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INVOLUTIONS IN ALGEBRAS RELATED TO SECOND DUALS OF HYPERGROUP ALGEBRAS

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Abstract. Let K be a hypergroup. The purpose of this article is to study the question of involutions on algebras $M(K)^{**}$, $L(K)^{**}$, and $L_c(K)^{**}$. We show that the natural involution of $M(K)$ has the canonical extension to $M(K)^{**}$ if and only if the natural involution of $L(K)$ has the canonical extension to $L(K)^{**}$. Also, we give necessary and sufficient conditions for $M(K)^{**}$ and $L(K)^{**}$ to admit an involution extending the natural involution of $M(K)$ when K is left amenable. Finally, we find the necessary and sufficient conditions for $L_c(K)^{**}$ to admit an involution.

1. Introduction and preliminaries

For a locally compact Hausdorff space K, let $M(K)$ be the Banach space of all bounded complex regular Borel measures on K. For $x \in K$, δ_x will denote the unit point mass at x. Let $M_p(K)$ be the set of all probability measures on K, and let $C_b(K)$ be the Banach space of all continuous bounded complex-valued functions on K. We denote by $C_0(K)$ the space of all continuous functions on K vanishing at infinity, and we note by $C_c(K)$ the space of all continuous functions on K with compact support.

The space K is called a *hypergroup* if there is a map $\lambda : K \times K \longrightarrow M_p(K)$ with the following properties:

(i) for every $x, y \in K$, the measure $\lambda_{(x,y)}$ (the value of λ at (x, y)) has a compact support;

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(ii) for each $\psi \in C_c(K)$, the map $(x, y) \longmapsto \int_K \psi(t)\, d\lambda_{(x,y)}(t)$ is in $C_b(K \times K)$, and $x \longmapsto \int_K \psi(t)\, d\lambda_{(x,y)}(t)$ is in $C_c(K)$, for every $y \in K$;

(iii) the convolution $(\mu, \nu) \longmapsto \mu * \nu$ of measures defined by

$$\int_K \psi(t)\, d(\mu * \nu)(t) = \int_K \int_K \int_K \psi(t)\, d\lambda_{(x,y)}(t)\, d\mu(x)\, d\nu(y)$$

is associative, where $\mu, \nu \in M(K)$, $\psi \in C_0(K)$ (note that $\lambda_{(x,y)} = \delta_x * \delta_y$);

(iv) there is a unique point $e \in K$ such that $\lambda_{(x,e)} = \delta_x$, for all $x \in K$.

When $\lambda_{(x,y)} = \lambda_{(y,x)}$, we say that K is a *commutative hypergroup* (for more details see [4], [13]). We do not assume that K has a Haar measure. We define

$$L(K) = \{\mu \mid \mu \in M(K), x \mapsto |\mu| * \delta_x, x \mapsto \delta_x * |\mu| \text{ are norm-continuous}\},$$

which is an ideal in $M(K)$, and we let K be foundation; that is, $K = \mathrm{cl}(\bigcup_{\mu \in L(K)} \operatorname{supp} \mu)$. Also, if K admits an invariant measure (Haar measure m), then $L(K) = L^1(K, m)$ (see [13]).

An involution on a hypergroup K is a homeomorphism $x \mapsto \tilde{x}$ in K such that $\tilde{\tilde{x}} = x$ and $e \in \operatorname{supp} \lambda_{(x,\tilde{x})}$, for all $x \in K$. There exist many hypergroups with an involution that are not group (see [4, Examples 4.4, 4.5]). For each $\mu \in M(K)$, define $\widetilde{\mu} \in M(K)$ by $\widetilde{\mu}(A) = \overline{\mu(\tilde{A})}$; that is, $\int_K f(x)\, d\widetilde{\mu}(x) = \int_K f(\tilde{x})\, d\overline{\mu}(x)$, for each $f \in C_c(K)$. Then $\mu \longrightarrow \tilde{\mu}$ is an involution on $M(K)$ such that $M(K)$ and $L(K)$ are Banach*-algebras (see [7]) and $\widetilde{\lambda}_{(x,y)} = \lambda_{(\tilde{y},\tilde{x})}$, whenever $x, y \in K$ (see [4]).

