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CENTRAL CALDERÓN–ZYGMUND OPERATORS ON HERZ-TYPE HARDY SPACES OF VARIABLE SMOOTHNESS AND INTEGRABILITY

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Abstract. In this article we use the atomic decomposition of a Herz-type Hardy space of variable smoothness and integrability to obtain the boundedness of the central Calderón–Zygmund operators on Herz-type Hardy spaces with variable smoothness and integrability.

1. Introduction

The classical versions of Herz spaces $K^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ were introduced by Herz in [4]. These spaces were studied in many papers (see, e.g., [5], [7], [8], [10], [15], and references therein). The topic of function spaces with variable exponents is a very active area of research nowadays (see, e.g., [2], [9], [13]), and one of the reasons is the wide variety of applications of such spaces (e.g., in the modeling of electro-rheological fluids, as in [16], and in differential equations with non-standard growth). Herz spaces with variable exponents were studied by several authors using different approaches. The Herz spaces $K^{\alpha}_{p(\cdot),q}$ and $\dot{K}^{\alpha}_{p(\cdot),q}$ were studied by Izuki in [6] and [7] and were improved in the variable case of the Herz spaces $K^{\alpha(\cdot)}_{p(\cdot),q}$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$ by Almeida and Drihem in [1], where they obtained the boundedness results for a class of sublinear operators. Izuki and Noi introduced

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the Herz spaces $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ in [8], where all parameters were taken as variable. Recently, in [3], Drihem and Seghiri studied the Herz spaces and Herz-type Hardy spaces with variable exponents which were introduced in [8] and obtained several results regarding equivalence of norms and the atomic decomposition of Herz-type Hardy spaces. Further, they studied the boundedness of some sublinear operators on Herz spaces with variable exponents. Another approach to studying variable exponent Herz spaces (with variable parameters), known as *continual Herz spaces*, was given by Samko in [17], where the boundedness of some sublinear operators was obtained. The Sobolev-type theorem on these spaces was studied by Rafeiro and Samko in [14].

In this paper, we obtain the boundedness of central Calderón–Zygmund operators on Herz-type Hardy spaces $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ by using the atomic decomposition of the Herz–Hardy space obtained in [3]. In the proofs of the main theorems we follow the ideas from [1]. The results for the case when $q = \text{const}$ were obtained in [19]. For the boundedness of such operators in the classical case, we refer to [12]. The present article has three sections. Section 2 deals with some basic notions regarding variable exponent Lebesgue and Herz spaces. In Section 3, we obtain the boundedness of central Calderón–Zygmund operators on Herz-type Hardy spaces. Throughout the paper, constants (often different constants in the same series of inequalities) will mainly be denoted by c or C; by the symbol $p'(x)$ we denote the function $\frac{p(x)}{p(x)-1}$, $1 < p(x) < \infty$; the relation $a \approx b$ means that there are positive constants c_1 and c_2 such that $c_1 a \le b \le c_2 a$; and by $\lfloor\cdot\rfloor$ we denote the floor function.

2. Preliminaries

2.1. Variable Lebesgue spaces. Let $x \in \mathbb{R}^n$ and $r > 0$, and we denote by $B(x, r)$ an open ball in $\mathbb{R}^n$ centered at x and of radius r. By $\operatorname{supp}(f)$ we denote the support of the function f. Let E be a measurable subset in $\mathbb{R}^n$; by $|E|$ we denote the Lebesgue measure of the set E, and χ_E denotes the characteristic function of the set E. Let E be a measurable set in $\mathbb{R}^n$ with positive measure. We denote

$$p^-(E) = \operatorname*{ess\,inf}_E p(x), \qquad p^+(E) := \operatorname*{ess\,sup}_E p(x)$$

for a measurable function p on E. A measurable function p belongs to the class $\mathcal{P}_0(E)$ ($p \in \mathcal{P}_0(E)$) if $0 < p^-(E) \le p^+(E) < \infty$. We say that $p \in \mathcal{P}(E)$ if $1 \le p^-(E) \le p^+(E) < \infty$. We say that a measurable function f on E belongs to $L^{p(\cdot)}(E)$ (or to $L^{p(x)}(E)$) if

$$S_{p(\cdot),E}(f) = \int_E |f(x)|^{p(x)}\,\mathrm{d}x < \infty.$$

The space $L^{p(\cdot)}(E)$ is a Banach space with respect to the norm (see, e.g., [11])

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\Big\{\eta > 0 : S_{p(\cdot),E}\Big(\frac{f}{\eta}\Big) \le 1\Big\}.$$

(For the following propositions, see [11], [2].)

