Ann. Funct. Anal. 10 (2019), no. 1, 46–59
https://doi.org/10.1215/20088752-2018-0007
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

ON THE STRUCTURE OF THE DUAL UNIT BALL OF STRICT u-IDEALS

JULIA MARTSINKEVITŠ* and MÄRT PÕLDVERE

Communicated by P. N. Dowling

ABSTRACT. It is known that if a Banach space Y is a u-ideal in its bidual Y^{**} with respect to the canonical projection on the third dual Y^{***} , then Y^* contains "many" functionals admitting a unique norm-preserving extension to Y^{**}—the dual unit ball B_{Y^*} is the norm-closed convex hull of its weak* strongly exposed points by a result of Å. Lima from 1995. We show that if Y is a strict u-ideal in a Banach space X with respect to an ideal projection P on X^* , and X/Y is separable, then B_{Y^*} is the τ_P-closed convex hull of functionals admitting a unique norm-preserving extension to X, where τ_P is a certain weak topology on Y^* defined by the ideal projection P.

1. Introduction

Throughout this article, all Banach spaces will be over the scalar field $\mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For a Banach space X, its dual space, closed unit ball, and unit sphere will be denoted, respectively, by X^*, B_X, and S_X. For a subset A of X, we denote its convex hull by $\operatorname{co}(A)$ and its linear span by $\operatorname{span}(A)$. The symbol $\mathcal{L}(X)$ will stand for the space of continuous linear operators from X to X.

Let X be a Banach space, and let Y be a closed subspace of X. According to the terminology in [5], Y is said to be an *ideal* in X if there exists a continuous linear projection $P \in \mathcal{L}(X^*)$ with $\ker P = Y^\perp := \{x^* \in X^*\colon x^*|_Y = 0\}$ and

Received Oct. 27, 2017; Accepted Feb. 24, 2018.
First published online Sep. 28, 2018.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46B20; Secondary 46A55, 46B22.
Keywords. strict ideal, norm-preserving extension, dentability, denting point.

$\|P\| = 1$. If $\operatorname{ran} P$ is norming for X in the sense that

$$\|x\| = \sup_{x^* \in B_{\operatorname{ran} P}} \left|x^*(x)\right| \quad \text{for all } x \in X,$$

then Y is called a *strict ideal*, and the ideal projection P is said to be *strict.*

It is straightforward to verify that if P is an ideal projection for Y in X, then, for every $x^* \in X^*$, the functional $Px^* \in X^*$ is a norm-preserving extension of the restriction $x^*|_Y \in Y^*$. It follows that the mapping $J_P\colon Y^* \ni y^* \mapsto Px^* \in X^*$, where $x^* \in X^*$ is any extension of y^*, is a linear isometry. In particular, $\operatorname{ran} J_P = \operatorname{ran} P$, and $\operatorname{ran} P$ is isometrically isomorphic to Y^*.

Suppose that Y is an ideal in X with respect to an ideal projection P. Then each $x \in X$ induces a functional $x_P \in Y^{**}$ defined by $x_P(y^*) = (J_P y^*)(x)$, $y^* \in Y^*$. If P is strict, then the mapping $x \mapsto x_P$ is an isometry and one can identify X with the closed subspace $X_P = \{x_P \in Y^{**}\colon x \in X\}$ of Y^{**}. In such a case, we will denote the weak topology $\sigma(Y^*, X_P)$ on Y^* by τ_P.

If an ideal projection $P \in \mathcal{L}(X^*)$ for Y in X satisfies $\|I - 2P\| = 1$, then it is called a *u-ideal* projection, and Y is said to be a *u-ideal* in X with respect to P. Our starting point is the following result of Lima and Lima.

Proposition 1.1 (cf. [7, Proposition 2.2])**.** *Let Y be a strict u-ideal in a Banach space X. Then every ideal projection for Y in X is strict.*

In [7], Proposition 1.1 was obtained using knowledge about centers of symmetry (see, e.g., [1, Proposition 2.2]). Denoting, whenever E is a subspace of X, by $\mathcal{C}_E$ the set of functionals in S_{Y^*} having a unique norm-preserving extension to $\overline{\operatorname{span}}(Y \cup E)$, we observe that the conclusion of Proposition 1.1 easily obtains under the assumption that the set $\mathcal{C}_E$ is big enough for every 1-dimensional subspace E of X.

Proposition 1.2. *Let Y be a strict ideal in a Banach space X, and suppose that for every 1-dimensional subspace E of X there is a strict ideal projection $P_E \in \mathcal{L}(X^*)$ for Y in X such that $B_{Y^*} = \overline{\operatorname{co}}^{\tau_{P_E}}(\mathcal{C}_E)$. Then every ideal projection for Y in X is strict.*

Proof. Let P be any ideal projection for Y in X, and let $x \in X$ and $\varepsilon > 0$ be arbitrary. It suffices to find a $y^* \in B_{Y^*}$ such that

$$\left|x_P(y^*)\right| = \left|(J_P y^*)(x)\right| > \|x\| - \varepsilon. \tag{1.1}$$

Set $E := \operatorname{span}(\{x\})$. Since P_E is strict, there is $v^* \in B_{Y^*}$ such that $|x_{P_E}(v^*)| = |(J_{P_E} v^*)(x)| > \|x\| - \varepsilon$. Since $B_{Y^*} = \overline{\operatorname{co}}^{\tau_{P_E}}(\mathcal{C}_E)$, there is $y^* \in \operatorname{co}(\mathcal{C}_E)$ such that $|x_{P_E}(y^*)| > \|x\| - \varepsilon$. For every $u^* \in \mathcal{C}_E$, the functionals $J_P u^* \in X^*$ and $J_{P_E} u^* \in X^*$ are norm-preserving extensions of u^*, and thus their restrictions to $\overline{\operatorname{span}}(Y \cup E)$ are also norm-preserving extensions of u^*; hence $x_P(u^*) = (J_P u^*)(x) = (J_{P_E} u^*)(x) = x_{P_E}(u^*)$. Since $y^* \in \operatorname{co}(\mathcal{C}_E)$, one has $x_P(y^*) = x_{P_E}(y^*)$, and (1.1) follows. □

In the light of Propositions 1.1 and 1.2, it is natural to ask whether the assumption of Proposition 1.2 holds if Y is a strict u-ideal in X with respect to a projection $P \in \mathcal{L}(X^*)$, and, moreover, whether in that case the projection P itself always fits in the role of P_E. The objective of this article is to answer these questions in the affirmative.

