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SOME INEQUALITIES OF JENSEN'S TYPE FOR LIPSCHITZIAN MAPS BETWEEN BANACH SPACES

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Abstract. In this article, we consider some Jensen-type inequalities for Lipschitzian maps between Banach spaces and functions defined by power series. We obtain as applications some inequalities of Levinson type for Lipschitzian maps. Applications for functions of norms in Banach spaces are provided as well.

1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

One of the central problems in perturbation theory is to find bounds for $\|f(A)-f(B)\|$ in terms of $\|A-B\|$ for different classes of measurable functions f for which the function of operator can be defined. It is known (see [2]) that in the infinite-dimensional case, the map $f(A) := |A|$ is not Lipschitz continuous on $\mathcal{B}(H)$ with the usual operator norm; that is, there is no constant $L > 0$ such that $\||A|-|B|\| \le L\|A-B\|$ for any $A, B \in \mathcal{B}(H)$. However, the following inequality holds

$$\big\||A|-|B|\big\| \le \frac{2}{\pi}\|A-B\|\Big(2+\log\Big(\frac{\|A\|+\|B\|}{\|A\|-\|B\|}\Big)\Big),$$

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for any $A, B \in \mathcal{B}(H)$ with $A \neq B$. Bhatia (in [2]) also obtained the Lipschitz-type inequality

$$\left\| f(A) - f(B) \right\| \leq f'(a)\|A - B\|, \tag{1.1}$$

where f is an operator monotone function on $(0, \infty)$ and $A, B \geq a1_H > 0$.

As an application of (1.1), Dragomir in [5] and [6] gave an Ostrowski-type inequality and a Hermite–Hadamard-type inequality for operator monotone functions (see also [7]). We note that the authors in [4] obtained some Jensen-type inequalities for a Lipschitz function. As corollaries, they got Jensen-type inequalities (see (1.1)) for a differentiable function on (m, M) such that $f'(u) \leq M$ for all $u \in (m, M)$. (For both Jensen inequality and Levinson inequality for Hilbert-space operators, see [9] and [10]; for further details on the subject, see [1] and [3].)

Motivated by the above results, in this paper we investigate some Jensen-type and Levinson-type inequalities for Lipschitzian maps between Banach spaces and functions defined by power series. As applications, some Levinson-type inequalities for Lipschitzian maps are obtained. Applications for functions of norms in Banach spaces are provided as well.

Now, we will give some more denotations. Let $\mathcal{F}(D(0,R))$ denote the set of all analytic functions given by the power series $f(z) = \sum_{i=0}^{\infty} \alpha_i z^i$ with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$ for some $R > 0$. If $f(z) := \sum_{i=0}^{\infty} \alpha_i z^i$, then we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series $f_a(z) := \sum_{i=0}^{\infty} |\alpha_i| z^i$. It is obvious that this new power series will have the same radius of convergence as the original series.

Let X and Y be complex linear spaces. An operator $\phi : X \to Y$ is linear if

$$\phi(\alpha x + \beta y) = \alpha\phi x + \beta\phi y \quad \text{for all } \alpha, \beta \in \mathbb{C} \text{ and all } x, y \in X.$$

A linear operator $\phi : X \to Y$ is *bounded* if there is a constant $M \geq 0$ such that

$$\|\phi x\| \leq M\|x\| \quad \text{for all } x \in X.$$

If $\phi : X \to Y$ is a bounded linear operator, then we define the *operator norm* or *uniform norm* $\|\phi\|$ of ϕ by

$$\|\phi\| = \inf\big\{M : \|\phi x\| \leq M\|x\|, \text{for all } x \in X\big\}.$$

We denote the set of all bounded linear operators $\phi : X \to Y$ by $\mathcal{B}(X, Y)$. When the domain and range spaces are the same, we write $\mathcal{B}(X, X) = \mathcal{B}(X)$.

2. Power series in Banach algebras

2.1. Preliminaries and previous results. Let B be an algebra. An *algebra norm* on B is a map $\|\cdot\| : B \to [0, \infty)$ such that $(B, \|\cdot\|)$ is a normed space. The normed algebra $(B, \|\cdot\|)$ is a Banach algebra if $\|\cdot\|$ is a *complete norm*. We assume that the Banach algebra is *unital*, which means that B has an identity 1 and that $\|1\| = 1$.

Let B be a unital algebra. The set of invertible elements of B is denoted by $\operatorname{Inv} B$. The *resolvent set* of $a \in \operatorname{Inv} B$ is $\rho(a) := \{z \in \mathbb{C} : z1 - a \in \operatorname{Inv} B\}$; the *spectrum* of a is $\sigma(a) = \mathbb{C} \setminus \rho(a)$; the *resolvent function* of a, $R_a : \rho(a) \to \operatorname{Inv} B$, is $R_a(z) := (z1 - a)^{-1}$; and the *spectral radius* of a is $\nu(a) := \sup\{|z| : z \in \sigma(a)\}$.

Let $f \in \mathcal{F}(D(0,R))$ given by the power series

$$f(z) := \sum_{i=0}^{\infty} \alpha_i z^i, \quad |z| < R.$$

If $\nu(a) < R$, then the series $\sum_{i=0}^{\infty} \alpha_i a^i$ converges in the Banach algebra B because $\sum_{i=0}^{\infty} |\alpha_i| \|a^i\| < \infty$, and we can define $f(a)$ to be its sum. Clearly, $f(a)$ is well defined. There are many examples of important functions on a Banach algebra B that can be constructed in this way. Furthermore, let $(B, \|\cdot\|)$ be a unital Banach algebra.

The following theorem is proved by Dragomir in [6, Theorem 4] and [5, Theorem 1].