Proposition A. *Let E be a measurable subset of $\mathbb{R}^n$. Suppose that $1 \le p^-(E) \le p^+(E) < \infty$. Then*

(i)

$$\|f\|^{p^+(E)}_{L^{p(\cdot)}(E)} \le S_{p(\cdot),E}(f) \le \|f\|^{p^-(E)}_{L^{p(\cdot)}(E)}, \quad \|f\|_{L^{p(\cdot)}(E)} \le 1;$$
$$\|f\|^{p^-(E)}_{L^{p(\cdot)}(E)} \le S_{p(\cdot),E}(f) \le \|f\|^{p^+(E)}_{L^{p(\cdot)}(E)}, \quad \|f\|_{L^{p(\cdot)}(E)} \ge 1.$$

(ii) *Hölder's inequality*

$$\Big|\int_E f(x)g(x)\,\mathrm{d}x\Big| \le \Big(\frac{1}{p^-(E)} + \frac{1}{(p^+(E))'}\Big)\|f\|_{L^{p(\cdot)}(E)}\|g\|_{L^{p'(\cdot)}(E)}$$

holds, where $f \in L^{p(\cdot)}(E)$ and $g \in L^{p'(\cdot)}(E)$.

(iii) *The generalized Hölder's inequality*

$$\|fg\|_{L^{r(\cdot)}(E)} \le c\|f\|_{L^{p(\cdot)}(E)}\|g\|_{L^{q(\cdot)}(E)}$$

holds, where $f \in L^{p(\cdot)}(E)$, $g \in L^{q(\cdot)}(E)$, and $\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$ for every $x \in E$.

Proposition B. *Let $1 \le r(x) \le p(x)$, and let E be a subset of $\mathbb{R}^n$ with $|E| < \infty$. Then the following inequality*

$$\|f\|_{L^{r(\cdot)}(E)} \le \big(|E|+1\big)\|f\|_{L^{p(\cdot)}(E)}$$

holds.

Definition 2.1. We say that p satisfies the local log-Hölder continuity condition ($p \in \mathcal{P}^{\log}(\mathbb{R}^n)$) if there is a positive constant A such that for all x and y in $\mathbb{R}^n$ the inequality

$$\big|p(x) - p(y)\big| \le A/\big(\ln\big(e + 1/|x-y|\big)\big)$$

holds.

Definition 2.2. We say that p satisfies the log-Hölder condition at the origin condition ($p \in \mathcal{P}^{\log}_0(\mathbb{R}^n)$) if there is a positive constant c_1 such that for all $x \in \mathbb{R}^n$ the inequality

$$\big|p(x) - p(0)\big| \le c_1/\big(\ln\big(e + 1/|x|\big)\big)$$

holds.

Definition 2.3. We say that p satisfies the log-Hölder condition at infinity ($p \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$) if there exist two positive constants c_2 and $p_\infty \in \mathbb{R}$ such that for all $x \in \mathbb{R}^n$ the inequality

$$\big|p(x) - p_\infty\big| \le c_2/\big(\ln\big(e + |x|\big)\big)$$

holds.

We denote $\mathcal{P}^{\log}_{0,\infty}(\mathbb{R}^n) := \mathcal{P}^{\log}_0(\mathbb{R}^n) \cap \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$. For the next lemma we refer to [9].