Theorem 1.3. *Let Y be a strict u-ideal in a Banach space X with respect to an ideal projection $P \in \mathcal{L}(X^*)$. Then, for every separable subspace E of X,*

(a) $B_{Y^*} = \overline{\mathrm{co}}^{\tau_P}(\mathcal{C}_E)$;

(b) *if Y is separable, then $B_{Y^*} = \overline{\mathrm{co}}^{\tau_P}(\mathcal{C}_E \cap \operatorname{ext} B_{Y^*})$, where $\operatorname{ext} B_{Y^*}$ is the set of extreme points of B_{Y^*}.*

Remark 1.4. Every Banach space Y is a strict ideal in its bidual Y^{**} with respect to the canonical projection $\pi := j_{Y^*}(j_Y)^*$, where $j_Y \colon Y \to Y^{**}$ and $j_{Y^*} \colon Y^* \to Y^{***}$ are canonical embeddings. If π happens to be a u-ideal projection, then, by a result of Lima [6, Proposition 4.1], B_{Y^*} is the norm-closed convex hull of its weak* strongly exposed points. Since the τ_π-topology on B_{Y^*} is the weak topology, and every weak* strongly exposed point of B_{Y^*} has a unique norm-preserving extension to Y^{**}, it follows that if Y is a u-ideal in its bidual Y^{**} with respect to the canonical projection π, then the dual unit ball B_{Y^*} is the τ_π-closed convex hull of functionals admitting a unique norm-preserving extension to Y^{**}.

Let us recall the notion of a *slice*. Let C be a nonempty bounded subset of a Banach space Z. Given $z^* \in Z^* \setminus \{0\}$ and $\alpha > 0$, the set

$$S(z^*, \alpha, C) := \big\{ z \in C \colon \operatorname{Re} z^*(z) > \sup \operatorname{Re} z^*(C) - \alpha \big\}$$

is called a *slice* of C. If τ is a locally convex topology on Z weaker than the norm topology, then slices of C whose defining functional comes from the topological dual $(Z, \tau)'$ (i.e., the linear space of all τ-continuous linear functionals on Z) are called *τ-slices*. In particular, if Z happens to be a dual space, say, $Z = E^*$, then slices of C whose defining functional comes from (the canonical image of) the predual E of Z are called *weak*-slices*.

In Section 2, letting τ_1 and τ_2 be two comparable locally convex topologies in a Banach space X, we consider (τ_1, τ_2)-dentability and (τ_1, τ_2)-denting points of bounded subsets of X. These concepts are a natural generalization due to Fundo [3, p. 1118] of the "ordinary" dentability and denting points. We prove that if Y is a strict u-ideal in X with respect to an ideal projection $P \in \mathcal{L}(X^*)$, then every τ_P-slice of B_{Y^*} contains a *weak**-slice of B_{Y^*}, and every (τ_P, τ_P)-denting point of B_{Y^*} is a $(weak^*, \tau_P)$-denting point of B_{Y^*} (see Proposition 2.5).

From [4, Theorem 1, Proposition 1] it follows that, for a subspace E of a separable Banach space X, every nonempty bounded subset of the dual space X^* is $(\sigma(X^*, E), \sigma(X^*, E))$-dentable. In Section 3, we prove this result without the assumption on the separability of X (see Theorem 3.1). It follows that if Y is a strict ideal in a Banach space X with respect to an ideal projection $P \in \mathcal{L}(X^*)$, then every nonempty bounded subset of Y^* is $(\sigma(Y^*, E_P), \sigma(Y^*, E_P))$-dentable, where $E_P := \{x_P \in Y^{**} \colon x \in E\}$ (see Corollary 3.3). In particular, every nonempty bounded subset of Y^* is (τ_P, τ_P)-dentable.

In Section 4, we prove that if Y is a strict u-ideal in a Banach space X with respect to an ideal projection $P \in \mathcal{L}(X^*)$, and X/Y is separable, then B_{Y^*} is the τ_P-closed convex hull of its elements admitting a unique norm-preserving extension to X. Moreover, if X itself is separable, then B_{Y^*} is the τ_P-closed convex hull of its $(weak^*, \tau_P)$-denting points (see Proposition 4.1). Since, by [9, Proposition 2.2, (iii)⇒(i)], every $(weak^*, \tau_P)$-denting point of B_{Y^*} is an extreme point of B_{Y^*} admitting a unique norm-preserving extension to X, Theorem 1.3 follows (by courtesy of Proposition 2.5).

2. Dentability in spaces with two comparable locally convex topologies

Let Z be a Banach space, and let τ be a locally convex topology on Z. Given a $z \in Z$, a seminorm p on Z, and an $\varepsilon > 0$, we define

$$\mathcal{U}_p(z,\varepsilon) := \big\{w \in Z \colon p(w-z) < \varepsilon\big\}.$$

Suppose that τ_1 and τ_2 are locally convex topologies on Z such that τ_1 is weaker than τ_2 and τ_2 is weaker than the norm topology. In this case, $(Z,\tau_1)' \subset (Z,\tau_2)' \subset Z^*$.

Definition 2.1 (cf. [3, Definitions 1, 3]). Let C be a nonempty bounded subset of Z. We say that

- the set C is (τ_1,τ_2)-*dentable* if, whenever p is a τ_2-continuous seminorm on Z and $\varepsilon > 0$, there is $x \in C$ such that
$$x \notin \overline{\mathrm{co}}^{\tau_1}\big(C \setminus \mathcal{U}_p(x,\varepsilon)\big); \tag{2.1}$$
- a point $x \in C$ is a (τ_1,τ_2)-*denting point* of C if, whenever p is a τ_2-continuous seminorm on Z and $\varepsilon > 0$, one has (2.1).

Dentability in locally convex spaces with two comparable topologies has been studied in [3] and [4].

Remark 2.2. If both τ_1 and τ_2 are the norm topology, then (τ_1,τ_2)-dentability and (τ_1,τ_2)-denting points are, respectively, the *ordinary dentability* and *ordinary denting points*. If Z is a dual space, say, $Z = E^*$, τ_1 is the weak* topology on $Z = E^*$, and τ_2 is the norm topology, then (τ_1,τ_2)-dentability and (τ_1,τ_2)-denting points are, respectively, *weak*-dentability* and *weak*-denting points*.

Let us introduce some more notation. For a finite family $\mathcal{F}$ of seminorms on Z, we define a seminorm $p_{\mathcal{F}}$ on Z by

$$p_{\mathcal{F}}(z) := \max_{p\in\mathcal{F}} p(z), \quad z \in Z,$$

and we write $\mathcal{U}_{\mathcal{F}}(z,\varepsilon) := \mathcal{U}_{p_{\mathcal{F}}}(z,\varepsilon)$. For a seminorm p on Z and a subset C of Z, we define

$$\operatorname{diam}_p C := \sup_{z,w\in C} p(z-w).$$

Remark 2.3. In general, $\operatorname{diam}_p C$ need not be finite. However, if C is bounded and p is norm-continuous (which is the case, e.g., whenever p is τ-continuous, where τ is a locally convex topology on Z weaker than the norm topology), then $\operatorname{diam}_p C$ is finite.

