a,b}\big(v/\sqrt{n}\big)\big| n\exp\left(-\frac{n-1}{4n}\Sigma^2 v\cdot v\right)\left[\frac{1}{n}g_1\left(\frac{v}{\sqrt{n}}\right)\Sigma^2 v\cdot v\right]\mathrm{d}v\\
&\quad\le \frac{\|\hat{H}_{a,b}\|_{L^\infty}}{n}\sup_{w\in B(0,\beta n^{-1/6})} g_1(w)\int_{\mathbb{R}^2}\exp\left(-\frac{1}{8}\Sigma^2 v\cdot v\right)\big[\Sigma^2 v\cdot v\big]\,\mathrm{d}v\\
&\quad\le \frac{4\varepsilon^2}{n}\sup_{w\in B(0,\beta n^{-1/6})} g_1(w)\int_{\mathbb{R}^2}\exp\left(-\frac{1}{8}\Sigma^2 v\cdot v\right)\big[\Sigma^2 v\cdot v\big]\,\mathrm{d}v \le \frac{\varepsilon^2}{n}\kappa_n,
\end{aligned}\tag{4}$$

with $\lim_{u\to 0} g_1(u) = 0$ and $\lim_{n\to+\infty}\kappa_n = 0$.

Second we have:

$$\begin{aligned}
&\frac{1}{n}\int_{B(0,\beta n^{1/3})}\exp\left(-\frac{1}{2}\Sigma^2 v\cdot v\right)\big|\hat{H}_{a,b}\big(v/\sqrt{n}\big) - \hat{H}_{a,b}(0)\big|\,\mathrm{d}v\\
&\quad\le \frac{1}{n}\sup_{w\in B(0,\beta n^{-1/6})}\big|\nabla\hat{H}_{a,b}(w)\big|_\infty \int_{B(0,\beta n^{1/3})}\frac{|v|_2}{\sqrt{n}}\exp\left(-\frac{1}{2}\Sigma^2 v\cdot v\right)\mathrm{d}v\\
&\quad\le \frac{1}{n}\frac{2\varepsilon\lambda(B(0,2\varepsilon))}{2\pi}\int_{\mathbb{R}^2}\frac{|v|_2}{\sqrt{n}}\exp\left(-\frac{1}{2}\Sigma^2 v\cdot v\right)\mathrm{d}v \le \frac{\varepsilon^2}{n}\kappa_n',
\end{aligned}\tag{5}$$

with $\lim_{n\to+\infty}\kappa_n' = 0$.

Third we have:

$$\frac{1}{n}\int_{\mathbb{R}^2\setminus B(0,\beta n^{1/3})}\exp\left(-\frac{1}{2}\Sigma^2 v\cdot v\right)\big|\hat{H}_{a,b}(0)\big|\,\mathrm{d}v \le \frac{\varepsilon^2}{n}\kappa_n'',\tag{6}$$

with $\lim_{n\to+\infty}\kappa_n'' = 0$.

Hence, since $\hat{H}_{a,b}(0) = \frac{1}{2\pi}\|H_{a,b}\|_{L^1}$, to estimate $\mathbb{E}(H_{a,b}(S_n - A))$ we are led to study the quantity

$$\frac{\|H_{a,b}\|_{L^1}}{n4\pi^2}\int_{\mathbb{R}^2}\exp\big(-\mathrm{i}vA/\sqrt{n}\big)\exp\left(-\frac{1}{2}\Sigma^2 v\cdot v\right)\mathrm{d}v = \frac{\|H_{a,b}\|_{L^1}}{n4\pi^2}\frac{2\pi\exp(-\Sigma^{-2}A\cdot A/(2n))}{\sqrt{\det\Sigma^2}}.$$

To conclude we notice that $|\|H_{a,b}\|_{L^1} - 4\varepsilon^2| \le 10\frac{\varepsilon^2}{n}$. □

4. Case of Euclidean extensions of subshifts of finite type

4.1. *Description of the $\mathbb{Z}^2$-extensions of a mixing subshift*

Let us fix a finite set $\mathcal{A}$ called alphabet. Let us consider a matrix M indexed by $\mathcal{A}\times\mathcal{A}$ with 0–1 entries. We suppose that there exists a positive integer n_0 such that each component of M^{n_0} is non-zero. We define the set of allowed sequences Σ as follows:

$$\Sigma := \big\{\omega := (\omega_n)_{n\in\mathbb{Z}}\colon \forall n\in\mathbb{Z},\ M(\omega_n,\omega_{n+1}) = 1\big\}.$$

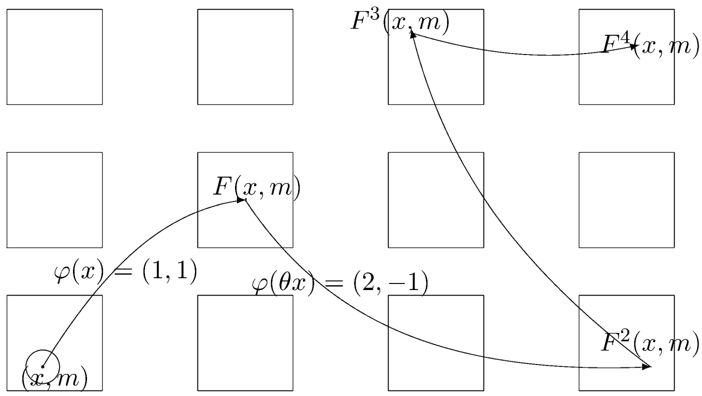

Fig. 1. Dynamics of the $\mathbb{Z}^2$-extension F of the shift.

