Ann. Funct. Anal. 9 (2018), no. 1, 30–40
https://doi.org/10.1215/20088752-2017-0021
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

# SCATTERED LOCALLY $C^*$-ALGEBRAS

MARIA JOIŢA

Communicated by R. B. V. Bhat

Abstract. In this article, we introduce the notion of a scattered locally $C^*$-algebra and we give the conditions for a locally $C^*$-algebra to be scattered. Given an action $\alpha$ of a locally compact group $G$ on a scattered locally $C^*$-algebra $A[\tau_\Gamma]$, it is natural to ask under what conditions the crossed product $A[\tau_\Gamma] \times_\alpha G$ is also scattered. We obtain some results concerning this question.

## 1. Introduction

A topological space $X$ is called *scattered* (or *dispersed*) if every nonempty subset of $X$ necessarily contains an isolated point. Rudin [16, p. 41, Theorem 6] showed that the linear functionals on $C(X)$, where $X$ is a compact Hausdorff space which is scattered, have a very simple structure. A compact Hausdorff space $X$ is scattered if and only if every Radon measure on $X$ is atomic. Pelczynski and Semadeni [13] gave several necessary and sufficient conditions for a compact Hausdorff space $X$ to be scattered in terms of $C(X)$. They showed that a compact Hausdorff space $X$ is scattered if and only if every linear functional $f$ on $C(X)$ is of the form

$$f(h) = \sum_{n=1}^{\infty} a_n h(x_n),$$

where $(x_n)_n$ is a fixed sequence of points in $X$ and $\sum_{n=1}^{\infty} |a_n| < \infty$. As a noncommutative generalization of a scattered compact Hausdorff space, the notion of a scattered $C^*$-algebra was introduced independently by Jensen [7] and Rothwell


Received Oct. 12, 2016; Accepted Jan. 30, 2017.
First published online Jul. 12, 2017.
2010 *Mathematics Subject Classification.* Primary 46L05; Secondary 46L55, 46L85.
*Keywords.* locally $C^*$-algebras, scattered locally $C^*$-algebras, crossed product of locally $C^*$-algebras.

[15]. A $C^*$-algebra $A$ is said to be *scattered* if every positive functional on $A$ is atomic (see [7, Definition 1.1]); or equivalently, any positive functional on $A$ is the sum of a finite or infinite sequence of pure functionals on $A$. (We refer the reader to [2], [5], [7], [8], [12], [10], [15] for other equivalent conditions on scattered $C^*$-algebras.)

The notion of a *"locally" $C^*$-algebra* is a generalization of the notion of a $C^*$-algebra. Instead of being given by a single $C^*$-norm, the topology on a locally $C^*$-algebra is defined by a directed family of $C^*$-seminorms. A locally $C^*$-algebra $A[\tau_\Gamma]$ is a complete Hausdorff topological $*$-algebra for which there exists an upward directed family $\Gamma$ of $C^*$-seminorms $\{p_\lambda\}_{\lambda\in\Lambda}$ defining the topology $\tau_\Gamma$. A Fréchet locally $C^*$-algebra is a locally $C^*$-algebra whose topology is given by a countable family of $C^*$-seminorms. A morphism of locally $C^*$-algebras is a continuous $*$-morphism $\Phi$ from a locally $C^*$-algebra $A[\tau_\Gamma]$ to another locally $C^*$-algebra $B[\tau_{\Gamma'}]$. Other terms with which locally $C^*$-algebras can be found in the literature are: pro-$C^*$-algebras, $b^*$-algebras, and LMC$^*$-algebras (see Phillips [14]).

Let $\{A_\lambda; \chi_{\lambda\mu}\}_{\lambda,\mu\in\Lambda,\lambda\geq\mu}$ be an inverse system of $C^*$-algebras. Then $\lim_{\leftarrow\lambda} A_\lambda$ with the topology given by the family of $C^*$-seminorms $\{p_{A_\lambda}\}_{\lambda\in\Lambda}, p_{A_\lambda}((a_\mu)_\mu) = \|a_\lambda\|_{A_\lambda}$, where $\|\cdot\|_{A_\lambda}$ denotes the $C^*$-norm on $A_\lambda$, is a locally $C^*$-algebra.

For a locally $C^*$-algebra $A[\tau_\Gamma]$, and every $\lambda \in \Lambda$, the quotient normed $*$-algebra $A_\lambda = A/\ker p_\lambda$, where $\ker p_\lambda = \{a \in A; p_\lambda(a) = 0\}$, is already complete, hence, it is a $C^*$-algebra in the norm $\|a + \ker p_\lambda\|_{A_\lambda} = p_\lambda(a), a \in A$ (see, e.g., [3, Theorem 10.24]). The canonical map from $A$ to $A_\lambda$ is denoted by $\pi_\lambda^A$. For $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, there is a canonical surjective $C^*$-morphism $\pi_{\lambda\mu}^A : A_\lambda \to A_\mu$ such that $\pi_{\lambda\mu}^A(a + \ker p_\lambda) = a + \ker p_\mu$ for all $a \in A$. Moreover, $\{A_\lambda; \pi_{\lambda\mu}^A\}_{\lambda,\mu\in\Lambda,\lambda\geq\mu}$ is an inverse system of $C^*$-algebras, called the Arens–Michael decomposition of the locally $C^*$-algebra $A[\tau_\Gamma]$. The Arens–Michael decomposition gives us a representation of $A[\tau_\Gamma]$ as an inverse limit of $C^*$-algebras; namely, $A[\tau_\Gamma] = \lim_{\leftarrow\lambda} A_\lambda$, up to a topological $*$-isomorphism.

In this article, we introduce the notion of scattered locally $C^*$-algebra, and we give conditions for locally $C^*$-algebras to be scattered. Given an action $\alpha$ of a locally compact group $G$ on a scattered locally $C^*$-algebra $A[\tau_\Gamma]$, it is natural to ask under what condition the crossed product $A[\tau_\Gamma] \times_\alpha G$ is also scattered. We obtain some results concerning this question.

## 2. Scattered locally $C^*$-algebras

Let $A[\tau_\Gamma]$ be a locally $C^*$-algebra. A *continuous positive functional* on $A[\tau_\Gamma]$ is a continuous linear map $f : A \to \mathbb{C}$ with the property that $f(a^*a) \geq 0$ for all $a \in A$. If $f_\lambda$ is a positive functional on $A_\lambda$, then $f_\lambda \circ \pi_\lambda^A$ is a continuous positive functional on $A[\tau_\Gamma]$. Moreover, for any continuous positive functional $f$ on $A[\tau_\Gamma]$, there are $\lambda \in \Lambda$ and a positive functional $f_\lambda$ on $A_\lambda$, called the *positive functional associated to* $f$, such that $f = f_\lambda \circ \pi_\lambda^A$. A continuous positive functional $f$ on $A[\tau_\Gamma]$ is *pure* if $f \neq 0$, and if $g$ is another positive functional on $A[\tau_\Gamma]$ and $g \leq f$, then there is $\alpha \in [0, 1]$ such that $g = \alpha f$. A continuous positive functional $f$ on $A[\tau_\Gamma]$ is pure if and only if its associated positive functional $f_\lambda$ is pure.