Theorem A. *If $f \in \mathcal{F}(D(0,R))$, then for any $x, y \in B$ with $\|x\|, \|y\| < R$, we have*

$$\big\|f(y) - f(x)\big\| \le \|y - x\| \int_0^1 f_a'\big(\big\|(1-t)x + ty\big\|\big)\, dt. \tag{2.1}$$

2.2. Jensen-type inequalities. A combination $\sum_{j=1}^{n} p_j x_j$ of vectors x_j in a real linear space and real coefficients p_j is convex if $p_j \ge 0$ and $\sum_{j=1}^{n} p_j = 1$. If f is a convex function defined on a convex set in a real linear space, then the inequality

$$f\Big(\sum_{j=1}^{n} p_j x_j\Big) \le \sum_{j=1}^{n} p_j f(x_j) \tag{2.2}$$

holds for all convex combinations of vectors x_j belonging to the domain of f. This is the well-known Jensen inequality; the classic version can be seen in [8]. Next we give a Lipschitz-type inequality for power series in Banach algebras.

Theorem 2.1. *Let $f \in \mathcal{F}(D(0,R))$ be an analytic function, let $\bar{x} = \sum_{j=1}^{n} p_j x_j$ be a convex combination of vectors $x_j \in B$ such that $\|x_j\| < R$, and let $\phi \in \mathcal{B}(B)$ be a bounded linear operator such that $\|\phi x_j\| < R$. Then we have the inequality*

$$\begin{aligned}
&\Big\|\sum_{j=1}^{n} p_j f(\phi x_j) - f\Big(\sum_{j=1}^{n} p_j \phi x_j\Big)\Big\| \\
&\quad \le \|\phi\| \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \int_0^1 f_a'\big(\big\|(1-t)\phi x_j + t\phi\bar{x}\big\|\big)\, dt \\
&\quad \le \frac{1}{2}\|\phi\| \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \big(f_a'\big(\|\phi x_j\|\big) + f_a'\big(\|\phi \bar{x}\|\big)\big) \\
&\quad \le \|\phi\| \max_{1\le j\le n}\big\{f_a'\big(\|\phi x_j\|\big)\big\} \sum_{j=1}^{n} p_j \|x_j - \bar{x}\|.
\end{aligned} \tag{2.3}$$

Proof. Note that $\phi\bar{x} = \sum_{j=1}^n p_j \phi x_j$. Using the fact that $\sum_{j=1}^n p_j = 1$, and applying the Jensen inequality to the norm convexity, we get

$$\Big\|\sum_{j=1}^n p_j f(\phi x_j) - f(\phi\bar{x})\Big\| = \Big\|\sum_{j=1}^n p_j\big(f(\phi x_j) - f(\phi\bar{x})\big)\Big\| \leq \sum_{j=1}^n p_j \big\|f(\phi x_j) - f(\phi\bar{x})\big\|. \tag{2.4}$$

Applying Theorem A to ϕx_j and $\phi\bar{x}$, and using $\|\phi x_j - \phi\bar{x}\| \leq \|\phi\|\|x_j - \bar{x}\|$, we get

$$\big\|f(\phi x_j) - f(\phi\bar{x})\big\| \leq \|\phi\|\|x_j - \bar{x}\| \int_0^1 f_a'\big(\big\|(1-t)\phi x_j + t\phi\bar{x}\big\|\big)\, dt. \tag{2.5}$$

Note the function $f_a'(z) = \sum_{i=1}^\infty i|\alpha_i|z^{i-1}$. The restriction of f_a' to the real interval $[0, R)$ is nondecreasing and convex. The norm is convex. Therefore, their composition is convex. Then it follows that

$$f_a'\big(\big\|(1-t)\phi x_j + t\phi\bar{x}\big\|\big) \leq (1-t) f_a'\big(\|\phi x_j\|\big) + t f_a'\big(\|\phi\bar{x}\|\big).$$

Integrating the above inequality by t over the interval $[0, 1]$, we obtain

$$\int_0^1 f_a'\big(\big\|(1-t)\phi x_j + t\phi\bar{x}\big\|\big)\, dt \leq \frac{1}{2}\big(f_a'\big(\|\phi x_j\|\big) + f_a'\big(\|\phi\bar{x}\|\big)\big). \tag{2.6}$$

Applying the Jensen inequality to the composition of f_a' and norm, we get

$$f_a'\big(\|\phi\bar{x}\|\big) \leq \sum_{j=1}^n p_j f_a'\big(\|\phi x_j\|\big) \leq \max_{1\leq j\leq n}\big\{f_a'\big(\|\phi x_j\|\big)\big\},$$

and consequently

$$\frac{1}{2}\big(f_a'\big(\|\phi x_j\|\big) + f_a'\big(\|\phi\bar{x}\|\big)\big) \leq \max_{1\leq j\leq n}\big\{f_a'\big(\|\phi x_j\|\big)\big\}. \tag{2.7}$$

Finally, combining the inequalities in (2.4)–(2.7), we obtain the multiple inequality in (2.3). $\square$

Remark 2.2. Let the assumptions of Theorem 2.1 be valid, and let $\|\phi x_j\| \leq M < R$ for any $x_j \in B$, $j = 1, \ldots, n$. Taking into account that f_a' is monotonic nondecreasing and then using (2.3), we obtain

$$\Big\|\sum_{j=1}^n p_j f(\phi x_j) - f\Big(\sum_{j=1}^n p_j \phi x_j\Big)\Big\| \leq \|\phi\| f_a'(M) \sum_{j=1}^n p_j \|x_j - \bar{x}\|.$$

Now, we define Jensen's map $\mathfrak{J} : \mathcal{F}(D(0, R)) \times \mathcal{B}(B) \times B^n \times \mathbb{R}_+^n \times [0, 1] \times \mathbb{N} \to B$ as

$$\mathfrak{J}(f, \phi, \mathbf{x}, \mathbf{p}, t, n) = \sum_{j=1}^n p_j f\Big((1-t)\phi x_j + t\sum_{k=1}^n p_k \phi x_k\Big), \tag{2.8}$$

where $f \in \mathcal{F}(D(0, R))$ is an analytic function, $\phi \in \mathcal{B}(B)$ is a bounded linear operator, $\mathbf{x} = (x_1, \ldots, x_n)$ is an n-tuple of elements $x_j \in B$, $\mathbf{p} = (p_1, \ldots, p_n)$ is an n-tuple of positive real numbers $p_j \in \mathbb{R}_+$ such that $\sum_{j=1}^n p_j = 1$, $t \in [0,1]$ and n is a natural number. In the following theorem, we show that the map (2.8) is a Lipschitzian map.

