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Planar Lorentz process in a random scenery

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Abstract. We consider the periodic planar Lorentz process with convex obstacles (and with finite horizon). In this model, a point particle moves freely with elastic reflection at the fixed convex obstacles. The random scenery is given by a sequence of independent, identically distributed, centered random variables with finite and non-null variance. To each obstacle, we associate one of these random variables. We suppose that each time the particle hits an obstacle, it wins the amount given by the random variable associated to the obstacle. We prove a convergence in distribution to a Wiener process for the total amount won by the particle (normalized by $\sqrt{n\log(n)}$) when the time n goes to infinity. Such a result has been established by Bolthausen [*Ann. Probab.* **17** (1989) 108–115)] in the case of random walks in $\mathbb{Z}^2$ given by sums of independent identically distributed random variables. We follow the scheme of his proof. The lack of independence will be compensated by some extensions of the local limit theorem proved by Szász and Varjú in [*Ergodic Theory Dynam. Systems* **24** (2004) 257–278]. This paper answers a question of Szász about the asymptotic behaviour of $\sum_{k=0}^{n-1}\zeta_{S_k}$ where $(\zeta_\ell)_\ell$ is a sequence of i.i.d. centered random variables (with finite and non-null variance) and where S_k is the number of the cell at the kth reflection.

Résumé. Nous considérons le processus de Lorentz dans le plan avec des obstacles convexes disposés de manière périodique (nous supposons de plus que l'horizon est fini). Dans ce modèle, une particule ponctuelle se déplace à vitesse unitaire et sa vitesse obéit à la loi de la réflexion de Descartes à l'instant d'un choc contre un obstacle. La scène aléatoire est donnée par une suite de variables aléatoires indépendantes de même loi, centrées, de variance finie non nulle. Chacune de ces variables aléatoires est associée à un obstacle. Nous associons à la particule une somme qui évolue avec le temps. Cette somme est nulle au départ. A chaque fois que la particule touche un obstacle, elle gagne la valeur de la variable aléatoire associée à cet obstacle. Nous montrons que la somme totale gagnée au temps n (normalisée par $\sqrt{n\log(n)}$) converge en loi vers un processus de Wiener lorsque n tend vers l'infini. Un tel résultat a été établi par Bolthausen [*Ann. Probab.* **17** (1989) 108–115)] dans le cas de marches aléatoires sur $\mathbb{Z}^2$ avec des pas indépendants et de même loi. Nous nous inspirons de son travail. Nous remplaçons l'hypothèse d'indépendance de [*Ann. Probab.* **17** (1989) 108–115)] par des extensions du théorème limite local établi par Szász and Varjú in [*Ergodic Theory Dynam. Systems* **24** (2004) 257–278]. Ce travail apporte une réponse à une question de Szász concernant le comportement asymptotique de $\sum_{k=0}^{n-1}\zeta_{S_k}$ où $(\zeta_\ell)_\ell$ est une suite de variables aléatoires indépendantes identiquement distribuées, centrées, de variance finie et non nulle et où S_k désigne le numéro de la cellule dans laquelle se trouve la particule à l'instant de la $k$ème reflexion.

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1. Introduction

A Lorentz process is a model in which a point particle moves freely (with unit speed) with elastic collisions at the surface of fixed obstacles. We consider a planar Lorentz gas with $\mathbb{Z}^2$-periodic configuration of strictly convex obstacles. We will moreover suppose that the horizon is finite which means that the time between two consecutive reflections is

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uniformly bounded. This model is a particular billiard model in an infinite-volume domain. The corresponding billiard flow and billiard transformation are introduced in Section 1.1.

Because of the $\mathbb{Z}^2$-periodicity, this model is naturally related to a billiard model in a domain of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (see Section 1.2). Since the early work of Sinai [23], this billiard system has been studied by many authors ([4–7,10] and others).

We associate with the point particle (of the Lorentz process) a null amount at the beginning. We suppose that, at each reflection time, the particle wins the (real) value associated with the obstacle met. We suppose that the values associated with the obstacles are independent identically distributed with null expectation, finite and non-null variance. We suppose moreover that these random variables are independent of the motion of the point particle of the Lorentz process. Our main result (stated precisely in Section 1.3) is that:

(i) If Z_n is the total amount won by the particle before the nth reflection, there exists $\beta_0 > 0$ such that $(\frac{Z_{\lfloor nt \rfloor}}{\beta_0\sqrt{n\log(n)}})_{t\ge 0}$ converges weakly to the standard Wiener process (as n goes to infinity).

(ii) If $\tilde{Z}_t$ is the total amount won by the particle before time t, there exists $\beta_1 > 0$ such that $(\frac{\tilde{Z}_{nt}}{\beta_1\sqrt{n\log(n)}})_{t\ge 0}$ converges weakly to the standard Wiener process (as n goes to infinity).

Now let us introduce precisely our model and state our exact results.

In $\mathbb{R}^2$, we consider a finite number of convex open sets $O_1, \dots, O_I$, with boundary C^3-smooth and with non-null curvature. We repeat these sets $\mathbb{Z}^2$-periodically by considering $U_{i,\ell} = \ell + O_i$ for all $i \in \{1, \dots, I\}$ and all $\ell \in \mathbb{Z}^2$. We suppose that the closures of the $U_{i,\ell}$ are pairwise disjoint. For any $\ell \in \mathbb{Z}^2$, we call ℓ-cell the union $\bigcup_{i=1}^I \partial U_{i,\ell}$. The random scenery is given by a sequence of independent identically distributed real-valued, centered random variables $(\zeta_{(i,\ell)})_{i\in\{1,\dots,I\},\ell\in\mathbb{Z}^2}$ with finite and non-null variance. The value of the random variable $\zeta_{(i,\ell)}$ is associated to the obstacle $U_{i,\ell}$. Let us consider a point particle moving in the domain $Q := \mathbb{R}^2 \setminus \bigcup_{i=1}^I \bigcup_{\ell\in\mathbb{Z}^2} U_{i,\ell}$ with unit speed and with elastic reflections off ∂Q. We associate with the particle an amount equal to 0 at time 0. This amount only changes at reflection times: the particle wins $\zeta_{(i,\ell)}$ each time it hits $U_{i,\ell}$. We are interested in the asymptotic behaviour of the amount when "the time" goes to infinity. We will envisage "the time" in two ways: continuous time and discrete time. We will define $\tilde{Z}_t$ as the total amount won by the particle before time t and Z_n as the total amount won by the particle before the nth reflection time.

1.1. *Billiard flow $(\mathcal{M}_1, \mu_1, (Y_t)_t)$ and billiard transformation (M, ν, T) in the plane*

We call configuration of a particle at some time its position-speed couple. When a reflection occurs, there is coexistence of two configurations: one corresponding to the incident vector and one corresponding to the reflected vector. To avoid ambiguity, we only consider reflected vectors. Hence the set of configurations (position-speed couples) will be:

$$\mathcal{M}_1 := \{(q, \vec{v}) \in Q \times \mathbb{R}^2\colon\ \|\vec{v}\| = 1;\, q \in \partial Q \Rightarrow \langle \vec{n}(q), \vec{v}\rangle \ge 0\},$$

with $\vec{n}(q)$ the unit vector normal to ∂Q at $q \in \partial Q$ oriented to the inside of Q. The billiard flow $(Y_t)_t$ is the flow on $\mathcal{M}_1$ such that $Y_t(q, \vec{v}) = (q_t, \vec{v}_t)$ is the configuration at time t of a particle with configuration $(q, \vec{v})$ at time 0. The billiard flow preserves the Lebesgue measure μ_1 on $\mathcal{M}_1$. Now we only consider reflection times. Let M be the set of reflected vectors off ∂Q:

$$M := \{(q, \vec{v}) \in \partial Q \times \mathbb{R}^2\colon\ \|\vec{v}\| = 1 \text{ and } \langle \vec{n}(q), \vec{v}\rangle \ge 0\}.$$

A point $(q, \vec{v}) \in M$ is parametrized by (i, r, φ, ℓ) if $q - \ell$ is the point of ∂O_i with curvilinear absciss r and if φ is the angular measure of $(\vec{n}(q), \vec{v})$ taken in $[-\frac{\pi}{2}; \frac{\pi}{2}]$. The billiard transformation T maps a configuration $y \in M$ at a reflection time to the configuration $T(y) = y'$ corresponding to the next reflection off ∂Q. This transformation preserves the measure ν given by $d\nu(q, \vec{v}) = \cos(\varphi)\,\mathrm{d}r\,\mathrm{d}\varphi$, with the parametrization (i, r, φ, ℓ) of $(q, \vec{v}) \in M$.

We define the function $\tau : M \to [0; +\infty[$ by: $\tau(q, \vec{v}) := \min\{s > 0\colon\ q + s\vec{v} \in \partial Q\}$. The quantity $\tau(q, \vec{v})$ corresponds to the distance to go until the next reflection off ∂Q. *Here, we suppose that the billiard system has finite horizon*, that is, $\sup \tau < +\infty$. We already know that this system is recurrent (see the works of Conze in [8], of Schmidt in [21] and of Szász and Varjú in [24]) and that it is totally ergodic (see [22] and [19]). Other results have been established by Dolgopyat, Szász and Varjú in [9]. The billiard flow $(\mathcal{M}_1, \mu_1, (Y_t)_t)$ can be represented as the special flow over (M, ν, T) with roof function τ. Let us specify this. Let us define $\tilde{\mathcal{M}}_1 := \{(y, s)\colon\ y \in M; 0 \le s < \tau(y)\}$ endowed with the measure $\tilde{\mu}_1$ given by: $d\tilde{\mu}_1(y, s) = \mathrm{d}\nu(y)\,\mathrm{d}s$. Let $(\tilde{Y}_t)_t$ be the flow defined on $\tilde{\mathcal{M}}_1$ by $\tilde{Y}_t(y, s) = (y, s + t)$ with the identifications $(y, \tau(y)) \equiv (T(y), 0)$. Let $\Delta : \tilde{\mathcal{M}}_1 \to \mathcal{M}_1$ be given by: $\Delta((q, \vec{v}), s) = (q + s\vec{v}, \vec{v})$. This bi-measurable function satisfies: $Y_t = \Delta \circ \tilde{Y}_t \circ \Delta^{-1}$ and $\Delta_*(\tilde{\mu}_1) = \mu_1$.

1.2. *Billiard transformation in the torus* $(\bar{M}, \bar{\nu}, \bar{T})$

Let us define $\bar{M} = \{(q, \vec{v}) \in M\colon\ q \in \bigcup_{i=1}^I \partial O_i\}$ and $\bar{T} : \bar{M} \to \bar{M}$ with $\bar{T}(q, \vec{v}) = (q', \vec{v}')$ if there exists $\ell \in \mathbb{Z}^2$ such that $T(q, \vec{v}) = (q' + \ell, \vec{v})$. Let $\bar{\nu}$ be the probability measure on $\bar{M}$ proportional to the restriction of ν to $\bar{M}$. The study of the dynamical system $(\bar{M}, \bar{\nu}, \bar{T})$ is complicated by the discontinuities of the transformation $\bar{T}$. But it is known that $\bar{T}$ is C^2-regular on $\bar{M} \setminus (R_0 \cup \bar{T}^{-1}(R_0))$, where the set $R_0 := \{(q, \vec{v}) \in \bar{M}\colon\ \langle \vec{v}, \vec{n}(q)\rangle = 0\}$ is the set of tangent vectors.

It is easy to see that the billiard system (M, ν, T) is a cylindrical extension of the billiard system $(\bar{M}, \bar{\nu}, \bar{T})$ by some function $\Phi : \bar{M} \to \mathbb{Z}^2$. For any $(q, \vec{v}) \in \bar{M}$ and any $\ell \in \mathbb{Z}^2$, we have $T(q + \ell, \vec{v}) = (q' + \ell + \Phi(q, \vec{v}), \vec{v}')$ with $(q', \vec{v}') = \bar{T}(q, \vec{v})$ and $T^n(q + \ell, \vec{v}) = (q_n + \ell + \sum_{k=0}^{n-1} \Phi(\bar{T}^k(q, \vec{v})), \vec{v}_n)$ with $(q_n, \vec{v}_n) = \bar{T}^n(q, \vec{v})$. In the sequel, we identify M with $\bar{M} \times \mathbb{Z}^2$ by the one-to-one map $\Pi_0 : \bar{M} \times \mathbb{Z}^2 \to M$ given by: $\Pi_0((q, \vec{v}), \ell) = (q + \ell, \vec{v})$. We notice that the image measure of ν by Π_0^{-1} is proportional to $\bar{\nu} \otimes \sum_{\ell \in \mathbb{Z}^2} \delta_\ell$ (where δ_ℓ is the Dirac measure in ℓ). Let us consider the asymptotic covariance matrix Σ^2 associated with Φ:

$$\Sigma^2 := \lim_{n \to +\infty} \mathrm{Cov}_{\bar{\nu}}\left(\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \Phi \circ \bar{T}^k\right).$$

Because of the recurrence of the Lorentz gas, the matrix Σ^2 is invertible. Let us write:

$$\forall x \in \bar{M}, \quad S_0(x) := 0 \quad \text{and} \quad S_n(x) := \sum_{k=0}^{n-1} \Phi \circ \bar{T}^k(x).$$

We will also define the random variable $\mathcal{I}_k$ on $\bar{M}$ equal to the index $i \in \{1, \dots, I\}$ of the obstacle O_i met by the particle at the kth reflection off an obstacle. We have: $\mathcal{I}_k = \mathcal{I}_0 \circ \bar{T}^k$.

1.3. *Lorentz gas walk in a random scenery*

Let us consider a sequence of independent identically distributed random variables $(\zeta_{i,\ell})_{i=1,\dots,I,\ell\in\mathbb{Z}^2}$ defined on some probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$. We suppose that $\zeta_{i,\ell}$ has zero mean and is square integrable with variance $\sigma^2 > 0$. We define the sequence of random variables $(Z_n)_n$ on the direct product $(\Omega, \mathcal{F}) := (M \times \Omega_0, \mathcal{B}(M) \otimes \mathcal{F}_0)$ by $Z_0 \equiv 0$ and, for all $n \ge 0$:

$$\forall (x, \ell_0) \in \bar{M} \times \mathbb{Z}^2, \forall \omega \in \Omega_0, \quad Z_n\big(\Pi_0(x, \ell_0), \omega\big) := \sum_{k=1}^{n} \zeta_{\mathcal{I}_k(x), \ell_0 + S_k(x)}(\omega).$$

The quantity $Z_n(X, \omega)$ corresponds to the total amount won at the nth reflection by a particle with initial configuration X, if the scenery is determined by ω. We are interested in the asymptotic behaviour of Z_n when n goes to infinity. We establish a result of convergence in distribution with respect to any probability measure $(h\nu) \otimes \mathbb{P}_0$ on $(\Omega, \mathcal{F})$.

Theorem 1. *Let $h : M \to \mathbb{R}$ be any positive ν-integrable function such that $\int_M h\,\mathrm{d}\nu = 1$. The sequence of processes $\left(\left(\sqrt{\frac{\pi\sqrt{\det(\Sigma^2)}(\sum_{i=1}^I \mathrm{length}(\partial O_i))^2}{\sum_{i=1}^I (\mathrm{length}(\partial O_i))^2 n \log(n) \sigma^2}}\, Z_{\lfloor nt\rfloor}\right)_{t\ge 0}\right)_{n\ge 1}$ converges weakly (in $D([0, \infty))$) to the Wiener process (with respect to the probability measure $(h\nu) \otimes \mathbb{P}_0$).*

With the same proof, we can answer the question of Szász by proving that the sequence of processes $((\sqrt{\frac{\pi\sqrt{\det(\Sigma^2)}}{n\log(n)\sigma^2}}\sum_{k=1}^{\lfloor nt\rfloor}\zeta_{1,S_k})_{t\geq 0})_{n\geq 1}$ converges weakly (in $D([0,\infty))$) to the Wiener process (with respect to the same probability measures $(h\nu)\otimes\mathbb{P}_0$).