*Definition* 2.1. A locally $C^*$-algebra $A[\tau_\Gamma]$ is *scattered* if any continuous positive functional $f$ on $A[\tau_\Gamma]$ is a countable sum $f = \sum_n f_n$ of pure functionals $f_n$ on $A$, in the pointwise convergence.

*Remark* 2.2. Let $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ be two isomorphic locally $C^*$-algebras. Then $A[\tau_\Gamma]$ is scattered if and only if $B[\tau_{\Gamma'}]$ is scattered.

**Proposition 2.3.** *Let $A[\tau_\Gamma]$ be a locally $C^*$-algebra. Then $A[\tau_\Gamma]$ is scattered if and only if the factors $A_\lambda, \lambda \in \Lambda$ in the Arens–Michael decomposition of $A[\tau_\Gamma]$, are scattered.*

*Proof.* First, we suppose that $A[\tau_\Gamma]$ is scattered. Let $\lambda \in \Lambda$, and let $f$ be a positive functional on $A_\lambda$. Then $f \circ \pi_\lambda^A$ is a continuous positive functional on $A[\tau_\Gamma]$, and since $A[\tau_\Gamma]$ is scattered, $f \circ \pi_\lambda^A = \sum_n f_n$, where the $f_n$, $n \in \mathbb{N}$, are pure. Let $n$ be a positive integer. Then, there are $\mu \in \Lambda$ and $f_{\mu,n}$ a positive functional on $A_\mu$ such that $f_n = f_{\mu,n} \circ \pi_\mu^A$. Furthermore,

$$\left|f_n(a)\right|^2 \leq \|f_{\mu,n}\| f_n(a^*a) \leq \|f_{\mu,n}\| f\left(\pi_\lambda^A(a^*a)\right) \leq \|f_{\mu,n}\| \|f\| p_\lambda(a)^2$$

for all $a \in A[\tau_\Gamma]$. Therefore, for each positive integer $n$, there is a positive functional $f_n^\lambda$ on $A_\lambda$ such that $f_n^\lambda \circ \pi_\lambda^A = f_n$. Moreover, since $f_n$ is pure, $f_n^\lambda$ is pure and $f = \sum_n f_n^\lambda$.

Conversely, suppose that $f$ is a continuous positive functional on $A[\tau_\Gamma]$. Then there are $\lambda \in \Lambda$ and a positive functional $f_\lambda$ on $A_\lambda$ such that $f = f_\lambda \circ \pi_\lambda^A$. Since $A_\lambda$ is scattered, $f_\lambda = \sum_n f_n$, where the $f_n$'s are pure. Then

$$f = f_\lambda \circ \pi_\lambda^A = \sum_n f_n \circ \pi_\lambda^A$$

and since for each positive integer $n$, $f_n \circ \pi_\lambda^A$ is pure, be a positive functional $A[\tau_\Gamma]$ is scattered. ☐

**Corollary 2.4.** *Any closed $*$-subalgebra of a scattered locally $C^*$-algebra is a scattered locally $C^*$-algebra.*

*Proof.* Let $A[\tau_\Gamma]$ be a scattered locally $C^*$-algebra, and let $B$ be a closed $*$-subalgebra of $A[\tau_\Gamma]$. Then $B$ is a locally $C^*$-algebra and the factors $B_\lambda, \lambda \in \Lambda$ in the Arens–Michael decomposition of $B$ can be identified with the $C^*$-subalgebras $\overline{\pi_\lambda^A(B)}$, the closure of the $*$-subalgebra $\pi_\lambda^A(B)$ in $A_\lambda$, of $A_\lambda, \lambda \in \Lambda$ which are scattered $C^*$-algebras. Then, $B_\lambda, \lambda \in \Lambda$, are scattered (see, e.g., [11, p. 677]) and so $B$ is scattered. ☐

A Hausdorff countably compactly generated topological space is a topological space $X$ which is the direct limit of a sequence of Hausdorff compact spaces $\{K_n\}_n$. The $*$-algebra $C(X)$ of all continuous complex-valued functions on $X$ has a structure of a locally $C^*$-algebra with respect to the topology given by $C^*$-seminorms $\{p_{K_n}\}_n$ with $p_{K_n}(f) = \sup\{|f(x)|; x \in K_n\}$. Moreover, for each $n, C(X)_n$ is isomorphic to $C(K_n)$, and for any commutative Fréchet locally $C^*$-algebra $A$, there is a Hausdorff countably compactly generated topological space $X$ such that $A$ is isomorphic with $C(X)$ (see [14, Theorem 5.7]).

**Corollary 2.5.** *A commutative Fréchet locally $C^*$-algebra $A[\tau_\Gamma]$ is scattered if and only if there is a Hausdorff countably compactly generated topological space $X$ which is the direct limit of a sequence of scattered Hausdorff compact spaces $\{K_n\}_n$ such that $A[\tau_\Gamma]$ is isomorphic with $C(X)$.*

**Corollary 2.6.** *Let $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ be two locally $C^*$-algebras. Then the maximal tensor product $A[\tau_\Gamma]\otimes_{\max}B[\tau_{\Gamma'}]$ of $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ is a scattered locally $C^*$-algebra if and only if the locally $C^*$-algebras $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ are scattered.*

*Proof.* The proof follows from Proposition 2.3, [2, Proposition 1], and [3, Theorem 31.7 and Corollary 31.11]. □

Recall that a continuous $*$-representation of a locally $C^*$-algebra $A[\tau_\Gamma]$ on a Hilbert space is a pair $(\varphi, \mathcal{H}_\varphi)$ consisting of a Hilbert space $\mathcal{H}_\varphi$ and a continuous $*$-morphism $\varphi$ from $A[\tau_\Gamma]$ to $L(\mathcal{H}_\varphi)$, the $C^*$-algebra of all bounded linear operators on $\mathcal{H}_\varphi$. If $(\varphi, \mathcal{H}_\varphi)$ is a representation of $A[\tau_\Gamma]$, then there exist $\lambda \in \Lambda$ and a $*$-representation $(\varphi_\lambda, \mathcal{H}_\varphi)$ of $A_\lambda$ such that $\varphi = \varphi_\lambda \circ \pi_\lambda^A$.

A continuous $*$-representation $(\varphi, \mathcal{H}_\varphi)$ of $A[\tau_\Gamma]$ is of type $I$ if the von Neumann algebra generated by $\varphi(A)$ is of type $I$ (i.e., the commutant of $\varphi(A)$ is an abelian $*$-subalgebra of $L(\mathcal{H}_\varphi)$). A locally $C^*$-algebra $A[\tau_\Gamma]$ is of type $I$ if each of its continuous $*$-representations is of type $I$.