Theorem 2.3. *Let $\mathfrak{J}(f,\phi,\mathbf{x},\mathbf{p},t,n)$ be the map defined by (2.8) and let $\bar{x} = \sum_{j=1}^n p_j x_j$ be a convex combination of vectors x_j. If $\|\phi x_j\|, \|x_j\| < R$, $j = 1,\dots,n$, then*

$$\begin{aligned}
&\big\|\mathfrak{J}(f,\phi,\mathbf{x},\mathbf{p},t_2,n) - \mathfrak{J}(f,\phi,\mathbf{x},\mathbf{p},t_1,n)\big\| \\
&\quad \le |t_2 - t_1|\|\phi\| \sum_{j=1}^n p_j \|x_j - \bar{x}\| \\
&\qquad \times \int_0^1 f_a'\Big(\big\|\big(1 - t_1 + (t_1 - t_2)s\big)\phi x_j + \big(t_1 - (t_1 - t_2)s\big)\phi\bar{x}\big\|\Big)\, ds \\
&\quad \le \frac{1}{2}|t_2 - t_1|\|\phi\| \sum_{j=1}^n p_j \|x_j - \bar{x}\| \Big((2 - t_1 - t_2) f_a'\big(\|\phi x_j\|\big) + (t_1 + t_2) f_a'\big(\|\phi\bar{x}\|\big)\Big) \\
&\quad \le |t_2 - t_1|\|\phi\| \max_{1\le j\le n}\big\{f_a'\big(\|\phi x_j\|\big)\big\} \sum_{j=1}^n p_j \|x_j - \bar{x}\|.
\end{aligned} \tag{2.9}$$

Proof. We use the same technique as in the proof of Theorem 2.1. We give a sketch of the proof:

$$\begin{aligned}
&\big\|\mathfrak{J}(f,\phi,\mathbf{x},\mathbf{p},t_2,n) - \mathfrak{J}(f,\phi,\mathbf{x},\mathbf{p},t_1,n)\big\| \\
&\quad \le \sum_{j=1}^n p_j \big\| f\big((1 - t_1)\phi x_j + t_1\bar{x}\big) - f\big((1 - t_2)\phi x_j + t_2\bar{x}\big)\big\| \\
&\quad \le |t_2 - t_1|\|\phi\| \sum_{j=1}^n p_j \|x_j - \bar{x}\| \\
&\qquad \times \int_0^1 f_a'\Big(\big\|\big(1 - t_1 + (t_1 - t_2)s\big)\phi x_j + \big(t_1 - (t_1 - t_2)s\big)\phi\bar{x}\big\|\Big)\, ds \\
&\quad \le |t_2 - t_1|\|\phi\| \sum_{j=1}^n p_j \|x_j - \bar{x}\| \\
&\qquad \times \Big(f_a'\big(\|\phi x_j\|\big) \int_0^1 |1 - t_1 + (t_1 - t_2)s|\, ds \\
&\qquad + f_a'\big(\|\phi\bar{x}\|\big) \int_0^1 |t_1 - (t_1 - t_2)s|\, ds \Big) \\
&\quad = |t_2 - t_1|\|\phi\| \sum_{j=1}^n p_j \|x_j - \bar{x}\| \Big(\Big(1 - \frac{t_1 + t_2}{2}\Big) f_a'\big(\|\phi x_j\|\big) + \frac{t_1 + t_2}{2} f_a'\big(\|\phi\bar{x}\|\big)\Big) \\
&\quad \le |t_2 - t_1|\|\phi\| \max_{1\le j\le n}\big\{f_a'\big(\|\phi x_j\|\big)\big\} \sum_{j=1}^n p_j \|x_j - \bar{x}\|.
\end{aligned}$$

$\square$

Corollary 2.4. *Let the assumptions of Theorem 2.3 be valid. Then*

$$
\begin{aligned}
&\Big\|\mathfrak{J}(f,\phi,\mathbf{x},\mathbf{p},t,n)-f\Big(\sum_{k=1}^{n}p_k\phi x_k\Big)\Big\| \\
&\quad\le (1-t)\|\phi\|\sum_{j=1}^{n}p_j\|x_j-\bar{x}\| \\
&\qquad\times\int_0^1 f_a'\big(\big\|(1-t)s\phi x_j+\big(1+(t-1)s\big)\phi\bar{x}\big\|\big)\,ds \\
&\quad\le \frac{1-t}{2}\|\phi\|\sum_{j=1}^{n}p_j\|x_j-\bar{x}\|\big((1-t)f_a'\big(\|\phi x_j\|\big)+(1+t)f_a'\big(\|\phi\bar{x}\|\big)\big) \\
&\quad\le (1-t)\|\phi\|\max_{1\le j\le n}\big\{f_a'\big(\|\phi x_j\|\big)\big\}\sum_{j=1}^{n}p_j\|x_j-\bar{x}\|,
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
&\Big\|\sum_{j=1}^{n}p_jf(\phi x_j)-\mathfrak{J}(f,\phi,\mathbf{x},\mathbf{p},t,n)\Big\| \\
&\quad\le t\|\phi\|\sum_{j=1}^{n}p_j\|x_j-\bar{x}\|\int_0^1 f_a'\big(\big\|\big(1-(1-s)t\big)\phi x_j+(1-s)t\phi\bar{x}\big\|\big)\,ds \\
&\quad\le \frac{t}{2}\|\phi\|\sum_{j=1}^{n}p_j\|x_j-\bar{x}\|\big((2-t)f_a'\big(\|\phi x_j\|\big)+tf_a'\big(\|\phi\bar{x}\|\big)\big) \\
&\quad\le t\|\phi\|\max_{1\le j\le n}\big\{f_a'\big(\|\phi x_j\|\big)\big\}\sum_{j=1}^{n}p_j\|x_j-\bar{x}\|,
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
&\Big\|\mathfrak{J}(f,\phi,\mathbf{x},\mathbf{p},t,n)-(1-t)\sum_{j=1}^{n}p_jf(\phi x_j)-tf\Big(\sum_{k=1}^{n}p_k\phi x_k\Big)\Big\| \\
&\le t(1-t)\|\phi\|\sum_{j=1}^{n}p_j\|x_j-\bar{x}\|\int_0^1\big(f_a'\big(\big\|(1-t)s\phi x_j+\big(1+(t-1)s\big)\phi\bar{x}\big\|\big) \\
&\quad+f_a'\big(\big\|\big(1-(1-s)t\big)\phi x_j+(1-s)t\phi\bar{x}\big\|\big)\big)\,ds \\
&\le \frac{t(1-t)}{2}\|\phi\|\sum_{j=1}^{n}p_j\|x_j-\bar{x}\|\big((3-2t)f_a'\big(\|x_j\|\big)+(1+2t)f_a'\big(\|\bar{x}\|\big)\big) \\
&\le 2t(1-t)\|\phi\|\max_{1\le j\le n}\big\{f_a'\big(\|x_j\|\big)\big\}\sum_{j=1}^{n}p_j\|x_j-\bar{x}\|.
\end{aligned}
\tag{2.12}
$$