If $\mathcal{F}$ is a finite family of seminorms on Z, then we write $\operatorname{diam}_{\mathcal{F}}$ instead of $\operatorname{diam}_{p_{\mathcal{F}}}$. Recall that a family $\mathcal{S}$ of seminorms on Z is said to *induce* the topology τ if, for every $z \in Z$, the family

$$\mathfrak{B}_{\mathcal{S}}(z) := \big\{\mathcal{U}_{\mathcal{F}}(z,\varepsilon)\colon \mathcal{F} \text{ is a finite subfamily of } \mathcal{S} \text{ and } \varepsilon > 0\big\}$$

is a basis of neighborhoods for z in τ, or, equivalently, the family $\mathfrak{B}_{\mathcal{S}}(0)$ is a basis of neighborhoods for 0 in τ.

The following proposition is an obvious generalization of a well-known characterization of *ordinary dentability* and *ordinary denting points*. We include its proof for the sake of completeness.

Proposition 2.4. *Let Z be a Banach space, let τ_1 and τ_2 be locally convex topologies on Z such that τ_1 is weaker than τ_2 and τ_2 is weaker than the norm topology, let $\mathcal{S}$ be a family of seminorms on Z inducing τ_2, and let C be a nonempty bounded subset of Z.*

(a) *The following assertions are equivalent:*
 (i) *C is (τ_1,τ_2)-dentable;*
 (i′) *whenever $\mathcal{F}$ is a finite subfamily of $\mathcal{S}$ and $\varepsilon > 0$, there is $x \in C$ such that*

$$x \notin \overline{\mathrm{co}}^{\tau_1}\big(C \setminus \mathcal{U}_{\mathcal{F}}(x,\varepsilon)\big); \tag{2.2}$$

 (ii) *whenever p is a τ_2-continuous seminorm on Z and $\varepsilon > 0$, there is a τ_1-slice $S(z^*,\alpha,C)$ such that*

$$\operatorname{diam}_p S(z^*,\alpha,C) < \varepsilon; \tag{2.3}$$

 (ii′) *whenever $\mathcal{F}$ is a finite subfamily of $\mathcal{S}$ and $\varepsilon > 0$, there is a τ_1-slice $S(z^*,\alpha,C)$ such that*

$$\operatorname{diam}_{\mathcal{F}} S(z^*,\alpha,C) < \varepsilon. \tag{2.4}$$

(b) *Let $x \in C$. The following assertions are equivalent:*
 (i) *x is a (τ_1,τ_2)-denting point of C;*
 (i′) *whenever $\mathcal{F}$ is a finite subfamily of $\mathcal{S}$ and $\varepsilon > 0$, one has (2.2);*
 (ii) *whenever p is a τ_2-continuous seminorm on Z and $\varepsilon > 0$, there is a τ_1-slice $S(z^*,\alpha,C)$ containing x and satisfying (2.3);*
 (ii′) *whenever $\mathcal{F}$ is a finite subfamily of $\mathcal{S}$ and $\varepsilon > 0$, there is a τ_1-slice $S(z^*,\alpha,C)$ containing x and satisfying (2.4).*

Proof. In both (a) and (b), both (i)⇒(i′) and (ii)⇒(ii′) follow from the fact that each $p \in \mathcal{S}$ is τ_2-continuous (and thus also $p_{\mathcal{F}}$ is τ_2-continuous). Also, in both (a) and (b), both (i′)⇒(i) and (ii′)⇒(ii) follow from the fact that, whenever p is a τ_2-continuous seminorm on Z, there is a finite subfamily $\mathcal{F}$ of $\mathcal{S}$ and a nonnegative real number M such that $p(z) \le M p_{\mathcal{F}}(z)$ for every $z \in Z$.

(a): (i)⇒(ii). Assume that C is (τ_1,τ_2)-dentable, let p be a τ_2-continuous seminorm on Z, and let $\varepsilon > 0$. Since C is (τ_1,τ_2)-dentable, there is $x \in C$ satisfying (2.1) with ε replaced by $\varepsilon/4$. By the Hahn–Banach separation theorem (see, e.g., [10, Theorem 2.2.28, p. 180]), there are $z^* \in (Z,\tau_1)'$ and $\alpha > 0$ such that

$$\operatorname{Re} z^*(x) - \alpha > \sup \operatorname{Re} z^*\big(\overline{\mathrm{co}}^{\tau_1}\big(C \setminus \mathcal{U}_p(x,\varepsilon/4)\big)\big).$$

For every $z \in S(z^*, \alpha, C)$, one has

$$\operatorname{Re} z^*(z) > \sup \operatorname{Re} z^*(C) - \alpha \geq \operatorname{Re} z^*(x) - \alpha;$$

thus $z \notin C \setminus \mathcal{U}_p(x, \varepsilon/4)$, and it follows that $S(z^*, \alpha, C) \subset \mathcal{U}_p(x, \varepsilon/4)$. Thus, whenever $z, w \in S(z^*, \alpha, C)$, one has

$$p(z-w) \leq p(z-x) + p(x-w) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},$$

and (2.3) follows.

(a): (ii)⇒(i). Assume that (ii) holds, let p be a τ_2-continuous seminorm on Z, and let $\varepsilon > 0$. By (ii), there is a τ_1-slice $S := S(z^*, \alpha, C)$ satisfying (2.3). Pick an arbitrary $x \in S$; then $S \subset \mathcal{U}_p(x, \varepsilon)$, and thus

$$\begin{aligned} C \setminus \mathcal{U}_p(x, \varepsilon) \subset C \setminus S &= \big\{ z \in C \colon \operatorname{Re} z^*(z) \leq \sup \operatorname{Re} z^*(C) - \alpha \big\} \\ &\subset \big\{ z \in \overline{\mathrm{co}}^{\tau_1}(C) \colon \operatorname{Re} z^*(z) \leq \sup \operatorname{Re} z^*(C) - \alpha \big\} =: K. \end{aligned}$$

Since K is convex and τ_1-closed, one has $\overline{\mathrm{co}}^{\tau_1}(C \setminus \mathcal{U}_p(x, \varepsilon)) \subset K$. Since $x \notin K$, (2.1) follows. We omit the proof of the equivalence (i)⇔(ii) in (b), which is not too different from that in (a). □

Our special interest in (τ_1, τ_2)-denting points lies in the case when Y is a strict ideal in a Banach space X with respect to an ideal projection $P \in \mathcal{L}(X^*)$, $Z = Y^*$, τ_1 and τ_2 are, respectively, the weak* topology and the τ_P-topology (i.e. the $\sigma(Y^*, X_P)$-topology) on $Y^* = Z$, and $C = B_{Y^*}$. In [9, Proposition 2.2, (iii)⇔(i)], we proved that y^* is a $(weak^*, \tau_P)$-denting point of B_{Y^*} if and only if y^* is an extreme point of B_{Y^*} having a unique norm-preserving extension to X. We conclude the section by proving the following proposition.