**Corollary 2.7.** *Let $A[\tau_\Gamma]$ be a locally $C^*$-algebra. If $A[\tau_\Gamma]$ is scattered, then $A[\tau_\Gamma]$ is of type $I$.*

*Proof.* Since $A[\tau_\Gamma]$ is scattered, the $A_\lambda$'s are scattered as well, and by [7, Theorem 2.3], the $A_\lambda, \lambda \in \Lambda$ are of type $I$. The corollary is proved since $A[\tau_\Gamma]$ is of type $I$ if and only if the $A_\lambda, \lambda \in \Lambda$ are of type $I$ (see [3, Proposition 30.8]). □

Let $\mathcal{I}$ be a closed two-sided $*$-ideal of $A[\tau_\Gamma]$. Then the quotient $*$-algebra $A[\tau_\Gamma]/\mathcal{I}$, equipped with the quotient topology, is a pre-locally $C^*$-algebra, and its completion $\overline{A[\tau_\Gamma]/\mathcal{I}}$ is a locally $C^*$-algebra. Moreover, for each $\lambda \in \Lambda$, $\mathcal{I}_\lambda = \overline{\pi_\lambda^A(\mathcal{I})}$, the closure of $\pi_\lambda^A(\mathcal{I})$ in the $C^*$-algebra $A_\lambda$ is a closed two-sided $*$-ideal of $A_\lambda$, and the $C^*$-algebras $A_\lambda/\mathcal{I}_\lambda$ and $(\overline{A[\tau_\Gamma]/\mathcal{I}})_\lambda$ are isomorphic (see, e.g., [3, Theorem 11.7]).

**Proposition 2.8.** *Let $A[\tau_\Gamma]$ be a locally $C^*$-algebra, and let $\mathcal{I}$ be a closed two-sided $*$-ideal of $A[\tau_\Gamma]$. Then $A[\tau_\Gamma]$ is scattered if and only if $\mathcal{I}$ and $\overline{A[\tau_\Gamma]/\mathcal{I}}$ are scattered.*

*Proof.* The proof follows from the above discussion, [7, Proposition 2.4], and Proposition 2.3. □

*Remark* 2.9. If $A[\tau_\Gamma]$ is a Fréchet locally $C^*$-algebra and $\mathcal{I}$ is a closed two-sided $*$-ideal of $A[\tau_\Gamma]$, then the quotient $*$-algebra $A[\tau_\Gamma]/\mathcal{I}$ is complete and so it is a Fréchet locally $C^*$-algebra (see, e.g., [3, Corollary 11.8]). Therefore, if $A[\tau_\Gamma]$ is a Fréchet locally $C^*$-algebra and $\mathcal{I}$ is a closed two-sided $*$-ideal of $A[\tau_\Gamma]$, then $A[\tau_\Gamma]$ is scattered if and only if $\mathcal{I}$ and $A[\tau_\Gamma]/\mathcal{I}$ are scattered.

An element $a$ in a locally $C^*$-algebra $A[\tau_\Gamma]$ is bounded if $\sup\{p_\lambda(a); \lambda \in \Lambda\} < \infty$. Put $b(A[\tau_\Gamma]) = \{a \in A[\tau_\Gamma]; a \text{ is bounded}\}$. The map $\|\cdot\|_\infty\colon b(A[\tau_\Gamma]) \to [0, \infty)$

defined by

$$\|a\|_\infty = \sup\{p_\lambda(a); \lambda \in \Lambda\}$$

is a $C^*$-norm, and $b(A[\tau_\Gamma])$, equipped with this $C^*$-norm, is a $C^*$-algebra which is dense in $A[\tau_\Gamma]$. Moreover, for each $\lambda \in \Lambda$, $\ker p_\lambda|_{b(A[\tau_\Gamma])} = \ker p_\lambda \cap b(A[\tau_\Gamma])$ is a closed two-sided $*$-ideal of $b(A[\tau_\Gamma])$, and the $C^*$-algebras $b(A[\tau_\Gamma])/\ker p_\lambda|_{b(A[\tau_\Gamma])}$ and $A_\lambda$ are isomorphic (see, e.g., [3, Theorem 10.24]).

**Proposition 2.10.** *Let $A[\tau_\Gamma]$ be a locally $C^*$-algebra.*

(1) *If the $C^*$-algebra $b(A[\tau_\Gamma])$ of all bounded elements is scattered, then $A[\tau_\Gamma]$ is scattered.*
(2) *If $A[\tau_\Gamma]$ is scattered and for some $\lambda \in \Lambda$, the closed two sided $*$-ideal $\ker p_\lambda|_{b(A[\tau_\Gamma])}$ of $b(A[\tau_\Gamma])$ is scattered, then $b(A[\tau_\Gamma])$ is scattered.*

*Proof.* (1) If $b(A[\tau_\Gamma])$ is scattered, then for each $\lambda \in \Lambda$, $\ker p_\lambda|_{b(A[\tau_\Gamma])}$ and $b(A[\tau_\Gamma])/\ker p_\lambda|_{b(A[\tau_\Gamma])}$ are scattered (see [7, Proposition 2.4]). Therefore, for each $\lambda \in \Lambda$, the $C^*$-algebra $A_\lambda$ is scattered, since $A_\lambda$ is isomorphic with $b(A[\tau_\Gamma])/\ker p_\lambda|_{b(A[\tau_\Gamma])}$, and by Proposition 2.3, $A[\tau_\Gamma]$ is scattered.

(2) If $A[\tau_\Gamma]$ is scattered, then $A_\lambda$ is scattered, and since $A_\lambda$ is isomorphic with $b(A[\tau_\Gamma])/\ker p_\lambda|_{b(A[\tau_\Gamma])}$ and the closed two sided $*$-ideal $\ker p_\lambda|_{b(A[\tau_\Gamma])}$ of $b(A[\tau_\Gamma])$ is scattered, $b(A[\tau_\Gamma])$ is scattered. □

Let $A[\tau_\Gamma]$ be a locally $C^*$-algebra, and let $Z(A[\tau_\Gamma]) = \{a \in A; ab = ba \text{ for all } b \in A\}$ be its center. Clearly, $Z(A[\tau_\Gamma])$ is a commutative locally $C^*$-subalgebra of $A$, and so it is a locally $C^*$-algebra with respect to the topology given by the family of $C^*$-seminorms $\{p_\lambda|_{Z(A[\tau_\Gamma])}\}_{\lambda\in\Lambda}$. For each $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu, \pi^A_{\lambda\mu}(Z(A_\lambda)) \subseteq Z(A_\mu)$ and so $\{Z(A_\lambda); \pi^A_{\lambda\mu}|_{Z(A_\lambda)}\}_{\lambda,\mu\in\Lambda,\lambda\geq\mu}$ is an inverse system of $C^*$-algebras.

