Ann. Funct. Anal. 9 (2018), no. 1, 123–136
https://doi.org/10.1215/20088752-2017-0033
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

ON THE p-SCHUR PROPERTY OF BANACH SPACES

MOHAMMAD B. DEHGHANI and S. MOHAMMAD MOSHTAGHIOUN*

Communicated by G. Androulakis

Abstract. We introduce the notion of the p-Schur property ($1 \le p \le \infty$) as a generalization of the Schur property of Banach spaces, and then we present a number of basic properties and some examples. We also study its relation with some geometric properties of Banach spaces, such as the Gelfand–Phillips property. Moreover, we verify some necessary and sufficient conditions for the p-Schur property of some closed subspaces of operator spaces.

1. Introduction

A sequence (x_n) in a Banach space X is called *weakly p-summable* with $1 \le p < \infty$ if, for each x^* in the dual space X^* of X, the sequence $(\langle x_n, x^* \rangle)$ is p-summable; that is,

$$\sum_{n=1}^{\infty} \bigl|\langle x_n, x^* \rangle\bigr|^p < \infty,$$

where $\langle x, x^* \rangle$ denotes the duality between $x \in X$ and $x^* \in X^*$. The space of all weakly p-summable sequences in X is denoted by $\ell_p^{\mathrm{weak}}(X)$, which is a Banach space with the norm

$$\bigl\|(x_n)\bigr\|_p^{\mathrm{weak}} = \sup\Bigl\{\Bigl(\sum_{n=1}^{\infty} \bigl|\langle x_n, x^* \rangle\bigr|^p\Bigr)^{\frac{1}{p}} : \|x^*\| \le 1\Bigr\}.$$

Received Jun. 25, 2016; Accepted Mar. 8, 2017.
First published online Oct. 5, 2017.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47L05; Secondary 46B25.
Keywords. Gelfand–Phillips property, Schur property, weakly p-compact set, weakly p-convergent sequence.

An operator T between Banach spaces X and Y is said to be *p-converging* if it transfers weakly p-summable sequences into norm-null sequences; that is, $\|Tx_n\| \to 0$ for all $(x_n) \in \ell_p^{\mathrm{weak}}(X)$. The class of all p-converging operators from X into Y is denoted by $C_p(X, Y)$. The Banach space of all (weakly) bounded sequences in X with supremum norm is denoted by $\ell_\infty^{\mathrm{weak}}(X)$. Moreover, by $c_0^{\mathrm{weak}}(X)$ we represent the closed subspace of $\ell_\infty^{\mathrm{weak}}(X)$ which contains all weakly null sequences of X.

For each $1 \le p < \infty$, a sequence (x_n) in a Banach space X is said to be *weakly p-convergent* to an $x \in X$ if the sequence $(x_n - x)$ is weakly p-summable; that is, $(x_n - x) \in \ell_p^{\mathrm{weak}}(X)$. The weakly ∞-convergent sequences are simply the weakly convergent sequences. Also, according to [4], a bounded set K in a Banach space is said to be *relatively weakly p-compact*, $1 \le p \le \infty$, if every sequence in K has a weakly p-convergent subsequence. If the limit point of each weakly p-convergent subsequence is in K, then we call K a *weakly p-compact set*. Also a Banach space X is weakly p-compact if the closed unit ball B_X of X is a weakly p-compact set. A bounded operator T from X into Y is called *weakly p-compact*, $1 \le p \le \infty$, if $T(B_X)$ is relatively weakly p-compact. The space of all weakly p-compact operators from X into Y is denoted by $W_p(X, Y)$, while the space of all bounded operators and weakly compact operators from X into Y are denoted by $L(X, Y)$ and $W(X, Y)$, respectively. Weakly ∞-compact operators are precisely those $T \in L(X, Y)$ for which $T(B_X)$ is relatively weakly compact; that is, $W_\infty(X, Y) = W(X, Y)$. The reader is referred to [4] for more information about these concepts.

We note that in [18], the authors used the same name "relatively weakly p-compact" for a subset K of a Banach space X such that for some sequence $(x_n) \in \ell_p^{\mathrm{weak}}(X)$,

$$K \subseteq \Big\{ \sum_j \lambda_j x_j : (\lambda_j) \in B_{\ell_q} \Big\},$$

where $p \ge 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Note that, by [5], this notion is stronger than the concept of a "relatively weakly p-compact set" that was considered by Castillo and Sánchez in [4].

We know that a Banach space X has the Dunford–Pettis (DP) property if every weakly compact operator T from X into any Banach space Y is a Dunford–Pettis operator; that is, T carries weakly convergent sequences to norm-convergent ones. Also, if $1 \le p \le \infty$, then the Banach space X has the Dunford–Pettis property of order p (DP_p) if for each Banach space Y, every weakly compact operator $T : X \to Y$ is p-converging; in other words, $W(X, Y) \subseteq C_p(X, Y)$. By definition, ∞-converging operators are equal to Dunford–Pettis ones. So the Dunford–Pettis property of order ∞ is the same as the DP property. Every Banach space with the DP property, such as the sequence spaces c_0, ℓ_1 and every Schur space, has the DP_p property. A Banach space X has the Schur property if every weakly null sequence in X converges in norm. The simplest Banach space with the Schur property is ℓ_1.

A subset K of a Banach space X is called *limited* (resp., *Dunford–Pettis* (DP)) if for each weak* null (resp., weak null) sequence (x_n^*) in X^*,

$$\lim_{n\to\infty}\sup_{x\in K}\left|\langle x, x_n^*\rangle\right| = 0.$$

In general, every relatively compact subset of X is limited and so is Dunford–Pettis. If every limited subset of a Banach space X is relatively compact, then X has the Gelfand–Phillips (GP) property. For example (see [2]), the classical Banach spaces c_0 and ℓ_1 have the GP property and every Schur space and spaces containing no copy of ℓ_1, such as reflexive spaces, have the same property. (The reader can find some useful and additional properties of limited and DP sets and Banach spaces with the Schur and GP properties in [2], [9], [13], and [17].)

