Ann. Funct. Anal. 9 (2018), no. 1, 87–100
https://doi.org/10.1215/20088752-2017-0026
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

ATOMIC DECOMPOSITION OF VARIABLE HARDY SPACES VIA LITTLEWOOD–PALEY–STEIN THEORY

JIAN TAN

Communicated by S. Barza

Abstract. The purpose of this paper is to give a new atomic decomposition for variable Hardy spaces via the discrete Littlewood–Paley–Stein theory. As an application of this decomposition, we assume that T is a linear operator bounded on L^q and $H^{p(\cdot)}$, and we thus obtain that T can be extended to a bounded operator from $H^{p(\cdot)}$ to $L^{p(\cdot)}$.

1. Introduction

Hardy spaces play a crucial role in the study of singular integral operators and their application to partial differential equations. In the classical case, Hardy space can be characterized by the Littlewood–Paley–Stein square functions, maximal functions, and atomic decompositions. Atomic decomposition is an especially significant tool in harmonic analysis and wavelet analysis for the study of function spaces and the operators acting on these spaces (see [2], [15]). Atomic decomposition was first introduced by Coifman [1] in 1-dimensional cases in 1974, and later it was extended to higher dimensions by Latter [14]. Atomic decompositions of Hardy spaces play an important role in the boundedness of operators on Hardy spaces, and it is usually sufficient to check that atoms are mapped into bounded elements of quasi-Banach spaces. The literature thus far suggests that atomic decomposition of the Hardy spaces is only one tool for proving the boundedness from H^p to L^p for singular integral operators. Recently, Zhao and Han [20] gave

Received Sep. 21, 2016; Accepted Feb. 23, 2017.
First published online Aug. 14, 2017.
2010 *Mathematics Subject Classification.* Primary 46E35; Secondary 47A30, 42B30.
Keywords. variable exponent, atomic decomposition, Hardy spaces, Littlewood–Paley–Stein functions.

a new atomic decomposition which converges in both $H^p(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$ rather than only in the distribution sense. Then Han, Lee, and Lin [11] showed that in weighted Hardy spaces $H^p(w)$ the atomic decomposition also converges in both $H^p(w)$ and $L^2(w)$.

In the present article, we study variable Hardy spaces. Note that the variable exponent spaces, such as the variable Lebesgue spaces and the variable Sobolev spaces, were studied by a substantial number of researchers (see, for example, [3], [4], [8], [13]). Recently the atomic decomposition of variable Hardy spaces was established independently by using maximal function characterizations (see [5, Theorem 6.3], [16, Theorem 4.5]).

The main purpose of this paper is to provide a new atomic decomposition of variable Hardy spaces via the discrete Littlewood–Paley–Stein theory. As an application, we derive the boundedness $H^{p(\cdot)} \to L^{p(\cdot)}$ via the boundedness from $H^{p(\cdot)} \to H^{p(\cdot)}$ for linear operators. To state the results, we begin with the definition of Lebesgue spaces with variable exponent, and we make some notations. For any Lebesgue measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ and for any measurable subset $E \subset \mathbb{R}^n$, we denote $p^-(E) = \inf_{x\in E} p(x)$ and $p^+(E) = \sup_{x\in E} p(x)$. We especially denote $p^- = p^-(\mathbb{R}^n)$ and $p^+ = p^+(\mathbb{R}^n)$. The symbols $\mathcal{S}$ and $\mathcal{S}'$ denote the class of Schwartz functions and tempered functions, respectively. As usual, for a function ψ on $\mathbb{R}^n$, $\psi_t(x) = t^{-n}\psi(t^{-1}x)$.

Definition 1.1. Let $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ be a Lebesgue measurable function. The variable Lebesgue space $L^{p(\cdot)}$ consists of all Lebesgue measurable functions f, for which the quantity $\int_{\mathbb{R}^n} |\varepsilon f(x)|^{p(x)}\, dx$ is finite for some $\varepsilon > 0$, and

$$\|f\|_{L^{p(\cdot)}} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}\, dx \le 1\Big\}.$$

Variable Lebesgue spaces were first established by Orlicz [18] in 1931. Two decades later, Nakano [17] systematically studied modular function spaces, including the variable Lebesgue spaces; modern development of the concept started in 1991 with Kováčik and Rákosník's paper [13]. As a special case of the theory of Nakano and Luxemberg (see [17]), we see that $L^{p(\cdot)}$ is a quasinormed space. This is especially true when $p^- \ge 1$, $L^{p(\cdot)}$ is a Banach space.

Let $p(\cdot)$: $\mathbb{R}^n \to (0, \infty)$ be a measurable function with $0 < p^- \le p^+ < \infty$, and let $\mathcal{P}^0$ be the set of all these $p(\cdot)$. Let $\mathcal{P}$ denote the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that $1 < p^- \le p^+ < \infty$. We now recall the following class of exponent functions, which can be found in [7, Theorem 8.1]. Let $\mathcal{B}$ be the set of $p(\cdot) \in \mathcal{P}$ such that the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}$.

An important subset of $\mathcal{B}$ is the log-Hölder (LH) condition. In the study of variable exponent function spaces it is common to assume that the exponent function $p(\cdot)$ satisfies the LH-condition. We say that $p(\cdot) \in \mathrm{LH}$ if $p(\cdot)$ satisfies

$$\big|p(x) - p(y)\big| \le \frac{C}{-\log(|x - y|)}, \quad |x - y| \le 1/2$$

and that

$$\left|p(x) - p(y)\right| \le \frac{C}{\log|x| + e}, \quad |y| \ge |x|.$$

It is well known that $p(\cdot) \in \mathcal{B}$ if $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}$. Moreover, various examples show that the above LH-conditions are necessary in a certain sense (see Pick and Růžička [19] for more details).

Definition 1.2 ([5, Section 3], [16, Section 3]). Let $f \in \mathcal{S}'$, let $\psi \in \mathcal{S}$, let $p(\cdot) \in \mathcal{P}^0$, and let $\psi_t(x) = t^{-n}\psi(t^{-1}x)$, $x \in \mathbb{R}^n$. Denote by $\mathcal{M}$ the grand maximal operator given by $\mathcal{M}f(x) = \sup\{|\psi_t * f(x)| : t > 0, \psi \in \mathcal{F}_N\}$ for any fixed large integer N, where $\mathcal{F}_N = \{\varphi \in \mathcal{S} : \int \varphi(x)\, dx = 1, \sum_{|\alpha| \le N} \sup(1+|x|)^N |\partial^\alpha \varphi(x)| \le 1\}$. The variable Hardy space $\mathcal{H}^{p(\cdot)}$ is the set of all $f \in \mathcal{S}'$ for which the quantity