For every real number $t\geq 0$, we define the random variable $\tilde{Z}_t$ on the direct product $(\Omega',\mathcal{F}') := (\mathcal{M}_1\times\Omega_0,\mathcal{B}(\mathcal{M}_1)\otimes\mathcal{F}_0)$ by:

$$\forall(q,\vec{v})\in M,\ \forall s\in\big[0,\tau(q,\vec{v})\big),\ \forall\omega\in\Omega_0\quad \tilde{Z}_t\big((q+s\vec{v},\vec{v}),\omega\big)=Z_{\tilde{n}(t+s,(q,\vec{v}))}\big((q,\vec{v}),\omega\big),$$

with

$$\tilde{n}\big(u,(q,\vec{v})\big) := \sup\left\{n\geq 0\colon \sum_{k=0}^{n-1}\tau\circ\bar{T}^k(q,\vec{v})\leq u\right\}$$

(representing the number of reflections before time u for a particle starting with configuration $(q,\vec{v})$ at time 0). The quantity $\tilde{Z}_t(Y,\omega)$ corresponds to the total amount won at time t by a particle with initial configuration $Y\in\mathcal{M}_1$ if the scenery is determined by ω.

Corollary 2. *Let $g:\mathcal{M}_1\to\mathbb{R}$ be any positive μ_1-integrable function such that $\int_{\mathcal{M}_1} g\,\mathrm{d}\mu_1=1$. The sequence of processes $((\sqrt{\frac{\pi\sqrt{\det(\Sigma^2)}(\sum_{i=1}^I \mathrm{length}(\partial O_i))^2\int_{\bar{M}}\tau\,\mathrm{d}\bar{\nu}}{\sum_{i=1}^I(\mathrm{length}(\partial O_i))^2 n\log(n)\sigma^2}}\tilde{Z}_{nt})_{t\geq 0})_{n\geq 1}$ converges weakly (in $D[0,\infty)$) to the Wiener process (with respect to the probability measure $(g\mu_1)\otimes\mathbb{P}_0$ on $\mathcal{M}_1\times\Omega_0$).*

2. Proof of our results

2.1. *Tools*

In [24], Szász and Varjú establish a local limit theorem for Φ. In particular, they prove that: $\bar{\nu}(S_k=0)\sim(2\pi\sqrt{\det(\Sigma^2)}\,k)^{-1}$. As said briefly in the abstract, we use the scheme of the proof of Bolthausen [3]. We will compensate the lack of independence by two refinements of this local limit theorem (see Appendix A for the proofs).

Proposition 3. *There exist two real numbers $C>0$ and $\tau_1\in(0,1)$ such that, for all non-negative integers n, m and k and for all $i,j,i',j'\in\{1,\dots,I\}$ and for all $N_1,N_2\in\mathbf{Z}^2$, we have*:

$$\big|\mathrm{Cov}_{\bar{\nu}}(\mathbf{1}_{\{\mathcal{I}_0=i,S_n=N_1,\mathcal{I}_n=i'\}},\mathbf{1}_{\{\mathcal{I}_{n+m}=j,S_{n+m+k}-S_{n+m}=N_2,\mathcal{I}_{n+m+k}=j'\}})\big|\leq\frac{C\tau_1^m}{(n+1)(k+1)}.$$

Let us recall some hyperbolic properties of the billiard transformation. For $\bar{\nu}$-almost every point x in $\bar{M}$, there exist two unique maximal C^1-curves $\gamma^s(x)$ and $\gamma^u(x)$ such that:

$$\forall n\geq 0\quad \gamma^s(x)\subseteq\bar{M}\Big\backslash\bigcup_{n\geq 0}\bar{T}^{-n}(R_0)\quad\text{and}\quad\lim_{n\to+\infty}\mathrm{length}\big(\bar{T}^n\big(\gamma^s(x)\big)\big)=0$$

and

$$\forall n\geq 0\quad \gamma^u(x)\subseteq\bar{M}\Big\backslash\bigcup_{n\geq 0}\bar{T}^{n}(R_0)\quad\text{and}\quad\lim_{n\to+\infty}\mathrm{length}\big(\bar{T}^{-n}\big(\gamma^u(x)\big)\big)=0.$$

Moreover, there exist $\tilde{C}>0$ and $\tilde{\theta}\in(0,1)$ such that:

for $\bar{\nu}$-almost every x, $\forall n\geq 0$, $\mathrm{length}(\bar{T}^n(\gamma^s(x)))\leq\tilde{C}\tilde{\theta}^n$ and $\mathrm{length}(\bar{T}^{-n}(\gamma^u(x)))\leq\tilde{C}\tilde{\theta}^n$.

The curves $\gamma^s(x)$ are called *stable curves* and the curves $\gamma^u(x)$ are called *unstable curves*. Let us notice that, for all $n\geq 0$, S_n is constant on each stable curve.

Proposition 4. *Let any real number $p>1$. There exist $C>0$ and $K_0>0$ such that, for any positive integer k, any measurable set B such that, if $x\in B$ then $\gamma^s(x)\subseteq B$, for any integer $r\ge 0$ and any measurable set A union of connected components of $\bar M\setminus\bigcup_{i=0}^r \bar T^{-i}(R_0)$, for any $N\in\mathbb{Z}^2$, we have*:

$$\begin{aligned}&\left|\bar\nu\bigl(A\cap\{S_{k+r}-S_r=N\}\cap\bar T^{-(k+r)}(B)\bigr)-\frac{\bar\nu(A)\bar\nu(B)}{\sqrt{\det(\Sigma^2)}\,2\pi k}\mathrm{e}^{-\langle(\Sigma^2)^{-1}N,N\rangle/(2k)}\right|\\&\quad\le K_0\left(\frac{\bar\nu(B)+\bar\nu(A)\bar\nu(B)^{1/p}}{k^{3/2}}\left(\frac{|N|_2}{\sqrt{k}}+\frac{|N|_2^3}{k^{3/2}}\right)\mathrm{e}^{-\langle(\Sigma^2)^{-1}N,N\rangle/(2k)}+\frac{\bar\nu(B)^{1/p}}{k^2}\right),\end{aligned}\tag{1}$$

with $|N|_2=(n^2+m^2)^{1/2}$ if $N=(n,m)$.

The proofs of these results use Young's construction [25]. The fact that, by our method, we cannot take $p=1$ will complicate our calculations. In the independent case as in other friendly cases (such as subshifts of finite type), such estimations hold with $p=1$. The condition $p>1$ comes from the fact that the Young norm does not dominate $\|\cdot\|_\infty$ but can be chosen so that it dominates $\|\cdot\|_q$ for any arbitrary real number $q\ge 1$. Since p and q are such that $p^{-1}+q^{-1}=1$, the condition $1\le q<+\infty$ implies $1<p$. Hence the fact that we take $p\ne 1$ is due to the method used here and might certainly be improved by another approach.

2.2. *Scheme of the proof of Theorem 1*

Let us notice that, for every $(x,\ell_0)\in\bar M\times\mathbb{Z}^2$, $Z_n(\Pi_0(x,\ell_0),\cdot)$ has (with respect to $\mathbb{P}_0$) zero mean and its variance is $\sigma^2V_n(x)$ with $V_n(x):=\sum_{k,\ell=1}^n \mathbf{1}_{\{S_k=S_\ell\text{ and }\mathcal{I}_k=\mathcal{I}_\ell\}}(x)$. Hence the study of V_n will be useful.

Proposition 5. *We have*:

$$\mathbb{E}_{\bar\nu}[V_n]\sim_{n\to+\infty} c_0 n\log(n)\quad \textit{with } c_0:=\frac{\sum_{i=1}^I(\mathrm{length}(\partial O_i))^2}{(\sum_{i=1}^I\mathrm{length}(\partial O_i))^2\pi\sqrt{\det(\Sigma^2)}}.$$

Proof. We have: $\mathbb{E}_{\bar\nu}[V_n]=n+2\sum_{i=1}^I\sum_{k=1}^{n-1}(n-k)\bar\nu(\mathcal{I}_0=i,S_k=0,\mathcal{I}_k=i)$. But, according to Proposition 4, $\bar\nu(\mathcal{I}_0=i,S_k=0,\mathcal{I}_k=i)$ is equivalent to $\frac{(\bar\nu(\mathcal{I}_0=i))^2}{2\pi\sqrt{\det(\Sigma^2)}k}$ when k goes to infinity. □

The following technical result is proved in Section 2.4:

Proposition 6. *We have*: $\mathrm{Var}_{\bar\nu}(V_n)=\mathrm{O}(n^2\log(n))$.

These two propositions ensure the convergence in probability (with respect to $h\nu$) of $(V_n/(n\log n))_n$ to c_0 as n goes to infinity. For all $(x,\ell_0)\in\bar M\times\mathbb{Z}^2$ and all $\omega\in\Omega_0$, we have:

$$Z_n\bigl(\Pi_0(x,\ell_0),\omega\bigr)=\sum_{i=1}^I\sum_{\ell\in\mathbb{Z}^2}\zeta_{i,\ell+\ell_0}(\omega)\mathcal{N}_{i,\ell}(n)(x),$$

with $\mathcal{N}_{i,\ell}(n)(x):=\sum_{k=1}^n\mathbf{1}_{\{S_k=\ell\text{ and }\mathcal{I}_k=i\}}(x)$. We have: $V_n(x)=\sum_{i=1}^I\sum_{\ell\in\mathbb{Z}^2}(\mathcal{N}_{i,\ell}(n)(x))^2$. For all $\ell\in\mathbb{Z}^2$, let us define:

$$\mathcal{N}_\ell(n)(x):=\sum_{k=1}^n\mathbf{1}_{\{S_k=\ell\}}(x).$$

We clearly have: $\mathcal{N}_\ell(n)(x)=\sum_{i=1,\dots,I}\mathcal{N}_{i,\ell}(n)(x)$. This will be useful in our calculations:

(1) *Convergence of the finite-dimensional distributions.*
Let $m \geq 1$, $a_1, \dots, a_m \in \mathbb{R}$ and $0 = t_0 < t_1 < \cdots < t_m$. We have:

$$\sum_{j=1}^{m} a_j (Z_{\lfloor nt_j \rfloor} - Z_{\lfloor nt_{j-1} \rfloor}) = \sum_{j=1}^{m} \sum_{i=1}^{I} \sum_{\ell \in \mathbb{Z}^2} a_j \big(\mathcal{N}_{i,\ell}\big(\lfloor nt_j \rfloor\big) - \mathcal{N}_{i,\ell}\big(\lfloor nt_{j-1} \rfloor\big)\big) \zeta_{i,\ell+\ell_0}.$$

Following [3], we will apply the Lindeberg theorem (see Theorem 7.2 in [2]). For (x, ℓ_0) fixed in $\bar{M} \times \mathbb{Z}^2$, this random variable taken at $(\Pi_0(x, \ell_0), \cdot)$ is a sum of independent (but not identically distributed) random variables (with respect to $\mathbb{P}_0$). We will prove in Section 3 that we have:

Proposition 7. *For $\bar{\nu}$-almost every $x \in \bar{M}$, for any $a > 0$, $\sup_{i=1,\dots,I;\ell \in \mathbb{Z}^2} \mathcal{N}_{i,\ell}(n)(x) = \mathrm{o}(n^a)$.*

Hence, according to the Lindeberg theorem, for $\bar{\nu}$-almost every $x \in \bar{M}$, for all $\ell_0 \in \mathbb{Z}^2$, the random variable:

$$\hat{Z}_n\big(\Pi_0(x, \ell_0), \cdot\big) = \frac{\sum_{j=1}^{m} a_j (Z_{\lfloor nt_j \rfloor}(\Pi_0(x, \ell_0), \cdot) - Z_{\lfloor nt_{j-1} \rfloor}(\Pi_0(x, \ell_0), \cdot))}{\sqrt{\sum_{i=1}^{I} \sum_{\ell \in \mathbb{Z}^2} (\sum_{j=1}^{m} a_j (\mathcal{N}_{i,\ell}(\lfloor nt_j \rfloor)(x) - \mathcal{N}_{i,\ell}(\lfloor nt_{j-1} \rfloor)(x)))^2 \sigma^2}}$$

converges in distribution (with respect to $\mathbb{P}_0$) to a Gaussian centered random variable with variance 1 (as n goes to infinity). Hence, by Lebesgue's dominated convergence theorem, $\hat{Z}_n$ converges in distribution (with respect to $(h\nu) \otimes \mathbb{P}_0$) to a Gaussian centered random variable with variance 1 (as n goes to infinity). Moreover:

Proposition 8. *The sequence of random variables*

$$\left(\frac{\sum_{i=1}^{I} \sum_{\ell \in \mathbb{Z}^2} (\sum_{j=1}^{m} a_j (\mathcal{N}_{i,\ell}(\lfloor nt_j \rfloor) - \mathcal{N}_{i,\ell}(\lfloor nt_{j-1} \rfloor)))^2 \sigma^2}{c_0 \sigma^2 n \log(n)} \right)_{n \geq 1}$$

converges in probability (for $h\nu$) to $\sum_{j=1}^{m} (a_j)^2 (t_j - t_{j-1})$ as n goes to infinity.

Proof. Since the random variables considered here only depend on $x \in \bar{M}$ and not on the number ℓ_0 of the cell, it is enough to prove the result for the measure $\bar{\nu}$. Let us notice that if $m = 1$, this means that $\frac{\sigma^2 V_n}{c_0 \sigma^2 n \log(n)}$ converges in probability to 1, which is an immediate consequence of Propositions 5 and 6. But it is more complicated when $m \geq 2$. Let us notice that:

$$\sum_{i=1}^{I} \sum_{\ell \in \mathbb{Z}^2} \left(\sum_{j=1}^{m} a_j \big(\mathcal{N}_{i,\ell}\big(\lfloor nt_j \rfloor\big) - \mathcal{N}_{i,\ell}\big(\lfloor nt_{j-1} \rfloor\big)\big) \right)^2 = \Delta + \Gamma,$$

with $\Delta := \sum_{i=1}^{I} \sum_{\ell \in \mathbb{Z}^2} \sum_{j=1}^{m} (a_j)^2 (\mathcal{N}_{i,\ell}(\lfloor nt_j \rfloor) - \mathcal{N}_{i,\ell}(\lfloor nt_{j-1} \rfloor))^2$ and

$$\begin{aligned} \Gamma &:= 2 \sum_{i=1}^{I} \sum_{\ell \in \mathbb{Z}^2} \sum_{1 \leq j < j' \leq m} a_j a_{j'} \sum_{k=\lfloor nt_{j-1} \rfloor + 1}^{\lfloor nt_j \rfloor} \sum_{k'=\lfloor nt_{j'-1} \rfloor + 1}^{\lfloor nt_{j'} \rfloor} \mathbf{1}_{\{S_k = \ell, \mathcal{I}_k = i, S_{k'} = \ell, \mathcal{I}_{k'} = i\}} \\ &= 2 \sum_{1 \leq j < j' \leq m} a_j a_{j'} \sum_{k=\lfloor nt_{j-1} \rfloor + 1}^{\lfloor nt_j \rfloor} \sum_{k'=\lfloor nt_{j'-1} \rfloor + 1}^{\lfloor nt_{j'} \rfloor} \mathbf{1}_{\{S_k = S_{k'}, \mathcal{I}_k = \mathcal{I}_{k'}\}}. \end{aligned}$$

- First, let us notice that we have: $\mathbb{E}_{\bar{\nu}}[\Delta] = \sum_{j=1}^{m} (a_j)^2 \mathbb{E}_{\bar{\nu}}[V_{\lfloor nt_j \rfloor - \lfloor nt_{j-1} \rfloor}]$ and

$$\mathbb{E}_{\bar{\nu}}[\Gamma] \leq \big(|a_1| + \cdots + |a_m|\big)^2 \sum_{k=1}^{\lfloor nt_m \rfloor - \lfloor nt_1 \rfloor} k \mathbb{P}(S_k = 0) \leq \mathrm{O}(n),$$

since $\mathbb{P}(S_k=0)=\mathrm{O}(\frac{1}{k})$ (according to the local limit theorem of Szász and Varjú [24] or to our Proposition 4). Hence, we have proven that:

$$\mathbb{E}_{\bar{\nu}}\left[\sum_{i=1}^{I}\sum_{\ell\in\mathbb{Z}^2}\left(\sum_{j=1}^{m}a_j\big(\mathcal{N}_{i,\ell}\big(\lfloor nt_j\rfloor\big)-\mathcal{N}_{i,\ell}\big(\lfloor nt_{j-1}\rfloor\big)\big)\right)^2\right]\sigma^2$$

is equivalent to $c_0\sigma^2 n\log(n)\sum_{j=1}^m(a_j)^2(t_j-t_{j-1})$.