Lemma 2.4. *Let* $p \in \mathcal{P}_{0,\infty}^{\log}(\mathbb{R}^n)$*. Then*

$$\|\chi_{B(0,r)}\|_{L^{p(\cdot)}} \le c_0 r^{\frac{n}{p(0)}} \quad \text{for } 0<r\le 1 \tag{2.1}$$

and

$$\|\chi_{B(0,r)}\|_{L^{p(\cdot)}} \le c_\infty r^{\frac{n}{p_\infty}} \quad \text{for } r \ge 1, \tag{2.2}$$

respectively, where $c_0 \ge 1$ *and* $c_\infty \ge 1$ *does not depend on* r*.*

Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The mixed variable exponent Lebesgue space $l^{q(\cdot)}(L^{p(\cdot)})$ is defined on a sequence of $L^{p(\cdot)}$-functions by the modular

$$S_{p(\cdot),q(\cdot)}\big((f_\nu)_\nu\big) = \sum_\nu \inf\Big\{\lambda_\nu > 0 : S_{p(\cdot)}\Big(\frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}}\Big) \le 1\Big\}.$$

The space is equipped with the quasi-norm

$$\big\|(f_\nu)_\nu\big\|_{l^{q(\cdot)}(L^{p(\cdot)})} = \inf\Big\{\mu > 0 : S_{p(\cdot),q(\cdot)}\Big(\frac{1}{\mu}(f_\nu)_\nu\Big) \le 1\Big\}.$$

2.2. Variable Herz-type Hardy spaces. In this subsection, we introduce variable Herz spaces and Herz-type Hardy spaces. The classical Herz spaces were first considered in [4]. (For the following definitions and statements, see [3].) Let us set $B_k := B(0, 2^k)$, $R_k := B_k \setminus B_{k-1}$, and $\chi_k = \chi_{R_k}$ for $k \in \mathbb{Z}$.

Definition 2.5. Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \mapsto \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The inhomogeneous Herz space $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n)$ such that

$$\|f\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)} := \|f\chi_{B_0}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \big\|(2^{k\alpha(\cdot)} f\chi_k)_{k\ge 1}\big\|_{l^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{2.3}$$

The homogeneous Herz space $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)} := \big\|(2^{k\alpha(\cdot)} f\chi_k)_{k\in\mathbb{Z}}\big\|_{l^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{2.4}$$

If α, p, q are constant, then these spaces coincide with the classical Herz spaces, first considered in [4]. Let us denote

$$\big\|\{g_k\}\big\|_{l^q_>(L^{p(\cdot)})} := \Big(\sum_{k=0}^{\infty} \|g_k\|_{L^{p(\cdot)}}^q\Big)^{1/q}$$

and

$$\big\|\{g_k\}\big\|_{l^q_<(L^{p(\cdot)})} := \Big(\sum_{k=-\infty}^{-1} \|g_k\|_{L^{p(\cdot)}}^q\Big)^{1/q}$$

for a sequence $\{g_k\}_{k\in\mathbb{Z}}$ of measurable function. For the following proposition we refer to [3].

Proposition C. *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. If $\alpha, q \in \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, then*

$$K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = K_{p(\cdot),q_\infty}^{\alpha_\infty}(\mathbb{R}^n).$$

Further, if α, q also satisfy the log-Hölder condition at the origin, then

$$\|f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)} = \left\|\{2^{k\alpha(0)} f\chi_k\}\right\|_{l_<^{q(0)}(L^{p(\cdot)})} + \left\|\{2^{k\alpha_\infty} f\chi_k\}\right\|_{l_>^{q_\infty}(L^{p(\cdot)})} < \infty.$$

Let $G_N f$ be the grand maximal function of f defined by

$$G_N f(x) = \sup_{\phi \in \mathcal{A}_N} \left|\phi_N^*(f)(x)\right|,$$

where

$$\mathcal{A}_N = \left\{\phi \in \mathscr{S}(\mathbb{R}^n) : \sup_{|\alpha|\le N, |\beta|\le N} \left|x^\alpha \partial^\beta \phi(x)\right| \le 1\right\}$$

and

$$\phi_N^*(f)(x) = \sup_{t>0} \left|\phi_t * f(x)\right|,$$

with $\phi_t(\cdot) = t^{-n}\phi(\frac{\cdot}{t})$ and where $\mathscr{S}(\mathbb{R}^n)$ denotes the class of Schwartz functions.