The main aim of the present article is to introduce the p-Schur property of Banach spaces as a generalization of the Schur property. We give some basic facts and some examples of this concept. We obtain some sufficient conditions which illustrate the relation between the Gelfand–Phillips property and the p-Schur property of Banach spaces. Finally, we characterize closed subspaces of some operator spaces with the p-Schur property relative to the p-converging of some special operators, the so-called *evaluation operators*.

2. Main results

Recall that a series $\sum_{n=1}^{\infty} x_n$ in X is said to be *unconditionally convergent* if $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for every permutation π of the natural numbers. Also, a series $\sum_{n=1}^{\infty} x_n$ in X is *weakly unconditionally Cauchy* or *weakly unconditionally convergent* (WUC) if for every $x^* \in X^*$, $\sum_{n=1}^{\infty} |\langle x_n, x^*\rangle| < \infty$.

Definition 2.1 ([8, p. 37]). Let X and Y be two Banach spaces. An operator $T : X \to Y$ is called *unconditionally converging* if $\sum_{n=1}^{\infty} Tx_n$ is unconditionally convergent whenever $\sum_{n=1}^{\infty} x_n$ is WUC.

Theorem 2.2 ([1, Theorem 2.4.11]). *For every WUC series in a Banach space X to be unconditionally convergent, it is necessary and sufficient that X contains no copy of c_0.*

Definition 2.3. A Banach space X has the p-Schur property ($1 \le p \le \infty$) if every weakly p-compact subset of X is compact.

In other words, if $1 \le p < \infty$, a Banach space X has the p-Schur property if and only if every sequence $(x_n) \in \ell_p^{\mathrm{weak}}(X)$ is a norm-null sequence, and X has the ∞-Schur property if and only if every sequence in $c_0^{\mathrm{weak}}(X)$ is norm-null. So the ∞-Schur property coincides with the Schur property. Also, one can note that every Schur space has the p-Schur property for all $p \ge 1$.

We note that a quantitative version of the Schur property, namely, the "C-Schur property," was introduced and studied in [12]. In fact, a Banach space X is said to have the C-Schur property (where $C \ge 0$) if

$$\mathrm{ca}(x_k) \le C\delta(x_k),$$

for any bounded sequence (x_k) in X, for which

$$\mathrm{ca}(x_k) = \inf_{n\in\mathbb{N}} \mathrm{diam}\{x_k : k \geq n\}$$

and

$$\delta(x_k) = \sup_{x^*\in B_{X^*}} \inf_{n\in\mathbb{N}} \mathrm{diam}\big\{x^*(x_k) : k \geq n\big\}.$$

In this notion, if X has the C-Schur property for some $C \geq 1$, then X has the Schur property. Moreover, in [12, Example 1.4] Banach spaces with the Schur property and without the C-Schur property for any $C > 0$ have been constructed. Therefore, our definition of the p-Schur property is not equivalent to this quantitative version of the Schur property.

Theorem 2.4. *Let X be a Banach space which contains no copy of c_0. Then X has the 1-Schur property.*

Proof. Assume that $(x_n) \in \ell_1^{\mathrm{weak}}(X)$. Then (x_n) is WUC. By Theorem 2.2, $\sum_{n=1}^{\infty} x_n$ is convergent and so $\|x_n\| \to 0$. □

Corollary 2.9 shows that the converse of this theorem is also valid.

Example 2.5. It is well known that ℓ_1 has the Schur property. Then it has the p-Schur property for all $p \geq 1$. However, it is known that ℓ_p $(p > 1)$, $L_1(\mu)$, and weakly sequentially complete Banach spaces and reflexive spaces are not necessarily Schur spaces. But they contain no copy of c_0. Therefore, they have the 1-Schur property.

Example 2.6. Suppose that $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then ℓ_p does not have the q-Schur property. Indeed, if (e_n) is the standard basis of ℓ_p, then we have

$$\sum_{n=1}^{\infty} \big|\langle x^*, e_n\rangle\big|^q = \sum_{n=1}^{\infty} |x_n|^q < \infty$$

for all $x^* = (x_1, x_2, \ldots) \in \ell_q$. But clearly, $\|e_n\| = 1$ for all n. It is known that $\ell_p^{\mathrm{weak}}(X) \subseteq \ell_q^{\mathrm{weak}}(X)$ for all $1 \leq p \leq q$. Therefore, we conclude that if X has the q-Schur property, then X has the p-Schur property. This fact and the above example show that ℓ_2 does not have the p-Schur property for all $p \geq 2$.

Theorem 2.7. *For each $1 \leq p \leq \infty$, the following statements are equivalent.*

(i) *X has the p-Schur property.*
(ii) *Every subspace of X has the p-Schur property.*
(iii) *Every separable subspace Z of X is contained in a subspace $Y \subseteq X$ which is p-Schur and is complemented in X.*
(iv) *X is the direct sum of two p-Schur spaces.*

Proof. If Y is a subspace of X, since $\ell_p^{\mathrm{weak}}(Y) \subseteq \ell_p^{\mathrm{weak}}(X)$, then the implication (i)⇒(ii) is obvious.

(ii)⇒(iii) This implication is clear.

(iii)⇒(i) Suppose that E is a weakly p-compact subset of X and that $(x_n) \subseteq E$. Then there is a subsequence (x_{n_k}) of (x_n) that is weakly p-convergent to some

$x \in E$. On the other hand, the sequence (x_n) is contained in a p-Schur subspace Y and so $x_{n_k} \to x$ in norm.