- Now, let us prove that $\mathrm{Var}(\sum_{i=1}^I\sum_{\ell\in\mathbb{Z}^2}(\sum_{j=1}^m a_j(\mathcal{N}_{i,\ell}(\lfloor nt_j\rfloor)-\mathcal{N}_{i,\ell}(\lfloor nt_{j-1}\rfloor)))^2)$ is in $\mathrm{o}((n\log(n))^2)$. We have: $\mathrm{Var}(\Delta+\Gamma)\le 2(\mathrm{Var}(\Delta)+\mathrm{Var}(\Gamma))$. Let us start with bounding $\mathrm{Var}(\Delta)$. Let us notice that we have $\Delta=\sum_{j=1}^m(a_j)^2\Delta_j$ with

$$\Delta_j=\sum_{i=1}^{I}\sum_{\ell\in\mathbb{Z}^2}\big(\mathcal{N}_{i,\ell}\big(\lfloor nt_j\rfloor\big)-\mathcal{N}_{i,\ell}\big(\lfloor nt_{j-1}\rfloor\big)\big)^2.$$

Let $j=1,\dots,m$. We have: $\mathrm{Var}(\Delta_j)=\mathrm{Var}(V_{\lfloor nt_j\rfloor-\lfloor nt_{j-1}\rfloor})$. Hence, according to Proposition 6, $\mathrm{Var}(\Delta)$ is in $\mathrm{O}(n^2\log(n))$. Now we have to bound $\mathrm{Var}(\Gamma)$. Since $\mathbb{E}_{\bar{\nu}}[\Gamma]=\mathrm{O}(n)$, it is enough to bound $\mathbb{E}_{\bar{\nu}}[\Gamma^2]$. We have:

$$\begin{aligned}\mathbb{E}_{\bar{\nu}}\big[\Gamma^2\big]&\le 4m^4\sum_{1\le j<j'\le m}(a_ja_{j'})^2\mathbb{E}_{\bar{\nu}}\left[\left(\sum_{k=\lfloor nt_{j-1}\rfloor+1}^{\lfloor nt_j\rfloor}\sum_{k'=\lfloor nt_{j'-1}\rfloor+1}^{\lfloor nt_{j'}\rfloor}\mathbf{1}_{\{S_k=S_{k'},\mathcal{I}_k=\mathcal{I}_{k'}\}}\right)^2\right]\\&\le 4m^4\sum_{1\le j<j'\le m}(a_ja_{j'})^2\sum_{k_1,k_2=\lfloor nt_{j-1}\rfloor+1}^{\lfloor nt_j\rfloor}\sum_{k_1',k_2'=\lfloor nt_{j'-1}\rfloor+1}^{\lfloor nt_{j'}\rfloor}\bar{\nu}\big(\{S_{k_1}=S_{k_1'}\text{ and }S_{k_2}=S_{k_2'}\}\big).\end{aligned}$$

In the proof of Proposition 6, we estimate this quantity and prove that it is in $\mathrm{O}(n^2\log(n))$ (see the estimate of the terms A_2 and A_3 in Section 2.4). □

This ends the proof of the convergence of the finite dimensional distributions (for the probability measure $(h\nu)\otimes\mathbb{P}_0$).

(2) *Tightness.* Let us notice that the distribution of $(Z_n(\Pi_0(x,\ell_0),\cdot))_n$ with respect to $\mathbb{P}_0$ does not depend on ℓ_0 in $\mathbb{Z}^2$. Hence, the tightness with respect to $(h\nu)\otimes\mathbb{P}_0$ follows from the tightness for $\bar{\nu}\otimes\mathbb{P}_0$. Following [3] and, according to Theorem 8.4 of [2], let us prove that, for every $\varepsilon>0$, there exists $\lambda>0$ such that, if n is large enough, then we have:

$$(\bar{\nu}\otimes\mathbb{P}_0)\Big(\sup_{i\le n}|Z_i|\ge\lambda\sqrt{n\log(n)}\Big)\le\frac{\varepsilon}{\lambda^2}.$$

Let $\varepsilon>0$. For any $n\ge1$, let us define: $Z_n^*:=\max_{i=0,\dots,n}Z_i$. Let us recall the general argument given by Bolthausen ([3], pp. 114–115). We will be able to use this argument since $\mathrm{Var}(Z_m)=\mathrm{O}(m\log(m))$ and since $\frac{V_m}{m\log(m)}$ converges in probability to $c_0>0$. For any real number $\rho>\sqrt{2}$, any integer $m\ge1$ and any $(x,\ell_0)\in\bar{M}\times\mathbb{Z}^2$, since $\mathrm{Var}_{\mathbb{P}_0}[Z_m(\Pi_0(x,\ell_0),\cdot)]=\sigma^2V_m(x)$, we have:

$$\begin{aligned}\mathbb{P}_0\big(Z_m^*\big(\Pi_0(x,\ell_0),\cdot\big)\ge\rho\sqrt{\sigma^2V_m(x)}\big)&\le\mathbb{P}_0\big(Z_m\big(\Pi_0(x,\ell_0),\cdot\big)\ge\big(\rho-\sqrt{2}\big)\sqrt{\sigma^2V_m(x)}\big)\\&\quad+\mathbb{P}_0\big(Z_{m-1}^*\big(\Pi_0(x,\ell_0),\cdot\big)\ge\rho\sqrt{\sigma^2V_m(x)}\big)\\&\quad\times\mathbb{P}_0\big(Z_{m-1}^*\big(\Pi_0(x,\ell_0),\cdot\big)-Z_m\big(\Pi_0(x,\ell_0),\cdot\big)\ge\sqrt{2}\sqrt{\sigma^2V_m(x)}\big)\\&\le\mathbb{P}_0\big(Z_m\big(\Pi_0(x,\ell_0),\cdot\big)\ge\big(\rho-\sqrt{2}\big)\sqrt{\sigma^2V_m(x)}\big)\\&\quad+\frac{1}{2}\mathbb{P}_0\big(Z_m^*\big(\Pi_0(x,\ell_0),\cdot\big)\ge\rho\sqrt{\sigma^2V_m(x)}\big)\end{aligned}$$

(see [3], this comes from an argument of [17]). The same holds if we replace Z_n by $-Z_n$. Hence, we have:

$$(\bar{\nu}\otimes\mathbb{P}_0)\Big(\max_{j=1,\dots,m}|Z_j|\geq\rho\sqrt{\sigma^2V_m}\Big)\leq 2(\bar{\nu}\otimes\mathbb{P}_0)\big(|Z_m|\geq\big(\rho-\sqrt{2}\big)\sqrt{\sigma^2V_m}\big).$$

But, from Proposition 8, we know that $\frac{V_m}{m\log(m)}$ converges in probability to $c_0>0$. Let $\varepsilon>0$. According to Proposition 5, we have: $\mathbb{E}_{\bar{\nu}\otimes\mathbb{P}_0}[(Z_n)^2]\leq bn\log(n)$, for some $b>0$. Let $\rho>2\sqrt{2}$ and $\lambda=\rho\sqrt{c_0\sigma^2}$. Let us take $\delta=\frac{\varepsilon}{\lambda^2}$. There exists $m_1(\delta)$ such that, if $m\geq m_1(\delta)$, then we have: $\bar{\nu}(\{V_m<\frac{c_0m\log(m)}{2}\})<\frac{\delta}{4}$ and $\bar{\nu}(\{V_m>2c_0m\log(m)\})<\frac{\delta}{4}$. Hence, for sufficiently large ρ and every $m\geq m_1(\delta)$, we have:

$$\begin{aligned}
&(\bar{\nu}\otimes\mathbb{P}_0)\Big(\max_{j=1,\dots,m}|Z_j|\geq\rho\sqrt{\sigma^2c_0m\log(m)}\Big)\\
&\quad\leq\frac{\delta}{4}+(\bar{\nu}\otimes\mathbb{P}_0)\left(\max_{j=1,\dots,m}|Z_j|\geq\frac{1}{\sqrt{2}}\rho\sqrt{\sigma^2V_m}\right)\\
&\quad\leq 2(\bar{\nu}\otimes\mathbb{P}_0)\left(|Z_m|\geq\frac{1}{\sqrt{2}}\left(\frac{\rho}{\sqrt{2}}-\sqrt{2}\right)\sqrt{c_0\sigma^2m\log(m)}\right)+\frac{\varepsilon}{2\lambda^2}\\
&\quad\leq\frac{\varepsilon}{\lambda^2},
\end{aligned}$$

if ρ is large enough (since $\mathrm{Var}(Z_m)=O(m\log(m))$).

2.3. *Proof of Corollary* 2

Let us write $Z_t^{(n)}:=\sqrt{\frac{1}{c_0\sigma^2n\log(n)}}Z_{\lfloor nt\rfloor}$ with the constant c_0 defined in Proposition 5. According to Theorem 1, $Z^{(n)}$ converges in distribution to the Wiener process W in the sense of $D([0;+\infty))$ with respect to $h\nu\otimes\mathbb{P}_0$ with $h(y):=\int_0^{\tau(y)}g(\Delta(y,s))\,\mathrm{d}s$. For all $(q,\vec{v})\in M$, all $s\in[0;\tau(q,\vec{v})[$ and all $n\geq 1$, we have:

$$\tilde{Z}_{nt}\big((q+s\vec{v},\vec{v}),\omega\big)=Z_{n\tilde{n}(nt+s,(q,\vec{v}))/n}\big((q,\vec{v}),\omega\big).$$

Let us define: $\varphi_n(t)((q+s\vec{v}),\omega):=\frac{\tilde{n}(nt+s,(q,\vec{v}))}{n}$. We know that $(\frac{\tilde{n}(nt,\cdot)}{n})_{t\geq 0}$ converges in probability to $(\frac{t}{\int_{\bar{M}}\tau\,\mathrm{d}\bar{\nu}})_{t\geq 0}$ (in $\mathcal{D}([0;+\infty))$) with respect to $\bar{\nu}$ (see, e.g., [20]). Hence, $(\varphi_n(t))_t$ converges in probability to $(\frac{t}{\int_{\bar{M}}\tau\,\mathrm{d}\bar{\nu}})_{t\geq 0}$ (in $\mathcal{D}([0;+\infty))$) with respect to $g\mu_1\otimes\mathbb{P}_0$. This ends our proof, according to a classical argument (see [2], p. 145 and Theorem 4.4).

2.4. *Proof of Proposition* 6

Let σ_+^2 be the largest eigenvalue of Σ^2 and let a_0 be any real number satisfying $a_0\in(0,(\sigma_+^2)^{-1})$. We will use Proposition 4 and the fact that there exists $b_0>0$ such that, for every $x\in\mathbb{R}^2$, we have $(|x|_2+|x|_2^3)\mathrm{e}^{-\langle(\Sigma^2)^{-1}x,x\rangle/2}\leq b_0\mathrm{e}^{-a_0\langle x,x\rangle/2}$ and therefore:

$$\left(\frac{|N|_2}{\sqrt{k}}+\frac{|N|_2^3}{k^{3/2}}\right)\mathrm{e}^{-\langle(\Sigma^2)^{-1}N,N\rangle/(2k)}\leq b_0\mathrm{e}^{-a_0\langle N,N\rangle/2k}.$$

We have:

$$\begin{aligned}
\mathrm{Var}_{\bar{\nu}}(V_n)=4\sum_{0\leq k_1<\ell_1\leq n-1}\sum_{0\leq k_2<\ell_2\leq n-1}&\big[\bar{\nu}(S_{k_1}=S_{\ell_1},\mathcal{I}_{k_1}=\mathcal{I}_{\ell_1},S_{k_2}=S_{\ell_2}\text{ and }\mathcal{I}_{k_2}=\mathcal{I}_{\ell_2})\\
&-\bar{\nu}(S_{k_1}=S_{\ell_1}\text{ and }\mathcal{I}_{k_1}=\mathcal{I}_{\ell_1})\bar{\nu}(S_{k_2}=S_{\ell_2}\text{ and }\mathcal{I}_{k_2}=\mathcal{I}_{\ell_2})\big].
\end{aligned}$$

Let us define the event $E_{k,\ell} := \{S_k = S_\ell \text{ and } \mathcal{I}_k = \mathcal{I}_\ell\}$. The variance of V_n can be rewritten $8A_1 + 8A_2 + 8A_3 + 4A_4$ with:

$$A_1 := \sum_{0\le k_1<\ell_1\le k_2<\ell_2\le n-1} \big[\bar\nu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2}) - \bar\nu(E_{k_1,\ell_1})\bar\nu(E_{k_2,\ell_2})\big],$$

$$A_2 := \sum_{0\le k_1\le k_2<\ell_1<\ell_2\le n-1} \big[\bar\nu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2}) - \bar\nu(E_{k_1,\ell_1})\bar\nu(E_{k_2,\ell_2})\big],$$

$$A_3 := \sum_{0\le k_1<k_2<\ell_2\le \ell_1\le n-1} \big[\bar\nu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2}) - \bar\nu(E_{k_1,\ell_1})\bar\nu(E_{k_2,\ell_2})\big],$$

$$A_4 := \sum_{0\le k<\ell\le n-1} \big[\bar\nu(E_{k,\ell}) - (\bar\nu(E_{k,\ell}))^2\big].$$

- Control of A_1.
  Let $0\le k_1<\ell_1\le k_2<\ell_2\le n-1$. According to Proposition 3, we have:

$$\big|\bar\nu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2}) - \bar\nu(E_{k_1,\ell_1})\bar\nu(E_{k_2,\ell_2})\big| \le \frac{I^2 C{\tau_1}^{k_2-\ell_1}}{(\ell_1-k_1)(\ell_2-k_2)}.$$

  Hence: $|A_1| = \mathrm{O}(n\log^2(n))$.
- Control of A_2.
  - Let us start with the control of the product of the probabilities. We have:

$$\begin{aligned}\sum_{0\le k_1\le k_2<\ell_1<\ell_2\le n-1} \bar\nu(E_{k_1,\ell_1})\bar\nu(E_{k_2,\ell_2}) &\le c\sum_{0\le k_1\le k_2<\ell_1<\ell_2\le n}\frac{1}{\ell_1-k_1}\frac{1}{\ell_2-k_2}\\
&\le \sum_{\substack{m_1,m_2\ge 0\\ m_3,m_4\ge 1\\ m_1+m_2+m_3+m_4\le n}} \frac{c}{(m_2+m_3)(m_3+m_4)}\\
&\le \sum_{\substack{m_2\ge 0\\ m_3,m_4\ge 1\\ m_2+m_3+m_4\le n}} \frac{c(n-(m_2+m_3+m_4)+1)}{(m_2+m_3)(m_3+m_4)}\\
&\le \sum_{k,\ell=1}^{n}\sum_{\max(1,k+\ell-n)\le m_3\le\min(k,\ell)} \frac{c(n-(k+\ell-m_3)+1)}{k\ell}\\
&\le 2\sum_{k=1}^{n}\sum_{\ell=1}^{k}\frac{c(n-k+1)\ell}{k\ell} \le 2\sum_{k=1}^{n} c(n-k+1) = \mathrm{O}\big(n^2\big).\end{aligned}$$

