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SURJECTIVE ISOMETRIES ON VECTOR-VALUED DIFFERENTIABLE FUNCTION SPACES

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Abstract. In this article, we investigate the surjective linear isometries between the differentiable function spaces $C_0^p(X,E)$ and $C_0^q(Y,F)$, where X, Y are open subsets of $\mathbb{R}$ and E, F are strictly convex Banach spaces with dimension greater than 1. We show that such isometries can be written as weighted composition operators.

1. Introduction

The classical Banach–Stone theorem gives the first characterization of the surjective linear isometries between spaces of scalar-valued continuous functions. Several researchers have derived many extensions of this theorem and applied them to a variety of different settings (for a survey of this topic, we refer the reader to [5]). Cambern and Pathak (see [3], [4]) considered the surjective linear isometries on the spaces of scalar-valued differentiable functions on the locally compact subsets of $\mathbb{R}$, and gave the representation for such isometries. Pathak [8] and Koshimizu [6] considered isometries on the space $C^n[0,1]$ and obtained their representations. Then Botelho and Jamison [2] extended these results to vector-valued continuously differentiable function spaces $C^1([0,1],H)$, where H is a finite-dimensional Hilbert space. Moreover, Wang [10] worked on the scalar-valued differentiable function spaces $C_0^n(X)$ with open subset $X\subset\mathbb{R}^n$. Recently, the first author and Wang [7] investigated the surjective isometries on the space

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$C_0^p(X, E)$ whenever X is an open subset of Euclidean space and E is a reflexive strictly convex space. From the above-mentioned literature, the characterization of the extreme points of the dual unit ball of the differentiable function spaces played a crucial role in our proofs. On the other hand, Wang [9] used different tools to investigate the surjective isometries between the unit spheres of $C_0^n(X)$ whenever X is a locally compact subset of $\mathbb{R}$ without isolated points.

In this article, we use the "parallel relation" (see Definition 2.1) of elements in Banach spaces to investigate the surjective isometries on the spaces of vector-valued continuously differentiable functions on an open subset of $\mathbb{R}$. We show that such isometries can be written in the form of canonical weighted composition operators (see Theorem 2.13). This result extends the main results of [2]–[4], [8], and [11] and gives a smooth version of the Banach–Stone theorem.

2. Main results

Throughout this paper, we assume that $p, q \in \mathbb{N}$, X, Y are open subsets of $\mathbb{R}$ and that E, F are Banach spaces with strictly convex norm. Let ρ be an ℓ^1-norm on $\mathbb{R}^{p+1}$. We use $C_0^p(X, E)$ to denote the Banach space consisting of all E-valued functions which have up to pth-times continuous derivatives on X and vanish at infinity; that is, the set

$$\{x \in X : \rho(\|f(x)\|, \dots, \|f^{(p)}(x)\|) \geq \varepsilon\}$$

is compact in X for any $\varepsilon > 0$, with the norm

$$\|f\| = \max_{x \in X} (\rho f)(x) = \max_{x \in X} \rho(\|f(x)\|, \dots, \|f^{(p)}(x)\|) \quad \text{for all } f \in C_0^p(X, E).$$

Similarly, let σ be an ℓ^1-norm on $\mathbb{R}^{q+1}$, and define

$$\|g\| = \max_{y \in Y} (\sigma g)(y) = \max_{y \in Y} \sigma(\|g(y)\|, \dots, \|g^{(q)}(y)\|) \quad \text{for all } g \in C_0^q(Y, F).$$

Normalize the norms ρ and σ by assuming that

$$\rho(0, \dots, 0, 1) = \sigma(0, \dots, 0, 1) = 1.$$

A function $f \in C_0^p(X, E)$ is said to *peak* at $x_0 \in X$ if it attains its norm at x_0 and nowhere else.

Definition 2.1. Suppose that E is a Banach space and that $u, v \in E$. We say that $u \mid v$ if there exists $\alpha \geq 0$ such that $u = \alpha v$ or $v = \alpha u$. For $\lambda = (\lambda_i)_{1 \leq i \leq n} \in E^n$ and $\beta = (\beta_i)_{1 \leq i \leq n} \in E^n$, we write $\lambda \parallel \beta$ if $\lambda_i \mid \beta_i$ for each $1 \leq i \leq n$.