We endow Σ with the metric d given by

$$d(\omega, \omega') := \mathrm{e}^{-m},$$

where m is the greatest integer such that $\omega_i = \omega'_i$ whenever $|i| < m$. We define the shift $\theta : \Sigma \to \Sigma$ by $\theta((\omega_n)_{n\in\mathbb{Z}}) = (\omega_{n+1})_{n\in\mathbb{Z}}$. For any function $f : \Sigma \to \mathbb{R}$ we denote by $S_n f = \sum_{\ell=0}^{n-1} f \circ \theta^\ell$ its ergodic sum. Let us consider an Hölder continuous function $\varphi : \Sigma \to \mathbb{Z}^2$. We define the $\mathbb{Z}^2$-extension F of the shift θ by (see Fig. 1)

$$F : \Sigma \times \mathbb{Z}^2 \to \Sigma \times \mathbb{Z}^2,$$
$$(x, m) \mapsto \big(\theta x, m + \varphi(x)\big).$$

We want to know the time needed for a typical orbit starting at $(x, m) \in \Sigma \times \mathbb{Z}^2$ to return ε-close to the initial point after iterations of the map F. By translation invariance we can assume that the orbit starts in the cell $m = 0$. More precisely, let

$$\tau_\varepsilon(x) = \min\big\{n \ge 1:\ F^n(x, 0) \in B(x, \varepsilon) \times \{0\}\big\}.$$

Observe that $F^n(x, m) = (\theta^n x, m + S_n\varphi(x))$, thus

$$\tau_\varepsilon(x) = \min\big\{n \ge 1:\ S_n\varphi(x) = 0 \text{ and } d\big(\theta^n x, x\big) < \varepsilon\big\}.$$

Let ν be the Gibbs measure associated to some Hölder continuous potential h, and denote by σ_h^2 the asymptotic variance of h under the measure ν. Recall that σ_h^2 vanishes if and only if h is cohomologous to a constant, and in this case ν is the unique measure of maximal entropy.

We know that there exists a positive integer m_0 such that the function φ is constant on each m_0-cylinder.

Let us denote by σ_φ^2 the asymptotic covariance matrix of φ:

$$\sigma_\varphi^2 = \lim_{n\to+\infty} \mathrm{Cov}_\nu\left(\frac{1}{\sqrt{n}} S_n\varphi\right).$$

We suppose that $\int_\Sigma \varphi \, \mathrm{d}\nu = 0$.

We add the following hypothesis of non-arithmeticity on φ: We suppose that, for any $u \in [-\pi; \pi]^2 \setminus \{(0, 0)\}$, the only solutions (λ, g), with $\lambda \in \mathbb{C}$ and $g : \Sigma \to \mathbb{C}$ measurable with $|g| = 1$, of the functional equation

$$g \circ \sigma \overline{g} = \lambda \mathrm{e}^{\mathrm{i}u\cdot\varphi}$$

is the trivial one $\lambda = 1$ and $g = \mathit{const}$.

Note that this condition implies that σ_φ^2 is invertible. If this non-arithmeticity condition was not satisfied, then the range of $S_n\varphi$ would be essentially contained in a sub-lattice and we could make a change of variable (and eventually of dimension) to reduce our study to the corresponding twisted $\mathbb{Z}^a$-extension (where a is the rank of σ_φ^2). Moreover, we emphasize that this non-arithmeticity condition is equivalent to the fact that all the circle extensions T_u defined by $T_u(x,t)=(\theta(x), t+u\cdot\varphi(x))$ are weakly-mixing for $u\in[-\pi;\pi]^2\setminus\{(0,0)\}$.

In this context, we prove the following results:

Theorem 8. *The sequence of random variables* $\frac{\log\log\tau_\varepsilon}{-\log\varepsilon}$ *converges almost surely as* $\varepsilon\to 0$ *to the Hausdorff dimension* d *of the measure* ν.

Theorem 9. *The sequence of random variables* $\nu(B_\varepsilon(\cdot))\log\tau_\varepsilon(\cdot)$ *converges in distribution as* $\varepsilon\to 0$ *to a random variable with distribution function* $t\mapsto\frac{\beta t}{1+\beta t}\mathbf{1}_{(0;+\infty)}(t)$, *with* $\beta:=\frac{1}{2\pi\sqrt{\det\sigma_\varphi^2}}$.

Corollary 10. *If the measure* ν *is not the measure of maximal entropy, then the sequence of random variables* $\frac{\log\log\tau_\varepsilon+d\log\varepsilon}{\sqrt{-\log\varepsilon}}$ *converges in distribution as* $\varepsilon\to 0$ *to a centered Gaussian random variable of variance* $2\sigma_h^2$.

In the case where ν *is the measure of maximal entropy, then the sequence of random variables* $\varepsilon^d\log\tau_\varepsilon$ *converges in distribution to a finite mixture of the law found in the previous theorem, that is, there exists some probability vector* $\alpha=(\alpha_n)$ *and positive constants* β_n *such that the sequence of random variables* $\varepsilon^d\log\tau_\varepsilon$ *converges in distribution to a random variable with distribution function* $\sum_n\alpha_n\frac{\beta_n t}{1+\beta_n t}\mathbf{1}_{(0;+\infty)}(t)$.

Example 11. *We provide an example where the function* $\varphi(x)$ *only depends on the first coordinate* x_0, *that is,* $\varphi(x)=\varphi(x_0)$. *On the shift* $\Sigma=\{I,E,N,W,S\}^{\mathbb{Z}}$ *the function* φ *given by* $\varphi(I)=(0,0)$, $\varphi(E)=(1,0)$, $\varphi(N)=(0,1)$, $\varphi(W)=(-1,0)$ *and* $\varphi(S)=(0,-1)$ *fulfills the hypothesis.*

The remaining part of the section is devoted to the proof of these results. In Section 4.2 we recall some preliminary results and prove a uniform conditional local limit theorem. In Section 4.3 we prove Theorem 8 and in Section 4.4 we prove Theorem 9 and Corollary 10.

4.2. *Spectral analysis of the Perron–Frobenius operator and local limit theorem*

In order to exploit the spectral properties of the Perron–Frobenius operator we quotient out the "past." We define:

$$\hat\Sigma:=\bigl\{\omega:=(\omega_n)_{n\in\mathbb{N}}:\ \forall n\in\mathbb{N},\ M(\omega_n,\omega_{n+1})=1\bigr\},$$
$$\hat d\bigl((\omega_n)_{n\ge 0},(\omega'_n)_{n\ge 0}\bigr):=e^{-\hat r(\omega,\omega')}$$

with $\hat r((\omega_n)_{n\ge0},(\omega'_n)_{n\ge0})=\inf\{m\ge 0:\ \omega_m\ne\omega'_m\}$ and

$$\hat\theta\bigl((\omega_n)_{n\ge0}\bigr)=(\omega_{n+1})_{n\ge0}.$$

Let us define the canonical projection $\Pi:\Sigma\to\hat\Sigma$ by $\pi((\omega_n)_{n\in\mathbb{Z}})=(\omega_n)_{n\ge0}$. Let $\hat\nu$ be the image probability measure (on $\hat\Sigma$) of ν by Π. There exists a function $\psi:\hat\Sigma\to\mathbb{Z}^2$ such that $\psi\circ\Pi=\varphi\circ\theta^{m_0}$.