**Proposition 2.11.** *Let $A[\tau_\Gamma]$ be a locally $C^*$-algebra. Then $Z(A[\tau_\Gamma]) = \lim_{\leftarrow\lambda} Z(A_\lambda)$, up to an isomorphism of locally $C^*$-algebras.*

*Proof.* Consider the map $\Phi : Z(A[\tau_\Gamma]) \to \lim_{\leftarrow\lambda} Z(A_\lambda)$ given by

$$\Phi(a) = \left(\pi^A_\lambda(a)\right)_\lambda.$$

Clearly, $\Phi$ is a $*$-morphism and $p_{Z(A_\lambda)}(\Phi(a)) = p_\lambda|_{Z(A[\tau_\Gamma])}(a)$ for all $a \in Z(A[\tau_\Gamma])$ and for all $\lambda \in \Lambda$. If $(a_\lambda)_\lambda$ is a coherent sequence in $\{Z(A_\lambda); \pi^A_{\lambda\mu}|_{Z(A_\lambda)}\}_{\lambda,\mu\in\Lambda,\lambda\geq\mu}$, then there is $a \in A$ such that $\pi^A_\lambda(a) = a_\lambda$ for all $\lambda \in \Lambda$. Take $b \in A$. From $\pi^A_\lambda(ab) = \pi^A_\lambda(a)\pi^A_\lambda(b) = \pi^A_\lambda(b)\pi^A_\lambda(a) = \pi^A_\lambda(ba)$ for all $\lambda \in \Lambda$, we deduce that $ab = ba$, and so $a \in Z(A[\tau_\Gamma])$. Therefore, $\Phi$ is an isomorphism of locally $C^*$-algebras. □

*Remark* 2.12. We remark that, in general, the isometric $C^*$-morphism $\varphi_\lambda : (Z(A[\tau_\Gamma]))_\lambda \to Z(A_\lambda), \varphi_\lambda(a + \ker(p_\lambda|_{Z(A[\tau_\Gamma])})) = a + \ker p_\lambda$ is not onto.

An inverse system $\{A_i; \chi_{ij}\}_{i,j\in I,i\geq j}$ of topological algebras is called *perfect* if the restrictions to the inverse limit algebra $A = \lim_{\leftarrow i} A_i$ of the canonical projections $\pi_i : \prod_{i\in I} A_i \to A_i, i \in I$, namely, the continuous morphisms $\pi_i|_A : A \to A_i, i \in I$, are onto maps. The resulting inverse limit algebra $A = \lim_{\leftarrow i} A_i$ is called a *perfect topological algebra* (see [4, Definition 2.7]).

*Definition* 2.13. We say that a locally $C^*$-algebra $A[\tau_\Gamma]$ is *with perfect center* if the inverse system of $C^*$-algebras $\{Z(A_\lambda); \pi^A_{\lambda\mu}|_{Z(A_\lambda)}\}_{\lambda,\mu\in\Lambda,\lambda\geq\mu}$ is perfect.

Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a directed family of Hilbert spaces such that for each $\lambda,\mu\in\Lambda$ with $\lambda\geq\mu$, $H_\mu$ is a closed subspace of $H_\lambda$ and $\langle\cdot,\cdot\rangle_\mu=\langle\cdot,\cdot\rangle_\lambda|_{H_\mu}$. Then $H=\lim_{\lambda\to}H_\lambda$ with the inductive limit topology is called a *locally Hilbert space.* $L(H)$ denotes all linear maps $T:H\to H$ such that for each $\lambda\in\Lambda$, $T|_{H_\lambda}\in L(H_\lambda)$, the $C^*$-algebra of all bounded linear operators on $H_\lambda$, and $P_{\lambda\mu}T|_{H_\lambda}=T|_{H_\lambda}P_{\lambda\mu}$ for all $\lambda,\mu\in\Lambda$ with $\lambda\geq\mu$, where $P_{\lambda\mu}$ is the projection of $H_\lambda$ on $H_\mu$. If $T\in L(H)$, then there is $T^*\in L(H)$ such that $T^*|_{H_\lambda}=(T|_{H_\lambda})^*$ for all $\lambda\in\Lambda$. Then $L(H)$ has a structure of locally $C^*$-algebra with the topology given by the family of $C^*$-seminorms $\{p_\lambda\}_{\lambda\in\Lambda}$ with $p_\lambda(T)=\|T|_{H_\lambda}\|_{L(H_\lambda)}$ (see, e.g., [6, Theorem 5.1]).

*Example* 2.14. Let $H=\lim_{\lambda\to}H_\lambda$ be a locally Hilbert space. Then $H$ is a pre-Hilbert space with the inner product given by $\langle\xi,\eta\rangle=\langle\xi,\eta\rangle_\lambda$ if $\xi,\eta\in H_\lambda$. Let $\widetilde{H}$ be the Hilbert space obtained by the completion of $H$. For each $\lambda\in\Lambda$, $H_\lambda$ is a closed subspace of $\widetilde{H}$. The projection of $\widetilde{H}$ on $H_\lambda$ is denoted by $P_\lambda$. Clearly, the restriction $P_\lambda|_H$ of $P_\lambda$ on $H$ is an element in $L(H)$. It is easy to check that $Z(L(H))$ is the locally $C^*$-subalgebra of $L(H)$ generated by $\{P_\lambda|_H,\lambda\in\Lambda\}$ and that for each $\lambda\in\Lambda$, $(Z(L(H)))_\lambda$ is isomorphic with the $C^*$-subalgebra of $L(H_\lambda)$ generated by $\{P_\lambda|_{H_\mu},\mu\in\Lambda,\mu\leq\lambda\}$.

On the other hand, $L(H)_\lambda$ is isomorphic with the $C^*$-subalgebra of $L(H_\lambda)$, the $C^*$-algebra of all bounded linear operators on $H_\lambda$ generated by $\{T\in L(H_\lambda);$ $P_{\lambda\mu}T=TP_{\lambda\mu},\mu\in\Lambda,\mu\leq\lambda\}$, and then $Z(L(H)_\lambda)$ is isomorphic with the $C^*$-subalgebra of $L(H_\lambda)$ generated by $\{P_\lambda|_{H_\mu},\mu\in\Lambda,\mu\leq\lambda\}$.

Therefore, for each $\lambda\in\Lambda$, the $C^*$-algebras $(Z(L(H)))_\lambda$ and $Z(L(H)_\lambda)$ are isomorphic and $L(H)$ is a locally $C^*$-algebra with perfect center.

If the Hilbert spaces $H_\lambda,\lambda\in\Lambda$, are finite-dimensional, then the $C^*$-algebras $L(H_\lambda),\lambda\in\Lambda$, are scattered (see [2]). Therefore the factors $L(H)_\lambda,\lambda\in\Lambda$, in the Arens–Michael decomposition of $L(H)$ are scattered, and by Proposition 2.3, $L(H)$ is a scattered locally $C^*$-algebra with perfect center.

It is known that a $C^*$-algebra $A$ is scattered if and only if it is of type $I$ and its center $Z(A)$ is a scattered $C^*$-algebra (see [10, Theorem 2.2]). The following result is a generalization of [10, Theorem 2.2].