From now on, we consider the linear surjective isometry T from $C_0^p(X, E)$ onto $C_0^q(Y, F)$. For $0 < \delta < 1$, there exists $h_\delta \in C^p(\mathbb{R})$ determined by the conditions $h_\delta^{(i)}(x_0) = 0$ for any $0 \leq i < p$ and

$$h_\delta^{(p)}(x) = \Big(1 - \frac{|x - x_0|}{\delta}\Big)^+.$$

Then one can derive that

$$h_\delta^{(i)}(x) = \int_{x_0}^{x} \frac{(x - t)^{p-1-i}}{(p - 1 - i)!} h_\delta^{(p)}(t)\, dt,$$

which implies that

$$\begin{aligned}\left|h_\delta^{(i)}(x)\right| &\leq \int_{\min\{x,x_0\}}^{\max\{x,x_0\}} \frac{|I|^{p-1-i}}{(p-1-i)!} h_\delta^{(p)}(t)\,dt \\ &\leq \frac{\delta |I|^{p-i-1}}{(p-i-1)!}, \quad \forall 0 \leq i \leq p-1, x \in \mathbb{R}. \end{aligned} \tag{2.1}$$

Lemma 2.2. *Suppose that $x_0 \in I \subset X$, where I is an open set in X, and let λ_0 be an element in E^{p+1} whose final coordinate is nonzero. There exist an open neighborhood U of λ_0 in E^{p+1} and a continuous map $\lambda \mapsto H_\lambda$ from U into $C_0^p(X, E)$ such that* $\operatorname{supp} H_\lambda \subset I$*, H_λ peaks at x_0 and $(H_\lambda^{(i)}(x_0))_{0\leq i\leq p} \parallel \lambda$. Furthermore, for $\lambda = (a_0, \dots, a_p)$ and for each $i = 0, 1, \dots, p$, if $a_i \neq 0$, then $H_\lambda^{(i)}(x_0) \neq 0$.*

Proof. We may assume that I is a bounded interval. Let φ be a function in $C_0^p(\mathbb{R})$ such that $\operatorname{supp}(\varphi) \subset I$ and such that $\varphi = 1$ on a neighborhood of x_0. Let U be a bounded open neighborhood of λ_0 so that there exists $\varepsilon > 0$ such that $\|a_p\| > \varepsilon$ for any $\lambda = (a_0, \dots, a_p) \in U$.

For each $\lambda = (a_0, \dots, a_p) \in U$, define

$$f_\lambda(x) = \sum_{k=0}^{p-1} \frac{a_k}{k!}(x - x_0)^k$$

and set

$$H_\lambda = (\delta f_\lambda + a_p h_\delta)\varphi,$$

where $\delta > 0$ is to be chosen. Clearly, $\lambda \mapsto H_\lambda$ is a continuous function such that $\operatorname{supp} H_\lambda \subset I$ and $(H_\lambda^{(i)}(x_0))_{0\leq i\leq p} \parallel \lambda$. Furthermore, if $a_i \neq 0$, then $H_\lambda^{(i)}(x_0) \neq 0$.

We will show that for sufficiently small $\delta > 0$ (independent of λ), H_λ peaks at x_0. Observe that $\|a_p\||h_\delta^{(p)}(x_0)| = \|a_p\| > \varepsilon$, $\sup_{\lambda\in U} \|f_\lambda \varphi\| < \infty$ since U is bounded, and

$$\inf_{\delta>0} \inf_{\lambda\in U} \|H_\lambda\| \geq \inf_{\lambda\in U} (\rho H_\lambda)(x_0) \geq \varepsilon > 0.$$

Clearly, $H_\lambda = 0$ outside I. By choosing δ_0 to be small, we may assume that $\varphi = 1$ on $B(x_0, \delta_0) = (x_0 - \delta_0, x_0 + \delta_0)$. If $x \in I \setminus B(x_0, \delta_0)$, then $h_\delta^{(p)}(x) = 0$ for $\delta < \delta_0$. Since I is bounded, it follows from (2.1) that

$$\lim_{\delta\to 0} (h_\delta \varphi)^{(i)}(x) = 0 \quad \text{uniformly on } I \setminus B(x_0, \delta_0), 0 \leq i \leq p.$$

Then one can derive that

$$\lim_{\delta\to 0} \rho(h_\delta \varphi)(x) = 0 \quad \text{uniformly on } I \setminus B(x_0, \delta_0).$$

For sufficiently small $\delta > 0$, we thus have

$$(\rho H_\lambda)(x) \leq \delta \|f_\lambda \varphi\| + \|a_p\| \rho(h_\delta \varphi)(x) < \|a_p\| \leq \|H_\lambda\|$$

for all $x \in I \setminus B(x_0, \delta_0)$. Hence H_λ does not attain its norm in $I \setminus B(x_0, \delta_0)$ if δ is small enough.