Let us denote by $P:L^2(\hat\nu)\to L^2(\hat\nu)$ the Perron–Frobenius operator such that:

$$\forall f,g\in L^2(\hat\nu)\qquad\int_{\hat\Sigma}Pf(x)g(x)\,d\hat\nu(x)=\int_{\hat\Sigma}f(x)g\circ\hat\theta(x)\,d\hat\nu(x).$$

Let $\eta\in(0,1)$. Let us denote by $\mathcal{B}$ the set of bounded η-Hölder continuous function $g:\hat\Sigma\to\mathbb{C}$ endowed with the usual Hölder norm:

$$\|g\|_{\mathcal{B}}:=\|g\|_\infty+\sup_{x\ne y}\frac{|g(y)-g(x)|}{\hat d(x,y)^\eta}.$$

We denote by $\mathcal{B}^*$ the topological dual of $\mathcal{B}$. For all $u \in \mathbb{R}^2$, we consider the operator P_u defined on $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ by:

$$P_u(f) := P\big(\mathrm{e}^{\mathrm{i}u\psi} f\big).$$

Note that the hypothesis of non-arithmeticity of φ is equivalent to the following one on ψ: *for any* $u \in [-\pi; \pi]^2 \setminus \{(0,0)\}$, *the operator* P_u *has no eigenvalue on the unit circle.*

We will use the method introduced by Nagaev in [11,12], adapted by Guivarc'h and Hardy in [8] and extended by Hennion and Hervé in [10]. It is based on the family of operators $(P_u)_u$ and their spectral properties expressed in these two propositions.

Proposition 12 (Uniform contraction). *There exist* $\alpha \in (0,1)$ *and* $c > 0$ *such that, for all* $u \in [-\pi; \pi]^2 \setminus [-\beta; \beta]^2$, *for all integers* $n \ge 0$ *and for all* $f \in \mathcal{B}$, *we have*:

$$\big\| P_u^n(f)\big\|_{\mathcal{B}} \le c\alpha^n \|f\|_{\mathcal{B}}.$$

This property easily follows from the fact that the spectral radius is smaller than 1 for $u \neq 0$. In addition, since P is a quasicompact operator on $\mathcal{B}$ and since $u \mapsto P_u$ is a regular perturbation of $P_0 = P$, we have:

Proposition 13 (Perturbation result). *There exist* $\alpha > 0$, $\beta > 0$, $c_1 > 0$, $c_2 > 0$, $\theta \in (0,1)$, $u \mapsto \lambda_u$ *belonging to* $C^3([-\beta; \beta]^2 \to \mathbb{C})$, $u \mapsto v_u$ *belonging to* $C^3([-\beta; \beta]^2 \to \mathcal{B})$ *and* $u \mapsto \phi_u$ *belonging to* $C^3([-\beta; \beta]^2 \to \mathcal{B}')$ *such that, for all* $u \in [-\beta; \beta]^2$, *for all* $f \in \mathcal{B}$ *and for all* $n \ge 0$, *we have the decomposition*:

$$P_u{}^n(f) = \lambda_u{}^n \phi_u(f) v_u + N_u^n(f),$$

with:

(1) $\|N_u{}^n(f)\|_{\mathcal{B}} \le c_2 \alpha^n \|f\|_{\mathcal{B}}$,
(2) $|\lambda_u| \le \mathrm{e}^{-c_1|u|^2}$ *and* $c_1|u|^2 \le \sigma_\phi^2 u \cdot u$,
(3) *with initial values*: $v_0 = \mathbf{1}$, $\phi_0 = \hat{\nu}$, $\nabla\lambda_{u=0} = 0$ *and* $D^2\lambda_{u=0} = -\sigma_\varphi^2$.

This result is a multidimensional version of IV-8, IV-11, IV-12 of [10], in this context.

The next proposition is essential to our work. It may be viewed as a doubly local version of the central limit theorem: first, it is local in the sense that we are looking at the probability that $S_n\varphi = 0$ while the classical central limit theorem is only concerned with the probability that $|S_n\varphi| \le \varepsilon\sqrt{n}$; second, it is local in the sense that we are looking at this probability conditioned to the fact that we are starting from a set A and landing at a set B on the base. For any integer $q \ge 0$, we call q-cylinder of Σ any set of the form $\{y \in \Sigma\colon\ d(x,y) < \mathrm{e}^{-q}\}$ (i.e., $\{y \in \Sigma\colon\ \forall i = -q, \dots, q,\ y_i = x_i\}$) for some $x \in \Sigma$.

Proposition 14. *There exists a constant* $c_3 > 0$ *such that, for all integers* $n > k > m_0$ *and all* $q > 0$, *all* q*-cylinders* A *of* Σ *and all measurable subsets* B *of* $\hat{\Sigma}$, *we have*:

$$\left|\nu\big(A \cap \{S_n\varphi = 0\} \cap \theta^{-n}\big(\theta^k\big(\Pi^{-1}(B)\big)\big)\big) - \frac{\nu(A)\hat{\nu}(B)}{2\pi(n-k)\sqrt{\det(\sigma_\varphi^2)}}\right| \le c_3 \frac{\hat{\nu}(B)k\mathrm{e}^{\eta q}}{(n-k)^{3/2}}.$$