Proposition 2.5. *Let Y be a strict u-ideal in X with respect to an ideal projection $P \in \mathcal{L}(X^*)$. Then*

(a) *every τ_P-slice of B_{Y^*} contains a weak*-slice of B_{Y^*};*
(b) *every (τ_P, τ_P)-denting point of B_{Y^*} is a $(weak^*, \tau_P)$-denting point.*

For the proof of Proposition 2.5, it is convenient to state (and prove) the following two lemmas which are, respectively, a partial case (with $s = 1$) of [8, Lemma 4.3] and an obvious generalization of [8, Lemma 4.4]. We include their proofs for the sake of completeness.

Lemma 2.6 (cf. [8, Lemma 4.3])**.** *Suppose that $x \in S_X$, $y \in S_Y$, and $\gamma > 0$ are such that*

$$\|x - 2y\| < 1 + 2\gamma. \tag{2.5}$$

Then, whenever $\alpha > \gamma$ and P is an ideal projection for Y in X,

$$S(y, \alpha - \gamma, B_{Y^*}) \subset S(x_P, 2\alpha, B_{Y^*}). \tag{2.6}$$

Proof. Let $\alpha > \gamma$, and let P be an ideal projection for Y in X. Since, by (2.5), for all $y^* \in B_{Y^*}$,

$$\operatorname{Re} x_P(y^*) > 2 \operatorname{Re} y^*(y) - 1 - 2\gamma,$$

and the inequality

$$2\operatorname{Re} y^*(y) - 1 - 2\gamma > 1 - 2\alpha$$

is equivalent to $\operatorname{Re} y^*(y) > 1 - (\alpha - \gamma)$, the inclusion (2.6) follows. □

Lemma 2.7 (cf. [8, Lemma 4.4]). *Let C be a nonempty bounded convex subset of a Banach space Z, let $z^* \in S_{Z^*}$, $\alpha > 0$, and $K \geq 1$, and let p be a (norm-)continuous seminorm on Z. Then*

$$\operatorname{diam}_p S(z^*, K\alpha, C) \leq K \operatorname{diam}_p S(z^*, \alpha, C).$$

Proof. Set $d := \operatorname{diam}_p S(z^*, \alpha, C)$, and let $z_1, z_2 \in S(z^*, K\alpha, C)$. It suffices to show that $p(z_1 - z_2) \leq Kd$. To this end, set $M := \sup \operatorname{Re} z^*(C)$ and pick a $z_0 \in S(z^*, \alpha, C)$ such that

$$\begin{aligned}\Big(1 - \frac{1}{K}\Big)\big(M - \operatorname{Re} z^*(z_0)\big) &< \frac{1}{K}\big(\operatorname{Re} z^*(z_j) - (M - K\alpha)\big)\\ &= \alpha - \frac{1}{K}\big(M - \operatorname{Re} z^*(z_j)\big), \quad j = 1, 2.\end{aligned}$$

Now put

$$u_j = \Big(1 - \frac{1}{K}\Big) z_0 + \frac{1}{K} z_j, \quad j = 1, 2.$$

Since

$$M - \operatorname{Re} z^*(u_j) = \Big(1 - \frac{1}{K}\Big)\big(M - \operatorname{Re} z^*(z_0)\big) + \frac{1}{K}\big(M - \operatorname{Re} z^*(z_j)\big) < \alpha,$$

one has $u_j \in S(z^*, \alpha, C)$, $j = 1, 2$, and thus

$$p(z_1 - z_2) = p\big(K(u_1 - u_2)\big) = Kp(u_1 - u_2) \leq Kd,$$

as desired. □

In the proof of Proposition 2.5, we also use the following theorem which is the partial case (with $K = \{2\}$, $a = 1$) of [5, Lemma 2.2].

Theorem 2.8 (cf. [5, Lemma 2.2]). *Let Y be an ideal in X with respect to a projection $P \in \mathcal{L}(X^*)$. The following assertions are equivalent:*

(i) *P is a u-ideal projection;*
(ii) *whenever $x \in S_X$, there is a net (y_α) in B_Y converging to x in the $\sigma(X, \operatorname{ran} P)$-topology such that*

$$\limsup_\alpha \|x - 2y_\alpha\| \leq 1.$$

Remark 2.9. Suppose that, in Theorem 2.8, P is a strict ideal projection. Then, in the assertion (ii), $\lim_\alpha \|y_\alpha\| = 1$, because, letting, for each $n \in \mathbb{N}$, an element $x_n^* \in \operatorname{ran} P$, $\|x_n^*\| \leq 1$, be such that $|x_n^*(x)| > 1 - 1/n$, one has $\liminf_\alpha \|y_\alpha\| \geq \liminf_\alpha |x_n^*(y_\alpha)| > 1 - 1/n$.

Proof of Proposition 2.5. (a). Let $S(x_P, \alpha, B_{Y^*})$ be a τ_P-slice of B_{Y^*}. We may assume that $\|x\| = 1$. Letting $0 < \gamma < \alpha/2$, by Theorem 2.8 and Remark 2.9, there is $y \in S_Y$ such that $\|x - 2y\| < 1 + 2\gamma$. By Lemma 2.6, $S(y, \alpha/2 - \gamma, B_{Y^*}) \subset S(x_P, \alpha, B_{Y^*})$.

(b). Let y^* be a (τ_P, τ_P)-denting point of B_{Y^*}, let p be a τ_P-continuous seminorm on Y^*, let $\varepsilon > 0$, and let $x \in S_X$ and $\alpha > 0$ be such that $y^* \in S(x_P, \alpha, B_{Y^*})$ and $\operatorname{diam}_p S(x_P, \alpha, B_{Y^*}) < \varepsilon/2$. By Proposition 2.4 and Lemma 2.7, it suffices to show that there are $y \in S_Y$ and $\beta > 0$ with $y^* \in S(y, \beta, B_{Y^*}) \subset S(x_P, 2\alpha, B_{Y^*})$. To this end, choose $\gamma \in (0, \alpha)$ such that $y^* \in S(x_P, \alpha - \gamma, B_{Y^*})$. By Theorem 2.8 and Remark 2.9, there is $y \in S_Y$ with

$$y^* \in S(y, \alpha - \gamma, B_{Y^*}) \qquad \text{and} \qquad \|x - 2y\| < 1 + 2\gamma.$$

By Lemma 2.6, $S(y, \alpha - \gamma, B_{Y^*}) \subset S(x_P, 2\alpha, B_{Y^*})$. □

3. Bounded sets in a dual Banach space X^* are $(\sigma(X^*, E), \sigma(X^*, E))$-dentable for every subspace E of X

In this section, we prove the following theorem.