On the other hand, $H_\lambda = \delta f_\lambda + a_p h_\delta$ on $B(x_0, \delta_0)$. Observe that for $x \in B(x_0, \delta_0)$ with $x \neq x_0$,

$$\big\|H_\lambda^{(p)}(x_0)\big\| - \big\|H_\lambda^{(p)}(x)\big\| = \|a_p\| \frac{|x - x_0|}{\delta}, \tag{2.2}$$

while for $0 \leq i < p$,

$$\begin{aligned}\big|\big\|H_\lambda^{(i)}(x_0)\big\| - \big\|H_\lambda^{(i)}(x)\big\|\big| &\leq \big\|H_\lambda^{(i)}(x) - H_\lambda^{(i)}(x_0)\big\| \\ &\leq \delta \sup_{z \in B(x_0,\delta_0)} \big\|f_\lambda^{(i+1)}(z)\big\| |x - x_0| \\ &\quad + \|a_p\| \sup_{z \in B(x_0,\delta_0)} \big|h_\delta^{(i+1)}(z)\big| |x - x_0| \\ &\leq \big(\delta K + C\|a_p\|\big)|x - x_0|, \end{aligned} \tag{2.3}$$

where

$$K = \max_{0 \leq i < p} \sup_{\lambda \in U} \sup_{z \in I} \big\|f_\lambda^{(i+1)}(z)\big\| < \infty \qquad \text{and} \qquad C = \max_{0 \leq i < p} \frac{|I|^{p-i-1}}{(p-i-1)!},$$

independent of δ. For $x_0 - \delta_0 < x < x_0$, since ρ is the ℓ^1-norm and by (2.2)–(2.3), one can derive that

$$\begin{aligned} &\frac{\rho(\|H_\lambda(x_0)\|, \ldots, \|H_\lambda^{(p)}(x_0)\|) - \rho(\|H_\lambda(x)\|, \ldots, \|H_\lambda^{(p)}(x)\|)}{x_0 - x} \\ &= \frac{\rho(0, \ldots, 0, \|H_\lambda^{(p)}(x_0)\|) - \rho(0, \ldots, 0, \|H_\lambda^{(p)}(x)\|)}{x_0 - x} \\ &\quad + \frac{\rho(\|H_\lambda(x_0)\|, \ldots, \|H_\lambda^{(p-1)}(x_0)\|, 0) - \rho(\|H_\lambda(x)\|, \ldots, \|H_\lambda^{(p-1)}(x)\|, 0)}{x_0 - x} \\ &\geq \frac{\varepsilon}{\delta} - \frac{\rho(|\|H_\lambda(x_0)\| - \|H_\lambda(x)\||, \ldots, |\|H_\lambda^{(p-1)}(x_0)\| - \|H_\lambda^{(p-1)}(x)\||, 0)}{x_0 - x} \\ &\geq \frac{\varepsilon}{\delta} - \big(\delta K + C\|a_p\|\big)\rho(1, \ldots, 1, 0), \end{aligned}$$

which implies that $(\rho H_\lambda)(x_0) > (\rho H_\lambda)(x)$ for small enough δ. Similarly, we can show that if $x_0 < x < x_0 + \delta_0$, then $(\rho H_\lambda)(x_0) > (\rho H_\lambda)(x)$ for small enough δ. These two estimates combine to show that for small enough δ, if $x \in B(x_0, \delta_0) \setminus \{x_0\}$, then we have

$$(\rho H_\lambda)(x_0) > (\rho H_\lambda)(x). \qquad \square$$

Remark 2.3. Suppose that I is an open neighborhood of $x_0 \in X$ and that $\lambda = (\lambda_0, \ldots, \lambda_p) \in E^{p+1}$. Fix $0 \neq a \in E$, let

$$\mu_i = \begin{cases} \lambda_i & \text{if } \lambda_i \neq 0, \\ a & \text{if } \lambda_i = 0. \end{cases}$$

Set $\mu = (\mu_0, \ldots, \mu_p)$. By Lemma 2.2, there exists $H \in C_0^p(X, E)$, supported in I, such that H peaks at x_0 and $0 \neq H^{(i)}(x_0) \mid \mu_i$ for all $0 \leq i \leq p$. Then $0 \neq H^{(i)}(x_0) \mid \lambda_i$ for all $0 \leq i \leq p$ as well.

Lemma 2.4. *Suppose that Tf and Tg attain their norms at y_0. Assume that $(Tf)^{(j)}(y_0) \mid (Tg)^{(j)}(y_0)$ for all $0 \le j \le q$. Then there exists x_0 such that both f and g attain their norms at x_0 and $f^{(i)}(x_0) \mid g^{(i)}(x_0)$ for $0 \le i \le p$.*

Proof. Observe that

$$\begin{aligned}\|f+g\| = \|Tf+Tg\| &\ge \sigma\big(\big(\big\|(Tf+Tg)^{(j)}(y_0)\big\|\big)_{0\le j\le q}\big)\\ &= \sigma\big(\big(\big\|(Tf)^{(j)}(y_0)\big\|\big)_{0\le j\le q}\big) + \sigma\big(\big(\big\|(Tg)^{(j)}(y_0)\big\|\big)_{0\le j\le q}\big)\\ &= \|Tf\| + \|Tg\| = \|f\| + \|g\| \ge \|f+g\|.\end{aligned}$$