Proof. We want to estimate the measure of the set $Q = A \cap \{S_n\varphi = 0\} \cap \theta^{-n}(\theta^k \Pi^{-1} B)$. Since A is a q-cylinder, $\theta^{-q} A = \Pi^{-1}\hat{A}$ for the cylinder set $\hat{A} = \Pi\theta^{-q}A$. Next, since $\varphi \circ \theta^{m_0} = \psi \circ \Pi$ we have the identity $\{S_n\varphi \circ \theta^{m_0} = 0\} = \{S_n\psi \circ \Pi = 0\}$. Thus using the semi-conjugacy $\hat{\theta} \circ \Pi = \Pi \circ \theta$

$$\begin{aligned}\theta^{-q-m_0} Q &= \theta^{-m_0}\big(\Pi^{-1}\hat{A}\big) \cap \big\{S_n\psi \circ \Pi \circ \theta^q = 0\big\} \cap \theta^{-n-q+(k-m_0)}\big(\Pi^{-1}B\big) \\ &= \Pi^{-1}\big(\hat{\theta}^{-m_0}(\hat{A}) \cap \big\{S_n\psi \circ \hat{\theta}^q = 0\big\} \cap \hat{\theta}^{-n-q+(k-m_0)}(B)\big).\end{aligned}$$

Since ψ is integer-valued, the relation $\mathbf{1}_{\{0\}}(k) = \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} e^{iu\cdot k}\,du$ for any $k \in \mathbb{Z}^2$ gives, by invariance of ν,

$$\nu(Q) = \mathbb{E}_{\hat\nu}\left(\mathbf{1}_{\hat A}\circ\hat\theta^{m_0}\mathbf{1}_B\circ\hat\theta^{q+n-(k-m_0)}\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2}\exp\bigl(iu\cdot S_n\psi\circ\hat\theta^q\bigr)\,du\right)$$
$$= \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2}\mathbb{E}_{\hat\nu}\bigl(\mathbf{1}_{\hat A}\circ\hat\theta^{m_0}\mathbf{1}_B\circ\hat\theta^{q+n-(k-m_0)}\exp\bigl(iu\cdot S_n\psi\circ\hat\theta^q\bigr)\bigr)\,du.$$

We then estimate the expectation $a(u) = \mathbb{E}_{\hat\nu}(\cdots)$. Using the fact that the Perron–Frobenius operator P is the dual of $\hat\theta$ we get

$$a(u) = \mathbb{E}_{\hat\nu}\bigl(P^q\bigl(\mathbf{1}_{\hat A}\circ\hat\theta^{m_0}\bigr)\exp(iu\cdot S_n\psi)\mathbf{1}_B\circ\hat\theta^{n-(k-m_0)}\bigr)$$
$$= \mathbb{E}_{\hat\nu}\bigl(P_u^n\bigl(P^q(\mathbf{1}_{\hat A}\circ\hat\theta^{m_0})\mathbf{1}_B\circ\hat\theta^{n-(k-m_0)}\bigr)\bigr)$$
$$= \mathbb{E}_{\hat\nu}\bigl(P_u^{k-m_0}\bigl(\mathbf{1}_B P_u^{n-(k-m_0)}P^q\bigl(\mathbf{1}_{\hat A}\circ\hat\theta^{m_0}\bigr)\bigr)\bigr).$$

Let us denote for simplicity $\ell = n-(k-m_0)$. We first show that for large u, the quantity $a(u)$ is negligeable. Using the contraction inequality given in Proposition 12 applied to ${P_u}^\ell(\mathbf{1})$, the fact that $\|P^q(\mathbf{1}_{\hat A}\circ\hat\theta^{m_0})\|_{\mathcal{B}} \le 1+e^{\eta(q+m_0)}$, and the fact that $|\mathbb{E}_{\hat\nu}[{P_u}^{k-m_0}(\mathbf{1}_B g)]| \le \hat\nu(B)\|g\|_{\mathcal{B}}$, we get whenever $u \notin [-\beta,\beta]^2$,

$$\bigl|a(u)\bigr| \le \mathbb{E}_{\hat\nu}\bigl(\mathbf{1}_B P^\ell P^q\bigl(\mathbf{1}_{\hat A}\circ\hat\theta^{m_0}\bigr)\bigr) = O\bigl(\hat\nu(B)\alpha^\ell e^{\eta q}\bigr). \tag{7}$$

We then estimate the main term, coming from small values of u. The decomposition given in Proposition 13 gives for any $u \in [-\beta,\beta]^2$

$$a(u) = \underbrace{\lambda_u^\ell\phi_u\bigl(P^q\bigl(\mathbf{1}_{\hat A}\circ\hat\theta^{m_0}\bigr)\bigr)\mathbb{E}_{\hat\nu}\bigl[{P_u}^{k-m_0}(\mathbf{1}_B v_u)\bigr]}_{a_1(u)} + \underbrace{\mathbb{E}_{\hat\nu}\bigl[P_u^{k-m_0}\bigl(\mathbf{1}_B N_u^\ell\bigl(P^q\bigl(\mathbf{1}_{\hat A}\circ\hat\theta^{m_0}\bigr)\bigr)\bigr)\bigr]}_{a_2(u)}.$$

Notice that the second term is, by inequality (1) in Proposition 13, of order

$$a_2(u) = O\bigl(\hat\nu(B)\alpha^\ell e^{\eta q}\bigr). \tag{8}$$

Moreover, since $u \mapsto v_u$ and $u \mapsto \phi_u$ are C^1-regular with $v_0 = 1$ and $\phi_0 = \hat\nu$, the first term has the estimate

$$a_1(u) = \lambda_u^\ell\hat\nu(\hat A)\mathbb{E}_{\hat\nu}\bigl[{P_u}^{k-m_0}(\mathbf{1}_B)\bigr] + O\bigl(\lambda_u^\ell|u|\hat\nu(B)e^{\eta q}\bigr)$$
$$= \lambda_u^\ell\hat\nu(\hat A)\hat\nu(B) + O\bigl(\lambda_u^\ell|u|\hat\nu(B)ke^{\eta q}\bigr),$$

where the second estimate is obtained by reintroducing the unperturbed Perron–Frobenius operator P in P_u, $|\mathbb{E}_{\hat\nu}[{P_u}^{k-m_0}(\mathbf{1}_B)] - \hat\nu(B)| = |\mathbb{E}_{\hat\nu}((e^{iu\cdot S_{k-m_0}\psi}-1)\mathbf{1}_B)| \le |u|(k-m_0)\|\psi\|_\infty\hat\nu(B)$.