$$\|f\|_{H^{p(\cdot)}} = \|\mathcal{M}f\|_{L^{p(\cdot)}} < \infty.$$

Throughout this paper, C or c will denote a positive constant which may vary at each occurrence but which is independent of the essential variables, and $A \sim B$ means that there are constants $C_1 > 0$ and $C_2 > 0$ independent of the essential variables such that $C_1 B \le A \le C_2 B$. Given a measurable set $S \subset \mathbb{R}^n$, $|S|$ denotes the Lebesgue measure and χ_S means the characteristic function. We also use the notation $j \wedge j' = \min\{j, j'\}$ and $j \vee j' = \max\{j, j'\}$. In what follows, we recall the new atoms for variable Hardy spaces, which were first introduced in [16]. Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, $0 < p^- \le p^+ \le 1 < q \le \infty$. Fix an integer $d \ge d_{p(\cdot)} \equiv \min\{d \in \mathbb{N} \cup \{0\} : p^-(n+d+1) > n\}$. A function a on $\mathbb{R}^n$ is called a $(p(\cdot), q)$-*atom* if there exists a cube Q such that $\operatorname{supp} a \subset Q$; $\|a\|_{L^q} \le \frac{|Q|^{1/q}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$; $\int_{\mathbb{R}^n} a(x) x^\alpha\, dx = 0$ for $|\alpha| \le d$.

Now, let us state our main results.

Theorem 1.3. *Let $p(\cdot) \in \mathrm{LH} \cap \mathcal{P}^0, 1 < q < \infty$. If $f \in L^q \cap H^{p(\cdot)}$, there exist sequences of $(p(\cdot), q)$-atoms $\{a_j\}$ and scalars $\{\lambda_j\}$ such that $f = \sum_j \lambda_j a_j$, where the series converges to f in both $H^{p(\cdot)}$- and L^q-norms. Next write*

$$\mathcal{A}\big(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\big) = \left\| \left\{ \sum_j \left(\frac{|\lambda_j| \chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right)^{p^-} \right\}^{\frac{1}{p^-}} \right\|_{L^{p(\cdot)}}.$$

Then we have

$$\mathcal{A}\big(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\big) \le C\|f\|_{H^{p(\cdot)}}$$

As an application of this new atomic decomposition, we obtain the following.

Corollary 1.4. *Let $p(\cdot) \in \mathrm{LH} \cap \mathcal{P}^0, 1 < q < \infty$. Suppose that a linear operator T is bounded on L^q and $H^{p(\cdot)}$. Then T can be extended to a bounded operator from $H^{p(\cdot)}$ to $L^{p(\cdot)}$.*

Remark 1.5. It is well known that Calderón–Zygmund operators are bounded on $L^q, 1 < q < \infty$ and from H^p to L^p for certain p. The regularities of kernels of Calderón–Zygmund operators play a crucial role in the proof. However, Corollary 1.4 gives a similar but general result, which does not use the regularity condition directly. In fact, we only require the boundedness results of linear operators T and atomic decomposition for variable Hardy spaces $H^{p(\cdot)}$ here. Moving in another direction, in order to prove the H^p to L^p, the regularity of kernels of Calderón–Zygmund operators T is required to deal with $\|Ta\|_{L^p} \le C$, where a is classical (p,q)-atom for H^p.

2. Proof of Theorem 1.3 and Corollary 1.4

To show Theorem 1.3, we will apply the Littlewood–Paley–Stein theory. To be precise, let $\psi \in \mathcal{S}$ satisfy

$$\operatorname{supp} \widehat{\psi} \subset \big\{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\big\},$$

and let

$$\sum_{j\in\mathbb{Z}} \big|\widehat{\psi}(2^{-j}\xi)\big|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

Denote by $\mathcal{S}_\infty$ the set of functions $f \in \mathcal{S}$ satisfying $\int_{\mathbb{R}^n} f(x)x^\alpha\,dx = 0$ for all muti-indices $\alpha \in \mathbb{Z}_+^n := (\{0,1,2,\dots\})^n$. Denote by $\mathcal{S}'_\infty$ its topological dual space. For $f \in \mathcal{S}'_\infty$, we recall the definition of the Littlewood–Paley–Stein square function of f,

$$\mathcal{G}(f)(x) := \Big(\sum_{j\in\mathbb{Z}} \big|\psi_j * f(x)\big|^2\Big)^{1/2},$$

and the discrete Littlewood–Paley–Stein square function,

$$\mathcal{G}^d(f)(x) := \Big(\sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^n} \big|\psi_j * f(2^{-j}\mathbf{k})\big|^2 \chi_Q(x)\Big)^{1/2},$$

where Q here denotes dyadic cube in $\mathbb{R}^n$ with side length 2^{-j} and the lower-left corner of Q is $2^{-j}\mathbf{k}$.

First we recall the well-known discrete Calderón identity, which can be found in [9, Lemma 2.1].

Proposition 2.1. *Let ψ be the function mentioned above. Then for all $f \in \mathcal{S}'_\infty$,*

$$f(x) = \sum_{j\in\mathbb{Z}} 2^{-jn} \sum_{\mathbf{k}\in\mathbb{Z}^n} \psi_j * f(2^{-j}\mathbf{k})\psi_j(x - 2^{-j}\mathbf{k}),$$

where the series converges in L^2, $\mathcal{S}_\infty$, and $\mathcal{S}'_\infty$.

We also need the following boundedness of the vector-valued maximal operator M, whose proof can be found in [4, Corollary 2.1].

Proposition 2.2. *Let $p(\cdot) \in \mathrm{LH} \cap \mathcal{P}$. Then for any $q > 1$, $f = \{f_i\}_{i\in\mathbb{Z}}$, $f_i \in L_{\mathrm{loc}}$, $i \in \mathbb{Z}$,*

$$\big\|\big\|\mathbb{M}(f)\big\|_{l^q}\big\|_{L^{p(\cdot)}} \le C\big\|\|f\|_{l^q}\big\|_{L^{p(\cdot)}},$$

where $\mathbb{M}(f) = \{M(f_i)\}_{i\in\mathbb{Z}}$.

The following proposition gives equivalent characterizations of $H^{p(\cdot)}$.