Let x_0 be a point at which $f+g$ attains its norm. Then one can derive that

$$\begin{aligned}\|f\| + \|g\| = \|f+g\| &= \rho\big(\big(\big\|(f+g)^{(i)}(x_0)\big\|\big)_{0\le i\le p}\big)\\ &\le \rho\big(\big(\big\|f^{(i)}(x_0)\big\|\big)_{0\le i\le p}\big) + \rho\big(\big(\big\|g^{(i)}(x_0)\big\|\big)_{0\le i\le p}\big)\\ &\le \|f\| + \|g\|.\end{aligned}$$

Clearly, both f and g attain their norms at x_0. From the above, we also have

$$\begin{aligned}\rho\big(\big(\big\|(f+g)^{(i)}(x_0)\big\|\big)_{0\le i\le p}\big) &= \rho\big(\big(\big\|f^{(i)}(x_0)\big\|\big)_{0\le i\le p}\big) + \rho\big(\big(\big\|g^{(i)}(x_0)\big\|\big)_{0\le i\le p}\big)\\ &= \rho\big(\big(\big\|f^{(i)}(x_0)\big\| + \big\|g^{(i)}(x_0)\big\|\big)_{0\le i\le p}\big)\\ &\ge \rho\big(\big(\big\|(f+g)^{(i)}(x_0)\big\|\big)_{0\le i\le p}\big).\end{aligned}$$

Therefore, $\|f^{(i)}(x_0) + g^{(i)}(x_0)\| = \|f^{(i)}(x_0)\| + \|g^{(i)}(x_0)\|$ for all $0 \le i \le p$. By strict convexity of the norm of E, we can derive that $f^{(i)}(x_0) \mid g^{(i)}(x_0)$ for all $0 \le i \le p$. □

Lemma 2.5. *Suppose that f peaks at x_0 and that $f^{(i)}(x_0) \ne 0$ for all $0 \le i \le p$. Then there exists y_0 such that Tf peaks at y_0. Moreover, $(Tf)^{(j)}(y_0) \ne 0$ for any $0 \le j \le q$.*

Proof. Suppose on the contrary that there exist distinct points y_1 and y_2 such that Tf attains its norm at both y_1 and y_2. By Lemma 2.2, there exist nonzero functions g_1 and g_2 with disjoint support such that g_k peaks at y_k and such that $g_k^{(j)}(y_k) \mid (Tf)^{(j)}(y_k)$ for all $0 \le j \le q$ and $k = 1, 2$. We may assume that $\|g_1\| = \|g_2\| = 1$. By Lemma 2.4, $T^{-1}g_k$ attains its norm at x_0 and $f^{(i)}(x_0) \mid (T^{-1}g_k)^{(i)}(x_0)$ for any $0 \le i \le p$ and $k = 1, 2$. Since $f^{(i)}(x_0) \ne 0$, one can derive that

$$(T^{-1}g_1)^{(i)}(x_0) \mid (T^{-1}g_2)^{(i)}(x_0) \quad \text{for any } 0 \le i \le p.$$

Now

$$\begin{aligned}1 &= \|g_1 + g_2\| = \|T^{-1}g_1 + T^{-1}g_2\|\\ &\ge \rho\big(\big(\big\|(T^{-1}g_1)^{(i)}(x_0) + (T^{-1}g_2)^{(i)}(x_0)\big\|\big)_{0\le i\le p}\big)\\ &= \rho\big(\big(\big\|(T^{-1}g_1)^{(i)}(x_0)\big\|\big)_{0\le i\le p}\big) + \rho\big(\big(\big\|(T^{-1}g_2)^{(i)}(x_0)\big\|\big)_{0\le i\le p}\big)\\ &\ge \rho\big(\big(\big\|(T^{-1}g_1)^{(i)}(x_0)\big\|\big)_{0\le i\le p}\big)\\ &= \|T^{-1}g_1\| = \|g_1\| = 1.\end{aligned}$$

Thus $\|(T^{-1}g_2)^{(i)}(x_0)\| = 0$ for all $i = 0, 1, \ldots, p$. But since $T^{-1}g_2$ attains its norm at x_0, it follows that $\|g_2\| = 0$, contrary to its choice. This proves that there exists y_0 such that Tf peaks at y_0.