Let $B = L(K)^* L(K)$. In [13], it was shown that B^* (dual of B) is a Banach algebra by an Arens-type product and that $L(K) \subseteq B^*$. If K admits an invariant measure (Haar measure) m, then

$$B = LUC(K) = \{f \in C_b(K) \mid x \to l_x f \text{ from } K \text{ into } C_b(K) \text{ is continuous}\},$$

where $l_x f(y) = f(x * y) = \int_K f(t)\, d\lambda_{(x,y)}$, for $y \in K$. Most of our notation in this paper comes from [13], where $E \in L(K)^{**}$ is the weak*-limit of (e_α), a bounded approximate identity in $L(K)$, and E is in fact a right identity for $L(K)^{**}$. Also, $\varepsilon(K) = \pi^{-1}(\delta_e)$ and $\varepsilon_1(K) = \{E \in \varepsilon(K) \mid \|E\| = 1\}$, where $\pi : L(K)^{**} \longrightarrow B^*$ is the adjoint of the embedding of B in $L(K)^*$. Following [13, Definition 8] a compact set $Z \subseteq K$ is called a *compact carrier* for $m \in L(K)^{**}$ if for all $f \in L(K)^*$, $\langle m, f \rangle = \langle m, f\chi_z \rangle$, where $f\chi_z$ is defined by $\langle f\chi_z, \mu \rangle = \langle f, \chi_z \mu \rangle$, for all $\mu \in L(K)$. Now, let

$$L_c(K)^{**} = \mathrm{cl}_{L(K)^{**}}\{m \mid m \in L(K)^{**}, m \text{ has a compact carrier}\}.$$

If $K = G$ is a locally compact group, then $L_c(K)^{**} = L_0^\infty(G)^*$, where $L_0^\infty(G)$ is the closed ideal of $L^\infty(G)$ consisting of all $f \in L^\infty(G)$ such that, for given $\epsilon > 0$, there exists a compact set $K \subset G$ such that $\|f\|_{G \setminus K} < \epsilon$.

This paper is organized as follows. In Section 2, by Theorem 2.2, we extend the result of Grosser [12, Theorem 2] to hypergroups. We show that the natural involution of $M(K)$ has the canonical extension to $M(K)^{**}$ if and only if the natural involution of $L(K)$ has the canonical extension to $L(K)^{**}$ (see Theorem 2.4). In Theorem 2.5, we also find the necessary and sufficient conditions for $L_c(K)^{**}$ to admit an involution, and, as an application, we answer in the affirmative a question raised by Farhadi and Ghahramani [5, Problem 5.3] when we replace $L_1(G)^{**}$ by $L_0^\infty(G)^*$. Also, in Theoerm 2.9, we prove that $M(K)^{**}$ has an involution extending the natural involution of $M(K)$ (say, γ) such that $\gamma(L_c(K)^{**}) = L_c(K)^{**}$ if and only if $L(K)^{**}$ has an involution extending the natural involution of $L(K)$ when K is left amenable. Indeed, we show that the second dual algebra $L(K)^{**}$ and $M(K)^{**}$ do not admit an involution extending the natural involution of $M(K)$ when K is an infinite amenable. Thus, for the above classes of hypergroups we answer in the negative a question raised by Duncan and Hosseiniun [3]. Finally, in Section 3, we give some sufficient conditions under which second dual Banach algebras $L(K)^{**}$ and $L_c(K)^{**}$ admit trivolutions.

2. Existence of involutions $L(K)^{**}$ and $L_c(K)^{**}$

Throughout this paper, K is a foundation hypergroup without a Haar measure and the second dual $L(K)^{**}$ with the first Arens product denoted by $(L(K)^{**}, \Box)$.