Proposition 2.3. *Let $p(\cdot) \in \mathrm{LH} \cap \mathcal{P}^0$. Then for all $f \in \mathcal{S}'_\infty$, let*

$$\|f\|_{H^{p(\cdot)}} \sim \big\|\mathcal{G}(f)\big\|_{L^{p(\cdot)}} \sim \big\|\mathcal{G}^d(f)\big\|_{L^{p(\cdot)}}.$$

Proof. The equivalence of the first two norms was proved in [21, Theorem 1.4]. To complete the proof of Proposition 2.3, we only need to prove that

$$\big\|\mathcal{G}(f)\big\|_{L^{p(\cdot)}} \sim \big\|\mathcal{G}^d(f)\big\|_{L^{p(\cdot)}}.$$

First, we show that

$$\big\|\mathcal{G}(f)\big\|_{L^{p(\cdot)}} \leq \big\|\mathcal{G}^d(f)\big\|_{L^{p(\cdot)}}.$$

To do this, let $f \in \mathcal{S}'_\infty$. The discrete Calderón identity and the almost-orthogonality estimates yield that, for any $v \in Q$ and $\frac{n}{M+n} < \delta < (p^- \wedge 1)$,

$$\begin{aligned}
&\big|(\psi_j * f)(x)\big| \\
&\quad = \sum_{j' \in \mathbb{Z}} 2^{-j'n} \sum_{\mathbf{k} \in \mathbb{Z}^n} \psi_{j'} * f(2^{-j'}\mathbf{k} \in \mathbb{Z}^{\mathbf{n}}) \psi_j * \psi_{j'}(x - 2^{-j'}\mathbf{k}) \\
&\quad \leq C \sum_{j' \in \mathbb{Z}} 2^{-|j-j'|L} \sum_{\mathbf{k}} \frac{2^{(j\wedge j')n}}{(1+2^{j\wedge j'}|x-2^{-j'}\mathbf{k}|)^{M+n}} (\psi_{j'} * f)(2^{-j'}\mathbf{k}) \\
&\quad \leq C \sum_{j' \in \mathbb{Z}} 2^{-|j-j'|L} 2^{n(\frac{1}{\delta}-1)((j'-j)\vee 0)} \Big\{ M\Big(\sum_{\mathbf{k}} \big|(\psi_{j'} * f)(2^{-j'}\mathbf{k})\big|^2 \chi_{Q'} \Big)^{\frac{\delta}{2}}(v) \Big\}^{\frac{1}{\delta}},
\end{aligned}$$

where L and M are large enough and Q, Q' denote dyadic cubes in $\mathbb{R}^n$ with side lengths 2^{-j} and $2^{-j'}$, respectively. Then, applying Hölder's inequality, we have

$$\begin{aligned}
&\big|\mathcal{G}(f)(x)\big|^2 \\
&\quad = \sum_{j \in \mathbb{Z}} \big|\psi_j * f(x)\big|^2 \\
&\quad \leq C \sum_{j' \in \mathbb{Z}} \Big(\sum_{j \in \mathbb{Z}} 2^{-|j-j'|L} 2^{n(\frac{1}{\delta}-1)((j'-j)\vee 0)} \Big\{ M\Big(\sum_{\mathbf{k}} \big|(\psi_{j'} * f)(2^{-j'}\mathbf{k})\big|^2 \chi_{Q'} \Big)^{\frac{\delta}{2}}(v) \Big\}^{\frac{1}{\delta}} \Big)^2 \\
&\quad \leq C \sum_{j' \in \mathbb{Z}} \Big\{ M\Big(\sum_{\mathbf{k}} \big|(\psi_{j'} * f)(2^{-j'}\mathbf{k})\big|^2 \chi_{Q'} \Big)^{\frac{\delta}{2}}(v) \Big\}^{\frac{2}{\delta}}.
\end{aligned}$$

Thus by the vector-valued maximal operator M on $L^{p(\cdot)/\delta}(l^{2/\delta})$, we have

$$\begin{aligned}
\big\|\mathcal{G}(f)\big\|_{L^{p(\cdot)}} &= \Big\| \Big(\sum_{j \in \mathbb{Z}} |\psi_j * f(x)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p(\cdot)}} \\
&\leq C \Big\| \Big(\sum_{j' \in \mathbb{Z}} \Big\{ M\Big(\sum_{\mathbf{k}} \big|(\psi_{j'} * f)(2^{-j'}\mathbf{k})\big|^2 \chi_{Q'} \Big)^{\frac{\delta}{2}} \Big\}^{\frac{2}{\delta}} \Big)^{\frac{\delta}{2}} \Big\|^{\frac{1}{\delta}}_{L^{\frac{p(\cdot)}{\delta}}} \\
&\leq \Big\| \Big(\sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \big|\psi_j * f(2^{-j}\mathbf{k})\big|^2 \chi_Q(x) \Big)^{\frac{1}{2}} \Big\|_{L^{p(\cdot)}} = \big\|\mathcal{G}^d(f)\big\|_{L^{p(\cdot)}},
\end{aligned}$$

where $\frac{n}{M+n} < \delta < (p^- \wedge 1)$.

On the other hand, repeating the same method used by Frazier and Jawerth in [10], we can obtain that

$$\left\|\mathcal{G}(f)\right\|_{L^{p(\cdot)}} \sim \left\|\Big(\sum_{Q}\big(\sup_Q(f)\tilde{\chi_Q}\big)^2\Big)^{1/2}\right\|_{L^{p(\cdot)}} \sim \left\|\Big(\sum_{Q}\big(\inf_{Q,\gamma}(f)\tilde{\chi_Q}\big)^2\Big)^{1/2}\right\|_{L^{p(\cdot)}},$$

where Q dyadic with $l(Q)=2^{-j}$, $\tilde{\chi_Q}=|Q|^{-1/2}\chi_Q$, $\sup_Q(f)=|Q|^{1/2}\sup_{y\in Q}|\psi_j * f(y)|$, and $\inf_{Q,\gamma}(f)=|Q|^{1/2}\max\{\inf_{y\in\tilde{Q}}|\psi * f(y)| : l(\tilde{Q})=2^{-\gamma}l(Q), \tilde{Q}\subset Q\}$ for $\gamma\in\mathbb{Z}$ with $\gamma>0$. Besides, we have the fact that

$$\left\|\mathcal{G}^d(f)\right\|_{L^{p(\cdot)}} \le C\left\|\Big(\sum_{Q}\big(\sup_Q(f)\tilde{\chi_Q}\big)^2\Big)^{1/2}\right\|_{L^{p(\cdot)}}.$$

Thus we complete the proof of Proposition 2.3. □

We would like to point out that functions $\psi_j(x)$ used in the discrete Calderón-type identity do not have compact support. To prove the atomic decomposition for $H^{p(\cdot)}$, we derive the following new discrete Calderón-type identity, which, for the spaces of homogeneous type, was first used in [6].