Theorem 3.1. *Let X be a Banach space, and let E be a subspace of X. Then every nonempty bounded subset of the dual space X^* is $(\sigma(X^*, E), \sigma(X^*, E))$-dentable.*

Remark 3.2. If, in Theorem 3.1, the space X is separable, then the assertion follows from [4, Theorem 1, Proposition 1], because, under that assumption, every weak* compact convex subset of the dual space X^* is separable in the $(\sigma(X^*, E), \sigma(X^*, E))$-topology.

Our interest in Theorem 3.1 lies in the following corollary.

Corollary 3.3. *Let Y be a strict ideal in a Banach space X with respect to an ideal projection $P \in \mathcal{L}(X^*)$, and let E be a subspace of X. Set $E_P := \{x_P \in Y^{**} \colon x \in E\}$, and denote by σ_E the weak topology $\sigma(Y^*, E_P)$ on Y^*. Then every nonempty bounded subset of Y^* is (σ_E, σ_E)-dentable. In particular, every nonempty bounded subset of Y^* is (τ_P, τ_P)-dentable.*

Proof. Letting D be a nonempty bounded subset of Y^*, it suffices to observe that D is (σ_E, σ_E)-dentable if and only if $J_P(D)$ is $(\sigma(X^*, E), \sigma(X^*, E))$-dentable (in X^*), and that $\tau_P = \sigma_E$ for $E = X$. □

Proof of Theorem 3.1. Our proof is an adaption of the proof that a Banach space, which contains a bounded non-dentable subset, does not have the Radon–Nikodým property [2, Theorem 4, p. 133].

Suppose for contradiction that X^* contains a nonempty bounded subset D which is not $(\sigma(X^*, E), \sigma(X^*, E))$-dentable. Then, by Proposition 2.4(a), there are $\varepsilon > 0$ and a finite subset $\mathcal{F}$ of S_E such that, for every $x^* \in D$, one has $x^* \in \overline{\operatorname{co}}^{\sigma(X^*, E)}(D \setminus \mathcal{U}_p(x^*, \varepsilon))$, where the seminorm p on X^* is defined by $p(x^*) = \max_{x \in \mathcal{F}} |x^*(x)|$ for $x^* \in X^*$. In other words,

(♯) whenever $x^* \in D$, $\delta > 0$, and $\mathcal{G}$ is a finite subset of E, there are $N \in \mathbb{N}$, real numbers $\alpha_1[x^*], \dots, \alpha_N[x^*] > 0$ with $\sum_{j=1}^N \alpha_j[x^*] = 1$, and elements $x_1^*[x^*], \dots, x_N^*[x^*] \in D \setminus \mathcal{U}_p(x^*, \varepsilon)$ such that

$$\Big|\Big(x^* - \sum_{j=1}^N \alpha_j[x^*] x_j^*[x^*]\Big)(x)\Big| < \delta \quad \text{for all } x \in \mathcal{G}.$$

It suffices to construct partitions $\pi_n := \{I_1^n, \dots, I_{K_n}^n\}$ ($K_n \in \mathbb{N}$) of the half-open interval $[0,1)$ into half-open intervals $I_1^n, \dots, I_{K_n}^n$ and functions $f_n\colon [0,1) \to X^*$, $n \in \mathbb{N}$, such that

(1) each f_n is of the form $f_n = \sum_{k=1}^{K_n} \chi_{I_k^n} x_{n,k}^*$, where $x_{n,1}^*, \dots, x_{n,K_n}^* \in D$;
(2) π_{n+1} refines π_n in the sense that each interval in π_n is a finite union of intervals in π_{n+1};
(3) $\max_{x \in \mathcal{F}} |x(f_n(t) - f_{n+1}(t))| \geq \varepsilon$ for all $n \in \mathbb{N}$ and all $t \in [0,1)$;
(4) $|\int_I x(f_n - f_m)\, d\mu| < \mu(I)/2^n$ for all $n \in \mathbb{N}$, all $m \geqslant n$, all $x \in \mathcal{F}$, and all $I \in \pi_n$ (here μ is the Lebesgue measure).

Indeed, assume that functions f_n and partitions π_n, $n \in \mathbb{N}$, satisfying conditions (1)–(4) have been constructed. Define, for every $I \in \bigcup_{n=1}^\infty \pi_n$, a functional $F[I]$ on $\operatorname{span}(\mathcal{F}) =: X_0$ by

$$F[I](x) = \lim_n \int_I x f_n \, d\mu, \quad x \in X_0$$

(note that (4) guarantees, for every $x \in X_0$, the sequence $(\int_I x f_n\, d\mu)$ to be Cauchy); then $F[I]$ can easily be seen to be linear and bounded with $\|F[I]\| \leq M\mu(I)$, where the real number $M \geq 0$ is such that $\|x^*\| \leq M$ for every $x^* \in D$; thus $F[I] \in X_0^*$. For every $n \in \mathbb{N}$, let $\mathcal{B}_n$ be the σ-algebra of subsets of $[0,1)$ generated by $\pi_n = \{I_1^n, \dots, I_{K_n}^n\}$, and define a function

$$g_n := \sum_{k=1}^{K_n} \chi_{I_k^n} \frac{F[I_k^n]}{\mu(I_k^n)} \colon [0,1) \to X_0^*;$$

then $(g_n, \mathcal{B}_n)$ is a martingale in $L_1([0,1), X_0^*)$. Indeed, let $m, n \in \mathbb{N}$, $m > n$, and let $g_m = \sum_{k=1}^{K_n} \sum_{j=1}^{N_k} \chi_{I_j^{n,k}} \frac{F[I_j^{n,k}]}{\mu(I_j^{n,k})}$, where $I_k^n = \bigcup_{j=1}^{N_k} I_j^{n,k}$ with $I_j^{n,k} \in \pi_m$ for every $k \in \{1, \dots, K_n\}$; if $k \in \{1, \dots, K_n\}$, then

$$\begin{aligned}\int_{I_k^n} g_m \, d\mu &= \int_{I_k^n} \sum_{j=1}^{N_k} \chi_{I_j^{n,k}} \frac{F[I_j^{n,k}]}{\mu(I_j^{n,k})}\, d\mu = \sum_{j=1}^{N_k} \int_{I_j^{n,k}} \frac{F[I_j^{n,k}]}{\mu(I_j^{n,k})}\, d\mu \\ &= \sum_{j=1}^{N_k} F[I_j^{n,k}] = F[I_k^n] = \int_{I_k^n} g_n \, d\mu.\end{aligned}$$