In this section, we investigate the existence of involutions on $L(K)^{**}$ and $L_c(K)^{**}$. The starting point of this section is the following proposition, which is an assistance for the proof of results of this paper.

Proposition 2.1. *Let K be a hypergroup. Then $L(K)^{**}$ with the first Arens product has an identity if and only if K is discrete.*

Proof. Suppose that $L(K)^{**}$ with the first Arens product has an identity E. Since $L(K)$ has a bounded approximate identity, we have $L(K)^* = L(K)^*L(K) = B$ (see [10, Proposition 2.2]). It follows that the natural embedding of B in $L(K)^*$ is the identity map and so is π, and, consequently, $M(K) = E \Box L_c(K)^{**}$ by [13, Proposition 13(a), Theorem 14(b)]. Therefore, by [13, Theorem 14(c)], we have

$$M(K) = \bigcap_{E \in \varepsilon_1(K)} E \Box L_c(K)^{**} = L(K).$$

So, $M(K) = L(K)$; that is, $\delta_e \in L(K)$. Thus K is discrete.

Conversely, if K is discrete, then $L(K)$ has an identity and so $L(K)^{**}$ has an identity. $\Box$

In [12, Theorem 2], Grosser showed that for the group algebra $L^1(G)$, a necessary condition for $L^1(G)^{**}$ to have an involution is that G is discrete. The next theorem is a generalization of Grosser's theorem for hypergroups.

Theorem 2.2. *Suppose that K is a hypergroup. If $L(K)^{**}$ admits an involution with respect to the first Arens product, then K is discrete.*

Proof. Since $L(K)^{**}$ admits an involution and $L(K)$ has a bounded approximate identity, $L(K)^*L(K) = L(K)^*$ (see [12, Theorem 1]), and, consequently, $L(K)^{**}$ has an identity (see [10, Proposition 2.2]). Therefore, by Proposition 2.1, K is discrete. $\Box$

Definition 2.3. Let A be a Banach algebra with an involution $\rho : A \longrightarrow A$. Then the second adjoint $\rho^{**} : A^{**} \longrightarrow A^{**}$ is called the *canonical extension* of ρ if ρ^{**} is an involution on A^{**} (with respect to either of the Arens products).

Now, we give a necessary and sufficient condition for the existence of a canonical extension of the natural involution of $L(K)$ and $M(K)$.

Theorem 2.4. *Let K be a hypergroup with an involution $\sim : K \to K$. Then the following statements are equivalent:*

(i) *The natural involution of $M(K)$ has the canonical extension to $M(K)^{**}$.*
(ii) *K is finite.*
(iii) *The natural involution of $L(K)$ has the canonical extension to $L(K)^{**}$.*

Proof. (i) $\Rightarrow$ (ii). Since the natural involution of $M(K)$ has the canonical extension to $M(K)^{**}$, $M(K)$ is Arens regular by [1, Theorem 6.2], and, hence, e is isolated in $K = \operatorname{supp} L(K)$ by [9, Theorem 3.1], because the convolution on $M(K)$ is *weak** separately continuous in $\sigma(M(K), C_0(K))$ and $M(K)$ is Arens-regular. Therefore, there exists an element $\mu \in L(K)$ such that $e \in \operatorname{supp} \mu$ and $|\mu|(\{e\}) \neq 0$ (since the set $\{e\}$ is open in K). Define

$$\nu = \frac{\chi_e|\mu|}{|\mu|(\{e\})}.$$

We see that $\nu \in L(K)$ and is a multiple of δ_e. It follows that K is discrete. In this case, the concept of Jewett's hypergroup [7] and the concept of Dunkl's hypergroup coincide. So, K has a Haar measure, and $L^1(K) = M(K)$ (see [7, Theorem 7.1.A]). Now, since $L^1(K) = M(K)$ is Arens-regular, K is finite (see [8, Corollary 4.2.7]).