Proposition 2.4. *Suppose that $p(\cdot)\in \mathrm{LH}\cap\mathcal{P}^0$. Let ϕ be a Schwartz function with support on the unit ball satisfying the conditions: for all $\xi\in\mathbb{R}^n$,*

$$\sum_{j\in\mathbb{Z}}\left|\widehat{\phi}(2^{-j}\xi)\right|^2=1,$$

and $\int_{\mathbb{R}^n}\phi(x)x^\alpha\,dx=0$ for all $0\le|\alpha|\le M$. Then for all $f\in H^{p(\cdot)}\cap L^q$, $1<q<\infty$, there exists a function $g\in H^{p(\cdot)}\cap L^q$ with

$$\|f\|_{L^q}\sim\|g\|_{L^q} \qquad \text{and} \qquad \|f\|_{H^{p(\cdot)}}\sim\|g\|_{H^{p(\cdot)}}$$

such that, for some large integer N depending on ϕ and $p(\cdot),q,M$,

$$f(x)=\sum_{j\in\mathbb{Z}}\sum_{Q}|Q|\phi_j * g(x_Q)\phi_j(x-x_Q),$$

where Q represents dyadic cubes with side length 2^{-j-N}, and the series converges in both norms of L^q and $H^{p(\cdot)}$.

Proof. By using the classical Calderón identity on L^2 and applying Coifman's decomposition of the identity, we have

$$\begin{aligned}
f(x)&=\sum_{j\in\mathbb{Z}}\phi_j * \phi_j * f(x)\\
&=\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\phi_j(x-u)\phi_j * f(u)\,du\\
&=\sum_{j\in\mathbb{Z}}\sum_{Q}\int_{Q}\phi_j(x-u)\phi_j * f(u)\,du\\
&=\sum_{j\in\mathbb{Z}}\sum_{Q}|Q|\phi_j(x-x_Q)\phi_j * f(x_Q)+R_N(f)(x),
\end{aligned}$$

where $Q \subset \mathbb{R}^n$, $l(Q) = 2^{-j-N}$ for some large integer N which will be chosen later, and x_Q is the lower-left point of Q, and

$$R_N(f)(x) = \sum_{j\in\mathbb{Z}} \sum_Q \int_Q \big[\phi_j(x-u)\phi_j * f(u) - \phi_j(x-x_Q)\phi_j * f(x_Q)\big]\,du.$$

Now we prove that

$$\big\|R_N(f)\big\|_{L^2} \le C2^{-N}\|f\|_{L^2}$$

and that

$$\big\|R_N(f)\big\|_{H^{p(\cdot)}} \le C2^{-N}\|f\|_{H^{p(\cdot)}}.$$

To do this, first we rewrite $R_N(f)(x)$ as

$$\begin{aligned} R_N(f)(x) &= \sum_{j\in\mathbb{Z}} \sum_Q \int_Q \big[\phi_j(x-u) - \phi_j(x-x_Q)\big]\phi_j * f(u) \\ &\qquad + \phi_j(x-x_Q)\big[\phi_j * f(u) - \phi_j * f(x_Q)\big]\,du \\ &=: \int_{\mathbb{R}^n} R_N(x,y) f(y)\,dy, \end{aligned}$$

where the kernel of $R_N(f)(x)$ is given by

$$\begin{aligned} R_N(x,y) &= \sum_{j\in\mathbb{Z}} \sum_Q \int_Q \big[\phi_j(x-u) - \phi_j(x-x_Q)\big]\phi_j(u-y)\,dy \\ &\quad + \sum_{j\in\mathbb{Z}} \sum_Q \int_Q \phi_j(x-x_Q)\big[\phi_j(u-y) - \phi_j(x_Q-y)\big]\,dy \\ &=: R_N^1(x,y)f(y)\,dy + R_N^2(x,y)f(y)\,dy. \end{aligned}$$

Next, we need to verify that $R_N(x,y)$ is a Calderón–Zygmund kernel. For $R_N^1(x,y)$, since ϕ is a Schwartz function, we have

$$\big|\partial_x^\alpha \phi_j(x-u) - \partial_x^\alpha \phi_j(x-x_Q)\big| \le C2^{-N} \frac{2^{j|\alpha|2^{-j}}}{(2^{-j}+|x_Q-u|)^{n+2M+1}}.$$

Thus for $0 \le |\alpha|, |\beta| \le M$ we get

$$\begin{aligned} \big|\partial_x^\alpha \partial_y^\beta R_N^1(x,y)\big| &\le C2^{-N} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \frac{2^{j|\alpha|2^{-j}}}{(2^{-j}+|x_Q-u|)^{n+2M+1}} \big|\partial_y^\beta \phi_j(u-y)\big|\,dy \\ &\le C2^{-N} \frac{1}{|x-y|^{n+|\alpha|+|\beta|}}. \end{aligned}$$

The same results hold for $R_N^2(x,y)$; hence these also hold for $R_N(x,y)$ with the constant $C2^{-N}$.

Let $R_N^1(f) = \sum_j R_j(f)$, where $R_j(f)(x) = \sum_Q \int_Q [\phi_j(x-u) - \phi_j(x-x_Q)]\phi_j(u-y)\,dy$. By the proof of the estimates for the kernel of R_N given above, we have

$\|R_j(f)\|_{L^2} \le C2^{-N}\|f\|_{L^2}$. Applying an almost orthogonality argument in [12, Lemmas 2.18] yields that

$$\left|R_jR_k^*(f)(x,y)\right| \le C2^{-N}2^{|j-k|}\frac{2^{-(j\wedge k)}}{(2^{-(j\wedge k)}+|x-y|)^{n+1}},$$

where $j\wedge k$ is the minimum of j and k. Then we have

$$\left\|R_jR_k^*(f)(f)\right\|_{L^2} \le C2^{-N}2^{|j-k|}\|f\|_{L^2}.$$

We can also obtain similar results for $R_k^*R_j(f)$. Thus we prove that R_N is bounded on L^2 with constant $C2^{-N}$; hence R_N is also bounded on $L^q, 1<q<\infty$ with constant $C2^{-N}$ since R_N is a Calderón–Zygmund operator. Theorem 5.3 in [16] yields that R_N is bounded on $H^{p(\cdot)}$ with constant $C2^{-N}$.

Choose N large enough so that $C2^{-N}<1$. Noting that $I=T_N+R_N$, then $T_N^{-1}=\sum_{n=0}^{\infty}(R_N)^n$ is also bounded on L^2 and $H^{p(\cdot)}$, where

$$T_N(f)(x)=\sum_{j\in\mathbb{Z}}\sum_{Q}|Q|\phi_j(x-x_Q)\phi_j*f(x_Q).$$

It follows that $\|g\|_{L^q}\sim\|f\|_{L^q}$ and $\|g\|_{H^{p(\cdot)}}\sim\|f\|_{H^{p(\cdot)}}$, $1<q<\infty$, where $g(x):=T_N^{-1}(f)(x)$. Moreover,

$$f(x)=\sum_{j\in\mathbb{Z}}\sum_{Q}|Q|\phi_j(x-x_Q)\phi_j*g(x_Q),$$

where the series converges in both L^q- and $H^{p(\cdot)}$-norms.