Now suppose that $J = \{j : (Tf)^{(j)}(y_0) = 0\} \neq \emptyset$. Fix $0 \neq u \in F$, and let

$$a_j^1 = \begin{cases} (Tf)^{(j)}(y_0) & \text{if } j \notin J, \\ u & \text{if } j \in J, \end{cases} \qquad a_j^2 = \begin{cases} (Tf)^{(j)}(y_0) & \text{if } j \notin J, \\ -u & \text{if } j \in J. \end{cases}$$

Choose $g_1, g_2 \in C_0^q(Y, F)$ such that g_k peaks at y_0 and such that $0 \neq g_k^{(j)}(y_0) \mid a_j^k$ for any $0 \leq j \leq q$ and $k = 1, 2$. Note that $g_k^{(j)}(y_0) \mid (Tf)^{(j)}(y_0)$ for all $0 \leq j \leq q$ and $k = 1, 2$. Thus

$$\begin{aligned} \|Tf + g_k\| &\geq \sigma\big(\big(\big\|(Tf + g_k)^{(j)}(y_0)\big\|\big)_{0 \leq j \leq q}\big) \\ &= \sigma\big(\big(\big\|(Tf)^{(j)}(y_0)\big\|\big)_{0 \leq j \leq q}\big) + \sigma\big(\big(\big\|g_k^{(j)}(y_0)\big\|\big)_{0 \leq j \leq q}\big) \\ &= \|Tf\| + \|g_k\| \geq \|Tf + g_k\|. \end{aligned}$$

Hence $Tf + g_k$ attains its norm at y_0, and $(Tf + g_k)^{(j)}(y_0) \mid (Tf)^{(j)}(y_0)$ for all $0 \leq j \leq q$ and $k = 1, 2$. By Lemma 2.4, $f + T^{-1}g_k$ and f attain their norms at a common point, which must be x_0, and $(f + T^{-1}g_k)^{(i)}(x_0) \mid f^{(i)}(x_0)$ for all $0 \leq i \leq p$. Since $f^{(i)}(x_0) \neq 0$ for all $0 \leq i \leq p$, it follows that

$$(f + T^{-1}g_1)^{(i)}(x_0) \mid (f + T^{-1}g_2)^{(i)}(x_0) \quad \text{for all } 0 \leq i \leq p.$$

Applying Lemma 2.4 to T^{-1}, we can see that $Tf + g_1$ and $Tf + g_2$ attain their norms at a common point y_1 and that $(Tf + g_1)^{(j)}(y_1) \mid (Tf + g_2)^{(j)}(y_1)$ for all $0 \leq i \leq q$. Since g_k peaks at y_0 and $\|Tf + g_k\| = \|Tf\| + \|g_k\|$, y_1 must be y_0. For any $j \in J$,

$$u \mid g_1^{(j)}(y_0) = (Tf + g_1)^{(j)}(y_0) \mid (Tf + g_2)^{(j)}(y_0) = g_2^{(j)}(y_0) \mid -u.$$

This is impossible since $g_k^{(j)}(y_0)$ and u are nonzero. □

Lemma 2.6. *Suppose that (f_n) converges to a nonzero function $f \in C_0^p(X, E)$ that peaks at some x_0. If (x_n) is a sequence so that f_n attains its norm at x_n for each $n \in \mathbb{N}$, then (x_n) converges to x_0.*

Proof. The sequence $(f_n, \ldots, f_n^{(p)})$ converges uniformly on X to $(f, \ldots, f^{(p)})$. There is a compact set K such that

$$\rho\big(\big\|f(x)\big\|, \ldots, \big\|f^{(p)}(x)\big\|\big) < \frac{\|f\|}{2} \quad \text{for all } x \notin K.$$

Then for sufficiently large n, we have that $\rho(\|f_n(x)\|, \ldots, \|f_n^{(p)}(x)\|) < \|f_n\|$ for all $x \notin K$. Thus we may assume that $x_n \in K$ for all $n \in \mathbb{N}$. For any convergent subsequence (x_{n_k}) of (x_n), we can assume that its limit is $z \in K$. Clearly, f must attain its norm at z, and thus $z = x_0$. This implies that (x_n) converges to x_0. □

Lemma 2.7. *Suppose that f_1 and f_2 peak at x_0, $f_k^{(i)}(x_0) \neq 0$ for all $0 \leq i \leq p, k = 1, 2$, and Tf_1 and Tf_2 peak at y_1 and y_2, respectively. If $f_1^{(i)}(x_0) \nmid f_2^{(i)}(x_0)$ for all $0 \leq i \leq p$, then $y_1 = y_2$.*

Proof. Suppose on the contrary that $y_1 \neq y_2$. Choose functions g_1 and g_2 with disjoint support such that g_k peaks at y_k, $0 \neq g_k^{(j)}(y_k) \mid (Tf_k)^{(j)}(y_k)$ for all $0 \leq j \leq q$, and $k = 1, 2$. We may also assume that $\|g_1\| = \|g_2\| = 1$.