(ii) $\Rightarrow$ (iii). Since K is finite, $L^1(K) = L(K)$ is finite-dimensional. Thus, $L(K) = L(K)^{**}$, and this completes the proof.

(iii) $\Rightarrow$ (i). From Theorem 2.2, K is discrete, and, consequently, (i) is proved. $\square$

In general, since A is not norm dense in A^{**}, involutions on A may have extensions to A^{**} which are different from the canonical extension. By the following theorem, we obtain sufficient and necessary conditions for $L_c(K)^{**}$ to have an involution extending the natural involution of $L(K)$.

Theorem 2.5. *Let K be a hypergroup with an involution $\sim : K \to K$ and endow $L_c(K)^{**}$ with the first Arens product. Then the following assertions are equivalent.*

(i) *The algebra $L_c(K)^{**}$ has an involution extending the natural involution of $L(K)$.*
(ii) *K is discrete.*
(iii) *$L_c(K)^{**}$ is semisimple.*

Proof. (i) $\Rightarrow$ (ii). By assumption, let ρ be an involution on $L_c(K)^{**}$. If K is compact, then $L_c(K)^{**} = L(K)^{**}$. So, ρ is an involution on $L(K)^{**}$, and, consequently, by Theorem 2.2, K is discrete.

Now, suppose that K is noncompact. $L(K)$ has a bounded approximate identity $(e_\alpha)_\alpha$ with $\|e_\alpha\| = 1$ [13]. Let E be a *weak** cluster point of $(e_\alpha)_\alpha$ in $L(K)^{**}$; it is clear that E is a right identity for $L(K)^{**}$ and $\|E\| = 1$ (see [13, Lemma 5]). By [13, Theorem 14(a)], $E \in L_c(K)^{**}$. On the other hand, $\rho(E)$ is a left identity for $L_c(K)^{**}$ and $\rho(E) = E$. So, E is an identity for $L_c(K)^{**}$. Therefore, by [13, Theorem 14(e)], we have

$$E \in Z_t\big(L_c(K)^{**}\big) = L(K),$$

where $Z_t(L_c(K)^{**}) = \{F \in L_c(K)^{**} \mid m \to F \,\Box\, m$ is weak*–weak* continuous$\}$. Hence, $L(K)$ has an identity and so K is discrete.

(ii) $\Rightarrow$ (i). Let K be a discrete hypergroup. Then the natural embedding of B in $L(K)^*$ is the identity map and so is π, and, consequently, $L_c(K)^{**} = M(K) = L(K)$ (see [13, Proposition 13(a)]). It follows that $L_c(K)^{**}$ has a natural involution.

(ii) $\Rightarrow$ (iii). Let K be a discrete hypergroup. Thus, $L_c(K)^{**} = M(K) = L(K)$, and K has a Haar measure. Now, the mapping

$$T : M(K) \longrightarrow B\big(L^2(K)\big), \qquad \mu \longmapsto T_\mu,$$

where $T_\mu(f) = \mu * f$, for all $f \in L^2(K)$. By [7, Theorem 6.2I], T is a faithful norm-decreasing unital $*$-representation of $M(K)$. It follows that $L_c(K)^{**} = M(K)$ is semisimple, because, by [2, Theorem 3.1.17], $L_c(K)^{**} = M(K)$ is $*$-semisimple, and, consequently, it is semisimple (see [2, p. 347]).