(i)$\Rightarrow$(iv) We have $X = X \oplus \{0\}$.

(iv)$\Rightarrow$(i) Let $X = Y \oplus Z$ such that Y and Z have the p-Schur property. Consider the projections $P_Y : X \to Y$ and $P_Z : X \to Z$. Assume that E is a weakly p-compact subset of X. Then $P_Y(E)$ is a weakly p-compact subset of Y and so is compact. Similarly, $P_Z(E)$ is a compact set. Any sequence $(x_n) \subseteq E$ can be written as $x_n = y_n + z_n$, where $y_n \in P_Y(E)$, $z_n \in P_Z(E)$. Then there are subsequences (y_{n_k}) and (z_{n_k}) and vectors $y \in P_Y(E)$ and $z \in P_Z(E)$ such that $y_{n_k} \to y$ and $z_{n_k} \to z$. Hence, $x_{n_k} = y_{n_k} + z_{n_k} \to y + z$. Since E is weakly p-compact, we conclude that $y + z \in E$. Therefore, E is compact. □

Corollary 2.8. *Every Banach space which contains a copy of c_0 does not have the p-Schur property.*

Proof. The Banach space c_0 does not have the 1-Schur property. Indeed, the standard basis $(e_n) \in \ell_1^{\mathrm{weak}}(c_0)$ is not a norm-null sequence. So c_0 is not p-Schur for all $p \geq 1$. □

Corollary 2.9. *The Banach space X has the* 1*-Schur property if and only if X contains no copy of c_0.*

Let $K(H)$ be the algebra of all compact operators on an infinite-dimensional Hilbert space H. It is known that c_0 can be identified with a closed commutative subalgebra of $K(H)$ (see [19]). It follows that $K(H)$ is not p-Schur. Also, clearly the Banach spaces $C(K)$ (of all continuous functions on a compact metric space) and ℓ_∞ contain a copy of c_0 and so do not have the p-Schur property.

Corollary 2.10. *Every Banach space X with an unconditional basis which is not isomorphic to a dual space fails to have the p-Schur property.*

Proof. By Theorems 3.2.10 and 3.3.2 of [1], X contains a copy of c_0. □

The following corollary refines a standard argument about the Schur property.

Corollary 2.11. *Let X be a Banach space. If X^* has the p-Schur property for some p, then X contains no copy of ℓ_1.*

Proof. Suppose that X contains a copy of ℓ_1. Then X^* contains a copy of c_0. But c_0 is not a p-Schur space. □

Theorem 2.12. *The Banach space X has the* 1*-Schur property if and only if every operator $T : X \to Y$ is unconditionally converging for all Banach spaces Y.*

Proof. First, suppose that X contains no copy of c_0 and that $\sum_{n=1}^{\infty} x_n$ is WUC in X. By Theorem 2.2, the series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent and so $\sum_{n=1}^{\infty} T x_n$ is unconditionally convergent. The converse also follows by Theorem 2.2, since the identity operator $i : X \to X$ is unconditionally converging. □

By similar techniques, we can conclude that the Banach space Y has the 1-Schur property if and only if every operator $T : X \to Y$ is unconditionally converging for all Banach spaces X. (We recall the property (V) of a Banach space which was introduced in [14] by Pelczynski: a Banach space X has the property (V) if for every Banach space Y, every unconditionally converging operator $T \in L(X,Y)$ is weakly compact.) Since ℓ_1 is a Schur space that contains no copy of c_0,the corollaries below follow from Theorem 2.12.

Corollary 2.13. *Let X be a Banach space with Pelczynski's property (V). Then every operator $T : X \to \ell_1$ is compact.*

Corollary 2.14. *Let X be a nonreflexive Banach space with the property (V). Then X contains a copy of c_0 and so fails to have the p-Schur property for all p.*

Proof. Assume that X contains no copy of c_0. By Theorem 2.2, we conclude that the identity operator $i : X \to X$ is unconditionally converging. Since X has the property (V), $i : X \to X$ is weakly compact. Therefore, X is reflexive. This is a contradiction. □

Corollary 2.15. *Let X be a Banach space with property (V) which has the 1-Schur property. Then $L(X,Y) = W(X,Y)$ for each Banach space Y.*

In the following we recall the definition of the DP^* property of order p from [11], which plays a critical role in the study of the p-Schur property of some Banach spaces.

Definition 2.16 ([11, Definition 2.3]). A Banach space X is said to have the DP^* property of order p (for $1 \leq p \leq \infty$) if all weakly p-compact sets in X are limited. In short, we say that X has the *DP_p^* property*.

Theorem 2.17 ([11, Theorem 2.4])**.** *Let $1 \leq p \leq \infty$. The Banach space X has the DP_p^* property if and only if $\langle x_n, x_n^* \rangle \to 0$ as $n \to \infty$ for all $(x_n) \in \ell_p^{\mathrm{weak}}(X)$ in X and all* weak* *null sequences (x_n^*) in X^*.*

Theorem 2.18. *If a Banach space X has the DP_p^* property and B_{X^*} is* weak* *sequentially compact, then X has the p-Schur property.*