Next we will prove that the series above also converges in L^q for any $1<q<\infty$. Since $L^q\cap L^2$ is dense in L^q, it suffices to show that the series converges in L^q for each function $f\in L^q\cap L^2$. For $f\in L^q\cap L^2$, set

$$B_l=\left\{Q: l(Q)=2^{-j-N}, Q\subset B(0,l), |j|\le l\right\},$$

where $B(0,l)$ is the ball centered at 0 with radius l in $\mathbb{R}^n$. Write $\phi_Q=\phi_j$; then we only need to show that for each function $g\in L^q\cap L^2$ and any positive integer L,

$$\Big\|\sum_{l>L}\sum_{Q\in B_l}|Q|\phi_Q(x-x_Q)\phi_Q*g(x_Q)\Big\|_{L^q}\to 0 \quad \text{as } L\to\infty.$$

Suppose that $h\in L^{q'}\cap L^2$ ($\frac{1}{q}+\frac{1}{q'}=1$). By duality argument, the Cauchy inequality and the Hölder inequality, we get that

$$\begin{aligned}
&\Big\|\sum_{l>L}\sum_{Q\in B_l}|Q|\phi_Q(x-x_Q)\phi*g(x_Q)\Big\|_{L^q}\\
&\quad=\sup_{\|h\|_{L^{q'}}\le 1}\Big|\Big\langle\sum_{l>L}\sum_{Q\in B_l}|Q|\phi_Q(x-x_Q)\phi*g(x_Q),h\Big\rangle\Big|\\
&\quad=\sup_{\|h\|_{L^{q'}}\le 1}\Big|\sum_{l>L}\sum_{Q\in B_l}|Q|\phi_Q*h(x_Q)\phi*g(x_Q)\Big|\\
&\quad\le\sup_{\|h\|_{L^{q'}}\le 1}\int_{\mathbb{R}^n}\sum_{l>L}\sum_{Q\in B_l}\big|\phi_Q*h(x_Q)\big|\big|\phi*g(x_Q)\big|\chi_Q(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le \sup_{\|h\|_{L^{q'}}\le 1} \int_{\mathbb{R}^n} \Big\{\sum_{l>L}\sum_{Q\in B_l} \big|\phi_Q * h(x_Q)\big|^2 \chi_Q(x)\Big\}^{1/2} \\
&\quad \times \Big\{\sum_{l>L}\sum_{Q\in B_l} \big|\phi_Q * h(x_Q)\big|^2 \chi_Q(x)\Big\}^{1/2} dx \\
&\le \sup_{\|h\|_{L^{q'}}\le 1} \Big\|\Big\{\sum_{l>L}\sum_{Q\in B_l} \big|\phi_Q * h(x_Q)\big|^2 \chi_Q\Big\}^{1/2}\Big\|_{L^{q'}} \\
&\quad \times \Big\|\Big\{\sum_{l>L}\sum_{Q\in B_l} \big|\phi_Q * h(x_Q)\big|^2 \chi_Q\Big\}^{1/2}\Big\|_{L^{q}} \\
&\le \Big\|\Big\{\sum_{l>L}\sum_{Q\in B_l} \big|\phi_Q * g(x_Q)\big|^2 \chi_Q\Big\}^{1/2}\Big\|_{L^{q}},
\end{aligned}
$$

which tends to zero as L goes to infinity. This ends the proof of Proposition 2.4. $\square$

Proof of Theorem 1.3. Let $f \in L^q \cap H^{p(\cdot)}$ $(0 < p^- \le p^+ < q < \infty$ with $q \ge 1)$, and let ϕ be the function mentioned in Proposition 2.4. Then by the discrete Calderón reproducing formula in Proposition 2.4, we have

$$
f(x) = \sum_{j\in\mathbb{Z}}\sum_{Q} |Q|\phi_j * g(x_Q)\phi_j(x - x_Q) \quad \text{in } L^q \cap H^{p(\cdot)}.
$$

Now we need the maximal square function defined by

$$
\mathcal{G}^d_\phi(f)(x) := \Big(\sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^n} \sup_{x_Q\in Q} \big|\phi_j * g(x_Q)\big|^2 \chi_Q(x)\Big)^{1/2}.
$$

From the proof of Proposition 2.3, we have

$$
\|f\|_{H^{p(\cdot)}} \sim \big\|\mathcal{G}^d_\phi(f)\big\|_{L^{p(\cdot)}}.
$$

Next, let

$$
\Omega_i = \big\{x \in \mathbb{R}^n : \mathcal{G}^d_\phi(f)(x) > 2^i\big\}
$$

and let

$$
\widetilde{\Omega}_i = \Big\{x \in \mathbb{R}^n : M(\chi_{\Omega_i})(x) > \frac{1}{10}\Big\},
$$

where M is the Hardy–Littlewood maximal operator. Then $\Omega_i \subset \widetilde{\Omega}_i$. By the L^2 boundedness of M, $|\widetilde{\Omega}_i| \le C|\Omega_i|$. Denote by $\mathcal{Q}$ the set of all dyadic cubes in $\mathbb{R}^n$. Let

$$
B_i = \Big\{Q \in \mathcal{Q} : |Q \cap \Omega_i| > \frac{1}{2}|Q|, |Q \cap \Omega_{i+1}| \le \frac{1}{2}|Q|\Big\}.
$$

We write $\phi_Q := \phi_l$, if $l(Q) = 2^{-l-N}$ and x_Q is the lower-left corner of Q. Following the discrete Calderón reproducing formula and denoting $\tilde{Q} \in B_i$ as the maximal dyadic cube in B_i, we rewrite

$$f(x) = \sum_{i\in\mathbb{Z}} \sum_{\tilde{Q}\in B_i} \sum_{Q\subset\tilde{Q}} |Q|\phi_Q * g(x_Q)\phi_Q(x - x_Q) =: \sum_i \sum_{\tilde{Q}\in B_i} \lambda_{\tilde{Q}} a_{\tilde{Q}}(x),$$

where

$$a_{\tilde{Q}} = \frac{1}{\lambda_{\tilde{Q}}} \sum_{Q\subset\tilde{Q}} |Q|\phi_Q * g(x_Q)\phi_Q(x - x_Q),$$

and

$$\lambda_{\tilde{Q}} = 2^i \|\chi_{5\tilde{Q}}\|_{L^{p(\cdot)}}.$$

From the definition of $a_{\tilde{Q}}$ and the support of ϕ, we get that $a_{\tilde{Q}}$ is supported in $5\tilde{Q}$. Then, to establish the atomic decomposition for variable Hardy spaces, we also need the following inequality:

$$\mathcal{A}\big(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\big) \le C\|f\|_{H^{p(\cdot)}}.$$