For every $I \in \bigcup_{n=1}^\infty \pi_n$, one has

$$\Big\|\frac{F[I]}{\mu(I)}\Big\| \leq \frac{M\mu(I)}{\mu(I)} = M.$$

Thus the martingale $(g_n, \mathcal{B}_n)$ is, in fact, L_∞-bounded. From (3) it follows that, for some $x \in \mathcal{F}$, the sequence (xf_n) is not Cauchy in $L_1([0,1),\mathbb{K})$. From (4) it follows that

$$\begin{aligned}
\int_{[0,1)} |xf_n - xg_n|\,d\mu &= \int_{[0,1)} \Big|\sum_{k=1}^{K_n} \chi_{I_k^n}\Big(x^*_{n,k}(x) - \frac{F[I_k^n](x)}{\mu(I_k^n)}\Big)\Big|\,d\mu \\
&= \sum_{k=1}^{K_n} \int_{I_k^n} \Big|x^*_{n,k}(x) - \frac{F[I_k^n](x)}{\mu(I_k^n)}\Big|\,d\mu \\
&= \sum_{k=1}^{K_n} \Big|x^*_{n,k}(x) - \frac{F[I_k^n](x)}{\mu(I_k^n)}\Big|\mu(I_k^n) \\
&= \sum_{k=1}^{K_n} \big|x^*_{n,k}(x)\mu(I_k^n) - F[I_k^n](x)\big| \\
&= \sum_{k=1}^{K_n} \Big|\int_{I_k^n} xf_n\,d\mu - \lim_m \int_{I_k^n} xf_m\,d\mu\Big| \\
&= \lim_m \sum_{k=1}^{K_n} \Big|\int_{I_k^n} x(f_n - f_m)\,d\mu\Big| \\
&\le \sum_{k=1}^{K_n} \frac{\mu(I_k^n)}{2^n} = \frac{1}{2^n}\mu\big([0,1)\big) = \frac{1}{2^n};
\end{aligned}$$

thus also the sequence (xg_n) is not Cauchy in $L_1([0,1),\mathbb{K})$. On the other hand, since $(xg_n, \mathcal{B}_n)$ is an L_∞-bounded martingale in $L_1([0,1),\mathbb{K})$, the sequence (xg_n) must converge in $L_1([0,1),\mathbb{K})$ (see, e.g., [2, Corollary 4, p. 126]), which is a contradiction.

To complete the proof, it remains to construct the functions $f_n\colon [0,1) \to X^*$ and partitions π_n of $[0,1)$, $n \in \mathbb{N}$, satisfying (1)–(4). Picking an arbitrary $x^* \in D$, define $f_1 := \chi_{[0,1)}x^*$ and $\pi_1 := \{[0,1)\}$. Next suppose that, for some $n \in \mathbb{N}$, the function $f_n = \sum_{k=1}^{K_n} \chi_{I_k^n} x^*_{n,k}$, where $K_n \in \mathbb{N}$, $x^*_{n,1}, \dots, x^*_{n,K_n} \in D$, and $I_1^n, \dots, I_{K_n}^n$ are pairwise disjoint half-open intervals with $\bigcup_{k=1}^{K_n} I_k^n = [0,1)$, and the partition $\pi_n := \{I_1^n, \dots, I_{K_n}^n\}$ have been defined. By ($\sharp$), for every $k \in \{1, \dots, K_n\}$, there are $N_k \in \mathbb{N}$, real numbers $\alpha_1[x^*_{n,k}], \dots, \alpha_{N_k}[x^*_{n,k}] > 0$ with $\sum_{j=1}^{N_k} \alpha_j[x^*_{n,k}] = 1$, and elements $x_1^*[x^*_{n,k}], \dots, x^*_{N_k}[x^*_{n,k}] \in D \setminus \mathcal{U}_p(x^*_{n,k}, \varepsilon)$ such that

$$\Big|\Big(x^*_{n,k} - \sum_{j=1}^{N_k} \alpha_j[x^*_{n,k}]x_j^*[x^*_{n,k}]\Big)(x)\Big| < \frac{1}{2^{n+1}} \quad \text{for all } x \in \mathcal{F}. \tag{3.1}$$

Now, if $k \in \{1, \dots, K_n\}$ and $I_k^n = [a,b)$, then, setting $\alpha_0[x^*_{n,k}] := 0$, define half-open intervals

$$I_j^{n,k} := \Big[a + (b-a)\sum_{i=0}^{j-1} \alpha_i[x^*_{n,k}], a + (b-a)\sum_{i=0}^{j} \alpha_i[x^*_{n,k}]\Big), \quad j = 1, \dots, N_k,$$

partitions $\pi_{n+1} := \{I_j^{n,k} : k \in \{1,\dots,K_n\}, j \in \{1,\dots,N_k\}\}$, and functions $f_{n+1} := \sum_{k=1}^{K_n}\sum_{j=1}^{N_k} \chi_{I_j^{n,k}} x_j^*[x_{n,k}^*]$. The functions f_n, $n\in\mathbb{N}$, defined as above satisfy (1) and (3), and the partitions π_n, $n\in\mathbb{N}$, satisfy (2). To prove (4), let $n\in\mathbb{N}$ and $k\in\{1,\dots,K_n\}$ be arbitrary. For every $x\in\mathcal{F}$, by (3.1),

$$\begin{aligned}\left|\int_{I_k^n} x(f_n - f_{n+1})\,d\mu\right| &= \left|\mu(I_k^n)x_{n,k}^*(x) - \sum_{j=1}^{N_k}\mu(I_j^{n,k})x_j^*[x_{n,k}^*](x)\right| \\ &= \left|x_{n,k}^*(x) - \sum_{j=1}^{N_k}\frac{\mu(I_j^{n,k})}{\mu(I_k^n)}x_j^*[x_{n,k}^*](x)\right|\mu(I_k^n) \\ &= \left|x_{n,k}^*(x) - \sum_{j=1}^{N_k}\alpha_j[x_{n,k}^*]x_j^*[x_{n,k}^*](x)\right|\mu(I_k^n) \\ &< \frac{\mu(I_k^n)}{2^{n+1}}.\end{aligned}$$

This establishes (4). □

4. Existence of functionals in the dual unit ball of strict u-ideals, which admit a unique norm-preserving extension

Theorem 1.3 follows quickly from the following proposition, Proposition 2.5, and [9, Proposition 2.2].

Proposition 4.1. *Let Y be a strict u-ideal in a Banach space X with respect to an ideal projection $P\in\mathcal{L}(X^*)$.*

(a) *If X/Y is separable, then $B_{Y^*} = \overline{\mathrm{co}}^{\tau_P}(\mathcal{C}_X)$, where $\mathcal{C}_X$ is the set of functionals in S_{Y^*} having a unique norm-preserving extension to X.*

(b) *If X is separable, then B_{Y^*} is the τ_P-closed convex hull of its $(weak^*,\tau_P)$-denting points.*

Proof of Theorem 1.3. Let E be a separable subspace of X. Set $Z := \overline{\mathrm{span}}(Y\cup E)$, and define an operator $P_0\colon Z^*\ni u^*\mapsto Px^*|_Z\in Z^*$, where $x^*\in X^*$ is any extension of u^*. One immediately verifies that P_0 is a strict u-ideal projection for Y in Z.