Definition 2.6. Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \mapsto \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$ and $N > n + 1$. The inhomogeneous Herz-type Hardy space $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ consists of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $G_N f \in K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, and we define

$$\|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)} = \|G_N f\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)}.$$

The homogeneous Herz-type Hardy space $H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ consists of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $G_N f \in \dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, and we define

$$\|f\|_{H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)} = \|G_N f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)}.$$

Definition 2.7. Let $\alpha \in L^\infty(\mathbb{R}^n)$, $q \in \mathcal{P}_0(\mathbb{R}^n)$, $p \in \mathcal{P}(\mathbb{R}^n)$, and $s \in \mathbb{N}_0$. A function a is said to be a *central* $(\alpha(\cdot), p(\cdot))$-*atom*, if

(1) $\operatorname{supp} a \subset \overline{B(0,r)} = \{x \in \mathbb{R}^n : |x| \le r\}$, $r > 0$;
(2) $\|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le |B(0,r)|^{-\alpha(0)/n}$, $0 < r < 1$;
(3) $\|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le |B(0,r)|^{-\alpha_\infty/n}$, $r \ge 1$;
(4) $\int_{\mathbb{R}^n} x^\beta a(x)\,\mathrm{d}x = 0$, $|\beta| \le s$.

A function a is said to be a *central* $(\alpha(\cdot), p(\cdot))$-*atom of restricted type*, if it satisfies conditions (iii) and (iv) above and $\operatorname{supp} a \subset \overline{B(0,r)}$, $r \ge 1$.

Now we present the atomic decomposition theorems. For these statements we refer to [18] in the case of α and q constant and [3] in the case of α and q are variable.

Proposition D. *Let $p \in \mathcal{P}(\mathbb{R}^n)$, $q \in \mathcal{P}_0(\mathbb{R}^n)$, and $\alpha : \mathbb{R}^n \mapsto \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. Let α, $q \in \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ and $p \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$. Suppose that $\alpha_\infty \ge n(1 - \frac{1}{p_\infty})$, and let s be a nonnegative integer such that $s \ge \lfloor \alpha_\infty + n(\frac{1}{p_\infty} - 1)\rfloor$. If $f \in$*

$HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, *then we have*

$$f=\sum_{k=0}^{\infty}\lambda_k a_k, \tag{2.5}$$

where the series converges in the sense of distribution, $\lambda_k\geq 0$, *each* a_k *is a central* $(\alpha(\cdot),p(\cdot))$*-atom of restricted type with* $\operatorname{supp} a_k\subset B_k$, *and*

$$\Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty}\leq c\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

Proposition E. *Let* $p\in\mathcal{P}(\mathbb{R}^n)$, $q\in\mathcal{P}_0(\mathbb{R}^n)$, *and* $\alpha:\mathbb{R}^n\mapsto\mathbb{R}$ *with* $\alpha\in L^\infty(\mathbb{R}^n)$. *Let* $\alpha,q\in\mathcal{P}^{\log}_{0,\infty}(\mathbb{R}^n)$ *and* $p\in\mathcal{P}^{\log}(\mathbb{R}^n)\cap\mathcal{P}^{\log}_{0,\infty}(\mathbb{R}^n)$. *Suppose that* $\alpha(\cdot)\geq n(1-\frac{1}{p^-})$, *and let* s *be a nonnegative integer such that* $s\geq\max\{\lfloor\alpha_\infty+n(\frac{1}{p_\infty}-1)\rfloor,\lfloor\alpha(0)+n(\frac{1}{p(0)}-1)\rfloor\}$. *If* $f\in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, *then we have*

$$f=\sum_{k=-\infty}^{\infty}\lambda_k a_k, \tag{2.6}$$

where the series converges in the sense of distribution, $\lambda_k\geq 0$, *each* a_k *is a central* $(\alpha(\cdot),p(\cdot))$*-atom with* $\operatorname{supp} a_k\subset B_k$, *and*

$$\Big(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\Big)^{1/q(0)}+\Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty}\leq c\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}. \tag{2.7}$$

3. Boundedness of central Calderón–Zygmund operators

In this section, we show that the central Calderón–Zygmund operator is bounded from $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. A similar conclusion holds for the inhomogeneous case. Our main result follows from a general result of a larger class of operators. We begin this section with the following definition.