Proof. Suppose that X is not p-Schur. Then there is $(x_n) \in \ell_p^{\mathrm{weak}}(X)$ such that $\|x_n\| = 1$ for all $n \geq 1$. Put

$$E_n = \operatorname{span}\{x_k : k \leq n\}.$$

Given a fixed k, it follows from Lemma 1.7 of [3] that there exists an index N_k such that $d(x_j, E_k) > \frac{1}{3}$ for all $j \geq N_k$. Thus, we may choose $x_{n_k} \notin E_k$ (where $n_k \geq N_k$) such that $d(x_{n_k}, E_k) > \frac{1}{3}$ (but, of course, $d(x_{n_k}, E_{n_k}) = 0$ since $x_{n_k} \in E_{n_k}$). There exists $y_k^* \in B_{X^*}$ such that $\langle y_k^*, x_{n_k} \rangle \geq \frac{1}{3}$ and $\langle y_k^*, x \rangle = 0$ for all $x \in E_k$. Thus, we construct a sequence $(y_k^*) \subseteq B_{X^*}$ such that $\langle y_k^*, x_{n_k} \rangle \geq \frac{1}{3}$ and $\langle y_k^*, x \rangle = 0$ for all $x \in E_k$ and for all k. Since B_{X^*} is weak* sequentially compact, there exists a subsequence $(y_{k_j}^*)$ such that $y_{k_j}^* \xrightarrow{w^*} y^* \in B_{X^*}$. Since for all j such that $k_j \geq k$ we have $E_k \subseteq E_{k_j}$ and thus $\langle y_{k_j}^*, x \rangle = 0$ for all $x \in E_k$, it follows that $\langle y^*, x \rangle = 0$ for all $x \in E_k$. Therefore, $\langle y^*, x_k \rangle = 0$ for all k. Let $x_{k_j}^* = y_{k_j}^* - y^*$. Then $x_{k_j}^* \xrightarrow{w^*} 0$. On the other hand, since $x_{n_{k_j}} \in E_{n_{k_j}}$, we have

$$\langle x_{k_j}^*, x_{n_{k_j}} \rangle = \langle y_{k_j}^*, x_{n_{k_j}} \rangle - \langle y^*, x_{n_{k_j}} \rangle \geq \frac{1}{3}$$

for all j. From Theorem 2.17, it follows that X does not have the DP_p^* property, which is a contradiction. $\square$

The Banach space c_0 illustrates that, in general, a GP space does not have the p-Schur property. The following theorem gives a sufficient condition for which a Banach space with the GP property could be a p-Schur space.

Theorem 2.19. *Let X be a Banach space with the* GP *and DP_p^* properties. Then X has the p-Schur property.*

Proof. Let $(x_n) \in \ell_p^{\mathrm{weak}}(X)$. It is well known that (x_n) is limited if and only if $\langle x_n, x_n^* \rangle \to 0$ for all w^*-null sequences $(x_n^*) \subseteq X^*$ (see [10]). Then by Theorem 2.17, since X has the DP_p^* property, we conclude that (x_n) is limited. On the other hand, we know that the Banach space X has the GP property if and only if every limited weakly null sequence in X is norm-null (see [9]), from which it follows that $\|x_n\| \to 0$. $\square$

Theorem 2.20. *For a Banach space X and each $1 \le p \le \infty$, the following are equivalent.*

(i) *X has the p-Schur property.*
(ii) *For each Banach space Y, $L(X,Y) = C_p(X,Y)$.*
(iii) *For each Banach space Y, $L(Y,X) = C_p(Y,X)$.*

Proof. (i)⇒(ii) Let $1 \le p < \infty$, and let X be a p-Schur space. If $T \in L(X,Y)$ and $(x_n) \in \ell_p^{\mathrm{weak}}(X)$, then $\|x_n\| \to 0$. Hence $\|Tx_n\| \to 0$. Therefore, $T \in C_p(X,Y)$.

(ii)⇒(i) If one considers the identity operator on X, then clearly (ii) implies (i).

The implication (i)⇔(iii) is proved similarly.

Note that if $p = \infty$, the ∞-Schur property coincides with the Schur property and the ∞-converging operators coincide with the Dunford–Pettis operators. $\square$

The following corollary proves that the validity of Theorem 2.20(ii) for ℓ_∞ instead of all Banach spaces Y is a sufficient condition for the p-Schur property of X.

Corollary 2.21. *A Banach space X has the p-Schur property if and only if $L(X,\ell_\infty) = C_p(X,\ell_\infty)$.*

Proof. Suppose that X does not have the p-Schur property. Then there exists a normalized sequence $(x_n) \in \ell_p^{\mathrm{weak}}(X)$. We can choose a sequence $(x_n^*) \in B_{X^*}$ such that $|\langle x_n, x_n^* \rangle| = 1$ for all integers n. Define $T : X \to \ell_\infty$ by $Tx = (\langle x_m^*, x \rangle)$. Then we have

$$\|Tx_n\| = \sup\bigl\{ \bigl|\langle x_m^*, x_n \rangle\bigr| : m \in \mathbb{N} \bigr\} \ge 1.$$

Therefore, $\|Tx_n\|$ is not a null sequence and T is not p-converging. $\square$

By Theorem 2.20, every weakly compact operator on a p-Schur space X is p-converging. Thus X has the DP_p property. But the converse is not true in general. For example, we know that for each compact Hausdorff space K, $C(K)$ and c_0 do not have the p-Schur property. However, they have the DP property and so have the DP_p property (see [8]). Theorem 2.31 shows that in reflexive

Banach spaces, the converse is true. Also, Theorem 2.30 gives another sufficient condition for the converse.

We recall that an operator $T : X \to Y$ between two Banach spaces X and Y is said to be *strictly singular* if there is no infinite-dimensional subspace $E \subseteq X$ such that $T|_E$ is an isomorphism onto its range (see [1, Definition 2.1.8]).