(a) Observing that Z/Y is separable, by Propositions 4.1(a) and 2.5(a),

$$B_{Y^*} = \overline{\mathrm{co}}^{\tau_{P_0}}(\mathcal{C}_E) \subset \overline{\mathrm{co}}^{weak^*}(\mathcal{C}_E) \subset \overline{\mathrm{co}}^{\tau_P}(\mathcal{C}_E) \subset B_{Y^*}.$$

(b) Assume that Y is separable. Then also Z is separable, and thus B_{Y^*} is the τ_{P_0}-closed convex hull of its $(weak^*,\tau_{P_0})$-denting points by Proposition 4.1(b). As in the proof of (a), from Proposition 2.5(a), it follows that B_{Y^*} is, in fact, the τ_P-closed convex hull of its $(weak^*,\tau_{P_0})$-denting points. It remains to observe that, by [9, Proposition 2.2, (iii)⇒(i)], every $(weak^*,\tau_{P_0})$-denting point of B_{Y^*} is an extreme point of B_{Y^*} having a unique norm-preserving extension to Z. □

Remark 4.2. Sufficient conditions in order that a convex set in a linear space with two locally convex topologies τ_1 and τ_2 were the closed convex hull of its

(τ_1, τ_2)-denting points, have been studied in [3, Section 4]. Proposition 4.1(b) cannot be derived directly from these results because the τ_P-topology on Y^* is not quasimetrizable (and neither is the weak* topology).

Proposition 4.1 is a quick consequence of the following proposition and the Hahn–Banach separation theorem.

Proposition 4.3. *Let Y, X, and $\mathcal{C}_X$ be as in Proposition 4.1.*

(a) *If X/Y is separable, then every τ_P-slice of B_{Y^*} contains a point in $\mathcal{C}_X$.*
(b) *If X is separable, then every τ_P-slice of B_{Y^*} contains a (weak$^*, \tau_P$)-denting point of B_{Y^*}.*

Proof of Proposition 4.1. We only prove (a). The proof of (b) is similar with some obvious changes.

Assume that X/Y is separable, and suppose for contradiction that there is $y^* \in B_{Y^*} \setminus \overline{\mathrm{co}}^{\tau_P}(\mathcal{C}_X)$. Then, by the Hahn–Banach separation theorem (see, e.g., [10, page 180, Theorem 2.2.28]), there are $x \in X$ and a real number $\beta > 0$ such that

$$\operatorname{Re} x_P(y^*) - \beta > \sup \operatorname{Re} x_P\big(\overline{\mathrm{co}}^{\tau_P}(\mathcal{C}_X)\big),$$

and thus $\mathcal{C}_X \cap S(x_P, \beta, B_{Y^*}) = \emptyset$. This is a contradiction to Proposition 4.3(a). □

Proposition 4.3 follows from the following lemma.

Lemma 4.4. *Let Y and X be as in Proposition 4.1, and let E be a subspace of X. Set $E_P := \{x_P \in Y^{**} \colon x \in E\}$, and denote by σ_E the weak topology $\sigma(Y^*, E_P)$ on Y^*. Then, whenever $S(z_P, \alpha, B_{Y^*})$ is a σ_E-slice with $z \in S_E$, p is a σ_E-continuous seminorm on Y^*, and $\varepsilon > 0$, there is a σ_E-slice $S(x_P, \beta, B_{Y^*})$ with $x \in S_E$ such that*

(1) *$S(x_P, \beta, B_{Y^*}) \subset S(z_P, \alpha, B_{Y^*})$;*
(2) *$\operatorname{diam}_p S(x_P, \beta, B_{Y^*}) < \varepsilon$.*

Proof of Proposition 4.3. Let $S(x_{0P}, \beta_0, B_{Y^*})$ with $x_0 \in S_X$ be a τ_P-slice of B_{Y^*}.

(a) Assume that X/Y is separable. Then there is a separable subspace E_0 of X such that $X = \overline{\mathrm{span}}(Y \cup E_0)$. Set $Z_1 := \overline{\mathrm{span}}(E_0 \cup \{x_0\})$, and proceed as follows. Given $n \in \mathbb{N}$ and a separable subspace Z_n of X, let A_n be a countable dense subset of S_{Z_n}. For every $x \in A_n$ and every $k \in \mathbb{N}$, by Proposition 2.5(a), there are $y^x_{n,k} \in S_Y$ and $\alpha^x_{n,k} > 0$ such that $S(y^x_{n,k}, \alpha^x_{n,k}, B_{Y^*}) \subset S(x_P, 1/k, B_{Y^*})$. Set

$$B_n := \{y^x_{n,k} \colon x \in A_n, k \in \mathbb{N}\} \qquad \text{and} \qquad Z_{n+1} := \overline{\mathrm{span}}(A_n \cup B_n).$$

Set $E := \overline{\bigcup_{n=1}^\infty Z_n}$. Since E is separable, there are finite subsets $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ of S_E such that the union $\bigcup_{n=1}^\infty \mathcal{F}_n$ is dense in S_E. For every $n \in \mathbb{N}$, define a seminorm $p_n \colon Y^* \ni y^* \mapsto \max_{x \in \mathcal{F}_n} |x_P(y^*)|$. By Lemma 4.4 and our construction, we can inductively find weak*-slices $S(y_n, \gamma_n, B_{Y^*})$ and σ_E-slices $S(x_{nP}, \beta_n, B_{Y^*})$ with $y_n \in S_{Y \cap E}$ and $x_n \in S_E$ such that, for every $n \in \mathbb{N}$,

(I) $S(x_{nP}, \beta_n, B_{Y^*}) \subset \{y^* \in B_{Y^*} \colon \operatorname{Re} y^*(y_n) \geq 1 - \gamma_n/2\} \subset S(y_n, \gamma_n, B_{Y^*}) \subset S(x_{n-1P}, \beta_{n-1}, B_{Y^*})$;
(II) $\operatorname{diam}_{p_n} S(x_{nP}, \beta_n, B_{Y^*}) < 1/n$.

There exists $y^* \in \bigcap_{n=1}^{\infty} S(y_n, \gamma_n, B_{Y^*}) = \bigcap_{n=1}^{\infty} S(x_{nP}, \beta_n, B_{Y^*})$, because B_{Y^*} is weak* compact. One can immediately verify that this y^* is a *weak**-to-σ_E point of continuity of the identity operator on B_{Y^*}, and thus a *weak**-to-$\sigma(Y^*, X_P)$ point of continuity. From [9, Proposition 2.1] it now follows that $y^* \in \mathcal{C}_X$.

(b) If X is separable, then set $E = X$ and follow the proof of (a) starting with picking the finite subsets $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ of S_E. The resulting y^* is now a $(weak^*, \tau_P)$-denting point of B_{Y^*}. $\square$

The proof of Lemma 4.4 makes use of the following lemma.