Definition 3.1. Let $p\in\mathcal{P}(\mathbb{R}^n)$, $q\in\mathcal{P}_0(\mathbb{R}^n)$, and $\alpha:\mathbb{R}^n\mapsto\mathbb{R}$ with $\alpha\in L^\infty(\mathbb{R}^n)$. Let $\alpha,q\in\mathcal{P}^{\log}_{0,\infty}(\mathbb{R}^n)$ and $p\in\mathcal{P}^{\log}(\mathbb{R}^n)\cap\mathcal{P}^{\log}_{0,\infty}(\mathbb{R}^n)$. Let s be the nonnegative integer given by $s=\max\{\lfloor\alpha_\infty+n(1/p_\infty-1)\rfloor,\lfloor\alpha(0)+n(1/p(0)-1)\rfloor\}$. We say that a sublinear operator T belongs to class $\mathcal{I}_{0,\infty}$ ($T\in\mathcal{I}_{0,\infty}$) if the following conditions hold:

(1) T is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$.
(2) There exists a constant $\delta>0$ such that $s+\delta>\max\{\alpha_\infty+n(1/p_\infty-1),\alpha(0)+n(1/p(0)-1)\}$, and for any compactly supported function f with

$$\int_{\mathbb{R}^n}f(x)x^\beta\,\mathrm{d}x=0,\quad|\beta|\leq s,$$

Tf satisfies the size condition

$$\big|Tf(x)\big|\leq c\big(\operatorname{diam}(\operatorname{supp}f)\big)^{s+\delta}|x|^{-(n+s+\delta)}\int_{\mathbb{R}^n}\big|f(y)\big|\,\mathrm{d}y, \tag{3.1}$$

if $\operatorname{dist}(x,\operatorname{supp}f)\geq|x|/2$.

Analogously, we can define the class $\mathcal{I}_\infty$, where all the exponents satisfy the log-Hölder condition near infinity, and replace the condition on s as above by $s = \lfloor \alpha_\infty + n(1/p_\infty - 1) \rfloor$.

The next statements are the main results of this section.

Theorem 3.2. *Let $p, \alpha \in \mathcal{P}_\infty(\mathbb{R}^n)$, $T \in \mathcal{I}_\infty$, and δ be as in Definition 3.1. If*

$$\frac{n}{p'_\infty} < \alpha_\infty < \frac{n}{p'_\infty} + \delta, \tag{3.2}$$

holds, then T is bounded from $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Theorem 3.3. *Let $p, \alpha \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$, $T \in \mathcal{I}_{0,\infty}$, and δ be as in Definition 3.1. If*

$$\frac{n}{p'(0)} < \alpha(0) < \frac{n}{p'(0)} + \delta, \qquad \frac{n}{p'_\infty} < \alpha_\infty < \frac{n}{p'_\infty} + \delta, \tag{3.3}$$

holds, then T is bounded from $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

We present the proof only for Theorem 3.3. The proof of Theorem 3.2 can be obtained by the same arguments.

Proof of Theorem 3.3. Now, by Proposition C, we have

$$\begin{aligned} \big\|T(f)\big\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} &\le c\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \big\|T(f)\chi_k\big\|^{q(0)}_{L^{p(\cdot)}}\Big)^{\frac{1}{q(0)}} \\ &\quad + \Big(\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \big\|T(f)\chi_k\big\|^{q_\infty}_{L^{p(\cdot)}}\Big)^{\frac{1}{q_\infty}} \\ &=: c(I_< + I_>). \end{aligned}$$