In addition, using C^3 smoothness of λ_u and Proposition 13 (the bounds (2) and initial values (3))

$$\left|\lambda_u^\ell - \exp\left(-\frac{\ell}{2}\sigma_\varphi^2 u\cdot u\right)\right| \le \ell\bigl(\exp\bigl(-c_1|u|^2\bigr)\bigr)^{\ell-1}\left|\lambda_u - \exp\left(-\frac{1}{2}\sigma_\varphi^2 u\cdot u\right)\right|$$
$$= O\bigl(\ell e^{-c_1\ell|u|^2}|u|^3\bigr) = O\bigl(e^{-c_4\ell|u|^2}|u|\bigr)$$

for the constant $c_4 = c_1/2$. Thus

$$a_1(u) = \exp\left(-\frac{\ell}{2}\sigma_\varphi^2 u\cdot u\right)\hat\nu(\hat A)\hat\nu(B) + O\bigl(e^{-c_4\ell|u|^2}|u|\hat\nu(B)ke^{\eta q}\bigr).$$

By the classical change of variable $v = u\sqrt{\ell}$ and Gaussian integral one easily sees that

$$\int_{[-\beta,\beta]^2}\exp\left(-\frac{\ell}{2}\sigma_\varphi^2 u\cdot u\right)du = \frac{1}{\ell}\int_{[-\beta\sqrt{\ell},\beta\sqrt{\ell}]^2}\exp\left(-\frac{1}{2}\sigma_\varphi^2 v\cdot v\right)dv = \frac{2\pi}{\ell\sqrt{\det\sigma_\varphi^2}} + O\left(\frac{1}{\ell^{3/2}}\right).$$

Proceeding similarly with the error term one gets as well

$$\int_{[-\beta,\beta]^2} |u|\mathrm{e}^{-c_4\ell|u|^2}\,\mathrm{d}u = \frac{1}{\ell^{3/2}}\int_{[-\beta\sqrt{\ell},\beta\sqrt{\ell}]^2} |v|\mathrm{e}^{-c_4|v|^2}\,\mathrm{d}v = \mathrm{O}\left(\frac{1}{\ell^{3/2}}\right).$$

Combining these two computations gives by integration of the approximation of $a_1(u)$ obtained above that

$$\int_{[-\beta,\beta]^2} a_1(u)\,\mathrm{d}u = \frac{2\pi}{\ell\sqrt{\det\sigma_\varphi^2}}\hat{\nu}(\hat{A})\hat{\nu}(B) + \mathrm{O}\left(\frac{\hat{\nu}(B)k\mathrm{e}^{\eta q}}{\ell^{3/2}}\right).$$

From this main estimate and (7) and (8) it follows immediately that

$$\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} a(u)\,\mathrm{d}u = \frac{1}{2\pi\ell\sqrt{\det\sigma_\varphi^2}}\hat{\nu}(\hat{A})\hat{\nu}(B) + \mathrm{O}\left(\frac{\hat{\nu}(B)k\mathrm{e}^{\eta q}}{(n-k)^{3/2}}\right).$$

□

4.3. *Proof of the pointwise convergence of the recurrence rate to the dimension*

Let us denote by $G_n(\varepsilon)$ the set of points for which n is an ε-return:

$$G_n(\varepsilon) := \left\{x \in \Sigma\colon\ S_n\varphi(x) = 0 \text{ and } d\left(\theta^n(x), x\right) < \varepsilon\right\}.$$

Let us consider the first return time in an ε-neighbourhood of a starting point $x \in \Sigma$:

$$\tau_\varepsilon(x) := \inf\left\{m \geq 1\colon\ S_m\varphi(x) = 0 \text{ and } d\left(\theta^m(x), x\right) < \varepsilon\right\} = \inf\left\{m \geq 1\colon\ x \in G_m(\varepsilon)\right\}.$$

Proof of Theorem 8. Let us denote by $\mathcal{C}_k$ the set of k-cylinders of Σ. For any $\delta > 0$ denote by $\mathcal{C}_k^\delta \subset \mathcal{C}_k$ the set of cylinders $C \in \mathcal{C}_k$ such that $\nu(C) \in (\mathrm{e}^{-(d+\delta)k}, \mathrm{e}^{-(d-\delta)k})$. For any $x \in \Sigma$ let $C_k(x) \in \mathcal{C}_k$ be the k-cylinder which contains x. Since d is twice[4] the entropy of the ergodic measure ν, by the Shannon–McMillan–Breiman theorem, the set

$$K_N^\delta = \left\{x \in \Sigma\colon\ \forall k \geq N, C_k(x) \in \mathcal{C}_k^\delta\right\}$$

has a measure $\nu(K_N^\delta) > 1 - \delta$ provided N is taken sufficiently large.

- Let us prove that, almost surely:

$$\liminf_{\varepsilon\to 0} \frac{\log\log\tau_\varepsilon}{-\log\varepsilon} \geq d.$$

Let $\alpha > \frac{1}{d}$ and $0 < \delta < d - \frac{1}{\alpha}$. Let us take $\varepsilon_n := \log^{-\alpha} n$ and $k_n := \lceil -\log\varepsilon_n \rceil$. According to Proposition 14, whenever $k_n \geq N$ we have:

$$\begin{aligned}\nu\left(K_N^\delta \cap G_n(\varepsilon_n)\right) &\leq \nu\left(\left\{x \in K_N^\delta\colon\ S_n\varphi(x) = 0 \text{ and } \theta^n(x) \in C_{k_n}(x)\right\}\right)\\ &\leq \sum_{C\in\mathcal{C}_{k_n}^\delta} \nu\left(C \cap \{S_n\varphi = 0\} \cap \theta^{-n}\theta^{k_n}\left(\theta^{-k_n}C\right)\right)\\ &\leq \beta \sum_{C\in\mathcal{C}_{k_n}^\delta}\left[\frac{\nu(C)\nu(C)}{n} + \mathrm{O}\left(\frac{\nu(C)k_n\mathrm{e}^{\eta k_n}}{n^{3/2}}\right)\right].\end{aligned}$$

[4]Note that we are working with the two-sided symbolic space Σ.