Theorem 2.22. *Suppose that $T \in C_p(X, Y)$ is not strictly singular. Then X and Y contain simultaneously some infinite-dimensional closed subspaces with the p-Schur property.*

Proof. Let T have a bounded inverse on the closed infinite-dimensional subspace Z of X. If $(x_n) \in \ell_p^{\mathrm{weak}}(Z)$, then since $(x_n) \in \ell_p^{\mathrm{weak}}(X)$, $\|Tx_n\| \to 0$ and so $\|x_n\| \to 0$. Hence Z has the p-Schur property. Therefore, $T(Z)$ also has the p-Schur property. □

By a similar argument, if $T \in W_p(X, Y)$ is not strictly singular, then X and Y contain simultaneously some infinite-dimensional weakly p-compact subspaces.

Corollary 2.23. *Any weakly p-compact operator which is p-converging is strictly singular.*

Proof. Let $T \in C_p(X, Y) \cap W_p(X, Y)$ and such that T is not strictly singular. Then X has an infinite-dimensional closed subspace Z such that the restriction operator $T|_Z$ is an into isomorphism. Then the technique of the proof of Theorem 2.22 shows that Z is weakly p-compact and has the p-Schur property. Hence B_Z is compact, which is impossible. □

Corollary 2.24. *Let X and Y be two Banach spaces, and let $1 \leq p \leq \infty$ and $T \in W_p(X, Y)$. If X has the DP_p property, then T is strictly singular and T^2 is compact.*

Proof. Since X has the DP_p property, we have $T \in C_p(X, Y)$ and so by Corollary 2.23 it is strictly singular. Moreover, by assumption, $T(B_X)$ is relatively weakly p-compact. Thus $T^2(B_X) = T(T(B_X))$ is relatively compact. □

Corollary 2.25. *Let X have the p-Schur property, and let Y be weakly p-compact. Then every $T \in L(X, Y)$ is strictly singular.*

Some necessary and sufficient conditions for the compactness of operators between Banach spaces have been given in [13]. We try to collect some other conditions for this situation.

Theorem 2.26. *Let X^* have the 1-Schur property, and assume that Y has the Schur property. Then every $T \in L(X, Y)$ is compact.*

Proof. Assume that there exists an operator $T \in L(X, Y)$ which is not compact. The Schur property of Y implies that there is a bounded sequence (x_n) in X with no weakly Cauchy subsequence. Then by Rosenthal's ℓ_1-theorem, X contains a copy of ℓ_1. It follows that X^* contains a copy of c_0, which is a contradiction, since X^* has the 1-Schur property. □

Theorem 2.27. *For given Banach spaces X and Y, if X is weakly p-compact and Y has the p-Schur property, then every $T \in L(X, Y)$ is compact.*

Proof. If B_X is weakly p-compact and $T \in L(X,Y)$, then $T(B_X)$ is also weakly p-compact and the p-Schur property of Y implies that $T(B_X)$ is compact. $\square$

The rest of this article establishes some relations between the DP_p and the p-Schur properties of Banach spaces. Recall that a Banach space X is said to be *smooth* if the norm $\|\cdot\|$ of X is Gâteaux-differentiable on X. Given a smooth Banach space X, a mapping $T : X \to X^*$ is a dual map if $\|Tx\| = \|x\|$ and $T(x)(x) = \|x\|^2$ for all $x \in X$. It is well known that the dual map is $\|\cdot\| - w^*$ continuous (see, e.g., [7]).

The following characterization of spaces having the DP_p property plays an essential role in achieving our subsequent results.

Theorem 2.28 ([4, Proposition 3.2])**.** *For a given Banach space X and $1 \le p \le \infty$, the following are equivalent.*

(i) *X has the DP_p property.*
(ii) *If $(x_n) \in \ell_p^{\text{weak}}(X)$ and $(x_n^*) \in c_0^{\text{weak}}(X^*)$, then $\langle x_n, x_n^* \rangle \to 0$.*

Theorem 2.29. *Let X be a smooth Banach space. Then the following statements are equivalent.*

(i) *X has the p-Schur property.*
(ii) *X has the DP_p property and every dual mapping on X is p-weak-norm sequentially continuous; that is, it maps all weakly p-convergent sequences into norm-convergent ones.*
(iii) *X has the DP_p property and there exists a dual mapping on X which is pw-w sequentially continuous; that is, it maps all weakly p-convergent sequences into weakly convergent ones.*

Proof. (i)⇒(ii) Let X be a p-Schur space, and let $(x_n) \in \ell_p^{\text{weak}}(X)$. Then $\|x_n\| \to 0$. If $T : X \to X^*$ is a dual mapping, then we conclude that $\|Tx_n\| \to 0$.

(ii)⇒(iii) This implication is clear.

(iii)⇒(i) In contrast, assume that X does not have the p-Schur property and that $T : X \to X^*$ is a pw-w sequentially continuous dual map. Then there is a normalized sequence $(x_n) \in \ell_p^{\text{weak}}(X)$. Since T is pw-w continuous, we conclude that (Tx_n) is a weakly null sequence in X^*. Theorem 2.28 implies that

$$\|x_n\|^2 = \langle x_n, Tx_n \rangle \to 0,$$

which is impossible. $\square$

Theorem 2.30. *If X has the DP_p property, then either X is not weakly p-compact or X^* has the Schur (p-Schur) property.*

Proof. Suppose that X has the DP_p property and that B_X is weakly p-compact. If X^* is not a Schur space, then there exists a normalized weakly null sequence (x_n^*) in X^*. Then we can find (x_n) in X with $\|x_n\| = 1$ and $x_n^*(x_n) > \frac{1}{2}$. Hence there is a subsequence (x_{n_k}) and $x_0 \in X$ such that $(x_{n_k} - x_0) \in \ell_p^{\text{weak}}(X)$. Now, from Theorem 2.28 it follows that $\langle x_{n_k}^*, x_{n_k} - x_0 \rangle \to 0$. On the other hand, $\langle x_{n_k}^*, x_0 \rangle \to 0$. Therefore, $\langle x_{n_k}^*, x_{n_k} \rangle \to 0$, which is a contradiction. $\square$

It is clear that every weakly p-compact space is reflexive, but the converse is not true in general. For example, ℓ_2 is a reflexive space, which is not a Schur space and has the DP_1 property. So by Theorem 2.30, ℓ_2 is not weakly 1-compact.