Let $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. By Proposition E the function f can be represented as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where a_j is a central $(\alpha(\cdot), p(\cdot))$-atom with $\operatorname{supp} a_j \subset B_j$. Hence,

$$\begin{aligned} I_< &\le c\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=-\infty}^{k-2} |\lambda_j| \big\|T(a_j)\chi_k\big\|_{L^{p(\cdot)}}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} \\ &\quad + c\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=k-1}^{\infty} |\lambda_j| \big\|T(a_j)\chi_k\big\|_{L^{p(\cdot)}}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} \\ &=: I^{(1)}_< + I^{(2)}_< \end{aligned}$$

and

$$\begin{aligned} I_> &\le c\Big(\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty}\Big(\sum_{j=-\infty}^{k-2} |\lambda_j| \big\|T(a_j)\chi_k\big\|_{L^{p(\cdot)}}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\ &\quad + c\Big(\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty}\Big(\sum_{j=k-1}^{\infty} |\lambda_j| \big\|T(a_j)\chi_k\big\|_{L^{p(\cdot)}}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\ &=: I^{(1)}_> + I^{(2)}_>. \end{aligned}$$

Estimate of $I^{(2)}_{<}$: By $(L^{p(\cdot)}, L^{p(\cdot)})$ boundedness of T and Minkowski's inequality, we have

$$\begin{aligned}
I^{(2)}_{<} &\leq c\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=k-1}^{\infty} |\lambda_j|\|a_j\|_{L^{p(\cdot)}}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} \\
&\leq c\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=k-1}^{-1} |\lambda_j|\|a_j\|_{L^{p(\cdot)}} + \sum_{j=0}^{\infty}|\lambda_j|\|a_j\|_{L^{p(\cdot)}}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} \\
&\leq c\Big(\sum_{k=-\infty}^{-1} \Big(\sum_{j=k-1}^{-1} |\lambda_j| 2^{(k-j)\alpha(0)}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} \\
&\quad + c\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=0}^{\infty}|\lambda_j| 2^{-j\alpha_\infty}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} \\
&=: cH_1 + cH_2.
\end{aligned}$$

Now for H_1, by Hölder's inequality with $(q(0), q'(0))$ and changing the order of summation, we have

$$\begin{aligned}
H_1 &\leq c\Big(\sum_{k=-\infty}^{-1}\sum_{j=k-1}^{-1} |\lambda_j|^{q(0)} 2^{(k-j)\frac{\alpha(0)q(0)}{2}}\Big)^{\frac{1}{q(0)}} \\
&\leq c\Big(\sum_{j=-\infty}^{-1} |\lambda_j|^{q(0)} \sum_{k=-\infty}^{j+1} 2^{(k-j)\frac{\alpha(0)q(0)}{2}}\Big)^{\frac{1}{q(0)}} \\
&\leq c\Big(\sum_{j=-\infty}^{-1} |\lambda_j|^{q(0)}\Big)^{\frac{1}{q(0)}} \\
&\leq c\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For H_2, first let $q_\infty > 1$; then by Hölder's inequality with exponents (q_∞, q'_∞) and Proposition E (the fact that $\alpha_\infty > 0$ together with inequality (2.7)) we have

$$\begin{aligned}
H_2 &= c\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big)^{\frac{1}{q(0)}}\Big(\sum_{j=0}^{\infty}|\lambda_j| 2^{-j\alpha_\infty}\Big) \\
&\leq c\Big(\sum_{j=0}^{\infty}|\lambda_j|^{q_\infty}\Big)^{1/q_\infty}\Big(\sum_{j=0}^{\infty} 2^{-j\alpha_\infty q'_\infty}\Big)^{1/q'_\infty} \\
&\leq c\Big(\sum_{j=0}^{\infty}|\lambda_j|^{q_\infty}\Big)^{1/q_\infty} \\
&\leq c\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Estimate of $I_>^{(2)}$: The estimate is similar to the one for H_1 with $q(0)$ replaced by q_∞, since $j \geq k-1 > 0$. Thus we have

$$I_>^{(2)} \leq c\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

Estimate of $I_>^{(1)}$: First, since $x \in R_k$, $y \in B_j$, and $j \leq k-2$ and $|x-y| \geq 2^k/2 \geq |x|/2$, we have, by the size condition (3.1) and Hölder's inequality,

$$\big|T(a_j)(x)\big| \leq \begin{cases} c2^{(j-k)(s+\delta)}2^{-kn}2^{\frac{nj}{p'_\infty}}2^{-j\alpha_\infty}, & j \geq 0, \\ c2^{(j-k)(s+\delta)}2^{-kn}2^{\frac{nj}{p'(0)}}2^{-j\alpha(0)}, & j < 0. \end{cases} \tag{3.4}$$