(iii) $\Rightarrow$ (ii). Let $L_c(K)^{**}$ be semisimple. Let $E \in \varepsilon_1(K) \subseteq L_c(K)^{**}$. Now, define the mapping

$$\varphi : L_c(K)^{**} \longrightarrow E \,\Box\, L_c(K)^{**}, \qquad m \longmapsto E \,\Box\, m.$$

Let $m \in \ker\varphi$, so $E \,\Box\, m = 0$. Now,

$$n \,\Box\, m = n \,\Box\, E \,\Box\, m = 0,$$

for all $n \in L_c(K)^{**}$. Hence, $L_c(K)^{**} \,\Box\, m = \{0\}$ and so

$$L_c(K)^{**} \,\Box\, m \subseteq q - \mathrm{Inv}\big(L_c(K)^{**}\big),$$

the set of all quasi-invertible elements of $L_c(K)^{**}$. Therefore,

$$m \in \mathrm{rad}\big(L_c(K)^{**}\big),$$

the intersection of the kernels of the irreducible representations of $L_c(K)^{**}$; this is because

$$\mathrm{rad}\big(L_c(K)^{**}\big) = \big\{n \in L_c(K)^{**} \;\big|\; L_c(K)^{**} \,\Box\, n \subseteq q - \mathrm{Inv}\big(L_c(K)^{**}\big)\big\}.$$

Thus, $m = 0$. It follows that φ is an isomorphism. Therefore,

$$L_c(K)^{**} = E \,\Box\, L_c(K)^{**}.$$

By [13, Theorem 14(c)], we have

$$L_c(K)^{**} = \bigcap_{E \in \varepsilon_1(K)} E \,\Box\, L_c(K)^{**} = L(K).$$

Thus, $L(K)$ has a right identity. It follows that K is discrete. $\square$

In [5, Problem 5.3], Farhadi and Ghahramani ask the following question.

Question 2.6. Let G be discrete, and suppose that $l^1(G)^{**}$ admits an involution that extends the natural involution of $l^1(G)$. Does it then follow that $\mathrm{rad}(l^1(G)^{**}) = \{0\}$?

Although we have not answered the above question, we are able to show the following.

Corollary 2.7. *Let G be a locally compact group. Then,* $\mathrm{rad}(L_0^\infty(G)^*) = \{0\}$ *if and only if G is discrete.*

Let $f \in B$. Then, $f = g\mu$ $(g \in L^*(K), \mu \in L(K))$. For $x \in K$, we define the function $l_x f$ by $\langle l_x f, \nu\rangle = \langle f, \delta_x * \nu\rangle$, whenever $\nu \in L(K)$. We have

$$\langle l_x f, \nu\rangle = \langle g\mu, \delta_x * \nu\rangle = \langle g, \mu * \delta_x * \nu\rangle = \big\langle g(\mu * \delta_x), \nu\big\rangle.$$

Hence, $l_x f = g(\mu * \delta_x)$. It follows that $l_x f \in B$. Also, by [13, Proposition 2], $1 \in B$, where 1 is the constant function.

Definition 2.8. Let K be a hypergroup. A linear functional $m : B \longrightarrow \mathbb{C}$ is called a *mean* if $m(1) = \|m\| = 1$. A mean on B is called a *left invariant mean* if $m(l_x f) = m(f)$, for $f \in B$ and $x \in K$. A hypergroup K is called *left amenable* if there exists a left invariant mean on B.

Theorem 2.9. *Suppose that K is a left amenable hypergroup with an involution $\sim : K \to K$. Then the following are equivalent:*

(i) *$M(K)^{**}$ has an involution extending the natural involution of $M(K)$ (say, γ) such that $\gamma(L_c(K)^{**}) = L_c(K)^{**}$.*
(ii) *$L(K)^{**}$ has an involution extending the natural involution of $L(K)$.*
(iii) *K is finite.*

Proof. (i) $\Rightarrow$ (ii). Since $\gamma|_{L_c(K)^{**}} : L_c(K)^{**} \to L_c(K)^{**}$ is an involution, by Theorem 2.5, K is discrete, and, consequently, $M(K) = L(K)$. Therefore, $L(K)^{**}$ has an involution extending the natural involution of $L(K)$.