We have now shown that every reflexive Banach space has the 1-Schur property. The following theorem gives us a sufficient condition for which a reflexive Banach space has the p-Schur property.

Theorem 2.31. *Let X be a reflexive Banach space with the DP_p property. Then X has the p-Schur property.*

Proof. Let X fail to have the p-Schur property. Then there is a normalized sequence $(x_n) \in \ell_p^{\text{weak}}(X)$. By the Hahn–Banach theorem, there is $x_n^* \in X^*$ with $\|x_n^*\| = 1$ and $\langle x_n^*, x_n\rangle = 1$ for all positive integers n. Since X is reflexive, we conclude that there is $x^* \in X^*$ such that for a subsequence $(x_{n_k}^*)$ of (x_n^*), $x_{n_k}^* \xrightarrow{w} x^*$. By Theorem 2.28, we have

$$\|x_{n_k}\| - \langle x^*, x_{n_k}\rangle = \langle x_{n_k}^* - x^*, x_{n_k}\rangle \to 0,$$

from which it follows that $\|x_{n_k}\| \to 0$. This is a contradiction. $\square$

3. Some operator spaces with the p-Schur property

In this section, we give some operator spaces which have the p-Schur property. The following theorem is a refinement of Theorem 3.3 of [15], which guarantees that the Schur property of X^* and Y is a sufficient condition for the Schur property of $L(X, Y)$.

Theorem 3.1. *If X^* has the p-Schur property and Y is a Schur space, then $L(X, Y)$ has the p-Schur property.*

Proof. In order to prove that $L(X, Y)$ has the p-Schur property, it is enough to show that every weakly p-convergent sequence $(T_n) \subseteq L(X, Y)$ is norm-convergent. Let $(T_n) \subseteq L(X, Y)$ be a weakly p-convergent sequence to $T_0 \in L(X, Y)$, and suppose that (x_n) is a sequence in X such that $\|x_n\| = 1$ and $\|T_n - T_0\| < \|(T_n - T_0)x_n\| + \frac{1}{n}$, for all n. For a fixed $y^* \in Y^*$, the sequence $(T_n^* - T_0^*)(y^*)$ is weakly p-summable in X^*, since for each $x^{**} \in X^{**}$ we have

$$\left\langle x^{**}, (T_n^* - T_0^*)(y^*)\right\rangle = \phi(T_n - T_0),$$

where $\phi(T) = x^{**}(T^* y^*)$ defines a bounded linear functional on $L(X, Y)$. Since X^* has the p-Schur property, we have $\|(T_n^* - T_0^*)(y^*)\| \to 0$. Then

$$\left|\left\langle (T_n - T_0)(x_n), y^*\right\rangle\right| \le \left\|(T_n^* - T_0^*)(y^*)\right\| \to 0.$$

It follows that $(T_n - T_0)x_n \xrightarrow{w} 0$ and so $\|(T_n - T_0)(x_n)\| \to 0$, by the Schur property of Y. Therefore, $\|T_n - T_0\| \to 0$. $\square$

Corollary 3.2. *Let X and Y be two Banach spaces. If X^* has the Schur property and Y^{**} has the p-Schur property, then $L(X, Y)$ has the p-Schur property.*

Proof. The mapping $T \mapsto T^*$ maps $L(X, Y)$ onto a closed subspace of $L(Y^*, X^*)$, which has the p-Schur property by virtue of Theorem 3.1. $\square$

In the following corollary, the projective tensor product of X and Y is denoted by $X \tilde{\otimes}_\pi Y$. (We refer the reader to [6] and [13] for undefined terminology.)

Corollary 3.3. *If X^* has the p-Schur property and Y^* is a Schur space, then*

(i) *$(X \tilde{\otimes}_\pi Y)^*$ has the p-Schur property;*

(ii) *every operator from $X \tilde{\otimes}_\pi Y$ into ℓ_1 is compact.*

Proof. (i) This follows easily from the fact that $L(X, Y^*) = (X \tilde{\otimes}_\pi Y)^*$ (see [16]). Now (ii) is a consequence of Theorem 2.26. ☐

As another corollary, one sees that the spaces $L(\ell_p, \ell_1)$ and $L(H, \ell_1)$ have the 1-Schur property, for each $1 < p < \infty$ and for each Hilbert space H.

Refer to [6] for the definition of a Banach operator ideal. Later in this article, $\mathcal{U}$ will be an arbitrary Banach operator ideal with ideal norm A. For all Banach spaces X and Y, the components $\mathcal{U}(X, Y)$ are Banach spaces with respect to the norm A. If M is a closed subspace of $\mathcal{U}(X, Y)$, then for arbitrary elements $x \in X$ and $y^* \in Y^*$, the evaluation operators $\phi_x : M \to Y$ and $\psi_{y^*} : M \to X^*$ on M are defined by

$$\phi_x(T) = Tx, \qquad \psi_{y^*}(T) = T^* y^* \quad (T \in M).$$

Also, the point evaluation sets related to $x \in X$ and $y^* \in Y^*$ are the images of the closed unit ball B_M of M, under the evaluation operators ϕ_x and ψ_{y^*}, and they are denoted by $M(x)$ and $M^*(y^*)$, respectively (see [13], [17]). As an easy consequence of Theorem 2.20, we have the following result.