**Theorem 2.15.** *Let $A[\tau_\Gamma]$ be a locally $C^*$-algebra with perfect center. Then the following statements are equivalent:*

(1) *$A[\tau_\Gamma]$ is a scattered locally $C^*$-algebra,*
(2) *$A[\tau_\Gamma]$ is of type $I$ and $Z(A[\tau_\Gamma])$ is a scattered locally $C^*$-algebra.*

*Proof.* (1) $\Rightarrow$ (2). This follows from Corrolaries 2.4 and 2.7.

(2) $\Rightarrow$ (1). Since $Z(A[\tau_\Gamma])$ is a scattered locally $C^*$-algebra, the factors $(Z(A[\tau_\Gamma]))_\lambda$, $\lambda\in\Lambda$, in the Arens–Michael decomposition of $Z(A[\tau_\Gamma])$ are scattered. On the other hand, $Z(A[\tau_\Gamma])=\lim_{\leftarrow\lambda}Z(A_\lambda)$, up to an isomorphism of locally $C^*$-algebras, and $\{Z(A_\lambda);\pi^A_{\lambda\mu}|_{Z(A_\lambda)}\}_{\lambda,\mu\in\Lambda,\lambda\geq\mu}$ is a perfect inverse system of $C^*$-algebras. Therefore, the $C^*$-algebras $Z(A_\lambda),\lambda\in\Lambda$, are scattered. Since $A[\tau_\Gamma]$ is of type $I$, the factors $A_\lambda,\lambda\in\Lambda$, in the Arens–Michael decomposition of $A[\tau_\Gamma]$ are of type $I$. Then, by [10, Theorem 2.2], the $C^*$-algebras $A_\lambda,\lambda\in\Lambda$, are scattered, and by Proposition 2.3, $A[\tau_\Gamma]$ is scattered. $\square$

## 3. Crossed products of scattered locally $C^*$-algebras

Let $G$ be a locally compact group, and let $A[\tau_\Gamma]$ be a locally $C^*$-algebra. An action of $G$ on $A[\tau_\Gamma]$ is a group morphism $\alpha$ from $G$ to $\mathrm{Aut}(A[\tau_\Gamma])$, the group of all automorphisms of $A[\tau_\Gamma]$, such that, for each $a \in A$, the map $g \mapsto \alpha_g(a)$ from $G$ to $A[\tau_\Gamma]$ is continuous. An action $\alpha$ of $G$ on $A[\tau_\Gamma]$ is an inverse limit action if there is a cofinal subset $\Gamma'$ of $\Gamma$ with the property that $p_\lambda(\alpha_g(a)) = p_\lambda(a)$ for all $a \in A$, for all $g \in G$, and for all $p_\lambda \in \Gamma'$. If $\alpha$ is an inverse limit action, we can suppose that $\Gamma' = \Gamma$, and then for each $\lambda \in \Lambda$ there is an action $\alpha^\lambda$ of $G$ on $A_\lambda$ such that $\alpha_g = \lim_{\leftarrow \lambda} \alpha_g^\lambda$ for all $g \in G$. If $G$ is compact, then any action of $G$ on $A[\tau_\Gamma]$ is an inverse limit action.

Recall that if $\alpha$ is an inverse limit action of $G$ on $A[\tau_\Gamma]$, then $L^1(G, \alpha, A[\tau_\Gamma]) = \{f : G \to A; \int_G p_\lambda(f(g))\, dg < \infty$ for all $\lambda \in \Lambda\}$, where $dg$ is the Haar measure on $G$, has a structure of locally $m$-convex $*$-algebra with the convolution as product and the involution given by $f^\#(g) = \Delta(g^{-1})\alpha_g(f(g^{-1})^*)$, where $\Delta$ is the modular function on $G$, and the topology given by the family of submultiplicative $*$-seminorms $\{N_\lambda\}_\lambda$, where $N_\lambda(f) = \int_G p_\lambda(f(g))\, dg$. The crossed product of $A[\tau_\Gamma]$ by $\alpha$, denoted by $A[\tau_\Gamma] \times_\alpha G$, is the enveloping locally $C^*$-algebra of the covariant algebra $L^1(G, \alpha, A[\tau_\Gamma])$. Moreover, for each $\lambda \in \Lambda$, the $C^*$-algebras $(A[\tau_\Gamma] \times_\alpha G)_\lambda$ and $A_\lambda \times_{\alpha^\lambda} G$ are isomorphic (see [9]).

As in the case of $C^*$-algebras, we have the following result.

**Proposition 3.1.** *Let $G$ be a locally compact group, and let $A[\tau_\Gamma]$ be a locally $C^*$-algebra. Then the crossed product $A[\tau_\Gamma] \times_\iota G$ of $A[\tau_\Gamma]$ by the trivial action $\iota$ of $G$ is scattered if and only if $A[\tau_\Gamma]$ and $C^*(G)$, the group $C^*$-algebras associated to $G$, are scattered.*

*Proof.* The assertion follows from Corollary 2.6 by taking into account that $A[\tau_\Gamma] \times_\iota G$ is isomorphic to the maximal tensor product of $A[\tau_\Gamma]$ and $C^*(G)$ (see, e.g., [9]). ☐

The following result extends [2, Proposition 6].

**Proposition 3.2.** *Let $G$ be a compact group, and let $\alpha$ be an action of $G$ on a locally $C^*$-algebra $A[\tau_\Gamma]$. If $A[\tau_\Gamma]$ is scattered, then $A[\tau_\Gamma] \times_\alpha G$ is scattered.*

*Proof.* If $A[\tau_\Gamma]$ is scattered, then, for each $\lambda \in \Lambda, A_\lambda$ is scattered (see Proposition 2.3), and by [2, Proposition 6] $A_\lambda \times_{\alpha^\lambda} G$ is scattered. From these facts and taking into account that for each $\lambda \in \Lambda$, the $C^*$-algebras $(A[\tau_\Gamma] \times_\alpha G)_\lambda$ and $A_\lambda \times_{\alpha^\lambda} G$ are isomorphic, we deduce that $A[\tau_\Gamma] \times_\alpha G$ is scattered. ☐

Let $\alpha = \lim_{\leftarrow \lambda} \alpha^\lambda$ be an inverse limit action of a locally compact group $G$ on a pro-$C^*$-algebra $A[\tau_\Gamma]$, and let $(A[\tau_\Gamma])^\alpha = \{a \in A; \alpha_g(a) = a$ for all $g \in G\}$ be the fixed point algebra of $A[\tau_\Gamma]$ under $\alpha$. Then $(A[\tau_\Gamma])^\alpha$ is a locally $C^*$-subalgebra of $A[\tau_\Gamma]$. Since, for each $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, $\pi_{\lambda\mu}^A((A_\lambda)^{\alpha^\lambda}) \subseteq (A_\mu)^{\alpha^\mu}$,

$$\left\{(A_\lambda)^{\alpha^\lambda}; \pi_{\lambda\mu}^A|_{(A_\lambda)^{\alpha^\lambda}}\right\}_{\lambda,\mu\in\Lambda,\lambda\geq\mu}$$

is an inverse system of $C^*$-algebras.