To prove it, we first observe that when $1 < q < \infty$,

$$\begin{aligned}\mathcal{A}\big(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\big) &= \bigg\|\bigg\{\sum_i \sum_{\tilde{Q}\in B_i} \Big(\frac{|\lambda_{\tilde{Q}}|\chi_{5\tilde{Q}}}{\|\chi_{5\tilde{Q}}\|_{L^{p(\cdot)}}}\Big)^{p^-}\bigg\}^{\frac{1}{p^-}}\bigg\|_{L^{p(\cdot)}} \\ &\le C\bigg\|\bigg\{\sum_i \sum_{\tilde{Q}\in B_i} (2^i\chi_{5\tilde{Q}})^{p^-}\bigg\}^{\frac{1}{p^-}}\bigg\|_{L^{p(\cdot)}}.\end{aligned}$$

Note that $5\tilde{Q} \subset \tilde{\Omega}_i$ when $\tilde{Q} \in B_i$. Since $\Omega_i \subset \tilde{\Omega}_i$ for each $i \in \mathbb{Z}$, and $|\tilde{\Omega}_i| \le C|\Omega_i|$ for all $x \in \mathbb{R}^n$, we have

$$\chi_{\tilde{\Omega}_i}(x) \le CM^{\frac{2}{p^-}}\chi_{\Omega_i}(x).$$

If we use these facts and apply the Fefferman–Stein vector-valued maximal inequality of variable Lebesgue spaces (see Proposition 2.2), we have

$$\begin{aligned}&\mathcal{A}\big(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\big) \\ &\quad\le C\bigg\|\bigg\{\sum_i (2^i\chi_{\tilde{\Omega}_i})^{p^-}\bigg\}^{\frac{1}{p^-}}\bigg\|_{L^{p(\cdot)}} \\ &\quad\le C\bigg\|\bigg\{\sum_i (2^i M^{\frac{2}{p^-}}\chi_{\Omega_i})^{p^-}\bigg\}^{\frac{1}{p^-}}\bigg\|_{L^{p(\cdot)}} = C\bigg\|\bigg\{\sum_i 2^{ip^-} M^2\chi_{\Omega_i}\bigg\}^{\frac{1}{2}}\bigg\|_{L^{\frac{2p(\cdot)}{p^-}}}^{\frac{2}{p^-}} \\ &\quad\le C\bigg\|\bigg\{\sum_i 2^{ip^-}\chi_{\Omega_i}^2\bigg\}^{\frac{1}{2}}\bigg\|_{L^{\frac{2p(\cdot)}{p^-}}}^{\frac{2}{p^-}} \le C\bigg\|\bigg\{\sum_i (2^i\chi_{\Omega_i})^{p^-}\bigg\}^{\frac{1}{p^-}}\bigg\|_{L^{p(\cdot)}}.\end{aligned}$$

If $\Omega_{i+1} \subset \Omega_i$ and $|\bigcap_{i=1}^\infty \Omega_i| = 0$, then for almost every $x \in \mathbb{R}^n$ we have

$$\sum_{i=-\infty}^\infty 2^i\chi_{\Omega_i}(x) = \sum_{i=-\infty}^\infty 2^i \sum_{j=i}^\infty \chi_{\Omega_j\setminus\Omega_{j+1}}(x) = 2\sum_{j=-\infty}^\infty 2^j\chi_{\Omega_j\setminus\Omega_{j+1}}(x).$$

Then we deduce from the definition of Ω_i that

$$\begin{aligned}
&\Big\|\Big\{\sum_i (2^i\chi_{\Omega_i})^{p^-}\Big\}^{\frac{1}{p^-}}\Big\|_{L^{p(\cdot)}}\\
&\le C\Big\|\Big\{\sum_i (2^i\chi_{\Omega_i\setminus\Omega_{i+1}})^{p^-}\Big\}^{\frac{1}{p^-}}\Big\|_{L^{p(\cdot)}}\\
&= C\inf\Big\{\lambda>0:\int_{\mathbb{R}^n}\Big(\sum_i\frac{2^i\chi_{\Omega_i\setminus\Omega_{i+1}}}{\lambda}\Big)^{p(x)}dx\le 1\Big\}\\
&= C\inf\Big\{\lambda>0:\sum_i\int_{\Omega_i\setminus\Omega_{i+1}}\Big(\frac{2^i}{\lambda}\Big)^{p(x)}dx\le 1\Big\}\\
&\le C\inf\Big\{\lambda>0:\int_{\mathbb{R}^n}\Big(\frac{\mathcal{G}_\phi^d f(x)}{\lambda}\Big)^{p(x)}dx\le 1\Big\}\le C\|f\|_{H^{p(\cdot)}}.
\end{aligned}$$

Thus we have proved the claim. To complete the proof of the theorem, next we only need to show that every $a_{\tilde{Q}}$ is a $(p(\cdot),q)$-atom.

By duality and the Hölder inequalities, for $1<q,q'<\infty$ and $\frac{1}{q}+\frac{1}{q'}=1$, we have

$$\begin{aligned}
&\Big\|\sum_{Q\subset\tilde{Q}}|Q|\phi_Q*g(x_Q)\phi_Q(\cdot-x_Q)\Big\|_{L^q}\\
&\le\sup_{\|h\|_{L^{q'}}\le 1}\int_{\mathbb{R}^n}\sum_{Q\subset\tilde{Q}}\phi_Q*g(x_Q)\phi_Q*h(x_Q)\chi_Q(x)\,dx\\
&\le\sup_{\|h\|_{L^{q'}}\le 1}\Big\|\Big\{\sum_{Q\subset\tilde{Q}}|\phi_Q*g(x_Q)|^2\chi_Q\Big\}^{\frac12}\Big\|_{L^q}\Big\|\Big\{\sum_{Q\subset\tilde{Q}}|\phi_Q*h(x_Q)|^2\chi_Q\Big\}^{\frac12}\Big\|_{L^{q'}}\\
&\le C\Big\|\Big\{\sum_{Q\subset\tilde{Q}}|\phi_Q*g(x_Q)|^2\chi_Q\Big\}^{\frac12}\Big\|_{L^q},
\end{aligned}$$

where the last inequality follows from the L^q-estimates of the discrete Littlewood–Paley square function. If $x\in Q\in B_i$, then $M\chi_{Q\cap\tilde{\Omega}_i\setminus\Omega_{i+1}}(x)>\frac12$. From this fact, we have

$$\chi_Q(x)\le 2M\chi_{Q\cap\tilde{\Omega}_i\setminus\Omega_{i+1}}(x)\Longrightarrow\chi_Q(x)\le 4M^2(\chi_{Q\cap\tilde{\Omega}_i\setminus\Omega_{i+1}})(x).$$