Corollary 3.4. *Let X and Y be Banach spaces. If the closed subspace M of $\mathcal{U}(X, Y)$ has the p-Schur property, then all evaluation operators ϕ_x and ψ_{y^*} are p-converging.*

Theorem 3.5. *Let X and Y be two Banach spaces such that Y has the Schur property. If M is a closed subspace of $\mathcal{U}(X, Y)$ such that each evaluation operator ψ_{y^*} is p-converging on M, then M has the p-Schur property.*

Proof. Suppose that M does not have the p-Schur property. Then there is a sequence $(T_n) \in \ell_p^{\text{weak}}(M)$ such that $\|T_n\| > \varepsilon$ for all positive integers n and some $\varepsilon > 0$. Choose a sequence (x_n) in B_X such that $\|T_n x_n\| > \varepsilon$. In addition, for each $y^* \in Y^*$, the evaluation operator ψ_{y^*} is p-converging. So $\|T_n^* y^*\| = \|\psi_{y^*}(T_n)\| \to 0$. Then $|\langle T_n x_n, y^* \rangle| \leq \|T_n^* y^*\| \|x_n\| \to 0$. Therefore, $(T_n x_n)$ is weakly null and so is norm-null, since Y has the Schur property. This is a contradiction. ☐

In the sequel, we give a sufficient condition for the p-Schur property of closed subspaces of the $L_{w^*}(X^*, Y)$ of all bounded w^*-w continuous operators from X^* to Y. Note that if $T \in L_{w^*}(X^*, Y)$, then for every $y^* \in Y^*$ and every weak*-null net $(x_\alpha^*) \subseteq X^*$, one has $\langle T^* y^*, x_\alpha^* \rangle = \langle y^*, T x_\alpha^* \rangle \to 0$. This means that $T^* y^*$ belongs to $(X^*, w^*)^* = X$ and so T^* transfers Y^* into X.

Theorem 3.6. *Let X and Y be two Banach spaces such that X has the Schur property. If M is a closed subspace of $L_{w^*}(X^*, Y)$ such that each evaluation operator ϕ_{x^*} is p-converging on M, then M has the p-Schur property.*

Proof. Assume that one can choose $(T_n) \in \ell_p^{\text{weak}}(M)$ such that $\|T_n\| > \varepsilon$ for some $\varepsilon > 0$ and all positive integers n. Then by our assumption, for each $x^* \in X^*$, $\|T_n x^*\| = \|\phi_{x^*}(T_n)\| \to 0$ as $n \to \infty$. Since $\|T_n^*\| > \varepsilon$, there exists a sequence (y_n^*) in B_{Y^*} such that $\|T_n^* y_n^*\| > \varepsilon$ for all $n \geq 1$. Hence $|\langle T_n^* y_n^*, x^*\rangle| \leq \|T_n x^*\| \to 0$ for each $x^* \in X^*$ and so $(T_n^* y_n^*)$ is a weakly null sequence in the Schur space X. This yields a contradiction. □

By considering $p = \infty$ in Theorem 3.6, we have the following corollary.

Corollary 3.7. *Let X and Y be two Banach spaces such that X has the Schur property. If M is a closed subspace of $L_{w^*}(X^*, Y)$ such that each evaluation operator ϕ_{x^*} is completely continuous, then M has the Schur property.*

Definition 3.8. A bounded subset K of a Banach space X is *p-Limited* if

$$\lim_n \sup_{x \in K} \big|\langle x, x_n^*\rangle\big| = 0$$

for every $(x_n^*) \in \ell_p^{\text{weak}}(X^*)$.

Theorem 3.9. *Let X and Y be two Banach spaces, and let the dual M^* of a closed subspace $M \subseteq \mathcal{U}(X, Y)$ have the p-Schur property. Then all of the point evaluations $M(x)$ and $M^*(y^*)$ are p-Limited.*

Proof. Since M^* has the p-Schur property, by Theorem 2.20 the adjoint operators ϕ_x^* and $\psi_{y^*}^*$ are p-converging. Now suppose that $(y_n^*) \in \ell_p^{\text{weak}}(Y^*)$. Then $\|\phi_x^* y_n^*\| \to 0$ as $n \to \infty$ for all $x \in X$. On the other hand,

$$\begin{aligned}\|\phi_x^* y_n^*\| &= \sup\big\{\big|\langle \phi_x^* y_n^*, T\rangle\big| : T \in B_M\big\}\\ &= \sup\big\{\big|\langle y_n^*, Tx\rangle\big| : T \in B_M\big\}.\end{aligned}$$

This shows that $M(x)$ is p-Limited in Y for all $x \in X$. A similar proof shows that $M^*(y^*)$ is a p-Limited set in X^*. □

In the sequel, we give some conditions for which the point evaluations $M(x)$ and $M^*(y^*)$ are relatively compact for all $x \in X$ and all $y^* \in Y^*$. If X^* and Y are Schur spaces and M^* has the 1-Schur property, then Theorem 2.26 implies that $M(x)$ and $M^*(y^*)$ are relatively compact. In addition, we have the following theorem.