(ii) $\Rightarrow$ (iii). Since K is a hypergroup with an involution, $L(K)$ is a $*$-Banach algebra (with involution $\rho(\mu) = \widetilde{\mu}$). Let

$$\gamma : L(K)^{**} \to L(K)^{**}$$

be an involution on $L(K)^{**}$ extending the natural involution of $L(K)$. Thus, by Theorem 2.2, K is discrete, and so K has a Haar measure (see [7, Theorem 7.1.A]). Therefore, $L^1(K) = L(K)$ and $B = L_\infty(K)$. Let K be a noncompact hypergroup. Since K is amenable, by [15, Theorem 3.2], $TIM(L_\infty(K)) \neq \emptyset$ (topological two-sided invariant means on $L_\infty(K)$). Let m be a topological two-sided invariant mean on $L_\infty(K)$. By [15, Lemma 3.1], m is a two-sided invariant mean on $L_\infty(K)$. For $x \in K$ and $\mu \in M(K) = L^1(K)$, we have

$$\hat{\delta}_x \,\Box\, \gamma(m) = \gamma(\hat{\delta}_{\widetilde{x}}) \,\Box\, \gamma(m) = \gamma(m \,\Box\, \hat{\delta}_{\widetilde{x}}) = \gamma(m), \tag{4.1}$$

and also,

$$\gamma(m) \,\Box\, \hat{\delta}_x = \gamma(m) \,\Box\, \gamma(\hat{\delta}_{\widetilde{x}}) = \gamma(\hat{\delta}_{\widetilde{x}} \,\Box\, m) = \gamma(m).$$

It means that $\gamma(m)$ is two-sided translation invariant.

On the other hand, since $\|m\| = 1$, $m \geq 0$ and $L^1(K)^{**}$ has an identity, by [13, Theorem 18(d)], there exists a net $(\mu_\alpha) \subseteq M(K) = L^1(K)$ such that $\widehat{\mu}_\alpha \longrightarrow m$ in $\sigma(L^1(K)^{**}, L_\infty(K))$, where

$$\mu_\alpha = \sum_{i=1}^{t_\alpha} \lambda_{i,\alpha}\delta_{x_{i,\alpha}}, \qquad \sum_{i=1}^{t_\alpha} \lambda_{i,\alpha} = 1.$$

Now, for all $f \in L_\infty(K)$,

$$\begin{aligned}\big\langle m \,\Box\, \gamma(m), f\big\rangle &= \big\langle m, \gamma(m).f\big\rangle \\ &= \lim_\alpha \big\langle \hat{\mu}_\alpha, \gamma(m).f\big\rangle \\ &= \lim_\alpha \Big\langle \sum_{i=1}^{t_\alpha} \lambda_{i,\alpha}\hat{\delta}_{x_{i,\alpha}}, \gamma(m).f\Big\rangle \\ &= \lim_\alpha \Big\langle \Big(\sum_{i=1}^{t_\alpha} \lambda_{i,\alpha}\hat{\delta}_{x_{i,\alpha}}\Big) \,\Box\, \gamma(m), f\Big\rangle \\ &= \lim_\alpha \Big\langle \sum_{i=1}^{t_\alpha} \lambda_{i,\alpha}\gamma(m), f\Big\rangle \\ &= \big\langle \gamma(m), f\big\rangle.\end{aligned}$$

Thus, $m \,\Box\, \gamma(m) = \gamma(m)$. So,

$$m \,\Box\, \gamma(m) = \gamma\big(m \,\Box\, \gamma(m)\big) = \gamma\big(\gamma(m)\big) = m.$$

It follows that for all two-sided topological invariant means m on $L_\infty(K)$, we have

$$\gamma(m) = m.$$

Let m_1 and m_2 be any pair of two-sided topological invariant means on $L_\infty(K)$. Since $\hat{\delta}_x \,\Box\, m_2 = m_2$, by a similar argument as above, one can show that

$$m_1 \,\Box\, m_2 = m_2.$$

Thus, $\gamma(m_1 \,\Box\, m_2) = \gamma(m_2) = m_2$. Therefore, we have

$$m_2 = m_1 \,\Box\, m_2 = \gamma(m_1 \,\Box\, m_2) = \gamma(m_2) \,\Box\, \gamma(m_1) = m_2 \,\Box\, m_1 = m_1.$$

Thus, $|TIM(L_\infty(K))| = 1$. On the other hand, by [15, Theorem 5.5], $|TIM(L_\infty(K))| = 2^{2^d}$. Hence, this is a contradiction. Therefore, K is compact. Now, since K is compact and discrete, K is finite.