Lemma 4.5 (cf. [3, Lemma 1], [2, Lemma 2, p. 200])**.** *Let Z be a Banach space, let τ be a locally convex topology on Z weaker than the norm topology, let $x^* \in (Z, \tau)'$ with $\|x^*\| = 1$, and let K be a nonempty closed bounded convex subset of Z such that $K \subset \{x \in Z\colon\ \operatorname{Re} x^*(x) \geq 0\}$ and $K \cap \{x \in Z\colon\ \operatorname{Re} x^*(x) > 0\} \neq \emptyset$. Suppose that every nonempty bounded subset of Z is (τ, τ)-dentable. Then, whenever p is a τ-continuous seminorm on Z and $\varepsilon > 0$, there is a τ-slice $S(y^*, \beta, K)$ with $\|y^*\| = 1$ such that*

(1) *$S(y^*, \beta, K) \subset \{x \in K\colon\ \operatorname{Re} x^*(x) > 0\}$;*
(2) *$\operatorname{diam}_p S(y^*, \beta, K) < \varepsilon$.*

Proof. The lemma follows from [3, Lemma 1]. $\square$

Proof of Lemma 4.4. Let $S(z_P, \alpha, B_{Y^*})$ be a σ_E-slice with $z \in S_E$, let p be a σ_E-continuous seminorm on Y^*, and let $\varepsilon > 0$. Pick $v^* \in Y^*$ so that $z_P(v^*) = \alpha - 1$, and let $C := v^* + B_{Y^*}$. Then $\sup \operatorname{Re} z_P(C) = \alpha$ and thus $S(z_P, \alpha, C) = \{y^* \in C\colon\ \operatorname{Re} z_P(y^*) > 0\}$. Set $K := \{y^* \in C\colon\ \operatorname{Re} z_P(y^*) \geq 0\}$. Since, by Corollary 3.3, every nonempty bounded subset of Y^* is (σ_E, σ_E)-dentable, by Lemma 4.5, there is a σ_E-slice $S(x_P, \beta, K)$ with $x \in S_E$ such that

(I) $S(x_P, \beta, K) \subset \{y^* \in K\colon\ \operatorname{Re} z_P(y^*) > 0\} = S(z_P, \alpha, C)$;
(II) $\operatorname{diam}_p S(x_P, \beta, K) < \varepsilon$.

It now suffices to show that
$$S(x_P, \beta, C) \subset K, \tag{4.1}$$
because, in this case,
$$S(x_P, \beta, C) \subset S(x_P, \beta, K) \subset S(z_P, \alpha, C),$$
and therefore, since $C = v^* + B_{Y^*}$, one has $S(x_P, \beta, B_{Y^*}) \subset S(z_P, \alpha, B_{Y^*})$ and
$$\operatorname{diam}_p S(x_P, \beta, B_{Y^*}) = \operatorname{diam}_p S(x_P, \beta, C) \leq \operatorname{diam}_p S(x_P, \beta, K) < \varepsilon.$$

Suppose for contradiction that (4.1) fails; that is, let $y^* \in S(x_P, \beta, C) \setminus K$, that is, $y^* \in C$ with
$$\operatorname{Re} x_P(y^*) > \sup \operatorname{Re} x_P(C) - \beta \geq \sup \operatorname{Re} x_P(K) - \beta$$
and $\operatorname{Re} z_P(y^*) < 0$. Letting $u^* \in S(x_P, \beta, K)$ be arbitrary, one has $u^* \in K \subset C$ and $\operatorname{Re} x_P(u^*) > \sup \operatorname{Re} x_P(K) - \beta$. Since $\operatorname{Re} z_P(u^*) > 0$ by (I), there is $\lambda \in (0,1)$ such that, for $w^* := (1-\lambda) y^* + \lambda u^* \in C$, one has $\operatorname{Re} z_P(w^*) = 0$; thus $w^* \in K$. Since $\operatorname{Re} x_P(w^*) > \sup \operatorname{Re} x_P(K) - \beta$, that is, $w^* \in S(x_P, \beta, K)$, one must have $\operatorname{Re} z_P(w^*) > 0$ by (I), which is a contradiction. $\square$

Acknowledgments. The authors are grateful to the anonymous referees for enlightening remarks.

Martsinkevitš and Põldvere's work was partially supported by Estonian Ministry of Education and Research institutional research funding grant IUT20-57.

References

1. T. A. Abrahamsen, Å. Lima, and V. Lima, *Unconditional ideals of finite rank operators*, Czechoslovak Math. J. **58** (2008), no. 4, 1257–1278. Zbl 1174.46003. MR2471182. DOI 10.1007/s10587-008-0085-9. 47
2. J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys **15**, Amer. Math. Soc., Providence, 1977. Zbl 0369.46039. MR0453964. 53, 55, 58
3. M. Fundo, *The dentability in the spaces with two topologies*, Rocky Mountain J. Math. **27** (1997), no. 4, 1117–1130. Zbl 0907.46012. MR1627678. DOI 10.1216/rmjm/1181071864. 48, 49, 57, 58
4. M. Fundo, *Some aspects of dentability in bitopological and locally convex spaces*, Rocky Mountain J. Math. **29** (1999), no. 2, 505–518. Zbl 0954.46011. MR1705473. DOI 10.1216/rmjm/1181071649. 48, 49, 53
5. G. Godefroy, N. J. Kalton, and P. D. Saphar, *Unconditional ideals in Banach spaces*, Studia Math. **104** (1993), no. 1, 13–59. Zbl 0814.46012. MR1208038. 46, 52
6. Å. Lima, *Property* (wM^*) *and the unconditional metric compact approximation property*, Studia Math. **113** (1995), no. 3, 249–263. Zbl 0826.46013. MR1330210. DOI 10.4064/sm-113-3-249-263. 48
7. V. Lima and Å. Lima, *Strict u-ideals in Banach spaces*, Studia Math. **195** (2009), no. 3, 275–285. Zbl 1190.46015. MR2559177. DOI 10.4064/sm195-3-6. 47
8. J. Lippus and M. Põldvere, *Metric ideal structure in Banach spaces*, preprint. 51, 52
9. J. Martsinkevitš and M. Põldvere, *Uniqueness of norm-preserving extensions of functionals on the space of compact operators*, preprint, to appear in Math. Scand. 49, 51, 56, 58
10. R. E. Megginson, *An Introduction to Banach Space Theory*, Grad. Texts in Math. **183**, Springer, New York, 1998. Zbl 0910.46008. MR1650235. DOI 10.1007/978-1-4612-0603-3. 50, 57

Institute of Mathematics and Statistics, University of Tartu, J. Liivi 2, Tartu 50409, Estonia.

E-mail address: julia.martsinkevits@ut.ee; mart.poldvere@ut.ee