Theorem 3.10. *Suppose that X^{**} and Y^* are weakly p-compact and that $M \subseteq \mathcal{U}(X, Y)$ is a closed subspace. If the natural restriction operator $R : \mathcal{U}(X, Y)^* \to M^*$ is p-converging, then all of the point evaluations $M(x)$ and $M^*(y^*)$ are relatively compact.*

Proof. Let $T \in \mathcal{U}(X, Y)$. Since $\|T\| \leq A(T)$, the mapping $\Psi : X^{**} \tilde{\otimes}_\pi Y^* \to \mathcal{U}(X, Y)^*$, which is defined by

$$v \mapsto \Big(T \mapsto \operatorname{tr}(T^{**}v) := \sum_{n=1}^{\infty} \langle T^{**} x_n^{**}, y_n^*\rangle\Big),$$

is linear and continuous, where $v = \sum_{n=1}^{\infty} x_n^{**} \otimes y_n^* \in X^{**} \tilde{\otimes}_\pi Y^*$. So the operator $\phi = R \circ \Psi$ is p-converging. Now fix $x \in X$, and define the operator $U_x : Y^* \to$

$X^{**} \tilde{\otimes}_\pi Y^*$ by $U_x(y^*) = x \otimes y^*$. Since $\phi \circ U_x$ is p-converging and Y^* is weakly p-compact, we conclude that $\phi_x^* = \phi \circ U_x$ is compact. So ϕ_x is compact. Similarly, for $y^* \in Y^*$, define $V_{y^*} : X^{**} \to X^{**} \tilde{\otimes}_\pi Y^*$ by $V_{y^*}(x^{**}) = x^{**} \otimes y^*$. Then $(\Psi_{y^*})^*$ is p-converging and so is compact. This completes the proof. $\square$

Acknowledgment. The authors would like to express their sincere gratitude to the anonymous referees for helpful comments that improved the presentation of this article.

References

1. F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Grad. Texts in Math. **233**, Springer, New York, 2006. Zbl 1094.46002. MR2192298. 125, 127, 130
2. J. Bourgain and J. Diestel, *Limited operators and strict cosingularity*, Math. Nachr. **119** (1984), 55–58. Zbl 0601.47019. MR0774176. DOI 10.1002/mana.19841190105. 125
3. S. W. Brown, *Weak sequential convergence in the dual of an algebra of compact operators*, J. Operator Theory **33** (1995), no. 1, 33–42. Zbl 0838.47032. MR1342475. 128
4. J. Castillo and F. Sanchez, *Dunford–Pettis-like properties of continuous vector function spaces*, Rev. Mat. Complut. **6** (1993), no. 1, 43–59. Zbl 0807.46033. MR1245024. 124, 131
5. D. Chen, J. A. Chávez-Domínguez, and L. Li, *Unconditionally p-converging operators and Dunford–Pettis property of order p*, preprint, arXiv:1607.02161v1 [math.FA]. 124
6. A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Math. Stud. **176**, North-Holland, Amsterdam, 1993. Zbl 0774.46018. MR1209438. 133
7. R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monogr. Surv. Pure Appl. Math. **64**, Wiley, New York, 1993. Zbl 0782.46019. MR1211634. 131
8. J. Diestel, "A survey of results related to the Dunford–Pettis property" in *Integration, Topology, and Geometry in Linear Spaces (Chapel Hill, NC, 1979)*, Contemp. Math. **2**, Amer. Math. Soc., Providence, 1980, 15–60. Zbl 0571.46013. MR0621850. 125, 129
9. L. Drewnowski, *On Banach spaces with the Gelfand–Phillips property*, Math. Z. **193** (1986), no. 3, 405–411. Zbl 0629.46020. MR0862887. DOI 10.1007/BF01229808. 125, 129
10. G. Emmanuele, *On Banach spaces with the Gelfand–Phillips property, III*, J. Math. Pures Appl. (9) **72** (1993), no. 3, 327–333. Zbl 0830.46011. MR1225513. 129
11. J. H. Fourie and E. D. Zeekoei, *DP*-properties of order p on Banach spaces*, Quaest. Math. **37** (2014), no. 3, 349–358. Zbl 06695977. MR3285289. DOI 10.2989/16073606.2013.779611. 128
12. O. F. K. Kalenda and J. Spurný, *On a difference between quantitative weak sequential completeness and the quantitative Schur property*, Proc. Amer. Math. Soc. **140** (2012), no. 10, 3435–3444. Zbl 1283.46009. MR2929012. DOI 10.1090/S0002-9939-2012-11175-X. 125, 126
13. S. M. Moshtaghioun and J. Zafarani, *Weak sequential convergence in the dual of operator ideals*, J. Operator Theory **49** (2003), no. 1, 143–151. Zbl 1019.47047. MR1978326. 125, 130, 133
14. A. Pelczynski, *Banach spaces on which every unconditionally converging operator is weakly compact*, Bull. Acad. Pol. Sci. **10** (1962), 641–648. Zbl 0107.32504. MR0149295. 128
15. R. A. Ryan, *The Dunford–Pettis property and projective tensor products*, Bull. Pol. Acad. Sci. Math. **35** (1987), no. 11-12, 785–792. Zbl 0656.46057. MR0961717. 132
16. R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Monogr. Math., Springer, London, 2002. Zbl 1090.46001. MR1888309. DOI 10.1007/978-1-4471-3903-4. 133
17. M. Salimi and S. M. Moshtaghioun, *The Gelfand–Phillips property in closed subspaces of some operator spaces*, Banach J. Math. Anal. **5** (2011), no. 2, 84–92. Zbl 1235.47021. MR2792501. DOI 10.15352/bjma/1313363004. 125, 133

18. D. P. Sinha and A. K. Karn, *Compact operators whose adjoints factor through subspaces of ℓ_p*, Studia Math. **150** (2002), no. 1, 17–33. Zbl 1008.46008. MR1893422. DOI 10.4064/sm150-1-3. 124
19. A. Ülger, *Subspaces and subalgebras of $K(H)$ whose duals have the Schur property*, J. Operator Theory **37** (1997), no. 2, 371–378. Zbl 0894.47033. MR1452283. 127

Department of Mathematics, Yazd University, P.O. Box 89195-741, Yazd, Iran
E-mail address: m.b.deh91@gmail.com; moshtagh@yazd.ac.ir