(ii) $\Rightarrow$ (i). By assumption, K is finite. Hence, $L_c(K)^{**} = L(K)$ and $M(K) = L(K) = M(K)^{**}$. This completes the proof. $\square$

Duncan and Hosseiniun [3] ask whether there is an involution on $L^1(G)^{**}$ extending the natural involution of $L^1(G)$. The following corollary shows that if K is an infinite amenable hypergroup, the answer to this question is negative.

Corollary 2.10. *Let K be an infinite amenable hypergroup with an involution; then $L(K)^{**}$ does not admit an involution extending the natural involution of $L(K)$.*

Now, we are in a position to state and prove our other result. Parts of our proofs are adapted from [11, Theorem 1.3].

Theorem 2.11. *Let K be a hypergroup with an involution $\sim : K \to K$. Then the following assertions are equivalent:*

(i) *$L(K)^{**}$ is amenable.*
(ii) *K is finite.*
(iii) *$L(K)$ is Arens-regular.*

Proof. (i) $\Rightarrow$ (ii): Since $L(K)^{**}$ is amenable, $L(K)^{**}$ has a bounded approximate identity. Thus, by [11, Lemma 1.1], $L(K)^{**}$ has an identity. Therefore, by Proposition 2.1, K is discrete, and, consequently, $L(K) = M(K) = EL_c(K)^{**} = L_c(K)^{**}$. Now, since $L(K)^{**} = B^* = L(K) \oplus C_0(K)^{\perp}$ and $L(K)^{**}$ is amenable, $C_0(K)^{\perp}$ is amenable, and, therefore, $C_0(K)^{\perp}$ has a bounded approximate identity (e_α). Let $e_\alpha \longrightarrow e$ in $\sigma(L(K)^{**}, L(K)^*)$. Since (e_β) is a bounded approximate identity, $e_\alpha \,\square\, m \longrightarrow m$ in norm and $e_\alpha \,\square\, m \longrightarrow e \,\square\, m$ in $\sigma(L(K)^{**}, L(K)^*)$, for all $m \in C_0(K)^{\perp}$. It follows that $e \,\square\, m = m$, and, consequently, e is a left identity for $C_0(K)^{\perp}$. Therefore, e is an identity for $C_0(K)^{\perp}$. Now, let $n \in L(K)^{**}$. Since $C_0(K)^{\perp}$ is a closed ideal of $L(K)^{**}$, $e \,\square\, n, n \,\square\, e \in C_0(K)^{\perp}$ and $e \,\square\, n = (e \,\square\, n) \,\square\, e = e \,\square\, (n \,\square\, e) = n \,\square\, e$. Thus, $e \in Z_t(L(K)^{**})$. On the other hand, since K is discrete, $Z_t(L(K)^{**}) = L(K) = M(K)$ (see [8, Theorem 4.2.5]). It follows that $e \in M(K) \cap C_0(K)^{\perp} = \{0\}$. Therefore, $C_0(K)^{\perp} = \{0\}$, and, consequently, $L(K)^{**} = L(K) = L_c(K)^{**}$. Thus, by [13, Theorem 14(f)], K is compact. So, K is finite.

(ii) $\Rightarrow$ (iii). It is trivial.

(iii) $\Rightarrow$ (ii). Let $L(K)$ be Arens-regular. By [1, Theorem 6.2], the natural involution of L(K) has the canonical extension to $L(K)^{**}$. Therefore, by Theorem 2.4, K is finite.