  - Now it suffices to estimate:

$$\sum_{0\le k_1\le k_2<\ell_1<\ell_2\le n-1} \bar\nu(S_{\ell_1}-S_{k_1}=0 \text{ and } S_{\ell_2}-S_{k_2}=0).$$

    For any choice of $0\le k_1\le k_2<\ell_1<\ell_2\le n$, we have: $\bar\nu(S_{\ell_1}-S_{k_1}=0 \text{ and } S_{\ell_2}-S_{k_2}=0) = \sum_x \bar\nu(S_{k_2}-S_{k_1}=x, S_{\ell_1}-S_{k_2}=-x \text{ and } S_{\ell_2}-S_{\ell_1}=x)$. The sum is taken over $x\in\mathbb{Z}^2$ such that $|x|\le\|\Phi\|_\infty\min(k_2-k_1,\ell_1-k_2,\ell_2-\ell_1)$. According to Proposition 4 for $1<p<3/\sqrt{8}$, we have:

$$\bar\nu(S_{k_2}-S_{k_1}=x, S_{\ell_1}-S_{k_2}=-x \text{ and } S_{\ell_2}-S_{\ell_1}=x) \le \tilde K_0(A+B)$$

for some universal constant $\tilde{K}_0 > 0$ and with

$$A := \mathrm{e}^{-a_0\langle x,x\rangle/(2(k_2-k_1+1))}\frac{\bar{\nu}(S_{\ell_1}-S_{k_2}=-x \text{ and } S_{\ell_2}-S_{\ell_1}=x)}{k_2-k_1+1},$$

$$B := \frac{(\bar{\nu}(S_{\ell_1}-S_{k_2}=-x \text{ and } S_{\ell_2}-S_{\ell_1}=x))^{1/p}}{(k_2-k_1+1)^{3/2}}.$$

Analogously, we have:

$$\bar{\nu}(S_{\ell_1}-S_{k_2}=-x \text{ and } S_{\ell_2}-S_{\ell_1}=x) \le \tilde{K}_0\big(A'+B'\big)$$

with

$$A' := \mathrm{e}^{-a_0\langle x,x\rangle/(2(\ell_1-k_2))}\frac{\bar{\nu}(S_{\ell_2}-S_{\ell_1}=x)}{\ell_1-k_2} \quad \text{and} \quad B' := \frac{(\bar{\nu}(S_{\ell_2}-S_{\ell_1}=x))^{1/p}}{(\ell_1-k_2)^{3/2}}.$$

In the same way, we have:

$$\bar{\nu}(S_{\ell_2}-S_{\ell_1}=x) \le \tilde{K}_0\big(A''+B''\big)$$

with:

$$A'' := \frac{\mathrm{e}^{-a_0\langle x,x\rangle/(2(\ell_2-\ell_1))}}{\ell_2-\ell_1} \quad \text{and} \quad B'' := \frac{1}{(\ell_2-\ell_1)^{3/2}}.$$

* Terms with (A, A', A''). The sum of these terms over $(x, k_1, k_2, \ell_1, \ell_2)$ is less than:

$$\begin{aligned}\sum_x \sum_{0\le k_1\le k_2<\ell_1<\ell_2\le n-1} &\frac{\mathrm{e}^{-[1/(2(k_2-k_1+1))+1/(2(\ell_1-k_2))+1/(2(\ell_2-\ell_1))]a_0\langle x,x\rangle}}{(k_2-k_1+1)(\ell_1-k_2)(\ell_2-\ell_1)} \\ &\le c\sum_{k_1,k_2,\ell_1,\ell_2}\frac{\min(k_2-k_1+1,\ell_1-k_2,\ell_2-\ell_1)}{(k_2-k_1+1)(\ell_1-k_2)(\ell_2-\ell_1)} \\ &\le 6cn\sum_{k=1}^n\sum_{\ell\le p\le k}\frac{1}{kp} = \mathrm{O}\big(n^2\big).\end{aligned}$$

* Terms with (B, B', B''). The sum of these terms over $(x, k_1, k_2, \ell_1, \ell_2)$ is less than:

$$cn\sum_{k+\ell+m\le n}\frac{\min(k^2,l^2,m^2)}{k^{3/2}\ell^{3/(2p)}m^{3/(2p^2)}} \le 6cn\sum_{1\le k,\ell,m\le n}\frac{k^{2/3}\ell^{2/3}m^{2/3}}{k^{3/(2p^2)}\ell^{3/(2p^2)}m^{3/(2p^2)}}.$$

This is in $\mathrm{O}(n^2)$ since $1 < p < \frac{3}{\sqrt{8}}$.

* The remaining terms correspond to (A, A', B''), (A, B', A''), (A, B', B''), (B, A', A''), (B, A', B'') and (B, B', A''). The sum over $(x, k_1, k_2, \ell_1, \ell_2)$ of these terms is less (up to some fixed multiplicative constant) than:

$$n\sum_{k+\ell+m\le n}\frac{\min(k,\ell,m^2)}{k^{1/p}\ell^{1/p}m^{3/2}} + n\sum_{k+\ell+m\le n}\frac{\min(k,\ell^2,m^2)}{k^{1/(p^2)}\ell^{3/2}m^{3/(2p)}}.$$

(The first term corresponds to (A, A', B''), (A, B', A''), (B, A', A'') and the second one to the others.) We have:

$$n\sum_{1\le k,\ell,m\le n}\frac{\min(k,\ell,m^2)}{k^{1/p}\ell^{1/p}m^{3/2}} \le n\sum_{1\le k,\ell,m\le n}\frac{k^{1/p-1/2}\ell^{1/p-1/2}m^{4-4/p}}{k^{1/p}\ell^{1/p}m^{3/2}} = \mathrm{O}\big(n^2\big),$$

since $1 < p < 8/7$. Now let us estimate the second term:

$$n \sum_{k+\ell+m\le n} \frac{\min(k,\ell^2,m^2)}{k^{1/(p^2)}\ell^{3/2}m^{3/(2p)}} \le n \sum_{k+\ell+m\le n} \frac{k^{1-1/(2p^2)}\ell^{2(1/(4p^2))}m^{2(1/(4p^2))}}{k^{1/(p^2)}\ell^{3/2}m^{3/(2p)}} \le \mathrm{O}\big(n^2\big),$$

since $1 < p^2 < 5/4$.

- Control of A_3.
  - First we have:

$$\sum_{0\le k_1<k_2<\ell_2\le\ell_1\le n-1} \bar\nu(S_{k_1}=S_{\ell_1})\bar\nu(S_{k_2}=S_{\ell_2}) \le c \sum_{0\le k_1<k_2<\ell_2\le\ell_1\le n-1} \frac{1}{\ell_1-k_1}\frac{1}{\ell_2-k_2}$$
$$\le c\sum_{p=2}^{n-1}\sum_{\ell=1}^{p-1}\frac{n-p}{p}\frac{p-\ell}{\ell} \le c\sum_{p=2}^{n-1}\sum_{\ell=1}^{p-1}\frac{n}{\ell} \le \mathrm{O}\big(n^2\log(n)\big).$$

  - We have to estimate:

$$\sum_{0\le k_1<k_2<\ell_2\le\ell_1\le n-1} \bar\nu(S_{k_1}=S_{\ell_1} \text{ and } S_{k_2}=S_{\ell_2}).$$

We have:

$$\bar\nu(S_{k_1}=S_{\ell_1} \text{ and } S_{k_2}=S_{\ell_2}) = \bar\nu(0=S_{m+k+\ell-1} \text{ and } S_m=S_{m+k})$$
$$= \sum_x \bar\nu(S_m=x,\, S_{m+k}-S_m=0 \text{ and } S_{m+k+\ell-1}-S_{m+k}=-x)$$

with $m=k_2-k_1$, $k=\ell_2-k_2$ and $\ell=\ell_1-\ell_2+1$. The sum is taken over the x in $\mathbb{Z}^2$ such that: $|x|_\infty \le \|\Phi\|_\infty \min(m,\ell)$. According to Proposition 4, we have:

$$\bar\nu(S_m=x,\, S_{m+k}-S_m=0 \text{ and } S_{m+k+\ell-1}-S_{m+k}=-x) \le \tilde K_1(A+B+C),$$

for some universal constant $\tilde K_1$ and with:

$$A := \frac{\mathrm{e}^{-a_0\langle x,x\rangle/(2m)}\bar\nu(S_k=0 \text{ and } S_{k+\ell-1}-S_k=-x)}{m},$$

$$B := \frac{\mathrm{e}^{-a_0\langle x,x\rangle/(2m)}\bar\nu(S_k=0 \text{ and } S_{k+\ell-1}-S_k=-x)^{1/p}}{m^{3/2}},$$

$$C := \frac{\bar\nu(S_k=0 \text{ and } S_{k+\ell-1}-S_k=-x)^{1/p}}{m^2}.$$

Moreover, since $\Phi\circ\bar T^{-k}(q,\vec v) = -\Phi\circ\bar T^{k-1}(q,\vec v')$ (with $\vec v'$ such that $\langle\vec n(q),\vec v'\rangle = -\langle\vec n(q),\vec v\rangle$), $(\Phi\circ\bar T^{-k})_k$ has the same distribution as $(-\Phi\circ\bar T^k)_k$ (with respect to $\bar\nu$). Hence, we have:

$$\bar\nu(S_k=0 \text{ and } S_{k+\ell-1}-S_k=-x) = \bar\nu\left(\sum_{j=-k-\ell+2}^{-\ell+1}\Phi\circ\bar T^j=0 \text{ and } \sum_{i=-\ell+2}^{0}\Phi\circ\bar T^i=-x\right)$$
$$= \bar\nu(S_{\ell-1}=x \text{ and } S_{k+\ell-1}-S_{\ell-1}=0).$$

Hence according to Proposition 4 for $1<p<\sqrt{8/7}$ and since $\bar\nu(S_k=0)=\mathrm{O}(k^{-1})$, we have:

$$\bar\nu(S_k=0 \text{ and } S_{k+\ell-1}-S_k=-x) \le \tilde K_2\big(A'+B'+C'\big),$$

for some universal constant $\tilde{K}_2 > 0$ and with:

$$A' = \frac{\mathrm{e}^{-a_0\langle x,x\rangle/(2\ell)}}{\ell k}, \qquad B' = \frac{\mathrm{e}^{-a_0\langle x,x\rangle/(2\ell)}}{\ell^{3/2}k^{1/p}} \quad \text{and} \quad C' = \frac{1}{\ell^2 k^{1/p}}.$$

* The term with (A, A') is less than: $\frac{\mathrm{e}^{-a_0\langle x,x\rangle/(2m)}\mathrm{e}^{-a_0\langle x,x\rangle/(2\ell)}}{mk\ell}$. The sum of these quantities over (x, k_1, m, k, ℓ), is less than:

$$\mathrm{O}\biggl(n\sum_{m,k,\ell}\frac{\min(m,\ell)}{km\ell}\biggr) \le \mathrm{O}\biggl(n\sum_{m,k,\ell}\frac{m^{1/2}\ell^{1/2}}{km\ell}\biggr) = \mathrm{O}\bigl(n^2\log(n)\bigr).$$

* The term with (A, B') is: $\frac{\mathrm{e}^{-a_0\langle x,x\rangle/(2m)}\mathrm{e}^{-a_0\langle x,x\rangle/(2\ell)}}{m\ell^{3/2}k^{1/p}}$. The sum of these terms over (x, k_1, m, k, ℓ) is less than:

$$\begin{aligned}\mathrm{O}\biggl(n\sum_{m,k,\ell}\frac{\min(m,\ell)}{m\ell^{3/2}k^{1/p}}\biggr) &= \mathrm{O}\biggl(n\sum_{m,k,\ell}\frac{1}{m\sqrt{\ell}k^{1/p}}\biggr)\\ &= \mathrm{O}\bigl(nn^{1/2}\log(n)n^{1-1/p}\bigr) = \mathrm{O}\bigl(n^2\bigr),\end{aligned}$$

since $1 - 1/p < 1/2$.

* The term in (A, C') is: $\frac{\mathrm{e}^{-a_0\langle x,x\rangle/(2m)}}{m\ell^2k^{1/p}}$. The sum of these terms over (x, k_1, m, k, ℓ) is in:

$$\begin{aligned}\mathrm{O}\biggl(n\sum_{m,k,\ell}\frac{\min(m,\ell^2)}{m\ell^2k^{1/p}}\biggr) &= \mathrm{O}\biggl(n\sum_{m,k,\ell}\frac{(m)^{1/2}(\ell^2)^{1/2}}{m\ell^2k^{1/p}}\biggr)\\ &= \mathrm{O}\bigl(n\sqrt{n}\log(n)n^{1-1/p}\bigr) = \mathrm{O}\bigl(n^2\bigr)\end{aligned}$$

since $1 - 1/p < 1/2$.

* The terms with (B, A'), (B, B') and (B, C') are less than: $\frac{\mathrm{e}^{-a_0\langle x,x\rangle/(2m)}}{m^{3/2}\ell^{1/p}k^{1/(p^2)}}$. The sum of these terms over (x, k_1, m, k, ℓ) is in:

$$\mathrm{O}\biggl(n\sum_{m,k,\ell}\frac{m}{m^{3/2}\ell^{1/p}k^{1/(p^2)}}\biggr) = \mathrm{O}\bigl(n\sqrt{n}n^{1-1/p}n^{1-1/(p^2)}\bigr) = \mathrm{O}\bigl(n^2\bigr)$$

since $p^2 < 4/3$.

* The term with (C, A') is: $\frac{\mathrm{e}^{-a_0\langle x,x\rangle/(2\ell p)}}{m^2\ell^{1/p}k^{1/p}}$. The sum of these terms over (x, k_1, m, k, ℓ) is in:

$$\begin{aligned}\mathrm{O}\biggl(n\sum_{m,k,\ell}\frac{\min(\ell,m^2)}{m^2\ell^{1/p}k^{1/p}}\biggr) &= \mathrm{O}\biggl(n\sum_{m,k,\ell}\frac{(m^2)^{3/4}\ell^{1/4}}{m^2\ell^{1/p}k^{1/p}}\biggr)\\ &= \mathrm{O}\bigl(n\sqrt{n}n^{1+1/4-1/p}n^{1-1/p}\bigr) = \mathrm{O}\bigl(n^2\bigr)\end{aligned}$$

since $p < 8/7$.

* The terms with (C, B') and (C, C') are less than: $\frac{1}{m^2\ell^{3/2p}k^{1/p^2}}$. The sum of these terms over (x, k_1, m, k, ℓ) is less than:

$$\begin{aligned}\mathrm{O}\biggl(n\sum_{m,k,\ell}\frac{\min(\ell^2,m^2)}{m^2\ell^{3/2p}k^{1/p^2}}\biggr) &= \mathrm{O}\biggl(n\sum_{m,k,\ell}\frac{(m^2)^{3/4}(\ell^2)^{1/4}}{m^2\ell^{3/2p}k^{1/p^2}}\biggr)\\ &= \mathrm{O}\bigl(n\sqrt{n}n^{3/2-3/(2p)}n^{1-1/(p^2)}\bigr) = \mathrm{O}\bigl(n^2\bigr)\end{aligned}$$

since $p < 6/5$ and $p^2 < 4/3$.

- Control of A_4. We obviously have $A_4 = \mathrm{O}(n^2)$.

3. Proof of Proposition 7

Since, for all integers $m \geq 1$, $\bar{\nu}(\sup_{\ell\in\mathbb{Z}^2} \mathcal{N}_\ell(n) \geq \varepsilon n^a) \leq (2n \max \tau + 1)^2 \sup_{\ell\in\mathbb{Z}^2} \frac{\mathbb{E}_{\bar{\nu}}[(\mathcal{N}_\ell(n))^m]}{\varepsilon^m n^{am}}$, Proposition 7 will follow from the following lemma and from the first Borel–Cantelli lemma:

Lemma 9. *For all $k \geq 1$ and all $q > 0$, we have*: $\sup_{\ell\in\mathbb{Z}} \mathbb{E}_{\bar{\nu}}[(\mathcal{N}_\ell(n))^k] = \mathrm{o}(n^q)$.