Set $h_k = T^{-1}g_k$ for $k = 1, 2$, and $h = h_1 + h_2$ and $g = Th$. Observe that $\|h_k\| = \|g_k\| = 1$, $\|h\| = \|Th\| = \|g_1 + g_2\| = 1$ and $g^{(j)}(y_k) \mid g_k^{(j)}(y_k)$ for $0 \leq j \leq q$, $k = 1, 2$ since g_1 and g_2 are disjoint. Since $g_k = Th_k$ and Tf_k peak at y_k and $(Th_k)^{(j)}(y_k) \mid (Tf_k)^{(j)}(y_k)$ for $0 \leq j \leq q$, $k = 1, 2$, by Lemmas 2.4 and 2.5, we can derive that h_k and f_k peak at a common point, which must be x_0, and $h_k^{(i)}(x_0) \mid f_k^{(i)}(x_0)$ for $0 \leq i \leq p, k = 1, 2$. As Th and Th_k attain their norms at y_k and $(Th)^{(j)}(y_k) \mid (Th_k)^{(j)}(y_k)$ for any $0 \leq j \leq q, k = 1, 2$, by Lemma 2.4, we have that h_k and h attain their norms at a common point, which must be x_0, and $h^{(i)}(x_0) \mid h_k^{(i)}(x_0)$ for any $0 \leq i \leq p, k = 1, 2$.

Suppose that there exists i_0 such that $h^{(i_0)}(x_0) \neq 0$. Then we can derive that $h_1^{(i_0)}(x_0) \mid h_2^{(i_0)}(x_0)$. By Lemma 2.5 applied to T^{-1}, $h_k^{(i)}(x_0) \neq 0$ for all $0 \leq i \leq p$ and $k = 1, 2$. Therefore, $f_1^{(i_0)}(x_0) \mid f_2^{(i_0)}(x_0)$, contrary to the assumption. Thus $h^{(i)}(x_0) = 0$ for all $0 \leq i \leq p$. Since h attains its norm at x_0, it would follow that $\|h\| = 0$, contradicting the fact that $\|h\| = 1$. $\square$

Lemma 2.8. *Assume that* $\dim E > 1$. *Suppose that* f_1 *and* f_2 *peak at* x_0, $f_k^{(i)}(x_0) \neq 0$ *for all* $0 \leq i \leq p$ *and* $k = 1, 2$, *and* Tf_1 *and* Tf_2 *peak at* y_1 *and* y_2, *respectively. Then* $y_1 = y_2$.

Proof. By Lemma 2.2, choose sequences (h_{kn}) converging to h_k in $C_0^p(X, E)$, $k = 1, 2$, such that h_{kn} and h_k peak at x_0 for any $n \in \mathbb{N}$, $0 \neq h_k^{(i)}(x_0) \mid f_k^{(i)}(x_0)$ for $0 \leq i \leq p$ and $k = 1, 2$, and $h_{1n}^{(i)}(x_0) \nmid h_{2n}^{(i)}(x_0)$ for all $0 \leq i \leq p$ and $n \in \mathbb{N}$. By Lemma 2.7, Th_{1n} and Th_{2n} peak at the same point, which we will denote by z_n, while Th_k peaks at y_k for $k = 1, 2$. Since (Th_{kn}) converges to Th_k, by Lemma 2.6, (z_n) converges to both y_1 and y_2. Therefore, $y_1 = y_2$. $\square$

Lemma 2.9. *Assume that* $\dim E, \dim F > 1$. *There exists a homeomorphism* $\tau : X \to Y$ *such that if* $f \in C_0^p(X, E)$ *peaks at* x_0 *and* $f^{(i)}(x_0) \neq 0$ *for all* $0 \leq i \leq p$, *then* Tf *peaks at* $\tau(x_0)$ *and* $(Tf)^{(j)}(\tau(x_0)) \neq 0$ *for all* $0 \leq j \leq q$.

Proof. By Lemmas 2.5 and 2.8, there exists a mapping $\tau : X \to Y$ such that if $f \in C_0^p(X, E)$ peaks at x_0 and $f^{(i)}(x_0) \neq 0$ for all $0 \leq i \leq p$, then Tf peaks at $\tau(x_0)$ and $(Tf)^{(j)}(\tau(x_0)) \neq 0$ for all $0 \leq j \leq q$. Obviously, τ is a bijection. It suffices to show that τ is a homeomorphism from X onto Y.

Let $x_0 \in X$ and $r > 0$ such that $I = (x_0 - 2r, x_0 + 2r) \subseteq X$. Suppose that (x_n) is a sequence in X converging to x_0. By Lemma 2.2, there exists $H \in C_0^p(\mathbb{R}, E)$ such that $\operatorname{supp}(H) \subset (x_0 - r, x_0 + r)$, H peaks at x_0, and $H^{(i)}(x_0) \neq 0$ for all $0 \leq i \leq p$. Let $H_n(x) = H(x - x_n + x_0)$. Observe that $\operatorname{supp}(H_n) \subset (x_n - r, x_n + r) \subset I$ and $H_n \in C_0^p(X, E)$ for large n. Moreover, H_n peaks at x_n and $H_n^{(i)}(x_n) \neq 0$ for all $0 \leq i \leq p$. By definition of τ, TH_n peaks at $\tau(x_n)$ and TH peaks at $\tau(x_0)$. Since (H_n) converges to H in $C_0^p(X, E)$, (TH_n) converges to TH, which is a nonzero function. By Lemma 2.6, $(\tau(x_n))$ converges to $\tau(x_0)$. This proves that τ is continuous. By symmetry, τ is a homeomorphism from X onto Y. $\square$

Lemma 2.10. *Assume that* $\dim E, \dim F > 1$. *Let* $\tau : X \to Y$ *be the homeomorphism given in Lemma 2.9. If* $f \in C_0^p(X, E)$ *and* $f^{(i)}(x_0) = 0$ *for all* $0 \le i \le p$ *at some* $x_0 \in X$, *then* $(Tf)^{(j)}(\tau(x_0)) = 0$ *for all* $0 \le j \le q$.