Hence,

$$\big\|T(a_j)\chi_k\big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \begin{cases} c2^{\frac{-nk}{p'_\infty}}2^{(j-k)(s+\delta)}2^{\frac{nj}{p'_\infty}}2^{-j\alpha_\infty}, & j \geq 0, \\ c2^{\frac{-nk}{p'_\infty}}2^{(j-k)(s+\delta)}2^{\frac{nj}{p'(0)}}2^{-j\alpha(0)}, & j < 0, \end{cases} \tag{3.5}$$

$$\begin{aligned} I_>^{(1)} &\leq c\Big(\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty}\Big(\sum_{j=-\infty}^{k-2} |\lambda_j|\big\|T(a_j)\chi_k\big\|_{L^{p(\cdot)}}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\ &\leq c\Big(\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty}\Big(\sum_{j=-\infty}^{-1} |\lambda_j|\big\|T(a_j)\chi_k\big\|_{L^{p(\cdot)}}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\ &\quad + c\Big(\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty}\Big(\sum_{j=0}^{k-2} |\lambda_j|\big\|T(a_j)\chi_k\big\|_{L^{p(\cdot)}}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\ &\leq J_1 + J_2, \end{aligned}$$

where $J_2 = 0$ for $k = 0,1,2$. Let $q(0) > 1$; then, by using the estimate (3.5) for J_1 and Hölder's inequality with exponents $(q(0), q'(0))$ and the fact $s+\delta > \max\{\alpha_\infty + n(1/p_\infty - 1), \alpha(0) + n(1/p(0)-1)\}$, we have

$$\begin{aligned} J_1 &\leq c\Big(\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty}\Big(\sum_{j=-\infty}^{-1} |\lambda_j| 2^{\frac{-nk}{p'_\infty}}2^{(j-k)(s+\delta)}2^{\frac{nj}{p'(0)}}2^{-j\alpha(0)}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\ &\leq c\Big(\sum_{k=0}^{\infty} 2^{-kq_\infty(\frac{n}{p'_\infty}+s+\delta-\alpha_\infty)}\Big)^{\frac{1}{q_\infty}}\Big(\sum_{j=-\infty}^{-1} |\lambda_j| 2^{j(s+\delta+\frac{n}{p'(0)}-\alpha(0))}\Big) \\ &\leq c\Big(\sum_{j=-\infty}^{-1} |\lambda_j|^{q(0)}\Big)^{\frac{1}{q(0)}}\Big(\sum_{j=-\infty}^{-1} 2^{j(s+\delta+\frac{n}{p'(0)}-\alpha(0))q'(0)}\Big)^{\frac{1}{q'(0)}} \\ &\leq c\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Similarly for J_2, inserting the appropriate estimate from (3.5), it follows from Hölder's inequality with exponents (q_∞, q'_∞) and the condition $s+\delta >$