Proof. Let $\ell \in \mathbb{Z}^2$. We have:

$$\mathbb{E}_{\bar{\nu}}\big[\big(\mathcal{N}_\ell(n)\big)^k\big] \leq k! \sum_{0\leq j_1\leq\cdots\leq j_k\leq n-1} \bar{\nu}(S_{j_1} = \ell, S_{j_2} = S_{j_1}, \ldots, S_{j_k} = S_{j_{k-1}}).$$

But, according to Lemma 13 of Section A.4, for all $m \geq 2$, there exists $C_m > 1$ such that, for all $0 \leq k_1 \leq \cdots \leq k_m$ and for all $\alpha \in \mathbb{Z}^2$, we have:

$$\bar{\nu}(S_{k_1} = \alpha, S_{k_2} = S_{k_1}, \ldots, S_{k_m} = S_{k_{m-1}}) \leq \frac{C_m}{(k_1+1)(k_2-k_1+1)\cdots(k_m-k_{m-1}+1)}. \qquad \square$$

Appendix A. Proof of the extensions of the local limit theorem

Let $1 < p < 2$. We use the Young towers [25] and Nagaev's method [15,16]. Nagaev's method has been extended especially with the contribution of Le Page [14], Guivarc'h [11], and Guivarc'h and Hardy in [12]. It has been generalised by Hennion and Hervé in [13].

A.1. *Young towers*

In [25], Young constructs an integer d and two dynamical systems $(\tilde{M}, \tilde{\nu}, \tilde{T})$ and $(\hat{M}, \hat{\nu}, \hat{T})$ such that $(\tilde{M}, \tilde{\nu}, \tilde{T})$ is an extension of $(\bar{M}, \bar{\nu}, \bar{T}^d)$ and of $(\hat{M}, \hat{\nu}, \hat{T})$, that is, there exist two measurable functions $\pi : (\tilde{M}, \tilde{\nu}, \tilde{T}) \to (\bar{M}, \bar{\nu}, \bar{T}^d)$ and $\hat{\pi} : (\tilde{M}, \tilde{\nu}, \tilde{T}) \to (\hat{M}, \hat{\nu}, \hat{T})$ such that: $\pi \circ \tilde{T} = \bar{T}^d \circ \pi$, $\bar{\nu} = (\pi)_*(\tilde{\nu})$, $\hat{\pi} \circ \tilde{T} = \hat{T} \circ \hat{\pi}$ and $\hat{\nu} = (\hat{\pi})_*(\tilde{\nu})$. Let us give some useful details. Young constructs a well-chosen set $\Lambda = (\bigcup_{\gamma^u\in\Gamma^u_\Lambda} \gamma^u) \cap (\bigcup_{\gamma^s\in\Gamma^s_\Lambda} \gamma^s)$ where Γ^u_Λ is a set of unstable curves (of $\bar{T}$) and where Γ^s_Λ is a set of stable curves (of $\bar{T}$) such that each $\gamma^s \in \Gamma^s_\Lambda$ meets each $\gamma^u \in \Gamma^u_\Lambda$ at exactly one point. Then she constructs a well-chosen return time $R(\cdot)$ for $\bar{T}$ in Λ and a family $(\Lambda_i)_{i\geq 0}$ of pairwise disjoint subsets of Λ (with positive measure) such that:

- we have $\bar{\nu}(\Lambda \setminus \bigcup_{i\geq 0} \Lambda_i) = 0$ with $\Lambda_i = (\bigcup_{\gamma^u\in\Gamma^u_\Lambda} \gamma^u) \cap (\bigcup_{\gamma^s\in\Gamma^s_i} \gamma^s)$, with $\Gamma^s_i \subseteq \Gamma^s_\Lambda$;
- on Λ_i, the return time R is equal to a constant r_i;
- Λ_i is contained in a connected component of $\bar{M} \setminus \bigcup_{j=0}^{r_i} \bar{T}^{-j}(R_0)$ and we have: $\bar{T}^{r_i}(\Lambda_i) = (\bigcup_{\gamma^u\in\Gamma^u_i}) \cap (\bigcup_{\gamma^s\in\Gamma^s_\Lambda})$, with $\Gamma^u_i \subseteq \Gamma^u_\Lambda$.

Let us notice that $\sum_{j=0}^{R(\cdot)-1} \Phi \circ \bar{T}^j$ is equal to some constant L_i on each Λ_i. Using the fact that the billiard system in the plane (M, ν, T) is totally ergodic (see [8,19,21,22]) we can adapt Young's construction in such a way that $L_0 = (0, 0)$, $L_1 = (1, 0)$, $L_2 = (0, 1)$ and r_1 and r_2 and $r_3 - 1$ are multiples of r_0 (the idea is to adapt the construction of the first four sub-parallelograms Λ_0, Λ_1, Λ_2 and Λ_3 and then to follow Young's construction, cf. Appendix B). This observation gives another way to prove the non-arithmeticity than the proof given by Szász and Varjú in [24] (see our Lemma 11).

Now, we take d to be the biggest common divisor of the r_i. With our adaptation, d is equal to 1 (this fact is not essential here but it simplifies formulas). We take: $\tilde{M} := \{(x, \ell)\colon x \in \Lambda, \ell \in \mathbb{Z}_+, \ell < R(x)\}$ and $\tilde{T}(x, \ell) = (x, \ell + 1)$ if $\ell < R(x) - 1$ and $\tilde{T}(x, R(x) - 1) = (\bar{T}^{R(x)}(x), 0)$. We do not give the construction of $\tilde{\nu}$ here. The system $(\hat{M}, \hat{\nu}, \hat{T})$ is obtained from $(\tilde{M}, \tilde{\nu}, \tilde{T})$ by quotienting Λ along the stable curves: $\hat{M} = \{(x, \ell)\colon x \in \gamma^u_0, \ell \in \mathbb{Z}_+, \ell < R(x)\}$ for a fixed unstable curve γ^u_0 belonging to Γ^u_Λ and with $\hat{\pi} : (x, \ell) \mapsto (\gamma^s(x) \cap \gamma^u_0, \ell)$. Let us notice that if a measurable function $g : \bar{M} \to \mathbb{C}$ is constant along the stable curves, then there exists a unique $\hat{g} : \hat{M} \to \mathbb{C}$ such that $g \circ \pi = \hat{g} \circ \hat{\pi}$. In the following, we will consider the function $\psi : \hat{M} \to \mathbb{C}$ such that:

$$\psi \circ \hat{\pi} = \Phi \circ \pi.$$

Young defines a separation time $\hat{s}(\cdot, \cdot)$ on $\hat{M}$ such that if $\hat{s}(x, y) \geq n$, we have $\hat{s}(x, y) = n + \hat{s}(\hat{T}^n(x), \hat{T}^n(y))$ and the sets $\pi(\hat{\pi}^{-1}(\{x\}))$ and $\pi(\hat{\pi}^{-1}(\{y\}))$ are contained in the same connected component of $\bar{M} \setminus \bigcup_{j=0}^{n} \bar{T}^{-j}(R_0)$. Moreover, if x and y belong to the same $\hat{\pi}(\Lambda_i \times \{0\})$, then $\hat{s}(x, y) \geq r_i$. For any $\beta \in (0, 1)$ and any $\varepsilon \geq 0$, Young defines the functional space:

$$\mathcal{V}_{(\beta,\varepsilon)} := \{\hat{f} : \hat{M} \to \mathbb{C} \text{ measurable}, \|\hat{f}\|_{\mathcal{V}_{(\beta,\varepsilon)}} < +\infty\},$$

where $\|\hat{f}\|_{\mathcal{V}_{(\beta,\varepsilon)}} := \|\hat{f}\|_{(\beta,\varepsilon,\infty)} + \|\hat{f}\|_{(\beta,\varepsilon,h)}$, with

$$\|\hat{f}\|_{(\beta,\varepsilon,\infty)} := \sup_{\ell \geq 0} \|\hat{f}_{|\hat{\Delta}_\ell}\|_{L^\infty} \mathrm{e}^{-\ell\varepsilon} \quad \text{and} \quad \|\hat{f}\|_{(\beta,\varepsilon,h)} := \sup_{\ell \geq 0} \operatorname*{ess\,sup}_{\hat{x},\hat{y} \in \hat{\Delta}_\ell, \hat{s}(\hat{x},\hat{y}) \geq 0, \hat{x} \neq \hat{y}} \frac{|\hat{f}(\hat{x}) - \hat{f}(\hat{y})|}{\beta^{\hat{s}(\hat{x},\hat{y})}} \mathrm{e}^{-\ell\varepsilon},$$

where $\hat{\Delta}_\ell$ is the ℓth floor of the tower $\hat{M}$ (i.e., $\hat{\Delta}_\ell = \{(x, q) \in \hat{M}\colon\ q = \ell\}$). We suppose that $\beta \in (0, 1)$ and $\varepsilon > 0$ are such that there exists $C_0 > 0$ such that we have: $\|\cdot\|_{L^{p/(p-1)}(\hat{\nu})} \leq C_0 \|\cdot\|_{\mathcal{V}_{(\beta,\varepsilon)}}$ and such that the following strong ergodicity property of $\hat{P}$ is true (the existence of such β and ε is proven in [25]). It is easy to prove the following:

Lemma 10. *If g belongs to $\mathcal{V}_{(\beta,\varepsilon)}$ and if h belongs to $\mathcal{V}_{(\beta,0)}$, then gh belongs to $\mathcal{V}_{(\beta,\varepsilon)}$ and we have*:

$$\|gh\|_{\mathcal{V}_{(\beta,\varepsilon)}} \leq \|g\|_{\mathcal{V}_{(\beta,\varepsilon)}} \|h\|_{\mathcal{V}_{(\beta,0)}}.$$

A.2. Transfer operator

The adjoint operator $\hat{P}$ of $g \mapsto g \circ \hat{T}$ on $L^2(\hat{\nu})$ is a continuous linear operator on $\mathcal{V}_{(\beta,\varepsilon)}$ satisfying $\hat{P}\mathbf{1} = \mathbf{1}$. Moreover, $\hat{P}$ is strongly ergodic: there exist two real numbers $C_1 > 0$ and $\tau_1 \in (0, 1)$ such that, for all integers $n \geq 0$ and for all $\hat{f} \in \mathcal{V}_{(\beta,\varepsilon)}$ such that $\int_{\hat{M}} \hat{f} \, \mathrm{d}\hat{\nu} = 0$, we have: $\|\hat{P}^n \hat{f}\|_{\mathcal{V}_{(\beta,\varepsilon)}} \leq C_1 {\tau_1}^n \|\hat{f}\|_{\mathcal{V}_{(\beta,\varepsilon)}}$. Moreover, there exist inverse branches χ of $\hat{T}$ and a function κ such that $\hat{P}(g)(\hat{x}) = \sum_\chi \kappa(\chi(\hat{x}))g(\chi(\hat{x}))$. The inverse branches are such that $\hat{T}(\chi(\hat{x})) = \hat{x}$ and:

$$\text{if } \hat{s}(\hat{x}, \hat{y}) \geq 0, \text{ then} \quad \hat{s}(\chi(\hat{x}), \chi(\hat{y})) = 1 + \hat{s}(\hat{x}, \hat{y}). \tag{A.1}$$

There exists $c_\kappa > 0$ such that, for any $\hat{x}$ and any $\hat{y}$, we have:

$$0 < \kappa(\hat{x}) \leq 1, \quad \sum_\chi \kappa(\chi(\hat{x})) = 1 \quad \text{and} \quad \left|\log\left(\frac{\kappa(\hat{x})}{\kappa(\hat{y})}\right)\right| \leq c_\kappa \beta^{\hat{s}(\hat{x},\hat{y})}. \tag{A.2}$$

In the sequel, we use the operators family $(\hat{P}_u)_{u \in \mathbb{R}^2}$ given by $\hat{P}_u(h) := \hat{P}(\exp(\mathrm{i}\langle u, \psi\rangle)h)$. We write $\hat{S}_m$ for $\sum_{k=0}^{m-1} \psi \circ \hat{T}^k$. Since we have $\hat{P}^m(f \times g \circ \hat{T}^m) = g \times \hat{P}^m(f)$ (for all integers $m \geq 0$), it is easy to prove that:

$$\forall m \geq 0 \quad \hat{P}_u^m(h) = \hat{P}^m(\mathrm{e}^{\mathrm{i}\langle u, \hat{S}_m\rangle} h).$$

Let $\mathcal{B}$ be any complex Banach space. We define the set $\mathcal{B}'$ of continuous $\mathbb{C}$-linear maps from $\mathcal{B}$ in $\mathbb{C}$. We endow this set with the norm $\|\cdot\|_{\mathcal{B}'}$ given by: $\|A\|_{\mathcal{B}'} := \sup_{\|f\|_{\mathcal{B}}=1} |A(f)|$. We denote by $\mathcal{L}_{\mathcal{B}}$ the set of continuous $\mathbb{C}$-linear endomorphisms of $\mathcal{B}$. We endow this set with the norm $\|\cdot\|_{\mathcal{L}_{\mathcal{B}}}$ given by: $\|P\|_{\mathcal{L}_{\mathcal{B}}} := \sup_{\|f\|_{\mathcal{B}}=1} \|P(f)\|_{\mathcal{B}}$.

Theorem A.1 (Multidimensional version of Theorem IV.8 of [13]). *Let q be a positive integer. Let $\mathcal{B}$ be a complex Banach space. Let U_0 be an open subset of $\mathbb{R}^q$ containing 0. Let $m \geq 1$ be some integer. Let $(Q(t))_{t \in U_0}$ be a family of continuous linear operators on $\mathcal{B}$ such that the application $t \mapsto Q(t)$ is in $C^m(U_0, \mathcal{L}_{\mathcal{B}})$ and such that there exist* *two subspaces $\mathcal{F}$ and $\mathcal{H}$ of $\mathcal{B}$ with: $\mathcal{B} = \mathcal{F} \oplus \mathcal{H}$, $Q(0)(\mathcal{F}) \subseteq \mathcal{F}$, $Q(0)(\mathcal{H}) \subseteq \mathcal{H}$, $\dim(\mathcal{F}) = 1$ and $Q(0)_{|\mathcal{F}} \equiv \mathrm{id}_{|\mathcal{F}}$, the spectral radius of $Q(0)_{|\mathcal{H}}$ being strictly less than 1.*

Then there exists an open set U_1 containing 0 and contained in U_0, there exist three real numbers $\eta_1 > 0$, $\eta_2 > 0$, $c_1 \geq 0$ and four functions $\lambda \in C^m(U_1, \mathbb{C})$, $v \in C^m(U_1, \mathcal{B})$, $\phi \in C^m(U_1, \mathcal{B}')$ and $N \in C^m(U_1, \mathcal{L}_{\mathcal{B}})$ such that, for all $t \in U_1$, we have, for all $n \geq 1$,

$$Q(t)^n(h) = \lambda(t)^n \big(\phi(t)(h)\big) v(t) + N(t)^n(h),$$

with $Q(t)v(t) = \lambda(t)v(t)$, $Q(t)^*\phi(t) = \lambda(t)\phi(t)$, $(\phi(t))(v(t)) = 1$, $|\lambda(t)| \geq 1 - \eta_1$ and, for all $k = 0, \ldots, m$, for all $i_1, \ldots, i_k \in \{1, \ldots, q\}$ and all $n \geq 1$, $\|\frac{\partial^k}{\partial t_{i_1} \cdots \partial t_{i_k}}(N(t)^n)\|_{\mathcal{L}_{\mathcal{B}}} \leq c_1(1 - \eta_1 - \eta_2)^n$.