Proof. Let $y_0 = \tau(x_0)$. Assume that $(Tf)^{(j_0)}(y_0) \neq 0$ for some $0 \le j_0 \le q$. By Lemma 2.2, there exists $g \in C_0^q(Y, F)$ such that g peaks at y_0 and $0 \neq g^{(j)}(y_0) \mid (Tf)^{(j)}(y_0)$ for all $0 \le j \le q$. By the definition of τ, $T^{-1}g$ peaks at x_0 and $(T^{-1}g)^{(i)}(x_0) \neq 0$ for all $0 \le i \le p$.

Let I be an open neighborhood of x_0. By Lemma 2.2 again, there exists $h \in C_0^p(X, E)$, supported in I, such that h peaks at x_0 and $0 \neq h^{(i)}(x_0) \mid (T^{-1}g)^{(i)}(x_0)$ for all $0 \le i \le p$. We may assume that $\|h\| > \|f\|$. By Lemmas 2.4 and 2.5, Th peaks at y_0 and $(Th)^{(j)}(y_0) \mid g^{(j)}(y_0)$ for all $0 \le j \le q$. Thus $(Th)^{(j)}(y_0) \mid (Tf)^{(j)}(y_0)$ for all $0 \le j \le q$. We have

$$\begin{aligned}
\|f + h\| &= \|Tf + Th\| \\
&\ge \sigma\big(\big(\big\|(Tf)^{(j)}(y_0) + (Th)^{(j)}(y_0)\big\|\big)_{0\le j\le q}\big) \\
&= \sigma\big(\big(\big\|(Tf)^{(j)}(y_0)\big\|\big)_{0\le j\le q}\big) + \sigma\big(\big(\big\|(Th)^{(j)}(y_0)\big\|\big)_{0\le j\le q}\big) \\
&= \sigma\big(\big(\big\|(Tf)^{(j)}(y_0)\big\|\big)_{0\le j\le q}\big) + \|h\| \\
&> \|h\| > \|f\|.
\end{aligned}$$

Since $h(x) = 0$ for any $x \notin I$, $f + h$ must attain its norm at a point $x_1 \in I$. Then

$$\begin{aligned}
(\rho f)(x_1) + \|h\| \ge (\rho f)(x_1) + (\rho h)(x_1) &\ge \big(\rho(f + h)\big)(x_1) = \|f + h\| \\
&\ge \sigma\big(\big(\big\|(Tf)^{(j)}(y_0)\big\|\big)_{0\le j\le q}\big) + \|h\|.
\end{aligned}$$

Hence $(\rho f)(x_1) \ge \sigma((\|(Tf)^{(j)}(y_0)\|)_{0\le j\le q})$. Since I is an arbitrary neighborhood of x_0, we conclude that $0 = (\rho f)(x_0) \ge \sigma((\|(Tf)^{(i)}(y_0)\|)_{0\le j\le q})$ and $(Tf)^{(j)}(y_0) = 0$ for all $0 \le j \le q$. ◻

In the rest of the article, we would like to show that $p = q$ and τ is a C^p-diffeomorphism.

Lemma 2.11. *If* $f, g \in C_0^p(X, E)$ *and* $\|f(x)\| \cdot \|g(x)\| = 0$ *for all* $x \in X$, *then at any point* $x \in X$, *either* $f^{(i)}(x) = 0$ *for all* $0 \le i \le p$ *or* $g^{(i)}(x) = 0$ *for all* $0 \le i \le p$.

Proof. Let f and g be in $C_0^p(X, E)$ with $\|f(x)\| \cdot \|g(x)\| = 0$ for all $x \in X$. Set $V = \{x \in X : f(x) \neq 0\}$. Note that V is open in X and $g_{|V} = 0$. Hence $g_{|V}^{(i)} = 0$ for all $0 \le i \le p$. By continuity of $g^{(i)}$, we derive that $g_{|\overline{V}}^{(i)} = 0$ for all $0 \le i \le p$, where $\overline{V}$ is the closure of V. By the definition of V, it is clear that $f(x) = 0$ for all $x \notin \overline{V}$. Therefore, for each $x \notin \overline{V}$, we have $f^{(i)}(x) = 0$ for all $0 \le i \le p$. This completes the proof of the lemma. ◻

Let us recall that a map $S : C^p(X, E) \to C^q(Y, F)$ is said to be *disjointness-preserving* if $\|Sf(y)\| \cdot \|Sg(y)\| = 0$ for all $y \in Y$ whenever $f, g \in C^p(X, E)$ satisfy $\|f(x)\| \cdot \|g(x)\| = 0$ for all $x \in X$. A map S is called *biseparating* if it is a bijection and both S and S^{-1} are disjointness-preserving.