$\max\{\alpha_\infty + n(1/p_\infty - 1), \alpha(0) + n(1/p(0) - 1)\}$ that

$$
\begin{aligned}
J_2 &\le c\Big(\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty}\Big(\sum_{j=0}^{k-2} |\lambda_j| 2^{(j-k)(s+\delta)} 2^{\frac{-kn}{p'_\infty}} 2^{\frac{nj}{p'_\infty}} 2^{-j\alpha_\infty}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\
&\le c\Big(\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k-2} |\lambda_j| 2^{(j-k)(s+\delta+\frac{n}{p'_\infty}-\alpha_\infty)}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\
&\le c\Big(\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k-2} |\lambda_j|^{q_\infty} 2^{(j-k)(s+\delta+\frac{n}{p'_\infty}-\alpha_\infty)\frac{q_\infty}{2}}\Big)\Big(\sum_{j=0}^{k-2} 2^{(j-k)(s+\delta+\frac{n}{p'_\infty}-\alpha_\infty)\frac{q'_\infty}{2}}\Big)^{\frac{q_\infty}{q'_\infty}}\Big)^{\frac{1}{q_\infty}} \\
&\le c\Big(\sum_{j=0}^{\infty} |\lambda_j|^{q_\infty}\Big(\sum_{k=j+2}^{\infty} 2^{(j-k)(s+\delta+\frac{n}{p'_\infty}-\alpha_\infty)\frac{q_\infty}{2}}\Big)\Big)^{\frac{1}{q_\infty}} \\
&\le c\Big(\sum_{j=0}^{\infty} |\lambda_j|^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\
&\le c\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Estimate of $I_<^{(1)}$: The estimate is similar to the one for J_2 with q_∞ replaced by $q(0)$ since $j \le k-2 < 0$. The case when $0 < q < 1$ can be obtained in a similar way by using the inequality $(\sum x_i)^q \le (\sum x_i^q)$. □

3.1. Application. In this section, we apply our main result to obtain the boundedness of central Calderón–Zygmund operators, which are more general than the standard Calderón–Zygmund operators. We begin this section with the following definition.

Definition 3.4. Let K be a locally integrable function on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\}$. Then K is called a *central kernel* if there exist $\delta \in (0, 1]$ and $C > 0$, such that

$$\big|K(x, y) - K(x, 0)\big| \le C\frac{|y|^\delta}{|x|^{n+\delta}}, \quad |x| \ge 2|y|, \tag{3.6}$$

and

$$\big|K(x, 0)\big| \le \frac{c}{|x|^n}.$$

Definition 3.5. A linear operator $T_K : \mathscr{S}(\mathbb{R}^n) \mapsto \mathscr{S}'(\mathbb{R}^n)$ is said to be a *Calderón–Zygmund operator* associated with a central kernel K if

(1) T_K can be extended to be a bounded operator on $L^2(\mathbb{R}^n)$;
(2) for any $f \in L^2(\mathbb{R}^n)$ with compact support and almost every $x \notin \operatorname{supp} f$,

$$T_K f(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, \mathrm{d}y. \tag{3.7}$$

Now we present the boundedness of Calderón–Zygmund operators with a central kernel. We remark that the case for $q(\cdot) = \text{const}$ was proved in [19].

Theorem 3.6. *Let p, q, α be as in Theorem 3.2. Let T_K be a Calderón–Zygmund operator bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ and let it be associated with the central kernel K in the sense of (3.7). Then T_K is bounded from $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

Theorem 3.7. *Let p, q, α be as in Theorem 3.3. Let T_K be a Calderón–Zygmund operator bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ and let it be associated with the central kernel K in the sense of (3.7). Then T_K is bounded from $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

Proof. We only prove Theorem 3.6. The proof of Theorem 3.7 follows analogously. First, the condition $n/p'_\infty \le \alpha_\infty < n/p'_\infty + \delta$ implies that $s = \lfloor \alpha_\infty - n/p'_\infty \rfloor = 0$. Now let $f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$; then by Proposition D we have that $f(x) = \sum_{k=0}^{\infty} \lambda_k a_k(x)$ where a_k is central $(\alpha(\cdot), p(\cdot))$-atom. By sublinearity of T_K we have $|T_K f(x)| \le \sum_{k=0}^{\infty} |\lambda_k||T_K(a_k)(x)|$. Following the proof of Theorem 3.2, we have that, for each $x \in A_k$ and $\operatorname{supp}(f) \subset B_j$ $(j \le k-2)$ together with the condition of vanishing moments (i.e., $\int_{\mathbb{R}^n} a_k = 0$) and condition (3.6), the operator T_K satisfies condition (3.1) with $s = 0$. Hence the result follows by the same arguments as presented in Theorem 3.2. $\square$

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