Observe that for $C\in\mathcal{C}^\delta_{k_n}$ we have

$$\frac{k_n \mathrm{e}^{\eta k_n}}{\sqrt{n}}=\frac{\alpha\log\log n\log^{\alpha\eta} n}{\sqrt{n}}=\mathrm{O}\bigl(\varepsilon_n^{d+\delta}\bigr)=\mathrm{O}\bigl(\nu(C)\bigr),$$

hence it follows that

$$\nu\bigl(K_N^\delta\cap G_n(\varepsilon_n)\bigr)=\mathrm{O}\biggl(\sum_{C\in\mathcal{C}^\delta_{k_n}}\frac{\nu(C)^2}{n}\biggr)=\mathrm{O}\biggl(\frac{1}{n(\log n)^{(d-\delta)\alpha}}\biggr).$$

Hence, by the Borel–Cantelli lemma, for a.e. $x\in K_N^\delta$, if n is large enough, we have: $x\notin G_n(\varepsilon_n)$. Hence, if in addition x is not a periodic point, then for any n large enough, we have: $\tau_{\varepsilon_n}(x)>n$. This readily implies that:

$$\liminf_{n\to\infty}\frac{\log\log\tau_{\varepsilon_n}}{-\log\varepsilon_n}\ge\frac{1}{\alpha}\quad\text{a.e.},$$

which proves the lower bound on the lim inf since $(\varepsilon_n)_n$ decreases to zero and $\lim_{n\to+\infty}\frac{\varepsilon_n}{\varepsilon_{n+1}}=1$.

- Let us prove that, almost surely:

$$\limsup_{\varepsilon\to0}\frac{\log\log\tau_\varepsilon}{-\log\varepsilon}\le d.$$

Let $0<\alpha<\frac{1}{d}$ and $\delta>0$ such that $1-\alpha d-\alpha\delta>0$. Let us take $\varepsilon_n:=\log^{-\alpha} n$ and $k_n:=\lfloor-\log\varepsilon_n\rfloor+1$. For all $\ell=1,\dots,n$, we define:

$$A_\ell(\varepsilon):=G_\ell(\varepsilon)\cap\theta^{-\ell}\{\tau_\varepsilon>n-\ell\}.$$

Let us take $L_n:=\lceil\log^a n\rceil$, with $a>2\alpha(d+\delta+\eta)$. The sets $A_\ell(\varepsilon)$ are pairwise disjoint thus:

$$1\ge\sum_{\ell=0}^{n}\nu\bigl(A_\ell(\varepsilon_n)\bigr)\ge\sum_{\ell=L_n}^{n}\sum_{C\in\mathcal{C}^\delta_{k_n}}\nu\bigl(C\cap A_\ell(\varepsilon_n)\bigr).$$

According to Proposition 14, we have

$$\begin{aligned}\nu\bigl(C\cap A_\ell(\varepsilon_n)\bigr)&=\nu\bigl(C\cap\{S_\ell\varphi=0\}\cap\theta^{-\ell}\bigl(C\cap\{\tau_{\varepsilon_n}>n-\ell\}\bigr)\bigr)\\&=\biggl[\frac{\nu(C)}{2\pi\sqrt{\det\sigma^2}}+\mathrm{O}\biggl(\frac{k_n\mathrm{e}^{\eta k_n}}{\sqrt{\ell-k_n}}\biggr)\biggr]\frac{1}{\ell-k_n}\nu\bigl(C\cap\{\tau_{\varepsilon_n}>n\}\bigr)\\&\ge c'\varepsilon_n^{d+\delta}\frac{1}{\ell-k_n}\nu\bigl(C\cap\{\tau_{\varepsilon_n}>n\}\bigr)\end{aligned}$$

for any $C\in\mathcal{C}^\delta_{k_n}$ provided $k_n\ge N$; indeed, the error term is negligible since:

$$\frac{k_n\mathrm{e}^{\eta k_n}}{\sqrt{\ell-k_n}}=\mathrm{O}\biggl(\frac{(\log\log n)\log^{\alpha\eta}n}{\log^{a/2}(n)}\biggr)=\mathrm{o}\bigl(\varepsilon_n^d+\delta\bigr),$$

since $a>2\alpha(d+\delta+\eta)$. This chain of inequalities gives

$$\begin{aligned}\nu\bigl(K_N^\delta\cap\{\tau_\varepsilon>n\}\bigr)\le\sum_{C\in\mathcal{C}^\delta_{k_n}}\nu\bigl(C\cap\{\tau_\varepsilon>n\}\bigr)&\le\mathrm{O}\biggl(\biggl(\varepsilon_n^{d+\delta}\log\frac{n-k_n}{L_n-k_n}\biggr)^{-1}\biggr)\\&=\mathrm{O}\biggl(\frac{1}{\log^{1-\alpha d-\alpha\delta}n}\biggr).\end{aligned}$$

Now let us take $n_p := \lfloor \exp(p^{2/(1-\alpha d-\alpha\delta)}) \rfloor$. We have:

$$\sum_{p\geq 1} \nu\big(K_N^\delta \cap \{\tau_{\varepsilon_{n_p}} > n_p\}\big) < +\infty.$$

Hence, by the Borel–Cantelli lemma, almost surely $x \in K_N^\delta$, we have:

$$\limsup_{p\to+\infty} \frac{\log\log \tau_{\varepsilon_{n_p}}}{-\log \varepsilon_{n_p}} \leq \frac{1}{\alpha}.$$

This gives the estimate lim sup since $(\varepsilon_{n_p})_p$ decreases to zero and since $\lim_{p\to+\infty} \frac{\varepsilon_{n_p}}{\varepsilon_{n_{p+1}}} = 1$.