Thus by the Fefferman–Stein vector-valued inequality, for all $1<q<\infty$,

$$\begin{aligned}
&\Big\|\Big(\sum_{Q\subset\tilde{Q}}\sup_{x_Q\in Q}\big|\phi_Q*g(x_Q)\big|^2\chi_Q(x)\Big)^{1/2}\Big\|_{L^q}^q\\
&=\int_{\mathbb{R}^n}\Big(\sum_{Q\subset\tilde{Q}}\sup_{x_Q\in Q}\big|\phi_Q*g(x_Q)\big|^2\chi_Q(x)\Big)^{q/2}dx\\
&\le C\int_{\mathbb{R}^n}\Big(\sum_{Q\subset\tilde{Q}}\sup_{x_Q\in Q}\big|\phi_Q*g(x_Q)M(\chi_{Q\cap\tilde{\Omega}_i\setminus\Omega_{i+1}})(x)\big|^2\Big)^{q/2}dx
\end{aligned}$$

$$
\begin{aligned}
&\leq C \int_{\tilde{Q}\cap\tilde{\Omega}_i\setminus\Omega_{i+1}} \Big(\sum_{Q\subset\tilde{Q}} \sup_{x_Q\in Q} \big|\phi_Q * g(x_Q)\chi_Q(x)\big|^2\Big)^{q/2} dx \\
&\leq C2^{iq}|\tilde{Q}|.
\end{aligned}
$$

Then the estimate implies that, when $1 \leq q < \infty$,

$$
\begin{aligned}
\|a_{\tilde{Q}}\|_{L^q} &= \frac{1}{\lambda_{\tilde{Q}}} \Big\| \sum_{Q\subset\tilde{Q}} |Q|\phi_Q * g(x_Q))\phi_Q(\cdot - x_Q) \Big\|_{L^q} \\
&\leq \frac{1}{2^i\|\chi_{5\tilde{Q}}\|_{L^{p(\cdot)}}} \big(2^{iq}|\tilde{Q}|\big)^{1/q} \\
&\leq C \frac{|\tilde{Q}|^{1/q}}{\|\chi_{5\tilde{Q}}\|_{L^{p(\cdot)}}}.
\end{aligned}
$$

Hence, together with the cancellation conditions of ϕ, we have proved that $a_{\tilde{Q}}$ is a $(p(\cdot), q)$-atom. This ends the proof of Theorem 1.3. □

Before we prove Corollary 1.4, we need the following important inequality, which comes from [16, Lemma 4.10].

Proposition 2.5. *Let $p(\cdot) \in \mathrm{LH} \cap \mathcal{P}^0$, and let $p_- = \min\{1, p^-\}$. Let $\beta \in (0,1)$, and let $\delta \in (0, \frac{-\log_2 \beta}{n+1})$. If we are given sequences of nonnegative numbers $\{k_j\}_j$, measurable functions $\{b_j\}_j$ and cubes $\{Q\}_j$ such that $\operatorname{supp} b_j \subset Q_j$, and $\|b_j\|_{L^q(Q_j)} \neq 0$ for each j, then we have*

$$
\Big\| \Big\{ \sum_j \Big(\frac{k_j |b_j| |Q_j|^\delta}{\|b_j\|_{L^{1/\delta}(Q_j)} \|\chi_{Q_j}\|_{L^{p(\cdot)}}} \Big)^{p_-} \Big\}^{\frac{1}{p_-}} \Big\|_{L^{p(\cdot)}} \leq C\mathcal{A}\big(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\big).
$$

Proof of Corollary 1.4. Given that $f \in L^q \cap H^{p(\cdot)}$, the L^q-boundedness and $H^{p(\cdot)}$-boundedness of T yield that $Tf \in L^q \cap H^{p(\cdot)}$. By Theorem 1.3, for $Tf \in L^q \cap H^{p(\cdot)}$, there is a sequence of $(p(\cdot), q)$-atoms $\{a_j^*\}$ and a sequence of scalars $\{\lambda_i^*\}$ with

$$
\mathcal{A}\big(\{\lambda_j^*\}_{j=1}^\infty, \{Q_j^*\}_{j=1}^\infty\big) \leq C\|Tf\|_{H^{p(\cdot)}},
$$

such that $Tf = \sum_j \lambda_j^* a_j^*$, where the series converges to Tf in L^q.

Next we will prove that, when $Tf \in L^q \cap H^{p(\cdot)}$,

$$
\|Tf\|_{L^{p(\cdot)}} \leq C\mathcal{A}\big(\{\lambda_j^*\}_{j=1}^\infty, \{Q_j^*\}_{j=1}^\infty\big).
$$

To do this, we repeat the similar argument from Theorem 1.3. Then we rewrite

$$
Tf(x) = \sum_{i\in\mathbb{Z}} \sum_{\tilde{Q}^*\in B_i^*} \sum_{Q^*\subset\tilde{Q}^*} |Q^*|\phi_Q^* * g(x_{Q^*})\phi_{Q^*}(x - x_{Q^*}) =: \sum_i \sum_{\tilde{Q}^*\in B_i^*} \lambda_{\tilde{Q}^*} a_{\tilde{Q}^*}(x),
$$

where

$$
a_{\tilde{Q}^*} = \frac{1}{\lambda_{\tilde{Q}^*}} \sum_{Q^*\subset\tilde{Q}^*} |Q^*|\phi_{Q^*} * g(x_{Q^*})\phi_{Q^*}(x - x_{Q^*}),
$$

and

$$
\lambda_{\tilde{Q}^*} = 2^i \|\chi_{5\tilde{Q}^*}\|_{L^{p(\cdot)}}.
$$

We choose $q^* \in (0,\infty)$, which satisfies $\frac{1}{q} \in (0, -\log_2 \beta_0/(n+1))$. We observe that $\mathrm{supp} a_{\tilde{Q}^*} \subset 5\tilde{Q}^*$ and $\|a_{\tilde{Q}^*}\|_{L^{q^*}} \neq 0$ such that

$$1 \leq C \frac{|5\tilde{Q}^*|^{1/q^*}}{\|a_{\tilde{Q}^*}\|_{L^{q^*}(5\tilde{Q}^*)}\|\chi_{5\tilde{Q}^*}\|_{L^{p(\cdot)}}}.$$

In fact, the similar method in the proof of Theorem 1.3 yields

$$\begin{aligned}
\|a_{\tilde{Q}^*}\|_{L^{q^*}(5\tilde{Q}^*)} &\leq C\|a_{\tilde{Q}^*}\|_{L^{q^*}} \\
&\leq C2^{-i}\|\chi_{5\tilde{Q}^*}\|^{-1}_{L^{p(\cdot)}} \Big\| \sum_{Q^* \subset \tilde{Q}^*} |Q^*| \phi_{Q^*} * g(x_{Q^*}) \phi_{Q^*}(x - x_{Q^*}) \Big\|_{L^{q^*}} \\
&\leq C2^{-i}\|\chi_{5\tilde{Q}^*}\|^{-1}_{L^{p(\cdot)}} 2^i |5\tilde{Q}^*|^{1/q^*}.
\end{aligned}$$