Proof. The inequalities (2.10) and (2.11) follow from (2.9) by choosing $t_1=1$, $t_2=t$, and $t_1=t$, $t_2=0$, respectively. We obtain (2.12) using the triangle inequality and adding (2.10) multiplied by t and (2.11) multiplied by $1-t$. $\square$

Remark 2.5. Let the assumptions of Theorem 2.3 be valid. If $\|\phi x_j\| \le M < R$ for any $x_j \in B$, $j = 1, \dots, n$, then Theorem 2.3 gives

$$\left\|\mathfrak{J}(f,\phi,\mathbf{x},\mathbf{p},t_2,n) - \mathfrak{J}(f,\phi,\mathbf{x},\mathbf{p},t_1,n)\right\| \le |t_2 - t_1| \|\phi\| f_a'(M) \sum_{j=1}^{n} p_j \|x_j - \bar{x}\|.$$

Since the operator valued integral $\int_0^1 f((1-t)\phi x + t\phi y)\,dt$ exists for an analytic function $f \in \mathcal{F}(D(0,R))$, an operator $\phi \in \mathcal{B}(B)$, and $x, y \in B$, we can observe a map $\mathfrak{J}_1(f,\phi,\mathbf{x},\mathbf{p},n) = \int_0^1 \mathfrak{J}(f,\phi,\mathbf{x},\mathbf{p},t,n)\,dt$, where $\mathfrak{J}(f,\phi,\mathbf{x},\mathbf{p},t,n)$ is Jensen's map (2.8); that is,

$$\mathfrak{J}_1(f,\phi,\mathbf{x},\mathbf{p},n) = \sum_{j=1}^{n} p_j \int_0^1 f\Big((1-t)\phi x_j + t\sum_{k=1}^{n} p_k \phi x_k\Big)\,dt.$$

By using (2.10) and (2.11), we obtain the following theorem. We omit the proof.

Theorem 2.6. *Let the assumptions of Theorem 2.3 be valid. Then*

$$\begin{aligned}
&\Big\|\mathfrak{J}_1(f,\phi,\mathbf{x},\mathbf{p},n) - f\Big(\sum_{k=1}^{n} p_k\phi x_k\Big)\Big\| \\
&\quad\le \|\phi\| \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \Big(\frac{1}{6} f_a'\big(\|\phi x_j\|\big) + \frac{1}{3} f_a'\big(\|\phi\bar{x}\|\big)\Big) \\
&\quad\le \frac{1}{2}\|\phi\| \max_{1\le j\le n}\big\{f_a'\big(\|\phi x_j\|\big)\big\} \sum_{j=1}^{n} p_j\|x_j - \bar{x}\|.
\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\sum_{j=1}^{n} p_j f(\phi x_j) - \mathfrak{J}_1(f,\phi,\mathbf{x},\mathbf{p},n)\Big\| \\
&\quad\le \|\phi\| \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \Big(\frac{1}{3} f_a'\big(\|\phi x_j\|\big) + \frac{1}{6} f_a'\big(\|\phi\bar{x}\|\big)\Big) \\
&\quad\le \frac{1}{2}\|\phi\| \max_{1\le j\le n}\big\{f_a'\big(\|\phi x_j\|\big)\big\} \sum_{j=1}^{n} p_j\|x_j - \bar{x}\|.
\end{aligned}$$

Next, we define Jensen's map $\bar{\mathfrak{J}} : \mathcal{F}(D(0,R)) \times \mathcal{B}(B) \times B^n \times \mathbb{R}_+^n \times [0,1] \times \mathbb{N} \to B$ as

$$\bar{\mathfrak{J}}(f,\phi,\mathbf{x},\mathbf{p},t,n) = (1-t)\sum_{j=1}^{n} p_j f(\phi x_j) + t f\Big(\sum_{k=1}^{n} p_k\phi x_k\Big), \tag{2.13}$$

$f \in \mathcal{F}(D(0,R))$ is an analytic function, $\phi \in \mathcal{B}(B)$ be a bounded linear operator, $\mathbf{x} = (x_1, \dots, x_n)$ is an n-tuple of elements $x_j \in B$, $\mathbf{p} = (p_1, \dots, p_n)$ is an n-tuple of positive real numbers $p_j \in \mathbb{R}_+$ such that $\sum_{j=1}^{n} p_j = 1$, $t \in [0,1]$, and n is a natural number. We can obtain results similar to the above for the map (2.13) and its integral version; for example, we have the following theorem, which is similar to Theorem 2.3.