**Proposition 3.3.** *Let $\alpha = \lim_{\leftarrow\lambda} \alpha^\lambda$ be an inverse limit action of a locally compact group $G$ on a locally $C^*$-algebra $A[\tau_\Gamma]$. Then $(A[\tau_\Gamma])^\alpha = \lim_{\leftarrow\lambda}(A_\lambda)^{\alpha^\lambda}$, up to an isomorphism of locally $C^*$-algebras.*

*Proof.* Consider the map $\Psi : (A[\tau_\Gamma])^\alpha \to \lim_{\leftarrow\lambda}(A_\lambda)^{\alpha^\lambda}$given by

$$\Phi(a) = \left(\pi_\lambda^A(a)\right)_\lambda.$$

Clearly, $\Psi$ is a $*$-morphism and $p_{(A_\lambda)^{\alpha^\lambda}}(\Psi(a)) = p_\lambda|_{(A[\tau_\Gamma])^\alpha}(a)$ for all $a \in (A[\tau_\Gamma])^\alpha$ and for all $\lambda \in \Lambda$. If $(a_\lambda)_\lambda$ is a coherent sequence in $\{(A_\lambda)^{\alpha^\lambda}; \pi_{\lambda\mu}^A|_{(A_\lambda)^{\alpha^\lambda}}\}_{\lambda,\mu\in\Lambda,\lambda\geq\mu}$, then there is $a \in A$ such that $\pi_\lambda^A(a) = a_\lambda$ for all $\lambda \in \Lambda$. From $\pi_\lambda^A(\alpha_g(a) - a) = \alpha_g^\lambda(\pi_\lambda^A(a)) - \pi_\lambda^A(a) = 0$ for all $\lambda \in \Lambda$, we deduce that $a \in (A[\tau_\Gamma])^\alpha$, and so $\Psi$ is surjective. Therefore, $\Psi$ is an isomorphism of locally $C^*$-algebras. $\square$

*Remark* 3.4. We remark that, in general, the isometric $C^*$-morphism $\psi_\lambda : ((A[\tau_\Gamma])^\alpha)_\lambda \to (A_\lambda)^{\alpha^\lambda}, \psi_\lambda(a + \ker(p_\lambda|_{(A[\tau_\Gamma])^\alpha})) = a + \ker p_\lambda$ is not onto.

By [10, Theorem 3.2], the crossed product $A \times_\alpha G$ of a $C^*$-algebra $A$ by an action $\alpha$ of a compact abelian group $G$ is a scattered $C^*$-algebra if and only if $A^\alpha$ is a scattered $C^*$-algebra. We do not know if this result is true in the context of locally $C^*$-algebras, but we can prove the following results.

**Proposition 3.5.** *Let $G$ be a compact abelian group, and let $\alpha$ be an action of $G$ on a locally $C^*$-algebra $A[\tau_\Gamma]$. If $A \times_\alpha G$ is scattered, then $(A[\tau_\Gamma])^\alpha$ is scattered.*

*Proof.* If $A \times_\alpha G$ is scattered, then by Proposition 2.3 and [10, Theorem 3.2], the $(A_\lambda)^{\alpha^\lambda}, \lambda \in \Lambda$, are scattered. Since for each $\lambda \in \Lambda$, $((A[\tau_\Gamma])^\alpha)_\lambda$ is a $C^*$-subalgebra of $(A_\lambda)^{\alpha^\lambda}$, $((A[\tau_\Gamma])^\alpha)_\lambda$ is scattered and so $(A[\tau_\Gamma])^\alpha$ is scattered. $\square$

*Definition* 3.6. We say that an inverse limit action $\alpha = \lim_{\leftarrow\lambda} \alpha^\lambda$ of a locally compact group $G$ on a pro-$C^*$-algebra $A[\tau_\Gamma]$ is *perfect* if $\{(A_\lambda)^{\alpha^\lambda}; \pi_{\lambda\mu}^A|_{(A_\lambda)^{\alpha^\lambda}}\}_{\lambda,\mu\in\Lambda,\lambda\geq\mu}$ is a perfect inverse system of $C^*$-algebras.

*Example* 3.7. Let $G$ be a locally compact group, and let $A[\tau_\Gamma]$ be a locally $C^*$-algebra. The action $\delta$ of $G$ on the locally $C^*$-algebra $C_0(G, A)$ of all continuous functions from $G$ to $A$ vanishing to infinite, given by $\delta_g(f)(t) = f(tg)$ for all $g, t \in G$, is an inverse limit action $\delta = \lim_{\leftarrow\lambda} \delta^\lambda$, where $\delta^\lambda$ is the action of $G$ on $C_0(G, A_\lambda)$, given by $\delta_g^\lambda(f)(t) = f(tg)$ for all $g, t \in G$. Moreover, $\delta$ is perfect, since the fixed point algebra of $C_0(G, A)$ under $\delta$ is isomorphic with $A$.

Under perfectness of the action, we obtain the inverse statement of Proposition 3.5.

**Theorem 3.8.** *Let $G$ be a compact abelian group, and let $\alpha$ be a perfect action of $G$ on a locally $C^*$-algebra $A[\tau_\Gamma]$. Then the following statements are equivalent:*

(1) *$A \times_\alpha G$ is scattered,*
(2) *$(A[\tau_\Gamma])^\alpha$ is scattered.*

*Proof.* (1) $\Rightarrow$ (2). It follows from Proposition 3.5.

(2) $\Rightarrow$ (1). Since $(A[\tau_\Gamma])^\alpha$ is a scattered locally $C^*$-algebra, the factors $((A[\tau_\Gamma])^\alpha)_\lambda$, $\lambda \in \Lambda$, in the Arens–Michael decomposition of $(A[\tau_\Gamma])^\alpha$, are scattered. On the other hand, $(A[\tau_\Gamma])^\alpha = \lim_{\leftarrow\lambda}(A_\lambda)^{\alpha^\lambda}$, up to an isomorphism of locally $C^*$-algebras, and $\{(A_\lambda)^{\alpha^\lambda}; \pi^A_{\lambda\mu}|_{(A_\lambda)^{\alpha^\lambda}}\}_{\lambda,\mu\in\Lambda,\lambda\geq\mu}$ is a perfect inverse system of $C^*$-algebras. Therefore, the fixed point algebras $(A_\lambda)^{\alpha^\lambda}$ of $A_\lambda$ under $\alpha^\lambda, \lambda \in \Lambda$, are scattered. Then, by [10, Theorem 3.2], the factors $A_\lambda \times_{\alpha^\lambda} G, \lambda \in \Lambda$, in the Arens–Michael decomposition of $A \times_\alpha G$, are scattered, and by Proposition 2.3, $A \times_\alpha G$ is scattered. □