Thus we see that

$$\begin{aligned}
\|Tf\|_{L^{p(\cdot)}} &= \Big\| \sum_i \sum_{\tilde{Q}^* \in B_i^*} \lambda_{\tilde{Q}^*} a_{\tilde{Q}^*}(x) \Big\|_{L^{p(\cdot)}} \\
&\leq \Big\| \Big\{ \sum_i \sum_{\tilde{Q}^* \in B_i^*} \big(\lambda_{\tilde{Q}^*} a_{\tilde{Q}^*}(x)\big)^{p_-} \Big\}^{\frac{1}{p_-}} \Big\|_{L^{p(\cdot)}} \\
&\leq \Big\| \Big\{ \sum_{i,\tilde{Q}^*} \Big(\frac{\lambda_{\tilde{Q}^*} |a_{\tilde{Q}^*}| |5\tilde{Q}^*|^{1/q^*}}{\|a_{\tilde{Q}^*}\|_{L^{q^*}(5\tilde{Q}^*)} \|\chi_{5\tilde{Q}^*}\|_{L^{p(\cdot)}}} \Big)^{p_-} \Big\}^{\frac{1}{p_-}} \Big\|_{L^{p(\cdot)}} \\
&\leq C\mathcal{A}\big(\{\lambda_j^*\}_{j=1}^\infty, \{Q_j^*\}_{j=1}^\infty\big),
\end{aligned}$$

where the last inequality follows from Proposition 2.5. Then by the above estimates, we obtain, for $f \in L^q \cap H^{p(\cdot)}$,

$$\|Tf\|_{L^{p(\cdot)}} \leq C\mathcal{A}\big(\{\lambda_j^*\}_{j=1}^\infty, \{Q_j^*\}_{j=1}^\infty\big) \leq C\|Tf\|_{H^{p(\cdot)}} \leq C\|f\|_{H^{p(\cdot)}}.$$

Since $L^q \cap H^{p(\cdot)}$ is dense in $H^{p(\cdot)}$, T can be extended to a bounded operator from $H^{p(\cdot)}$ to $L^{p(\cdot)}$. ∎

Acknowledgments. The author wishes to express heartfelt thanks to the anonymous referees for many corrections and for valuable suggestions which improved this article significantly.

References

1. R. Coifman, *A real variable characterization of H^p*, Studia Math. **51** (1974), 269–274. Zbl 0289.46037. MR0358318. 87
2. R. Coifman and Y. Meyer, *Wavelets, Calderón–Zygmund and multilinear operators*, Cambridge Stud. Adv. Math. **48**, Cambridge Univ. Press, Cambridge, 1992. Zbl 0782.00087. MR1456993. 87
3. D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*, Birkhäuser, Basel, 2013. Zbl 1268.46002. MR3026953. DOI 10.1007/978-3-0348-0548-3. 88
4. D. Cruz-Uribe, A. Fiorenza, J. Martell, and C. Pérez, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **31** (2006), no. 1, 239–264. Zbl 1100.42012. MR2210118. 88, 90

5. D. Cruz-Uribe and L. Wang, *Variable Hardy spaces*, Indiana Univ. Math. J. **63** (2014), no. 2, 447–493. Zbl 1311.42053. MR3233216. DOI 10.1512/iumj.2014.63.5232. 88, 89
6. D. Deng and Y.-S. Han, *Harmonic Analysis on Spaces of Homogeneous Type*, Lecture Notes in Math. **1966**, Springer, Berlin, 2009. Zbl 1158.43002. MR2467074. 92
7. L. Diening, *Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces*, Bull. Sci. Math. **129** (2005), no. 8, 657–700. Zbl 1096.46013. MR2166733. DOI 10.1016/j.bulsci.2003.10.003. 88
8. L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Heidelberg, 2011. Zbl 1222.46002. MR2790542. DOI 10.1007/978-3-642-18363-8. 88
9. M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. **34** (1985), no. 4, 777–799. Zbl 0551.46018. MR0808825. DOI 10.1512/iumj.1985.34.34041. 90
10. M. Frazier and B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. **93** (1990), no. 1, 34–170. Zbl 0716.46031. MR1070037. DOI 10.1016/0022-1236(90)90137-A. 92
11. Y.-S. Han, M-Y. Lee, and C.-C. Lin, *Atomic decomposition and boundedness of operators on weighted Hardy spaces*, Canad. Math. Bull. **55** (2012), no. 2, 303–314. Zbl 1271.42030. MR2957246. DOI 10.4153/CMB-2011-072-7. 88
12. Y.-S. Han and E. T. Sawyer, *Littlewood-Paley theory on spaces of homogeneous type and the classical function spaces*, Mem. Amer. Math. Soc. **110** (1994), no. 530. Zbl 0806.42013. MR1214968. DOI 10.1090/memo/0530. 94
13. O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991), no. 4, 592–618. Zbl 0784.46029. MR1134951. 88
14. R. Latter, *A characterization of $H^p(\mathbb{R}^n)$ in terms of atoms*, Studia Math. **62** (1978), no. 1, 93–101. Zbl 0398.42017. MR0482111. 87
15. Y. Meyer, *Wavelets and Operators*, Cambridge University Press, Cambridge, 1992. Zbl 0776.42019. MR1228209. 87
16. E. Nakai and Y. Sawano, *Hardy spaces with variable exponents and generalized Campanato spaces*, J. Funct. Anal. **262** (2012), no. 9, 3665–3748. Zbl 1244.42012. MR2899976. DOI 10.1016/j.jfa.2012.01.004. 88, 89, 94, 98
17. H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Maruzen Co. Ltd., Tokyo, 1950. Zbl 0041.23401. MR0038565. 88
18. W. Orlicz, *Über konjugierte Exponentenfolgen*, Stud. Math. **3** (1931), 200–211. Zbl 0003.25203. 88
19. L. Pick and M. Růžička, *An example of a space $L^{p(x)}$ on which the Hardy–Littlewood maximal operator is not bounded*, Expo. Math. **19** (2001), no. 4, 369–371. Zbl 1003.42013. MR1876258. DOI 10.1016/S0723-0869(01)80023-2. 89
20. K. Zhao and Y.-S. Han, *Boundedness of operators on Hardy spaces*, Taiwanese J. Math. **14** (2010), no. 2, 319–327. Zbl 1209.42013. MR2655771. 87
21. C. Zhuo, D. Yang, and Y. Liang, *Intrinsic square function characterizations of Hardy spaces with variable exponents*, Bull. Malays. Math. Sci. Soc. (2) **39** (2016), no. 4, 1541–1577. Zbl 1356.42013. MR3549980. DOI 10.1007/s40840-015-0266-2. 91

College of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, People's Republic of China.

E-mail address: tanjian89@126.com