□

4.4. *Fluctuations of the rescaled return time*

Recall that $C_k(x) = \{y \in \Sigma\colon d(x,y) < \mathrm{e}^{-k}\}$. Let $R_k(y) = \min\{n \geq 1\colon \theta^n(y) \in C_k(y)\}$ denote the first return time of a point y into its k-cylinders $C_k(y)$, or equivalently the first repetition time of the sequence of symbols $y_{-k},\ldots,y_k$ of y. There have been many studies on this quantity; among all the results we will use the following:

Proposition 15 (Hirata [9]). *For ν-almost every point $x \in \Sigma$, the return times into the cylinders $C_k(x)$ are asymptotically exponentially distributed in the sense that*

$$\lim_{k\to\infty} \nu_{C_k(x)}\left(R_k(\cdot) > \frac{t}{\nu(C_k(x))}\right) = \mathrm{e}^{-t}$$

for a.e. x, where the convergence is uniform in t.

Lemma 16. *Let x be such that* $\lim_{k\to\infty} \nu_{C_k(x)}(R_k(\cdot) > \frac{t}{\nu(C_k(x))}) = \mathrm{e}^{-t}$ *for all $t > 0$. Then, for all $t > 0$, we have*:

$$\lim_{k\to+\infty} \nu_{C_k(x)}\left(\tau_{\mathrm{e}^{-k}} > \exp\left(\frac{t}{\nu(C_k(x))}\right)\right) = \frac{1}{1+\beta t},$$

with $\beta := \frac{1}{2\pi\sqrt{\det \sigma^2}}$.

Proof. We are inspired by the method used by Dvoretzky and Erdös in [7]. Let $k \geq m_0$ and n be some integers. We make a partition of a cylinder $C_k(x)$ according to the value $\ell \leq n$ of the last passage in the time interval $0,\ldots,n$ of the orbit of $(x,0)$ by the map F into $C_k(x)\times\{0\}$. This gives the following equality:

$$\nu\big(C_k(x)\big) = \sum_{\ell=0}^{n} \nu\big(C_k(x) \cap \{S_\ell = 0\} \cap \theta^{-\ell}\big(C_k(x) \cap \{\tau_{\mathrm{e}^{-k}} > n-\ell\}\big)\big). \tag{9}$$

Upper bound. Let $n_k = \lfloor \mathrm{e}^{t/\nu(C_k(x))} \rfloor$. First we claim that:

$$\limsup_{k\to+\infty} \nu_{C_k(x)}\big(\{\tau_{\mathrm{e}^{-k}} > n_k\}\big) \leq \frac{1}{1+\beta t}.$$

Let $a > 2\eta$ and $L_k = \mathrm{e}^{ak}$. According to the decomposition (9) and to Proposition 14, there exists $c_1' > 0$ such that, for k sufficiently large, we have:

$$\begin{aligned}\nu\big(C_k(x)\big) &\geq \nu\big(C_k(x)\cap\{\tau_{\mathrm{e}^{-k}}>n_k\}\big)+\sum_{\ell=L_k}^{n_k}\beta\frac{\nu(C_k(x))\nu(C_k(x)\cap\{\tau_{\mathrm{e}^{-k}}>n_k\})}{\ell-k}\\ &\quad - c_3\sum_{\ell=L_k}^{n_k}\frac{k\mathrm{e}^{\eta k}\nu(C_k(x)\cap\{\tau_{\mathrm{e}^{-k}}>n_k-\ell\})}{(\ell-k)^{3/2}}\\ &\geq \nu\big(C_k(x)\cap\{\tau_{\mathrm{e}^{-k}}>n_k\}\big)\left(1+\beta\nu\big(C_k(x)\big)\sum_{\ell=L_k}^{n_k}\frac{1}{\ell-k}\right)-c_1'\nu\big(C_k(x)\big)k\mathrm{e}^{\eta k}\mathrm{e}^{-ak/2}.\end{aligned}$$

Hence, we get:

$$\frac{\nu(C_k(x)\cap\{\tau_{\mathrm{e}^{-k}}>n_k\})}{\nu(C_k(x))}\leq\frac{1-c_1'k\mathrm{e}^{k(\eta-a/2)}}{1+\beta\nu(C_k(x))\sum_{\ell=L_k}^{n_k}1/(\ell-k)}.$$

The claim follows from the fact that $a>2\eta$.

Lower bound. Let $b=\liminf\frac{-1}{k}\log\nu(C_k(x))>0$. Without loss of generality we assume that the Hölder exponent η is such that $b>2\eta$. Let $q_k=\lfloor\mathrm{e}^{t/\nu(C_k(x))}\rfloor$, $n_k=\lfloor q_k\log(q_k)\rfloor$, $m_k=n_k-q_k$ and choose $\delta>0$ such that $2\eta<b(1-\delta)$. We now claim that:

$$\liminf_{k\to+\infty}\nu_{C_k(x)}\big(\{\tau_{\mathrm{e}^{-k}}>q_k\}\big)\geq\frac{1}{1+\beta t}.$$

Let us denote by $A_\ell(k,x)$ the sets involved in the decomposition (9):

$$A_\ell(k,x):=C_k(x)\cap\{S_\ell=0\}\cap\theta^{-\ell}\big(C_k(x)\cap\{\tau_{\mathrm{e}^{-k}}>n_k-\ell\}\big).$$

For $\ell=0$ we have

$$\nu\big(A_0(k,x)\big)\leq\nu\big(C_k(x)\cap\{\tau_{\mathrm{e}^{-k}}>q_k\}\big). \tag{10}$$

Let $M_k=\lfloor\nu(C_k(x))^{-1+\delta}\rfloor$. We first show that the contribution from small ℓ is negligible. According to the exponential statistics for return times, there exists ε_k, with $\lim_{k\to+\infty}\varepsilon_k=0$, such that we have (remember that the $A_\ell(k,x)$ are disjoints):

$$\begin{aligned}\sum_{\ell=1}^{M_k}\nu\big(A_\ell(k,x)\big) &\leq \nu\big(C_k(x)\cap\{R_k\leq M_k\}\big)\\ &\leq \nu\big(C_k(x)\big)\big(1-\exp\big(\nu\big(C_k(x)\big)^{\delta}\big)+\varepsilon_k\big)\\ &= \mathrm{o}\big(\nu\big(C_k(x)\big)\big).\end{aligned} \tag{11}$$