(ii) $\Rightarrow$ (i): Since K is discrete and compact, $L(K)^{**} = L_c(K)^{**}$, $L(K)^{**}$ is finite-dimensional, and, by Theorem 2.5, $L(K)^{**}$ is semisimple. Now, by the *Wedderburn structure theorem* (see [2, Theorem 1.5.9]), we have

$$L(K)^{**} \cong \bigodot_{j=1}^{N} M_{n_j},$$

where M_{n_j} is an $(n_j \times n_j)$-matrix. By [14, Theorem 2.2.4, Example 2.2.3], each M_{n_j} is amenable. A simple argument regarding adjoining the (virtual) diagonals together implies that the product $\bigodot_{j=1}^{N} M_{n_j}$ is amenable. Thus, $L(K)^{**}$ is amenable. $\square$

3. Existence of trivolution

In [6], Filali, Sangani Monfared, and Singh defined a trivolution on a complex algebra A. They obtained characterizations of trivolutions and showed with examples that they appear naturally on many Banach algebras, particularly those arising from group algebras. In this section, we give some sufficient conditions under which second dual Banach algebras $L(K)^{**}$ and $L_c(K)^{**}$ admit trivolutions.

Definition 3.1. A *trivolution* on a complex algebra A is a nonzero, conjugate linear, antihomomorphism $\tau : A \longrightarrow A$, such that $\tau^3 = \tau \circ \tau \circ \tau = \tau$. When A is a normed algebra, we shall assume that $\|\tau\| = 1$. The pair (A, τ) will be called a *trivolutive* algebra.

It follows from the definition that every involution on a nonzero complex algebra is a trivolution. Conversely, a trivolution which is either injective or surjective is an involution (for more details see [6]).

Theorem 3.2. *Let K be a compact hypergroup with an involution. Then for each $E \in \varepsilon(K)$, there are trivolutions of $L(K)^{**}$ onto $E \,\Box\, L(K)^{**}$.*

Proof. Let $E \in \varepsilon(K)$. The compactness of K implies that

$$L(K)^{**} = L_c(K)^{**} \qquad \text{and} \qquad M(K) = B^*.$$

Thus, B^* has an involution. Let ρ be any involution on B^*, and let $\sigma = \pi \mid_{E \Box L(K)^{**}}$. It follows from [13, Theorem 14(c)] that

$$\rho' = \sigma^{-1} \circ \rho \circ \sigma$$

is an involution on $EL(K)^{**}$. Define

$$\begin{aligned} \varphi : L(K)^{**} &\longrightarrow E \,\Box\, L(K)^{**}, \\ m &\longmapsto E \,\Box\, m. \end{aligned}$$

By [6, Theorem 2.8(iii)],

$$\theta := \rho' \circ \varphi$$

is a trivolution of $L(K)^{**}$ onto $E \,\Box\, L(K)^{**}$. ☐

Theorem 3.3. *Let K be a hypergroup with an involution. Then, for each $E \in \varepsilon_1(K)$, there exists a trivolution of $L_c(K)^{**}$ onto $E \,\Box\, L_c(K)^{**}$.*

Proof. By [13, Theorem 14], $\varepsilon_1(K) \subseteq L_c(K)^{**}$, and $E \,\Box\, L_c(K)^{**}$ is isometrically isomorphic to $M(K)$. Now, let ρ be an involution on $M(K)$. Thus,

$$\rho' := (\pi|_{L_c(K)^{**}})^{-1} \circ \rho \circ (\pi|_{L_c(K)^{**}})$$

is an involution on $L_c(K)^{**}$, and, consequently, by [6, Theorem 2.8(iii)],

$$\theta := \rho' \circ \varphi$$

is a trivolution of $L_c(K)^{**}$ onto $E \,\Box\, L_c(K)^{**}$, where

$$\begin{aligned} \varphi : L_c(K)^{**} &\longrightarrow E \,\Box\, L_c(K)^{**}, \\ m &\longmapsto E \,\Box\, m. \end{aligned}$$

☐

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