Idea of the proof. This is the multidimensional version of Theorem IV-8 of [13] which is based on the implicit functions theorem (see Chapter XIV of [13]). $\square$

Since the coordinates of $\psi = (\psi_1, \psi_2)$ belong to $\mathcal{V}_{(\beta,0)}$ and according to Lemma 10, we have:

Lemma A.2. *The map $t \mapsto \hat{P}_t$ is in $C^\infty(\mathbb{R}^2, \mathcal{L}_{\mathcal{V}_{(\beta,\varepsilon)}})$. Moreover, for all $t \in \mathbb{R}^2$, for all integers $m_1 \geq 0$ and $m_2 \geq 0$ with $m_1 + m_2 \geq 1$, we have: $\frac{\partial^{m_1+m_2}}{\partial^{m_1}_{t_1} \partial^{m_2}_{t_2}} \hat{P}_t(\cdot) = \hat{P}_t(\mathrm{i}^{m_1+m_2} \psi_1^{m_1} \psi_2^{m_2} \cdot) = \hat{P}(\mathrm{i}^{m_1+m_2} \mathrm{e}^{\mathrm{i}\langle t, \psi\rangle} \psi_1^{m_1} \psi_2^{m_2} \cdot)$.*

We apply Theorem A.1 to $Q(t) = \hat{P}_t$ and $\mathcal{B} = \mathcal{V}_{(\beta,\varepsilon)}$. We have $\lambda(0) = 1$, $v(0) = \mathbf{1}$ and $\phi(0) = \hat{\nu}$. Moreover, since $m \geq 2$, according to Corollaries III-11 and III-12 of [13], we get: $\nabla\lambda(0) = 0$ and $\mathrm{Hess}\,\lambda(0) = -\Sigma^2$, where Σ^2 is the limit of the covariance matrices sequence $(\mathrm{Cov}_{\bar{\nu}}(\frac{S_n}{\sqrt{n}}))_{n \geq 1}$. Let $b > 0$ be such that $[-b; b]^2$ is contained in the set U_1 given by the Theorem A.1 for $Q = \hat{P}$. We also suppose that there exists some constant $a > 0$ such that, for all $u \in [-b; b]^2$, $|\lambda_u| \leq \exp(-a\langle u, u\rangle)$ and $\frac{1}{2}\langle \Sigma^2 u, u\rangle > a\langle u, u\rangle$. Moreover, let us prove the following:

Lemma 11. *The spectral radia of $(\hat{P}_t)_{t \in [-\pi;\pi]^2 \setminus [-b;b]^2}$ are uniformly bounded by some constant strictly less than 1.*

Proof. To this purpose, as Szász and Varjú do in [24], we use Lemma 4.3 of Aaronson and Denker in [1]. According to this result, it is enough to prove that, for all $u \in [-\pi; \pi]^2 \setminus \{(0,0)\}$, $\hat{P}_u$ has no eigenvalue on the unit circle. Let $u = (u_1, u_2) \in [-\pi; \pi]^2$. Let us suppose that $\hat{P}_u$ has an eigenvalue of modulus 1. We will prove that $u = (0, 0)$. Let us suppose that there exists $f \in \mathcal{V}_{(\beta,\varepsilon)}$ non-identically equal to zero and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that: $\hat{P}_u(f) = \lambda f$ $\hat{\nu}$-a.s.:

- *The modulus of f is $\hat{\nu}$-a.s. equal to some constant* c_1. Indeed, $\hat{\nu}$-almost surely, for every $\ell \geq 0$, we have: $|f| = |(\hat{P}_u)^\ell(f)| \leq \hat{P}^\ell(|f|)$. Since $\lim_{N \to +\infty} \|\hat{P}^N(|f|) - \mathbb{E}_{\hat{\nu}}[|f|]\mathbf{1}\|_{\mathcal{V}_{(\beta,\varepsilon)}} = 0$, we conclude that, $\hat{\nu}$-almost surely, $|f| \leq \mathbb{E}_{\hat{\nu}}[|f|]$. Hence $g = \mathbb{E}_{\hat{\nu}}[|f|]\mathbf{1} - |f|$ is $\hat{\nu}$-a.s. non-negative with null expectation.
- *$\hat{\nu}$-a.s., $\forall n \geq 0$, $\mathrm{e}^{\mathrm{i}\langle u, \hat{S}_n\rangle} f = \lambda^n f \circ \hat{T}^n$.* Let E be the set of $y \in \hat{M}$ such that $|f(y)| = c_1$. Let $x \in \bigcap_{n \geq 0} \hat{T}^{-n}(E)$ be such that, for every integer $n \geq 0$, $(\hat{P}_u)^n(f)(\hat{T}^n(x)) = \lambda^n f(\hat{T}^n(x))$ and $(\hat{P}^n \mathbf{1}_E)(\hat{T}^n(x)) = 1$. For every integer $n \geq 1$, we have:

$$\lambda^n f\big(\hat{T}^n(x)\big) = \hat{P}^n\big(\mathrm{e}^{\mathrm{i}\langle u, \hat{S}_n\rangle} f\big)\big(\hat{T}^n(x)\big) = \sum_{y:\, \hat{T}^n(y) = \hat{T}^n(x)} \kappa_n(y) \mathrm{e}^{\mathrm{i}\langle u, \hat{S}_n(y)\rangle} f(y).$$

  Moreover, $\sum_{y:\, \hat{T}^n(y) = \hat{T}^n(x)} \kappa_n(y) = 1$, $|\lambda^n f(\hat{T}^n(x))| = c_1$ and, for all y such that $\hat{T}^n(y) = \hat{T}^n(x)$, we have: $\kappa_n(y) > 0$ and $|\mathrm{e}^{\mathrm{i}\langle u, \hat{S}_n(y)\rangle} f(y)| = c_1$. Hence, for all y such that $\hat{T}^n(y) = \hat{T}^n(x)$, we have: $\lambda^n f(\hat{T}^n(x)) = \mathrm{e}^{\mathrm{i}\langle u, \hat{S}_n(y)\rangle} f(y)$. In particular, we can take $y = x$.
- *f is $\hat{\nu}$-a.s. equal to some constant f_0 on $\hat{\Delta}_0$.* Let y and z be two density points of $\hat{\Delta}_0$ such that, for $\hat{\nu}$-almost every $w \in \hat{\Delta}_0$, we have: $|f(w) - f(y)| \leq \|f\|_{(\beta,\varepsilon,h)} \beta^{\hat{s}(w,y)}$ and $|f(w) - f(z)| \leq \|f\|_{(\beta,\varepsilon,h)} \beta^{\hat{s}(w,z)}$. For any $N \geq 1$, we consider two points y_N and z_N such that:
  - y_N and z_N belong to $\bigcap_{j=0}^N \hat{T}^{-jr_0}(\hat{\pi}(\Lambda_0 \times \{0\}))$ and hence $\hat{s}(y_N, z_N) \geq N r_0$;

– $|f(y_N) - f(z_N)| \le \|f\|_{(\beta,\varepsilon,h)}\beta^{\hat{s}(y_N,z_N)}$ and $f(y_N)\mathrm{e}^{\mathrm{i}\langle u,\hat{S}_{Nr_0}(y_N)\rangle} = \lambda^{Nr_0} f\circ\hat{T}^{Nr_0}(y_N)$ and $f(z_N)\mathrm{e}^{\mathrm{i}\langle u,\hat{S}_{Nr_0}(z_N)\rangle} = \lambda^{Nr_0} f\circ\hat{T}^{Nr_0}(z_N)$;
– $|f(\hat{T}^{Nr_0}(y_N)) - f(y)| \le 1/N$, $|f(\hat{T}^{Nr_0}(z_N)) - f(z)| \le 1/N$.
Since $\hat{S}_{Nr_0}(y_N) = \hat{S}_{Nr_0}(z_N)$, we have: $|f(y) - f(z)| \le \frac{2}{N} + |f(y_N) - f(z_N)|$ and therefore $f(y) = f(z)$.

- *Conclusion*. Let x in $\hat{\pi}(\Lambda_0 \times \{0\})$ be such that $f(x)\mathrm{e}^{\mathrm{i}\langle u,\hat{S}_{r_0}(x)\rangle} = \lambda^{r_0} f(\hat{T}^{r_0}(x))$ and $f(x) = f(\hat{T}^{r_0}(x)) = f_0$. Since $\hat{S}_{r_0} = L_0 = (0,0)$ on $\hat{\pi}(\Lambda_0 \times \{0\})$, we have: $f_0 = f_0\mathrm{e}^{\mathrm{i}\langle u,L_0\rangle} = \lambda^{r_0} f_0$ and so $\lambda^{r_0} = 1$.

  Let y be in $\hat{\pi}(\Lambda_1 \times \{0\})$ such that $f(y)\mathrm{e}^{\mathrm{i}\langle u,\hat{S}_{r_1}(y)\rangle} = \lambda^{r_1} f(\hat{T}^{r_1}(y))$ and $f(y) = f(\hat{T}^{r_1}(y)) = f_0$. We have: $f_0\mathrm{e}^{\mathrm{i}u_1} = f_0\mathrm{e}^{\mathrm{i}\langle u,L_1\rangle} = \lambda^{r_1} f_0$. Since r_1 is a multiple of r_0, we get: $\exp(\mathrm{i}u_1) = 1$ and so $u_1 = 0$ (u_1 is the first coordinate of u).

  Analogously, by taking Λ_2 instead of Λ_1, we conclude that $u_2 = 0$. Hence $u = (0,0)$. □

A.3. *Proof of Proposition 3*

Let us write A_u for $\{\mathcal{I}_0 = u\}$. For all integers $n \ge 1$, we have to estimate:

$$\operatorname{Cov}\big(\mathbf{1}_{A_i}\mathbf{1}_{\{S_n=N_1\}}\mathbf{1}_{A_{i'}}\circ\bar{T}^n, \big\{\mathbf{1}_{A_j}\mathbf{1}_{\{S_k=N_2\}}\mathbf{1}_{A_{j'}}\circ\bar{T}^k\big\}\circ\bar{T}^{n+m}\big).$$

Let us recall that we have $\Phi\circ\pi = \psi\circ\hat{\pi}$. Because of the construction of the separation time $\hat{s}$, if $\hat{s}(\hat{x},\hat{y}) \ge 1$, then $\psi(\hat{x}) = \psi(\hat{y})$. Hence, ψ is in the functional space $\mathcal{V}_{(\beta,\varepsilon)}$ and its norm is less than $3\|\Phi\|_\infty$. Moreover, there exist four measureable subsets $\hat{A}_i$, $\hat{A}_{i'}$, $\hat{A}_j$, $\hat{A}_{j'}$ of $\hat{M}$ such that, for all $u = i, i', j, j'$, we have: $\mathbf{1}_{\hat{A}_u}\circ\hat{\pi} = \mathbf{1}_{A_u}\circ\pi$ and $\mathbf{1}_{\hat{A}_u}$ belongs to $\mathcal{V}_{(\beta,0)}$ (indeed, $\mathbf{1}_{A_u}$ is constant on the stable curves and moreover if $\hat{s}(\hat{x},\hat{y}) \ge 1$ then $\mathbf{1}_{\hat{A}_u}(\hat{x}) = \mathbf{1}_{\hat{A}_u}(\hat{y})$). We use the operators $\hat{P}$ and $\hat{P}_u$ defined previously and the fact that:

$$\hat{P}_u^m\big(g\times h\circ\hat{T}^m\big) = h\times\hat{P}_u^m(g) \quad\text{and}\quad \mathbb{E}_{\hat{\nu}}\big[(\hat{P}_u)^m(g)\big] = \mathbb{E}_{\hat{\nu}}\big[\mathrm{e}^{\mathrm{i}\langle u,\hat{S}_m\rangle}g\big].$$

We have:

$$\begin{aligned}
&\mathbb{E}_{\bar{\nu}}\big[\mathbf{1}_{A_i}\mathbf{1}_{\{S_n=N_1\}}\mathbf{1}_{A_{i'}}\circ\bar{T}^n\big\{\mathbf{1}_{A_j}\mathbf{1}_{\{S_k=N_2\}}\mathbf{1}_{A_{j'}}\circ\bar{T}^k\big\}\circ\bar{T}^{n+m}\big]\\
&\quad= \frac{1}{(2\pi)^4}\int_{[-\pi;\pi]^2}\int_{[-\pi;\pi]^2}\mathrm{e}^{-\mathrm{i}\langle u,N_1\rangle}\mathrm{e}^{-\mathrm{i}\langle t,N_2\rangle}\\
&\qquad\times\mathbb{E}_{\hat{\nu}}\big[\mathbf{1}_{\hat{A}_i}\mathrm{e}^{\mathrm{i}\langle u,\hat{S}_n\rangle}\mathbf{1}_{\hat{A}_{i'}}\circ\hat{T}^n\mathbf{1}_{\hat{A}_j}\circ\hat{T}^{n+m}\mathrm{e}^{\mathrm{i}\langle t,\hat{S}_k\rangle}\circ\hat{T}^{n+m}\mathbf{1}_{\hat{A}_{j'}}\circ\hat{T}^{n+m+k}\big]\,\mathrm{d}u\,\mathrm{d}t\\
&\quad= \frac{1}{(2\pi)^4}\int_{[-\pi;\pi]^2}\int_{[-\pi;\pi]^2}\mathrm{e}^{-\mathrm{i}\langle u,N_1\rangle}\mathrm{e}^{-\mathrm{i}\langle t,N_2\rangle}\mathbb{E}_{\hat{\nu}}\big[(\mathbf{1}_{\hat{A}_{j'}})\hat{P}_t^k\big\{\mathbf{1}_{\hat{A}_j}\hat{P}^m\{\mathbf{1}_{\hat{A}_{i'}}\hat{P}_u^n(\mathbf{1}_{\hat{A}_i})\}\big\}\big]\,\mathrm{d}u\,\mathrm{d}t.
\end{aligned}$$

Lemma 12. *There exists $K > 0$ such that, for any function $h \in \mathcal{V}_{(\beta,\varepsilon)}$ and any non-negative integer k, we have*:

$$\int_{[-\pi;\pi]^2}\big\|\hat{P}_u^k(h)\big\|_{\mathcal{V}_{(\beta,\varepsilon)}}\,\mathrm{d}u \le K\frac{\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}}{k+1}.$$

Proof. According to Theorem A.1 applied to $Q = \hat{P}$ and to Lemma 11, we have:

$$\begin{aligned}
\int_{[-\pi;\pi]^2}\big\|\hat{P}_u^k(h)\big\|_{\mathcal{V}_{(\beta,\varepsilon)}}\,\mathrm{d}u &= \int_{[-b;b]^2}\big\|\lambda_u^k\phi_u(h)v_u\big\|_{\mathcal{V}_{(\beta,\varepsilon)}}\,\mathrm{d}u + \mathrm{O}\big(\delta^k\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}\big)\\
&= \mathrm{O}(1)\int_{[-b;b]^2}\mathrm{e}^{-ak\langle u,u\rangle}\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}\,\mathrm{d}u + \mathrm{O}\big(\delta^k\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}\big)\\
&= \mathrm{O}(1)\frac{1}{k+1}\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}\int_{[-b\sqrt{k};b\sqrt{k}]^2}\mathrm{e}^{-a\langle v,v\rangle}\,\mathrm{d}v + \mathrm{O}\big(\delta^k\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}\big).
\end{aligned}$$

□

We come back to the proof of Proposition 3. Let us write: $g_u = \mathbf{1}_{\hat{A}_{i'}} \hat{P}_u^n(\mathbf{1}_{\hat{A}_i})$. According to Lemma 12, to Lemma 10 and to the properties of $\hat{P}$, we have:

$$\begin{aligned}
&\left|\mathrm{Cov}\left(\mathbf{1}_{A_i}\mathbf{1}_{\{S_n=N_1\}}\mathbf{1}_{A_{i'}} \circ \bar{T}^n, \left\{\mathbf{1}_{A_j}\mathbf{1}_{\{S_k=N_2\}}\mathbf{1}_{A_{j'}} \circ \bar{T}^k\right\} \circ \bar{T}^{n+m}\right)\right| \\
&\quad \le \frac{1}{(2\pi)^4}\int_{[-\pi;\pi]^2}\int_{[-\pi;\pi]^2}\left|\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{A}_{j'}}\hat{P}_t^k\left\{\mathbf{1}_{\hat{A}_j}\left(\hat{P}^m(g_u) - \mathbb{E}_{\hat{\nu}}[g_u]\right)\right\}\right]\right| \mathrm{d}u\,\mathrm{d}t \\
&\quad \le \tilde{K}\frac{1}{k+1}\int_{[-\pi;\pi]^2}\left\|\hat{P}^m\left(g_u - \mathbb{E}_{\hat{\nu}}[g_u]\right)\right\|_{\mathcal{V}_{(\beta,\varepsilon)}}\mathrm{d}u \\
&\quad \le \hat{K}\frac{\tau_1^m}{k+1}\int_{[-\pi;\pi]^2}\left\|\mathbf{1}_{\hat{A}_{i'}}\hat{P}_u^n(\mathbf{1}_{\hat{A}_i})\right\|_{\mathcal{V}_{(\beta,\varepsilon)}}\mathrm{d}u \le \check{K}\frac{{\tau_1}^m}{(k+1)(n+1)}.
\end{aligned}$$

This ends the proof of Proposition 3.