Theorem 2.7. *Let $\bar{\mathfrak{J}}(f,\phi,\mathbf{x},\mathbf{p},t,n)$ be the map defined by (2.13), and let $\bar{x}=\sum_{j=1}^n p_jx_j$ be a convex combination of vectors x_j. If $\|\phi x_j\|, \|x_j\|<R$, $j=1,\dots,n$, then*

$$\begin{aligned}
&\big\|\bar{\mathfrak{J}}(f,\phi,\mathbf{x},\mathbf{p},t_2,n)-\bar{\mathfrak{J}}(f,\phi,\mathbf{x},\mathbf{p},t_1,n)\big\| \\
&\quad\le |t_2-t_1|\|\phi\|\sum_{j=1}^n p_j\|x_j-\bar{x}\| \\
&\qquad\times\int_0^1 f_a'\Big(\big\|\big(1-t_1+(t_1-t_2)s\big)\phi x_j+\big(t_1-(t_1-t_2)s\big)\phi\bar{x}\big\|\Big)\,ds \\
&\quad\le \frac{1}{2}|t_2-t_1|\|\phi\|\sum_{j=1}^n p_j\|x_j-\bar{x}\|\Big((2-t_1-t_2)f_a'\big(\|\phi x_j\|\big)+(t_1+t_2)f_a'\big(\|\phi\bar{x}\|\big)\Big) \\
&\quad\le |t_2-t_1|\|\phi\|\max_{1\le j\le n}\big\{f_a'\big(\|\phi x_j\|\big)\big\}\sum_{j=1}^n p_j\|x_j-\bar{x}\|.
\end{aligned}$$

The proof is similar to the one for Theorem 2.3. We omit the details.

Finally, we define Levinson's map $\mathfrak{L}_{\phi,\psi}(f,\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q},t,n)$ as the difference between the corresponding Jensen maps (2.8):

$$\mathfrak{L}_{\phi,\psi}(f,\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q},t,n)=\mathfrak{J}(f,\psi,\mathbf{y},\mathbf{q},t,n)-\mathfrak{J}(f,\phi,\mathbf{x},\mathbf{p},t,n),$$

or explicitly

$$\begin{aligned}
&\mathfrak{L}_{\phi,\psi}(f,\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q},t,n) \\
&\quad=\sum_{j=1}^n q_jf\Big((1-t)\psi y_j+t\sum_{k=1}^n q_k\psi y_k\Big) \\
&\qquad-\sum_{j=1}^n p_jf\Big((1-t)\phi x_j+t\sum_{k=1}^n p_k\phi x_k\Big),
\end{aligned}\tag{2.14}$$

where $f\in\mathcal{F}(D(0,R))$ is an analytic function, $\phi,\psi\in\mathcal{B}(B)$ are bounded linear operators, $\mathbf{x}=(x_1,\dots,x_n)$ and $\mathbf{y}=(y_1,\dots,y_n)$ are n-tuples of elements $x_j,y_j\in B$, $\mathbf{p}=(p_1,\dots,p_n)$ and $\mathbf{q}=(q_1,\dots,q_n)$ are n-tuples of positive real numbers p_j,q_j such that $\sum_{j=1}^n p_j=\sum_{j=1}^n q_j=1$, $t\in[0,1]$, and n is a natural number. By using Theorem 2.3 and the triangle inequality, we can show that (2.14) is a Lipschitzian map, as follows.

Theorem 2.8. *Let $\mathfrak{L}_{\phi,\psi}(f,\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q},t,n)$ be the map defined by (2.14), and let $\bar{x}=\sum_{j=1}^n p_jx_j$, $\bar{y}=\sum_{j=1}^n q_jy_j$. If $\|\phi x_j\|,\|x_j\|$, $\|\psi y_j\|,\|y_j\|<R$, $j=1,\dots,n$, then*

$$\begin{aligned}
&\big\|\mathfrak{L}_{\phi,\psi}(f,\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q},t_2,n)-\mathfrak{L}_{\phi,\psi}(f,\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q},t_1,n)\big\| \\
&\quad\le|t_2-t_1|C_1\cdot C_2\Big(\sum_{j=1}^n q_j\|y_j-\bar{y}\|+\sum_{j=1}^n p_j\|x_j-\bar{x}\|\Big),
\end{aligned}$$

where $C_1=\max\{\|\phi\|,\|\psi\|\}$ and $C_2=\max_{1\le j\le n}\{f_a'(\|\psi y_j\|),f_a'(\|\phi x_j\|)\}$.

Remark 2.9. Of course, we can obtain similar results for other versions of Levinson maps as the difference between the corresponding Jensen maps (2.13) or (3.3). We omit the details.

Now, we consider some simple examples.

Example 2.10. (1) If we consider the exponential function $\exp(z) = \sum_{i=0}^{\infty} \frac{1}{i!} z^i$, $z \in \mathbb{C}$, then we obtain special versions of the above inequalities. For example, by putting $t_1 = 1$ and $t_2 = 0$ in Theorem 2.8, we obtain

$$\Big\| \sum_{j=1}^{n} q_j \mathrm{e}^{\psi y_j} - \sum_{j=1}^{n} p_j \mathrm{e}^{\varphi x_j} - \mathrm{e}^{\sum_{j=1}^{n} q_j \phi y_j} + \mathrm{e}^{\sum_{j=1}^{n} p_j \phi x_j} \Big\| \\ \leq \mathrm{e}^{M} \max\big\{ \|\phi\|, \|\psi\| \big\} \Big(\sum_{j=1}^{n} q_j \|y_j - \bar{y}\| + \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \Big)$$

for all $x_j, y_j \in B$ such that $\|\phi x_j\|, \|\psi y_j\| \leq M < \infty$, $j = 1, \ldots, n$.

(2) If we consider the functions $\frac{1}{1-z} = \sum_{i=0}^{\infty} z^i$, $z \in D(0,1)$, then

$$\Big\| \sum_{j=1}^{n} q_j (1 + \psi y_j)^{-1} - \sum_{j=1}^{n} p_j (1 + \varphi x_j)^{-1} \\ - \Big(1 + \sum_{j=1}^{n} q_j \phi y_j \Big)^{-1} + \Big(1 + \sum_{j=1}^{n} p_j \phi x_j \Big)^{-1} \Big\| \\ \leq \frac{1}{(M-1)^2} \max\big\{ \|\phi\|^2, \|\psi\|^2 \big\} \Big(\sum_{j=1}^{n} q_j \|y_j - \bar{y}\| + \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \Big)$$

for all $x_j, y_j \in B$ such that $\|x_j\|, \|y_j\| < 1$ and $\|\phi x_j\|$, $\|\psi y_j\| \leq M < 1$, $j = 1, \ldots, n$.