Lemma 2.12. *Assume that E and F are strictly convex Banach spaces with* $\dim E$, $\dim F > 1$. *Let X and Y be open sets in $\mathbb{R}$, and let $p, q \in \mathbb{N}$. Then any surjective linear isometry $T : C_0^p(X, E) \to C_0^q(Y, F)$ can be extended to a linear biseparating map $\widetilde{T} : C^p(X, E) \to C^q(Y, F)$.*

Proof. For any $x \in X$ and any $a = (a_0, a_1, \ldots, a_p) \in E^{p+1}$, choose $h_{x,a} \in C_0^p(X, E)$ so that $h_{x,a}^{(i)}(x) = a_i$ for all $0 \leq i \leq p$. For any $f \in C^p(X, E)$, define $\widetilde{T}f$ on Y by

$$(\widetilde{T}f)(y) = Th_{x,a}(y),$$

where $x = \tau^{-1}(y)$ and $a_i = f^{(i)}(x)$ for all $0 \leq i \leq p$. It follows from Lemma 2.10 that $\widetilde{T}$ is well defined. Let $y_0 \in Y$ and $x_0 = \tau^{-1}(y_0)$. There are an open neighborhood U of x_0 and a function $g \in C_0^p(X, E)$ such that $g = f$ on U. For any $x \in U$,

$$g^{(i)}(x) = f^{(i)}(x) = h_{x,a}^{(i)}(x) \quad \text{for all } 0 \leq i \leq p,$$

where $a_i = f^{(i)}(x)$ for all $0 \leq i \leq p$.

By Lemma 2.10, $(Tg)(y) = (Th_{x,a})(y)$ for all $y \in \tau(U)$. Hence $\widetilde{T}f = Tg$ on $\tau(U)$ is q-times continuously differentiable on $\tau(U)$. Since y_0 is an arbitrary point in Y, this implies that $\widetilde{T}f \in C^q(Y, F)$. Using Lemmas 2.10 and 2.11, one can derive that $\widetilde{T}$ is a linear disjointness-preserving map that extends T. By symmetry, one may similarly define a linear disjointness-preserving map $\widetilde{S} : C^q(Y, F) \to C^p(X, E)$ such that $\widetilde{S}$ extends T^{-1}. By Lemma 2.10 and the definition of $\widetilde{T}$ and $\widetilde{S}$, we can verify that $\widetilde{T}$ and $\widetilde{S}$ are mutual inverses. This proves that $\widetilde{T}$ is a linear biseparating map. □

Theorem 2.13. *Assume that E and F are strictly convex Banach spaces with* $\dim E$, $\dim F > 1$. *Let $T : C_0^p(X, E) \to C_0^q(Y, F)$ be a surjective linear isometry, where X and Y are open sets in $\mathbb{R}$ and $p, q \in \mathbb{N}$. Then $p = q$, and there exist a C^p-diffeomorphism $\tau : X \to Y$ and surjective linear isomorphisms $J_y : E \to F$, $y \in Y$, such that*

$$Tf(y) = J_y\big(f\big(\tau^{-1}(y)\big)\big) \quad \textit{for all } f \in C_0^p(X, E), y \in Y.$$

Proof. By Lemma 2.12, T can be extended to a linear biseparating map $\widetilde{T} : C^p(X, E) \to C^q(Y, F)$. By [1, Theorem 6.2], we have $p = q$, and there are a C^p-diffeomorphism $\gamma : X \to Y$ and Banach space isomorphisms $J_y : E \to F$ for all $y \in Y$ such that

$$\widetilde{T}f(y) = J_y\big(f\big(\gamma^{-1}(y)\big)\big) \quad \text{for all } f \in C^p(X, E), y \in Y.$$

If $\gamma \neq \tau$, there exists $x \in X$ such that $y_1 = \gamma(x) \neq \tau(x) = y_2$. Choose $g \in C_0^q(Y, F)$ such that $g(y_1) \neq 0, g^{(i)}(y_2) = 0$ for all $0 \leq i \leq q$, and set $f = T^{-1}g$. By Lemma 2.10, $f(x) = 0$. Then, by the preceding formula,

$$g(y_1) = (Tf)(y_1) = (\widetilde{T}f)(y_1) = J_y\big(f(x)\big) = 0,$$

contrary to the choice of g. □

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