Let $\alpha = \lim_{\leftarrow\lambda} \alpha^\lambda$ be an inverse limit action of a locally compact group $G$ on a locally $C^*$-algebra $A[\tau_\Gamma]$. For each $g \in G$, the restriction of $\alpha_g$ on $b(A[\tau_\Gamma])$, the $C^*$-algebra of all bounded elements of $A[\tau_\Gamma]$, is an automorphism of $b(A[\tau_\Gamma])$ and the map $g \mapsto \alpha_g|_{b(A[\tau_\Gamma])}$ from $G$ to $\mathrm{Aut}(b(A[\tau_\Gamma]))$ is a group morphism. In general, the map $g \mapsto \alpha_g|_{b(A[\tau_\Gamma])}$ from $G$ to $\mathrm{Aut}(b(A[\tau_\Gamma]))$ is not an action of $G$ on $b(A[\tau_\Gamma])$, since for a fixed element $a \in b(A[\tau_\Gamma])$, the map $g \mapsto \alpha_g(a)$ from $G$ to $b(A[\tau_\Gamma])$ is not always continuous. We remark that if $\beta$ is an action of $G$ on $b(A[\tau_\Gamma])$ such that the closed two-sided $*$-ideals $\ker p_\lambda|_{b(A[\tau_\Gamma])}$, $\lambda \in \Lambda$, are $\beta$-invariant, then $\beta$ extends to an action $g \mapsto \widetilde{\beta}_g$ of $G$ on $A[\tau_\Gamma]$, where $\widetilde{\beta}_g$ is the extension of the automorphism $\beta_g$ of $b(A[\tau_\Gamma])$ to an automorphism of $A[\tau_\Gamma]$.

Suppose that for each $a \in b(A[\tau_\Gamma])$, the map $g \mapsto \alpha_g(a)$ from $G$ to $b(A[\tau_\Gamma])$ is continuous. Then $\alpha|_{b(A[\tau_\Gamma])}$ is an action of $G$ on $b(A[\tau_\Gamma])$ and the closed two-sided $*$-ideals $\ker p_\lambda|_{b(A[\tau_\Gamma])}$, $\lambda \in \Lambda$, are $\alpha|_{b(A[\tau_\Gamma])}$-invariant. Therefore, for each $\lambda \in \Lambda, \alpha|_{b(A[\tau_\Gamma])}$ induces an action of $G$ on $b(A[\tau_\Gamma])/\ker p_\lambda|_{b(A[\tau_\Gamma])}$, denoted by $\alpha|^\lambda_{b(A[\tau_\Gamma])}$, and the $C^*$-algebras $(b(A[\tau_\Gamma]) \times_{\alpha|_{b(A[\tau_\Gamma])}} G)/\ker p_\lambda|_{b(A[\tau_\Gamma])} \times_{\alpha|_{b(A[\tau_\Gamma])}} G$ and $b(A[\tau_\Gamma])/\ker p_\lambda|_{b(A[\tau_\Gamma])} \times_{\alpha|^\lambda_{b(A[\tau_\Gamma])}} G$ are isomorphic (see, e.g., [1, Chapter IV, Theorem 3.5.8]).

**Lemma 3.9.** *Let $\alpha = \lim_{\leftarrow\lambda} \alpha^\lambda$ be an inverse limit action of a locally compact group $G$ on a locally $C^*$-algebra $A[\tau_\Gamma]$ such that for each $a \in b(A[\tau_\Gamma])$, the map $g \mapsto \alpha_g(a)$ from $G$ to $b(A[\tau_\Gamma])$ is continuous. Then, for each $\lambda \in \Lambda$, the $C^*$-algebras $(b(A[\tau_\Gamma]) \times_{\alpha|_{b(A[\tau_\Gamma])}} G)/\ker p_\lambda|_{b(A[\tau_\Gamma])} \times_{\alpha|_{b(A[\tau_\Gamma])}} G$ and $A_\lambda \times_{\alpha^\lambda} G$ are isomorphic.*

*Proof.* Take $\lambda \in \Lambda$. The map $\varphi_\lambda : b(A[\tau_\Gamma])/\ker p_\lambda|_{b(A[\tau_\Gamma])} \to A[\tau_\Gamma])/\ker p_\lambda$, given by

$$\varphi_\lambda(a + \ker p_\lambda|_{b(A[\tau_\Gamma])}) = a + \ker p_\lambda,$$

is a $C^*$-isomorphism (see, e.g., [3, Theorem 10.24]). Furthermore,

$$\varphi_\lambda\big((\alpha|_{b(A[\tau_\Gamma])})_g(a + \ker p_\lambda|_{b(A[\tau_\Gamma])})\big) = \alpha^\lambda_g\big(\varphi_\lambda(a + \ker p_\lambda|_{b(A[\tau_\Gamma])})\big)$$

for all $a \in b(A[\tau_\Gamma])$ and for all $g \in G$. Thus, there is a $C^*$-isomorphism $\Phi_\lambda$ : $b(A[\tau_\Gamma])/\ker p_\lambda|_{b(A[\tau_\Gamma])} \times_{\alpha|^\lambda_{b(A[\tau_\Gamma])}} G \to A_\lambda \times_{\alpha^\lambda} G$ such that

$$\Phi_\lambda\big((a + \ker p_\lambda|_{b(A[\tau_\Gamma])}) \otimes f\big) = \varphi_\lambda(a + \ker p_\lambda|_{b(A[\tau_\Gamma])}) \otimes f$$

for all $a \in b(A[\tau_\Gamma])$ and for all $f \in C_c(G)$. Therefore, the $C^*$-algebras $A_\lambda \times_{\alpha^\lambda} G$ and $(b(A[\tau_\Gamma]) \times_{\alpha|_{b(A[\tau_\Gamma])}} G)/\ker p_\lambda|_{b(A[\tau_\Gamma])} \times_{\alpha|_{b(A[\tau_\Gamma])}} G$ are isomorphic. □

**Proposition 3.10.** *Let $\alpha = \lim_{\leftarrow\lambda} \alpha^\lambda$ be an inverse limit action of a locally compact group $G$ on a locally $C^*$-algebra $A[\tau_\Gamma]$ such that for each $a \in b(A[\tau_\Gamma])$, the map $g \mapsto \alpha_g(a)$ from $G$ to $b(A[\tau_\Gamma])$ is continuous. If $b(A[\tau_\Gamma]) \times_{\alpha|_{b(A[\tau_\Gamma])}} G$ is scattered, then $A \times_\alpha G$ is scattered.*