3. Norm inequalities

In this section we observe applications for functions of norms in Banach spaces and functions defined by power series in Banach algebra. Furthermore, let $(X, \|\cdot\|)$ be a Banach space. Similar to (2.1), we have that

$$\big| f\big(\|y\|\big) - f\big(\|x\|\big) \big| \leq \|y - x\| \int_0^1 f_a'\big(\big\|(1-t)x + ty\big\| \big)\, dt \tag{3.1}$$

holds for any $f \in \mathcal{F}(D(0,R))$ and $x, y \in X$ with $\|x\|, \|y\| < R$ (see also [6, Theorem 3]).

First, we give a Lipschitz type of the Jensen inequality for a function defined by a power series.

Theorem 3.1. *Let $f \in \mathcal{F}(D(0,R))$ be an analytic function, let $\bar{x} = \sum_{j=1}^{n} p_j x_j$ be a convex combination of vectors $x_j \in X$ such that $\|x_j\| < R$, and let $\phi \in \mathcal{B}(X, Y)$*

be a bounded linear operator such that $\|\phi x_j\| < R$. Then we have the inequality

$$\begin{aligned}
&\Big|\sum_{j=1}^{n} p_j f\left(\|\phi x_j\|\right) - f\Big(\Big\|\sum_{j=1}^{n} p_j \phi x_j\Big\|\Big)\Big| \\
&\quad\le \|\phi\| \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \int_0^1 f_a'\big(\big\|(1-t)\phi x_j + t\phi\bar{x}\big\|\big)\,dt \\
&\quad\le \frac{1}{2}\|\phi\| \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \big(f_a'\left(\|\phi x_j\|\right) + f_a'\left(\|\phi \bar{x}\|\right)\big) \\
&\quad\le \|\phi\| \max_{1\le j\le n}\big\{f_a'\left(\|\phi x_j\|\right)\big\} \sum_{j=1}^{n} p_j \|x_j - \bar{x}\|,
\end{aligned} \tag{3.2}$$

where $\bar{x} = \sum_{j=1}^{n} p_j x_j$.

Proof. Similarly to the proof of Theorem 2.1, by using linearity of ϕ and (3.1), we obtain the first inequality in (3.2). The second and the third inequalities in (3.2) follow from Theorem 2.1. □

Now, we can consider similar types of maps as in Section 2.2. We observe one of them as follows. We define the Jensen-type map $J : \mathcal{F}(D(0,R)) \times \mathcal{B}(X,Y) \times B^n \times \mathbb{R}_+^n \times [0,1] \times \mathbb{N} \to \mathbb{R}$ as

$$J(f,\phi,\mathbf{x},\mathbf{p},t,n) = \sum_{j=1}^{n} p_j f\Big(\Big\|t\phi x_j + (1-t)\sum_{k=1}^{n} p_k \phi x_k\Big\|\Big), \tag{3.3}$$

where $f \in \mathcal{F}(D(0,R))$ is an analytic function, $\phi \in \mathcal{B}(X,Y)$ is a bounded linear operator, $\mathbf{x} = (x_1,\dots,x_n)$ is an n-tuple of elements $x_j \in X$, $\mathbf{p} = (p_1,\dots,p_n)$ is an n-tuple of positive real numbers $p_j \in \mathbb{R}_+$ such that $\sum_{j=1}^{n} p_j = 1$, $t \in [0,1]$, and n is a natural number.

In the following theorem we show that (3.3) is a Lipschitzian map.

Theorem 3.2. *Let $J(f,\phi,\mathbf{x},\mathbf{p},t,n)$ be the map defined by (3.3), and let $\bar{x} = \sum_{j=1}^{n} p_j x_j$ be a convex combination of vectors x_j. If $\|\phi x_j\|, \|x_j\| < R$, $j = 1,\dots,n$, then*

$$\begin{aligned}
&\big|J(f,\phi,\mathbf{x},\mathbf{p},t_2,n) - J(f,\phi,\mathbf{x},\mathbf{p},t_1,n)\big| \\
&\quad\le |t_2 - t_1|\|\phi\| \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \\
&\qquad\times \int_0^1 f_a'\big(\big\|\big(1 - t_1 + (t_1 - t_2)s\big)\phi x_j + \big(t_1 - (t_1 - t_2)s\big)\phi\bar{x}\big\|\big)\,ds \\
&\quad\le \frac{1}{2}|t_2 - t_1|\|\phi\| \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \big((2 - t_1 - t_2) f_a'\left(\|\phi x_j\|\right) + (t_1 + t_2) f_a'\left(\|\phi\bar{x}\|\right)\big) \\
&\quad\le |t_2 - t_1|\|\phi\| \max_{1\le j\le n}\big\{f_a'\left(\|\phi x_j\|\right)\big\} \sum_{j=1}^{n} p_j \|x_j - \bar{x}\|.
\end{aligned}$$

Proof. We use the same technique as in the proof of Theorem 2.3. We omit the details. $\square$

By putting $t_1 = 1$, $t_2 = t$, or $t_1 = t$, $t_2 = 0$ in Theorem 3.2, we obtain the following corollary.