A.4. *Consequences*

Lemma 13. *For any $m \ge 2$, there exists $C_m > 1$ such that, for every $0 \le k_1 \le \cdots \le k_m$ and for any $\alpha \in \mathbb{Z}^2$, we have*:

$$\bar{\nu}(S_{k_1} = \alpha, S_{k_2} = S_{k_1}, \ldots, S_{k_m} = S_{k_{m-1}}) \le \frac{C_m}{(k_1+1)(k_2-k_1+1)\cdots(k_m-k_{m-1}+1)}.$$

Proof. We have:

$$\begin{aligned}
&\bar{\nu}(S_{k_1} = \alpha, S_{k_2} = S_{k_1}, \ldots, S_{k_m} = S_{k_{m-1}}) \\
&\quad = \frac{1}{(2\pi)^{2m}}\int_{([-\pi;\pi]^2)^m} \mathrm{e}^{-\mathrm{i}\langle u_1,\alpha\rangle}\mathbb{E}_{\hat{\nu}}\left[\mathrm{e}^{\mathrm{i}\langle u_1,\hat{S}_{k_1}\rangle}\prod_{j=2}^m \mathrm{e}^{\mathrm{i}\langle u_j,\hat{S}_{(k_j-k_{j-1})}\rangle}\circ\hat{T}^{k_{j-1}}\right]\mathrm{d}u_1\cdots\mathrm{d}u_m \\
&\quad \le \frac{C_0}{(2\pi)^{2m}}\int_{([-\pi;\pi]^2)^m}\left\|\hat{P}_{u_m}^{k_m-k_{m-1}}\left(\hat{P}_{u_{m-1}}^{k_{m-1}-k_{m-2}}\left(\cdots\left(\hat{P}_{u_2}^{k_2-k_1}\left(\hat{P}_{u_1}^{k_1}(\mathbf{1})\right)\right)\right)\right)\right\|_{\mathcal{V}_{(\beta,\varepsilon)}}\mathrm{d}u_1\cdots\mathrm{d}u_m.
\end{aligned}$$

We conclude with the use of Lemma 12 and with an easy induction. □

Let us notice that, since $u \mapsto v_u$ and $u \mapsto \phi_u$ are C^1 and since $u \mapsto \lambda_u$ is C^3 with $\nabla\lambda(0) = 0$ and $\mathrm{Hess}\,\lambda(0) = -\Sigma^2$, we have:

$$\left\|{\lambda_u}^k\mathbb{E}_{\hat{\nu}}[v_u]\phi_u(h) - \mathrm{e}^{-k\langle\Sigma^2u,u\rangle/2}\mathbb{E}_{\hat{\nu}}[h]\right\|_{\mathcal{V}_{(\beta,\varepsilon)}} = \mathrm{O}\left(\left(|u| + |u|^3k\right)\mathrm{e}^{-a(k-1)\langle u,u\rangle}\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}\right)$$

for every $u \in [-b;b]^2$ and every integer $\kappa \ge 1$. Hence, modifying slightly the proof of Lemma 12, we can get:

$$\left\|\frac{1}{(2\pi)^2}\int_{[-\pi;\pi]^2}\hat{P}_u^k(h)\,\mathrm{d}u - \frac{\mathbb{E}_{\hat{\nu}}[h]}{k2\pi\sqrt{\det(\Sigma^2)}}\right\|_{\mathcal{V}_{(\beta,\varepsilon)}} \le K\frac{\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}}{k\sqrt{k}}.$$

Therefore, according to the proof of Proposition 3 (with $m = 0$), this gives: $\bar{\nu}(S_n = 0, S_{n+k} - S_n = 0) \sim_{n,k\to+\infty} \frac{1}{2\pi\det(\Sigma^2)nk}$ used by Dolgopyat, Szász and Varjú in [9]. But, for general sets A and B as in the hypotheses of Proposition 4, with this estimation, we only get:

$$\left|\bar{\nu}\left(A \cap \bar{T}^{-(k+r)}(B) \cap \{S_{k+r} - S_r = N\}\right) - \frac{\bar{\nu}(A)\bar{\nu}(B)}{\sqrt{\det(\Sigma^2)}2\pi k}\mathrm{e}^{-\langle(\Sigma^2)^{-1}N,N\rangle/(2k)}\right| = \mathrm{O}\left(\left(\bar{\nu}(B)\right)^{1/p}k^{-3/2}\right).$$

This estimation is not sufficient for our purpose.

A.5. *Proof of Proposition 4*

Let A be a subset of $\bar{M}$ union of connected components of $\bar{M} \setminus \bigcup_{i=0}^{r} \bar{T}^{-i}(R_0)$. Let k be an integer larger than 3. Let B be a measurable set such that, if $x \in B$ then $\gamma^s(x) \subseteq B$. There exists a measurable subset $\hat{A}$ of $\hat{M}$ such that $\mathbf{1}_A \circ \pi = \mathbf{1}_{\hat{A}} \circ \hat{\pi}$. Let us prove that $\|\hat{P}^r(\mathbf{1}_{\hat{A}})\|_{\mathcal{V}_{(\beta,\varepsilon)}}$ is bounded (uniformly in r and in A). For any $\hat{x} \in \hat{M}$, we have: $\hat{P}^r(f)(\hat{x}) = \sum_{\chi_r} \{\prod_{k=0}^{r-1} \kappa(\hat{T}^k(\chi_r(\hat{x})))\} f(\chi_r(\hat{x}))$, where the sum is taken over the inverse branches χ_r of $\hat{T}^r$. Let $\hat{x}$ and $\hat{y}$ in $\hat{M}$ be such that: $\hat{s}(\hat{x},\hat{y}) \ge 0$. We have: $\hat{s}(\chi_r(\hat{x}),\chi_r(\hat{y})) = \hat{s}(\hat{x},\hat{y}) + r \ge r$ and $\chi_r(\hat{x})$ belongs to $\hat{A}$ if and only if $\chi_r(\hat{y})$ belongs to $\hat{A}$. Hence, according to (A.2) of Section A.2, we have: $|\log(\frac{\prod_{k=0}^{r-1}\kappa(\hat{T}^k(\chi_r(\hat{x})))}{\prod_{k=0}^{r-1}\kappa(\hat{T}^k(\chi_r(\hat{y})))})| \le c_\kappa \frac{\beta^{\hat{s}(\hat{x},\hat{y})}}{1-\beta}$ and:

$$\begin{aligned}\left|\hat{P}^r(\mathbf{1}_{\hat{A}})(\hat{x}) - \hat{P}^r(\mathbf{1}_{\hat{A}})(\hat{y})\right| &\le \sum_{\chi_r}\left(\prod_{k=0}^{r-1}\kappa\big(\hat{T}^k\big(\chi_r(\hat{x})\big)\big) + \prod_{k=0}^{r-1}\kappa\big(\hat{T}^k\big(\chi_r(\hat{y})\big)\big)\right) c_\kappa \frac{\beta^{\hat{s}(\hat{x},\hat{y})}}{1-\beta} \\ &\le 2c_\kappa \frac{\beta^{\hat{s}(\hat{x},\hat{y})}}{1-\beta}.\end{aligned}$$

Hence, we have: $\|\hat{P}^r(\mathbf{1}_{\hat{A}})\|_{\mathcal{V}_{(\beta,\varepsilon)}} \le 1 + 2\frac{c_\kappa}{1-\beta}$. There exists $\hat{B} \subseteq \hat{M}$ such that we have $\mathbf{1}_B \circ \pi = \mathbf{1}_{\hat{B}} \circ \hat{\pi}$. Let us recall that we have $\Phi \circ \pi = \psi \circ \hat{\pi}$. Because of the construction of the separation time $\hat{s}$, if $\hat{s}(\hat{x},\hat{y}) \ge 1$, then $\psi(\hat{x}) = \psi(\hat{y})$. Hence, the coordinates of ψ are in the functional space $\mathcal{V}_{(\beta,0)}$ and so in $\mathcal{V}_{(\beta,\varepsilon)}$ and its norm is less than $3\|\Phi\|_\infty$. For any integer $k \ge 1$ we have:

$$\begin{aligned}&\bar{\nu}\big(A \cap \{S_{k+r} - S_r = N\} \cap \bar{T}^{-(k+r)}(B)\big) \\ &\quad = \frac{1}{(2\pi)^2}\int_{[-\pi;\pi]^2} \mathrm{e}^{-\mathrm{i}\langle u,N\rangle}\mathbb{E}_{\hat{\nu}}\big[\mathbf{1}_{\hat{A}}\mathbf{1}_{\hat{B}} \circ \hat{T}^{k+r}\mathrm{e}^{\mathrm{i}\langle u,\hat{S}_k\rangle} \circ \hat{T}^r\big]\,\mathrm{d}u \\ &\quad = \frac{1}{(2\pi)^2}\int_{[-\pi;\pi]^2} \mathrm{e}^{-\mathrm{i}\langle u,N\rangle}\mathbb{E}_{\hat{\nu}}\big[\mathbf{1}_{\hat{B}}\hat{P}_u^k\big(\hat{P}^r(\mathbf{1}_{\hat{A}})\big)\big]\,\mathrm{d}u,\end{aligned}$$

with $\hat{P}_u(h) := \hat{P}(\exp(\mathrm{i}\langle u,\psi\rangle)h)$ and since $\hat{P}_u^k(f \circ \hat{T}^k \times g) = f \times \hat{P}_u^k(g)$. We will use the fact that: $|\mathbb{E}_{\hat{\nu}}[gh]| \le \|g\|_{L^p(\hat{\nu})}\|h\|_{L^{p/(p-1)}(\hat{\nu})} \le C_0\|g\|_{L^p(\hat{\nu})}\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}$ and that: $\|\mathbf{1}_{\hat{B}}\|_{L^p(\hat{\nu})} = \bar{\nu}(B)^{1/p}$. According to Theorem A.1 and Lemma 11, we have:

$$\begin{aligned}&\bar{\nu}\big(A \cap \{S_{(k+r)} - S_r = N\} \cap \bar{T}^{-(k+r)}(B)\big) \\ &\quad = \frac{1}{(2\pi)^2}\int_{[-b;b]^2} \mathrm{e}^{-\mathrm{i}\langle u,N\rangle}\lambda_u^k\phi_u\big(\hat{P}^r(\mathbf{1}_{\hat{A}})\big)\mathbb{E}_{\hat{\nu}}[\mathbf{1}_{\hat{B}}v_u]\,\mathrm{d}u + \mathrm{O}\big(\delta^k\big)\bar{\nu}(B)^{1/p} \\ &\quad = \frac{1}{k}\frac{1}{(2\pi)^2}\int_{[-b\sqrt{k};b\sqrt{k}]^2} \mathrm{e}^{-\mathrm{i}\langle u/\sqrt{k},N\rangle}\lambda_{u/\sqrt{k}}^k\phi_{u/\sqrt{k}}\big(\hat{P}^r(\mathbf{1}_{\hat{A}})\big)\mathbb{E}_{\hat{\nu}}[\mathbf{1}_{\hat{B}}v_{u/\sqrt{k}}]\,\mathrm{d}u + \mathrm{O}\big(\delta^k\big)\bar{\nu}(B)^{1/p}.\end{aligned}$$

- Let us estimate the error done when we replace $\lambda_{u/\sqrt{k}}^k$ by $\mathrm{e}^{-\langle\Sigma^2u,u\rangle/2}$ in this formula. Let us notice that we have:

$$\lambda_{u/\sqrt{k}}^k - \big(\mathrm{e}^{-\langle\Sigma^2u/\sqrt{k},u/\sqrt{k}\rangle/2}\big)^k = k\mathrm{e}^{-(1/2)\langle\Sigma^2u,u\rangle(k-1)/k}A_k(u) + B_k(u),$$

with $A_k(u) = \lambda_{u/\sqrt{k}} - \mathrm{e}^{-\langle\Sigma^2u/\sqrt{k},u/\sqrt{k}\rangle/2}$ and $|B_k(u)| \le \frac{k(k-1)}{2}\mathrm{e}^{-a\langle u,u\rangle(k-2)/k}(\lambda_{u/\sqrt{k}} - \mathrm{e}^{-\langle\Sigma^2u/\sqrt{k},u/\sqrt{k}\rangle/2})^2$. Since $\lambda_w - \mathrm{e}^{-\langle\Sigma^2w,w\rangle/2} = \mathrm{O}(|w|^3)$, we have:

$$\frac{1}{k}\frac{1}{(2\pi)^2}\int_{[-b\sqrt{k};b\sqrt{k}]^2}\big|B_k(u)\phi_{u/\sqrt{k}}\big(\hat{P}^r(\mathbf{1}_{\hat{A}})\big)\mathbb{E}_{\hat{\nu}}[\mathbf{1}_{\hat{B}}v_{u/\sqrt{k}}]\big|\,\mathrm{d}u = \mathrm{O}\bigg(\frac{1}{k^2}\bar{\nu}(B)^{1/p}\bigg).$$

We will approximate $A_k(u)$ by $A'_k(u) := \frac{1}{6}\sum_{i',j,j'} \frac{\partial^3\lambda}{\partial u_{i'}\partial u_j \partial u_{j'}}(0)\frac{u_{i'}u_j u_{j'}}{k^{3/2}}$. We have: $|A_k(u) - A'_k(u)| \le C\frac{|u|^4}{k^2}$. Hence, we have:

$$\frac{1}{(2\pi)^2}\int_{[-b\sqrt{k};b\sqrt{k}]^2}\left|\mathrm{e}^{-(1/2)\langle\Sigma^2 u,u\rangle(k-1)/k}\left(A_k(u)-A'_k(u)\right)\phi_{u/\sqrt{k}}\left(\hat{P}^r(\mathbf{1}_{\hat{A}})\right)\mathbb{E}_{\hat{\nu}}[\mathbf{1}_{\hat{B}}v_{u/\sqrt{k}}]\right|\mathrm{d}u$$
$$\le \mathrm{O}\left(k^{-2}\right)\bar{\nu}(B)^{1/p}.$$