*Proof.* If $b(A[\tau_\Gamma]) \times_{\alpha|_{b(A[\tau_\Gamma])}} G$ is scattered, then for each $\lambda \in \Lambda$, the $C^*$-algebra $(b(A[\tau_\Gamma]) \times_{\alpha|_{b(A[\tau_\Gamma])}} G)/\ker p_\lambda|_{b(A[\tau_\Gamma])} \times_{\alpha|_{b(A[\tau_\Gamma])}} G$ is scattered, and according to Lemma 3.9, for each $\lambda \in \Lambda$, the $C^*$-algebra $A_\lambda \times_{\alpha^\lambda} G$ is scattered. From this fact, [9, Corollary 1.3.7], and Proposition 2.3, we deduce that $A \times_\alpha G$ is scattered. □

**Theorem 3.11.** *Let $G$ be a compact abelian group, and let $\alpha$ be an action of $G$ on a locally $C^*$-algebra $A[\tau_\Gamma]$ such that for each $a \in b(A[\tau_\Gamma])$, the map $g \mapsto \alpha_g(a)$ from $G$ to $b(A[\tau_\Gamma])$ is continuous. If for some $\lambda \in \Lambda$ the closed two sided $*$-ideal $\ker p_\lambda|_{b(A[\tau_\Gamma])}$ of $b(A[\tau_\Gamma])$ is scattered, then the following statements are equivalent:*

(1) *$(b(A[\tau_\Gamma]))^{\alpha|_{b(A[\tau_\Gamma])}}$ is scattered,*
(2) *$b(A[\tau_\Gamma]) \times_{\alpha|_{b(A[\tau_\Gamma])}} G$ is scattered,*
(3) *$A \times_\alpha G$ is scattered,*
(4) *$(A[\tau_\Gamma]))^\alpha$ is scattered.*

*Proof.* (1) $\Leftrightarrow$ (2). By [10, Theorem 3.2], $(b(A[\tau_\Gamma]))^{\alpha|_{b(A[\tau_\Gamma])}}$ is scattered if and only if $b(A[\tau_\Gamma]) \times_{\alpha|_{b(A[\tau_\Gamma])}} G$ is scattered.

(2) $\Longrightarrow$ (3). This follows from Proposition 3.10.

(3) $\Longrightarrow$ (4). This follows from Proposition 3.5.

(4) $\Longrightarrow$ (1). If the closed two sided $*$-ideal $\ker p_\lambda|_{b(A[\tau_\Gamma])}$ of $b(A[\tau_\Gamma])$ is scattered, then $\ker p_\lambda|_{(b(A[\tau_\Gamma]))^{\alpha|_{b(A[\tau_\Gamma])}}}$ is scattered, and by Proposition 2.10, $(b(A[\tau_\Gamma]))^{\alpha|_{b(A[\tau_\Gamma])}}$ is scattered since $(b(A[\tau_\Gamma]))^{\alpha|_{b(A[\tau_\Gamma])}} = b((A[\tau_\Gamma]))^\alpha)$. □

**Acknowledgments.** We would like to thank M. Haralampidou of the University of Athens for valuable comments and suggestions during the preparation of this article.

Joiţa's work was partially supported by Romanian Ministry of Education grant CNCS-UEFISCDI, project number PN-II-ID-PCE-2012-4-0201.

## References

1. B. Blackadar, *Operator Algebras: Theory of $C^*$-Algebras and von Neumann Algebras*, Encyclopaedia Math. Sci. **122**, Springer, Berlin, 2006. Zbl 1092.46003. MR2188261. DOI 10.1007/3-540-28517-2. 38
2. C. H. Chu, *Crossed products of scattered $C^*$-algebras*, J. Lond. Math. Soc. (2) **26** (1982), no. 2, 317–324. Zbl 0468.46048. MR0675174. DOI 10.1112/jlms/s2-26.2.317. 31, 33, 35, 36
3. M. Fragoulopoulou, *Topological Algebras with Involution*, North-Holland Math. Stud. **200**, North-Holland, Amsterdam, 2005. Zbl 1197.46001. MR2172581. 31, 33, 34, 38
4. M. Haralampidou, "The Krull nature of locally $C^*$-algebras" in *Function Spaces (Edwardsville, Ill., 2002)*, Contemp. Math. **328**, Amer. Math. Soc., Providence, 2003, 195–200. Zbl 1048.46041. MR1990401. DOI 10.1090/conm/328/05781. 34
5. T. Huruya, *A spectral characterization of a class of $C^*$-algebras*, Sci. Rep. Niigata Univ. Ser. A **15** (1978), 21–24. Zbl 0375.46051. MR0482243. 31

6. A. Inoue, *Locally $C^*$-algebra*, Mem. Fac. Sci. Kyushu Univ. Ser. A, **25** (1971), 197–235. Zbl 0227.46060. MR0305089. DOI 10.2206/kyushumfs.25.197. 35
7. H. E. Jensen, *Scattered $C^*$-algebras*, Math. Scand. **41** (1977), no. 2, 308–314. Zbl 0372.46059. MR0482242. DOI 10.7146/math.scand.a-11723. 30, 31, 33, 34
8. H. E. Jensen, *Scattered $C^*$-algebras, II*, Math. Scand. **43** (1978), no. 2, 308–310. Zbl 0423.46050. MR0531308. DOI 10.7146/math.scand.a-11782. 31
9. M. Joiţa, *Crossed Products of Locally $C^*$-Algebras*, Editura Academiei Române, Bucharest, 2007. Zbl 1158.46001. MR2722292. 36, 39
10. M. Kusuda, *A characterization of scattered $C^*$-algebras and its application to $C^*$-crossed products*, J. Operator Theory **63** (2010), no. 2, 417–424. Zbl 1224.46116. MR2651922. 31, 35, 37, 38, 39
11. M. Kusuda, *$C^*$-algebras in which every $C^*$-subalgebra is AF*, Quart. J. Math. **63** (2012), no. 3, 675–680. Zbl 1258.46023. MR2967169. DOI 10.1093/qmath/har014. 32
12. A. J. Lazar, *On scattered $C^*$-algebras*, in preparation. 31
13. A. Pelczynski and Z. Semadeni, *Spaces of continuous functions, III: Spaces $C(\Omega)$ for $\Omega$ without perfect subsets*, Studia Math. **18** (1959), 211–222. Zbl 0091.27803. MR0107806. 30
14. N. C. Phillips, *Inverse limits of $C^*$-algebras*, J. Operator Theory **19** (1988), no. 1, 159–195. Zbl 0662.46063. MR0950831. 31, 32
15. M. L. Rothwell, *Scattered $C^*$-algebras*, in preparation. 31
16. W. Rudin, *Continuous functions on compact spaces without perfect subsets*, Proc. Amer. Math. Soc. **8** (1957), 39–42. Zbl 0077.31103. MR0085475. DOI 10.2307/2032807. 30

Department of Mathematics, Faculty of Applied Sciences, University Politehnica of Bucharest, 313 Spl. Independentei, 060042, Bucharest, Romania and Simion Stoilow Institute of Mathematics of the Roumanian Academy, 21 Calea Grivitei Street, 010702 Bucharest, Romania.

*E-mail address*: mjoita@fmi.unibuc.ro; maria.joita@mathem.pub.ro