Corollary 3.3. *Let the assumptions of Theorem 3.2 be valid. Then*

$$\begin{aligned}
&\Big|J(f,\phi,\mathbf{x},\mathbf{p},t,n) - f\Big(\Big\|\sum_{k=1}^{n} p_k \phi x_k\Big\|\Big)\Big| \\
&\quad \le (1-t)\|\phi\| \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \int_0^1 f_a'\big(\big\|(1-t)s\phi x_j + \big(1+(t-1)s\big)\phi\bar{x}\big\|\big)\, ds \\
&\quad \le \frac{1-t}{2}\|\phi\| \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \big((1-t) f_a'\big(\|\phi x_j\|\big) + (1+t) f_a'\big(\|\phi\bar{x}\|\big)\big) \\
&\quad \le (1-t)\|\phi\| \max_{1\le j\le n}\big\{f_a'\big(\|\phi x_j\|\big)\big\} \sum_{j=1}^{n} p_j \|x_j - \bar{x}\|, \\
&\Big|\sum_{j=1}^{n} p_j f\big(\|\phi x_j\|\big) - J(f,\phi,\mathbf{x},\mathbf{p},t,n)\Big| \\
&\quad \le t\|\phi\| \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \int_0^1 f_a'\big(\big\|\big(1-(1-s)t\big)\phi x_j + (1-s)t\phi\bar{x}\big\|\big)\, ds \\
&\quad \le \frac{t}{2}\|\phi\| \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \big((2-t) f_a'\big(\|\phi x_j\|\big) + t f_a'\big(\|\phi\bar{x}\|\big)\big) \\
&\quad \le t\|\phi\| \max_{1\le j\le n}\big\{f_a'\big(\|\phi x_j\|\big)\big\} \sum_{j=1}^{n} p_j \|x_j - \bar{x}\|,
\end{aligned}$$

and

$$\begin{aligned}
&\Big|J(f,\phi,\mathbf{x},\mathbf{p},t,n) - (1-t)\sum_{j=1}^{n} p_j f\big(\|\phi x_j\|\big) - t f\Big(\Big\|\sum_{k=1}^{n} p_k \phi x_k\Big\|\Big)\Big| \\
&\quad \le t(1-t)\|\phi\| \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \int_0^1 \Big(f_a'\big(\big\|(1-t)s\phi x_j + \big(1+(t-1)s\big)\phi\bar{x}\big\|\big) \\
&\qquad + f_a'\big(\big\|\big(1-(1-s)t\big)\phi x_j + (1-s)t\phi\bar{x}\big\|\big)\Big)\, ds \\
&\quad \le \frac{t(1-t)}{2}\|\phi\| \sum_{j=1}^{n} p_j \|x_j - \bar{x}\| \big((3-2t) f_a'\big(\|x_j\|\big) + (1+2t) f_a'\big(\|\bar{x}\|\big)\big) \\
&\quad \le 2t(1-t)\|\phi\| \max_{1\le j\le n}\big\{f_a'\big(\|x_j\|\big)\big\} \sum_{j=1}^{n} p_j \|x_j - \bar{x}\|.
\end{aligned}$$

Remark 3.4. Let the assumptions of Theorem 3.2 be valid. If $\|\phi x_j\| \leq M < R$ for any $x_j \in X$, $j = 1, \ldots, n$, then

$$\Big|J(f,\phi,\mathbf{x},\mathbf{p},t_2,n) - J(f,\phi,\mathbf{x},\mathbf{p},t_1,n)\Big| \leq |t_2 - t_1|\|\phi\| f'_a(M) \sum_{j=1}^{n} p_j \|x_j - \bar{x}\|.$$

Now, we observe the Levinson-type map $L_{\phi,\psi}(f,\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q},t,n)$ as the difference between the corresponding Jensen-type maps (3.3); that is,

$$L_{\phi,\psi}(f,\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q},t,n) = J(f,\phi,\mathbf{y},\mathbf{q},t,n) - J(f,\phi,\mathbf{x},\mathbf{p},t,n), \tag{3.4}$$

or explicitly

$$\begin{aligned} &L_{\phi,\psi}(f,\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q},t,n) \\ &\quad = \sum_{j=1}^{n} q_j f\Big(\Big\|(1-t)\psi y_j + t\sum_{k=1}^{n} q_k \psi y_k\Big\|\Big) \\ &\quad\quad - \sum_{j=1}^{n} p_j f\Big(\Big\|(1-t)\phi x_j + t\sum_{k=1}^{n} p_k \phi x_k\Big\|\Big), \end{aligned} \tag{3.5}$$

where $f \in \mathcal{F}(D(0,R))$ is an analytic function, $\phi, \psi \in \mathcal{B}(X,Y)$ are bounded linear operators, $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{y} = (y_1,\ldots,y_n)$ are n-tuples of elements $x_j, y_j \in X$, $\mathbf{p} = (p_1,\ldots,p_n)$ and $\mathbf{q} = (q_1,\ldots,q_n)$ are n-tuples of positive real numbers p_j, q_j such that $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} q_j = 1$, $t \in [0,1]$, and n is a natural number.

By using Theorem 3.2 and the triangle inequality, we can show that (3.5) is a Lipschitzian map, as follows.

Theorem 3.5. *Let $L_{\phi,\psi}(f,\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q},t,n)$ be the map defined by (3.4), and let $\bar{x} = \sum_{j=1}^{n} p_j x_j$, $\bar{y} = \sum_{j=1}^{n} q_j y_j$. If $\|\phi x_j\|, \|x_j\|$, $\|\psi y_j\|, \|y_j\| < R$, $j = 1,\ldots,n$, then*

$$\begin{aligned} &\Big|L_{\phi,\psi}(f,\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q},t_2,n) - L_{\phi,\psi}(f,\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q},t_1,n)\Big| \\ &\quad \leq |t_2 - t_1| C_1 \cdot C_2 \Big(\sum_{j=1}^{n} q_j \|y_j - \bar{y}\| + \sum_{j=1}^{n} p_j \|x_j - \bar{x}\|\Big), \end{aligned} \tag{3.6}$$

where $C_1 = \max\{\|\phi\|, \|\psi\|\}$ and $C_2 = \max_{1\leq j\leq n}\{f'_a(\|\psi y_j\|), f'_a(\|\phi x_j\|)\}$.

Applying Theorem 3.5, we obtain the following corollary.