Now, we notice that we have:

$$\frac{1}{(2\pi)^2}\int_{[-b\sqrt{k};b\sqrt{k}]^2}\mathrm{e}^{-(1/2)\langle\Sigma^2 u,u\rangle(k-1)/k}\left|A'_k(u)\right|\times\left|\phi_{u/\sqrt{k}}\left(\hat{P}^r(\mathbf{1}_{\hat{A}})\right)\mathbb{E}_{\hat{\nu}}[\mathbf{1}_{\hat{B}}v_{u/\sqrt{k}}]-\bar{\nu}(A)\bar{\nu}(B)\right|\mathrm{d}u$$
$$=\mathrm{O}\left(k^{-2}\right)\bar{\nu}(B)^{1/p}$$

and:

$$\frac{1}{(2\pi)^2 k^{3/2}}\int_{[-b\sqrt{k};b\sqrt{k}]^2}\mathrm{e}^{-\mathrm{i}\langle u/\sqrt{k},N\rangle}\mathrm{e}^{-(1/2)\langle\Sigma^2 u,u\rangle(k-1)/k}\bar{\nu}(A)\bar{\nu}(B)u_{i'}u_j u_{j'}\,\mathrm{d}u$$
$$=\frac{\bar{\nu}(A)\bar{\nu}(B)}{(2\pi)^2 k^{3/2}}\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}\langle u/\sqrt{k},N\rangle}\mathrm{e}^{-(1/2)\langle\Sigma^2 u,u\rangle(k-1)/k}u_{i'}u_j u_{j'}\,\mathrm{d}u+\mathrm{O}\left(\frac{\bar{\nu}(A)\bar{\nu}(B)}{k^2}\right)$$
$$=\frac{\bar{\nu}(A)\bar{\nu}(B)}{(2\pi)^2 k^{3/2}}\frac{1}{\mathrm{i}}\frac{\partial^3\Psi_k}{\partial X_{i'}\partial X_j\partial X_{j'}}\left(\frac{N}{\sqrt{k}}\right)+\mathrm{O}\left(\frac{\bar{\nu}(A)\bar{\nu}(B)}{k^2}\right),$$

with $\Psi_k(X) := \int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}\langle u,X\rangle}\mathrm{e}^{-(1/2)\langle\Sigma^2 u,u\rangle(k-1)/k}\,\mathrm{d}u$. We have:

$$\Psi_k(X)=\frac{2\pi}{\sqrt{\det(\Sigma^2)}}\left(\frac{k}{k-1}\right)\mathrm{e}^{-\langle k/(k-1)(\Sigma^2)^{-1}X,X\rangle/2}.$$

Since we have $|\frac{\partial^3\Psi_k}{\partial X_i\partial X_j\partial X_{j'}}(X)| \le C(|X|_2+|X|_2^3)\mathrm{e}^{-\langle(\Sigma^2)^{-1}X,X\rangle/2}$, we get:

$$\left|\frac{\partial^3\Psi_k}{\partial X_i\partial X_j\partial X_{j'}}\left(\frac{N}{\sqrt{k}}\right)\right|\le C\left(\frac{|N|_2}{\sqrt{k}}+\frac{|N|_2^3}{k^{3/2}}\right)\mathrm{e}^{-\langle(\Sigma^2)^{-1}N,N\rangle/(2k)}.$$

- Hence it remains to estimate:

$$\frac{1}{k}\frac{1}{(2\pi)^2}\int_{[-b\sqrt{k};b\sqrt{k}]^2}\mathrm{e}^{-\mathrm{i}\langle u/\sqrt{k},N\rangle}\mathrm{e}^{-\langle\Sigma^2 u,u\rangle/2}\phi_{u/\sqrt{k}}\left(\hat{P}^r(\mathbf{1}_{\hat{A}})\right)\mathbb{E}_{\hat{\nu}}[\mathbf{1}_{\hat{B}}v_{u/\sqrt{k}}]\,\mathrm{d}u.$$

- Using a Taylor expansion, we observe that if we replace $\phi_{u/\sqrt{k}}(\hat{P}^r(\mathbf{1}_{\hat{A}}))\mathbb{E}_{\hat{\nu}}[\mathbf{1}_{\hat{B}}v_{u/\sqrt{k}}]$ by

$$C_k(u)=\bar{\nu}(A)\bar{\nu}(B)+\left\langle\frac{u}{\sqrt{k}},\nabla\phi(0)\left(\hat{P}^r(\mathbf{1}_{\hat{A}})\right)\bar{\nu}(B)+\bar{\nu}(A)\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{B}}\nabla v(0)\right]\right\rangle,$$

we make an error in $\mathrm{O}(\frac{1}{k^2})\bar{\nu}(B)^{1/p}$.
- We have:

$$\frac{1}{k}\frac{1}{(2\pi)^2}\int_{\{|u|_\infty\ge b\sqrt{k}\}}|\mathrm{e}^{-\mathrm{i}\langle u/\sqrt{k},N\rangle}\mathrm{e}^{-\langle\Sigma^2 u,u\rangle/2}C_k(u)|\,\mathrm{d}u\le\mathrm{O}\left(k^{-2}\right)\bar{\nu}(B)^{1/p}.$$

- Hence we have to estimate:

$$\frac{1}{k}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}\langle u/\sqrt{k},N\rangle}\mathrm{e}^{-\langle\Sigma^2 u,u\rangle/2}C_k(u)\,\mathrm{d}u.$$

This quantity can be rewritten: $G+H$ with

$$G := \frac{\bar{\nu}(A)\bar{\nu}(B)}{(2\pi)^2 k}\Psi\left(\frac{N}{\sqrt{k}}\right)$$

and

$$H := \frac{\mathrm{i}}{(2\pi)^2 k^{3/2}}\left\langle \nabla\Psi\left(\frac{N}{\sqrt{k}}\right), \nabla\phi(0)\big(\hat{P}^r(\mathbf{1}_{\hat{A}})\big)\hat{\nu}(\hat{B}) + \bar{\nu}(A)\mathbb{E}_{\hat{\nu}}\big[\mathbf{1}_{\hat{B}}\nabla v(0)\big]\right\rangle$$

with

$$\Psi(X) := \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}\langle u,X\rangle}\mathrm{e}^{-\langle \Sigma^2 u,u\rangle/2}\,\mathrm{d}u = \frac{2\pi}{\sqrt{\det(\Sigma^2)}}\mathrm{e}^{-\langle(\Sigma^2)^{-1}X,X\rangle/2}.$$

But we have:

$$\nabla\Psi(X) = -\frac{2\pi}{\sqrt{\det(\Sigma^2)}}\mathrm{e}^{-\langle(\Sigma^2)^{-1}X,X\rangle/2}\big(\Sigma^2\big)^{-1}X.$$

Hence we have:

$$H = \frac{\mathrm{e}^{-\langle(\Sigma^2)^{-1}N,N\rangle/(2k)}}{\mathrm{i}2\pi k^{3/2}\sqrt{\det(\Sigma^2)}}\left\langle \frac{(\Sigma^2)^{-1}N}{\sqrt{k}}, \nabla\phi(0)\big(\hat{P}^r(\mathbf{1}_{\hat{A}})\big)\bar{\nu}(B) + \bar{\nu}(A)\mathbb{E}_{\hat{\nu}}\big[\mathbf{1}_{\hat{B}}\nabla v(0)\big]\right\rangle.$$

Therefore, there exists some constant $K_1>0$ such that:

$$|H| \le \frac{K_1}{k^{3/2}}\mathrm{e}^{-\langle(\Sigma^2)^{-1}N,N\rangle/(2k)}\frac{|N|_2}{\sqrt{k}}\big(\bar{\nu}(B) + \bar{\nu}(A)\bar{\nu}(B)^{1/p}\big).$$

This ends the proof of Proposition 4.

Appendix B. Adaptation of Young's construction

Young's construction for billiards uses the estimates of [7]. Because of the complexity of the construction, we only explain the needed changes without giving all the details.

(1) *Construction of Λ*. Let k_0 be some fixed integer large enough. Let us define: $\mathcal{K} = \{x\colon |\varphi(x)| \in \bigcup_{k\ge k_0}(\frac{\pi}{2}-\frac{1}{k^2})\}$. We take a well-chosen point $x^{(0)}$ and three well-chosen real numbers $\lambda_1>1$, $\delta>0$ and $\delta_1>0$. We consider the set Ω of points $z\in\gamma^u(x^{(0)})$ such that $\int_{\gamma^u(x^{(0)})_{|x^{(0)},z}}\cos(\varphi)\,\mathrm{d}r\le\delta$. For all $n\ge 1$, we define:

$$\Omega_n = \big\{y\in\Omega\colon \forall i=0,\dots,n;\ d\big(\bar{T}^i(y), R_0\cup\bar{T}^{-1}(R_0)\cup\mathcal{K}\big)\ge 2\delta_1{\lambda_1}^{-i}\big\}$$

and $\Omega_\infty = \bigcap_{n\ge1}\Omega_n$. We define $\Lambda = (\bigcup_{\gamma^s\in\Gamma^s_\Lambda}\gamma^s)\cap(\bigcup_{\gamma^u\in\Gamma^u_\Lambda}\gamma^u)$, where $\Gamma^s_\Lambda := \{\gamma^s_\delta(y),\ y\in\Omega_\infty\}$ and where $\gamma^s_\delta(y)$ is the set of points $z\in\gamma^s(y)$ such that $\int_{\gamma^s(y)_{|y,z}}\cos(\varphi)\,\mathrm{d}r\le\delta$ and where Γ^u_Λ is the set of unstable curves γ^u such that:

- γ^u intersects each $\gamma^s\in\Gamma^s_\Lambda$;
- for all $y\in\partial\gamma^u$ and for all $z\in\gamma^u\cap\bigcup_{\gamma^s\in\Gamma^s_\Lambda}\gamma^s$, we have: $\int_{(\gamma^u)_{|y,z}}\cos(\varphi)\,\mathrm{d}r\ge\delta$;
- for all $n\ge0$, $\bar{T}^{-n}(\gamma)$ is contained in at most three adjacent I_k's with, for all $k\ge k_0$,

$$I_k := \left\{(i,r,\varphi)\in\bar{M}\colon \frac{\pi}{2}-\frac{1}{k^2}<\varphi\le\frac{\pi}{2}-\frac{1}{(k+1)^2}\right\},$$

$$I_{-k} := \left\{(i,r,\varphi)\in\bar{M}\colon -\frac{\pi}{2}+\frac{1}{(k+1)^2}\le\varphi<-\frac{\pi}{2}+\frac{1}{k^2}\right\}$$

and $I_0 := \{(i, r, \varphi) \in \bar{M}\colon |\varphi| \le \frac{\pi}{2} - \frac{1}{(k_0)^2}\}$.

The set Λ is compact and has positive measure.

(2) There exists N_0 such that, for any $m \ge N_0$, we can follow Young's construction of a return time $R(\cdot)$ in Λ with values in $m\mathbb{Z}_+^*$. Let us notice that the construction is still true for a return time with values in $1 + \mathbb{Z}_+^* m$.

(3) Let a_0 be a positive integer such that $\Omega \setminus \bigcup_{k=0}^{a_0} \bar{T}^{-k}(R_0)$ is composed of at least five connected components with positive length. Let us write $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ three such components "away" from $\partial\Omega$.

(4) Let us give some explanations concerning our adaptation. Actually it consists essentially in an adaptation of the argument of Young in [25], p. 642:

- Modification of the result $(**)$ of [25], p. 642. For the proof of this result, Young refers to Theorem 3.13 in [7] (see also Lemma D.2.2.2 in [18]). Thanks to the total ergodicity of the billiard transformation in the plane [8,19,21,22], we can adapt the proof of this result to prove the following statement (in which we use some notations of [25]):

  $(***)$ Given $\varepsilon_0 \in (0, \delta)$ and $M \ge 1$, there exists $n_0 = n_0(\varepsilon_0, M) \ge 1$ such that, if γ is a homogeneous unstable curve with $p(\gamma) > \varepsilon_0$ and if ℓ belongs to $\{(0,0), (0,1), (1,0), (0,-1), (-1,0)\}$, then there exists $q \in \{1, \ldots, n_0\}$ such that $\bar{T}^{qM}(\gamma)$ contains a homogeneous segment γ' which u-crosses the middle half of Q with $> 2\delta$ sticking out from each side and such that $(S_{qM})_{|\bar{T}^{-qM}(\gamma')} = \ell$.
- The "technical nuisance" met by Young (see p. 642 of [25] and also Lemma D.2.2.5 in [18]) still exists and can be solved as in [25] by the use of:

  $(+++)$ Let $\varepsilon_0 \in (0, \delta)$ and M be some positive integer. There exists $R_1' = R_1'(\varepsilon_0, M) > 0$ such that, for every homogeneous curve γ contained in Ω_{nM} (with $n \ge R_1'$) such that $\bar{T}^{(n+q)M}\gamma$ is homogeneous (with $q \in \{1, \ldots, n_0(\varepsilon_0, M)\}$) and $p(\bar{T}^{nM}(\gamma)) > \varepsilon_0$, we have $p(\bar{T}^{(n+q)M}(\gamma) \setminus \bar{T}^{(n+q)M}(\gamma \cap \Omega_{(n+q)M})) \le \delta$.
- Let L be in $\mathbb{Z}^2$ and M be a positive integer. First let us get $\gamma \subseteq \Omega_{mM}$ such that $\bar{T}^{mM}(\gamma)$ is a homogeneous unstable curve (with $m \ge R_1'(M)$) such that $p(\bar{T}^{mM}(\gamma)) > \varepsilon_0$. We have $(S_{mM})_{|\gamma} = L + (a, b)$ for some $(a, b) \in \mathbb{Z}^2$. If $(a, b) = (0, 0)$, we apply $(***)$ with $\ell = (0, 0)$ and then $(+++)$. If $(a, b) \ne (0, 0)$, we apply "$(***)$ and $(+++)$" several times: $|a|$ times with $\ell = (-\mathrm{sign}(a), 0)$ and $|b|$ times with $\ell = (0, -\mathrm{sign}(b))$.

  This gives $\gamma' \subseteq \Omega_{m'M}$ such that $\bar{T}^{m'M}(\gamma')$ contains a homogeneous segment which u-crosses the middle half of Q with $> \delta$ sticking out from each side. This gives a part of Λ with a return time in Λ equal to $m'M$ and with $S_{m'M} = L$ on this part.

(5) Return time for the points in $B_0 := \bigcup_{\gamma^s \in \Gamma_\Lambda^s\colon \gamma^s \cap \mathcal{A}_0 \ne \varnothing} (\Lambda \cap \gamma^s)$. Using the preceding adaptation of [25], we construct $\Lambda_0 \subseteq B_0$ with positive measure and an integer $r_0 \ge \max(a_0, N_0)$ such that, on Λ_0, $R(\cdot) \equiv r_0$ and $S_{r_0}(\cdot) = (0, 0)$.

Following Young's construction, we construct a return time multiple of r_0 on $B_0 \setminus \Lambda_0$.

(6) Return time for the points in $B_1 := \bigcup_{\gamma^s \in \Gamma_\Lambda^s\colon \gamma^s \cap \mathcal{A}_1 \ne \varnothing} (\Lambda \cap \gamma^s)$. Analogously, we construct $\Lambda_1 \subseteq B_1$ with positive measure on which the return time is equal to an integer $r_1 \ge 1$ multiple of r_0 and on which we have: $S_{r_1}(\cdot) = (1, 0)$. We follow Young's construction with $m = r_1$ for the remaining part $B_1 \setminus \Lambda_1$.

(7) Let $B_2 := \bigcup_{\gamma^s \in \Gamma_\Lambda^s\colon \gamma^s \cap \mathcal{A}_2 \ne \varnothing} (\Lambda \cap \gamma^s)$. We construct $\Lambda_2 \subseteq B_2$ with positive measure and an integer $r_2 \ge 1$ multiple ot r_1 such that $R(\cdot) = r_2$ and $S_{r_2}(\cdot) = (0, 1)$ on Λ_2. We follow Young's construction with $m = r_2$ for the remaining part $B_2 \setminus \Lambda_2$.

(8) For the remaining part $\Lambda \setminus (B_0 \cup B_1 \cup B_2)$, we adapt Young's construction to get a return time with values in $1 + r_2\mathbb{Z}_+^*$.

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