We now estimate the measure of $A_\ell(k,x)$ for large values of ℓ. According to our local limit theorem (Proposition 14), for all $\ell=M_k+1,\dots,n_k$, we have:

$$\nu\big(A_\ell(k,x)\big)\leq\beta\underbrace{\frac{\nu(C_k(x))\nu(C_k(x)\cap\{\tau_{\mathrm{e}^{-k}}>n_k-\ell\})}{\ell-k}}_{\text{main term}}+c_3\underbrace{\frac{k\mathrm{e}^{\eta k}\nu(C_k(x)\cap\{\tau_{\mathrm{e}^{-k}}>n_k-\ell\})}{(\ell-k)^{3/2}}}_{\text{error term}}. \tag{12}$$

Observe that the sum of the error terms is controlled, for some constant $c_5>0$, by

$$\sum_{\ell\geq M_k+1}\frac{k\mathrm{e}^{\eta k}\nu(C_k(x))}{(\ell-k)^{3/2}}\leq c_5\nu\big(C_k(x)\big)k\mathrm{e}^{\eta k}\big(\nu\big(C_k(x)\big)^{-1+\delta}-k\big)^{-1/2}=\mathrm{o}\big(\nu\big(C_k(x)\big)\big). \tag{13}$$

On the other hand the sum of the main terms may be estimated for non-extremal values of ℓ by:

$$\sum_{\ell=M_k+1}^{m_k} \frac{\nu(C_k(x))\nu(C_k(x)\cap\{\tau_{\mathrm{e}^{-k}} > n_k-\ell\})}{\ell-k} \le \nu\big(C_k(x)\big)\nu\big(C_k(x)\cap\{\tau_{\mathrm{e}^{-k}} > q_k\}\big)\sum_{\ell=M_k+1}^{m_k}\frac{1}{\ell-k} \tag{14}$$

and for extremal values of ℓ the simple bound below holds:

$$\begin{aligned}\sum_{\ell=m_k+1}^{n_k} \frac{\nu(C_k(x))\nu(C_k(x)\cap\{\tau_{\mathrm{e}^{-k}} > n_k-\ell\})}{\ell-k} &\le \nu\big(C_k(x)\big)^2\sum_{\ell=m_k+1}^{n_k}\frac{1}{\ell-k}\\ &\le c_6\nu\big(C_k(x)\big)^2\log\left(\frac{n_k-k}{m_k-k}\right)\\ &= \mathrm{o}\big(\nu\big(C_k(x)\big)\big).\end{aligned} \tag{15}$$

Using the decomposition 9 and putting together formulas (10)–(15), we get:

$$\begin{aligned}\nu\big(C_k(x)\big) &\le \nu\big(C_k(x)\cap\{\tau_{\mathrm{e}^{-k}} > q_k\}\big)\left(1+\beta\nu\big(C_k(x)\big)\sum_{\ell=M_k+1}^{n_k}\frac{1}{\ell-k}\right)+\mathrm{o}\big(\nu\big(C_k(x)\big)\big)\\ &\le \nu\big(C_k(x)\cap\{\tau_{\mathrm{e}^{-k}} > q_k\}\big)\big(1+\beta\nu\big(C_k(x)\big)\log n_k\big)+\mathrm{o}\big(\nu\big(C_k(x)\big)\big).\end{aligned}$$

This proves the claim, which achieves the proof of the lemma. □

Proof of Theorem 9. Since the exponential statistics of return time holds a.e. by Proposition 15, Lemma 16 applies a.e. and by integration, using the Lebesgue dominated convergence theorem, we get that

$$\lim_{k\to\infty}\nu\left(\log\tau_{\mathrm{e}^{-k}}(\cdot) > \frac{t}{\nu(C_k(\cdot))}\right) = \frac{1}{1+\beta t}$$

for all $t \ge 0$. □

Proof of Corollary 10. *Non-zero variance*. Let us write:

$$Y_k := \frac{\log\log\tau_{\mathrm{e}^{-k}}(\cdot) - kd}{\sqrt{k}}.$$

In this case ν is a Gibbs measure with a non-degenerate Hölder potential h. The logarithm of the measure of the k-cylinder about x is, up to some constants, given by the Birkhoff sum $\sum_{j=-k}^{k} h\circ\sigma^j(x)$ of h on the orbit of x. It is well known that such sums follow a central limit theorem (e.g., [2]). This readily implies that $X_k = \frac{\log(\nu(C_k(\cdot))+kd}{\sqrt{k}}$ converges in distribution to a centered Gaussian random variable of variance $2\sigma_h^2$. It is enough to prove that $Y_k + X_k$ converges in probability to 0. This will be true if $Y_k + X_k$ converges in distribution to 0. This follows from Theorem 9 and from the formula:

$$Y_k + X_k = \frac{\log\log\tau_{\mathrm{e}^{-k}}(\cdot) + \log(\nu(C_k))}{\sqrt{k}}.$$

Zero variance. In this case the potential is cohomologous to a constant and the measure ν is the measure of maximal entropy, which is a Markov measure. Denote by π the transition matrix and by p the left eigenvector such that $p\pi = p$. The measure of a cylinder $C_k(x)$ is equal to $p_{x_{-k}}\prod_{j=-k}^{k-1}\pi_{x_jx_{j+1}}$. Since the function $\log\pi_{x_0x_1}$ has to be cohomologous to the entropy, the measure of a cylinder $C_k(x)$ simplifies down to

$$\nu\big(C_k(x)\big) = Q_{x_{-k}x_k}\mathrm{e}^{-(2k+1)d/2},$$

where $Q = (Q_{ij})$ is a (constant) matrix. Proceeding as in the proof of Theorem 9, we get that

$$\lim_{k\to\infty} \nu\big(\mathrm{e}^{-kd}\log\tau_{\mathrm{e}^{-k}} > t\big) = \sum_{i,j\in\mathcal{A}} \lim_{k\to\infty} \int_{\{x_{-k}=i,x_k=j\}} \mathbf{1}_{\{\mathrm{e}^{-kd}\log\tau_{\mathrm{e}^{-k}}>t\}}\,\mathrm{d}\nu = \sum_{i,j\in\mathcal{A}} p_i p_j \frac{1}{1+\beta Q_{ij} t}.$$

□

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