Corollary 3.6. *Let the assumptions of Theorem 3.5 be valid. Then*

$$\begin{aligned} &\Big|L_{\phi,\psi}(f,\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q},t,n) - f\Big(\Big\|\sum_{j=1}^{n} q_j \phi y_j\Big\|\Big) + f\Big(\Big\|\sum_{j=1}^{n} p_j \phi x_j\Big\|\Big)\Big| \\ &\quad \leq (1-t) C_1 \cdot C_2 \Big(\sum_{j=1}^{n} q_j \|y_j - \bar{y}\| + \sum_{j=1}^{n} p_j \|x_j - \bar{x}\|\Big), \end{aligned} \tag{3.7}$$

and

$$\Big|\sum_{j=1}^{n} q_j f\big(\|\phi y_j\|\big) - \sum_{j=1}^{n} p_j f\big(\|\phi x_j\|\big) - L_{\phi,\psi}(f,\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q},t,n)\Big|$$
$$\leq tC_1\cdot C_2\Big(\sum_{j=1}^{n} q_j\|y_j-\bar{y}\| + \sum_{j=1}^{n} p_j\|x_j-\bar{x}\|\Big). \tag{3.8}$$

Remark 3.7. Let the assumptions of Theorem 3.5 be valid. If $\|\phi x_j\|,\|\psi y_j\| \leq M < R$ for any $x_j, y_j \in B$, $j=1,\dots,n$, then (3.6)–(3.8) hold with $C_2 = f_a'(M)$.

Now, it is natural to consider some simple examples.

Example 3.8. (1) If we consider the logarithmic function $\log(1+z) = \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}\times z^i$, $z\in D(0,1)$, then we can obtain special versions of the above inequalities. So, putting $t_1 = 1$ and $t_2 = 0$ in Theorem 3.5, we obtain

$$\Big|\sum_{j=1}^{n} q_j \log\big(\|\psi(1+y_j)\|\big) - \sum_{j=1}^{n} p_j \log\big(\|\varphi(1+x_j)\|\big)$$
$$- \log\Big(\Big\|\sum_{j=1}^{n} q_j\phi y_j\Big\|\Big) + \log\Big(\Big\|\sum_{j=1}^{n} p_j\phi x_j\Big\|\Big)\Big|$$
$$\leq \frac{1}{1-M}\max\{\|\phi\|,\|\psi\|\}\Big(\sum_{j=1}^{n} q_j\|y_j-\bar{y}\| + \sum_{j=1}^{n} p_j\|x_j-\bar{x}\|\Big)$$

for any $x_j, y_j \in X$, $\phi,\psi\in\mathcal{B}(X,Y)$, such that $\|x_j\|,\|y_j\| < 1$ and $\|\phi x_j\|,\|\psi y_j\| \leq M < 1$, $j=1,\dots,n$.

(2) If we consider the function $\cos z = \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i)!}z^{2i}$, $z\in\mathbb{C}$, then $t_1 = 1$ and $t_2 = 0$ in Theorem 3.5, and we obtain

$$\Big|\sum_{j=1}^{n} q_j \cos\big(\|\psi y_j\|\big) - \sum_{j=1}^{n} p_j \cos\big(\|\varphi x_j\|\big) - \cos\Big(\Big\|\sum_{j=1}^{n} q_j\phi y_j\Big\|\Big) + \cos\Big(\Big\|\sum_{j=1}^{n} p_j\phi x_j\Big\|\Big)\Big|$$
$$\leq \sinh(M)\max\{\|\phi\|,\|\psi\|\}\Big(\sum_{j=1}^{n} q_j\|y_j-\bar{y}\| + \sum_{j=1}^{n} p_j\|x_j-\bar{x}\|\Big)$$

for any $x_j, y_j \in X$, $\phi,\psi\in\mathcal{B}(X,Y)$ such that $\|\phi x_j\|,\|\psi y_j\| \leq M < \infty$, $j = 1,\dots,n$.

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References

1. H. Araki and S. Yamagami, *An inequality for Hilbert-Schmidt norm*, Comm. Math. Phys. **81** (1981), no. 1, 89–96. Zbl 0468.47013. MR0630332. DOI 10.1007/BF01941801. 385

2. R. Bhatia, *Perturbation bounds for the operator absolute value*, Linear Algebra Appl. **226/228** (1995), 639–645. Zbl 0835.47009. MR1344590. DOI 10.1016/0024-3795(95)00201-2. 384, 385
3. S. S. Chang, Y. Q. Chen, and B. S. Lee, *On the semi-inner products in locally convex spaces*, Int. J. Math. Math. Sci. **20** (1997), no. 2, 219–224. Zbl 0899.46015. MR1444719. 385
4. Y. J. Cho, M. Matić, and J. Pečarić, *Inequalities of Jensen's type for Lipschitzian mappings*, Comm. Appl. Nonlinear Anal. **8** (2001), no. 3, 37–46. Zbl 0998.26015. MR1856070. 385
5. S. S. Dragomir, *Inequalities of Lipschitz type for power series in Banach algebras*, Ann. Math. Sil. **29** (2015), 61–83. Zbl 06751979. MR3495989. 385, 386
6. S. S. Dragomir, *Integral inequalities for Lipschitzian mappings between two Banach spaces and applications*, Kodai Math. J. **39** (2016), no. 1, 227–251. Zbl 1354.46018. MR3478280. 385, 386, 392
7. S. S. Dragomir and J. J. Koliha, *Two mappings related to semi-inner products and their applications in geometry of normed linear spaces*, Appl. Math. **45** (2000), no. 5, 337–355. Zbl 0996.46007. MR1777019. 385
8. J. L. W. V. Jensen, *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Math. **30** (1906), no. 1, 175–193. JFM 37.0422.02. MR1555027. 386
9. J. Mićić, J. Pečarić, and M. Praljak, *Levinson's inequality for Hilbert space operators*, J. Math. Inequal. **9** (2015), no. 4, 1271–1285. Zbl 1357.47018. MR3360169. 385
10. J. Pečarić, T. Furuta, J. Mićić Hot, and Y. Seo, *Mond–Pečarić Method in Operator Inequalities*, Monogr. Inequal. **1**, Element, Zagreb, 2005. Zbl 1135.47012. MR3026316. 385

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