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LOCAL LIE DERIVATIONS ON CERTAIN OPERATOR ALGEBRAS

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ABSTRACT. In this paper, we investigate local Lie derivations of a certain class of operator algebras and show that, under certain conditions, every local Lie derivation of such an algebra is a Lie derivation.

1. Introduction and preliminaries

A well-known and active direction in the study of derivations is the local derivations problem, which was initiated by Kadison [10] and by Larson and Sourour [11]. Recall that a linear map φ of an algebra $\mathcal{A}$ is called a *local derivation* if, for each $A \in \mathcal{A}$, there exists a derivation φ_A of $\mathcal{A}$ depending on A such that $\varphi(A) = \varphi_A(A)$. The question of determining under what conditions every local derivation must be a derivation has been studied by many authors (see [6], [7], [13], and [14]). Recently, Brešar [1] proved that each local derivation of algebras generated by all their idempotents is a derivation.

A linear map φ of an algebra $\mathcal{A}$ is called a *Lie derivation* if $\varphi([A, B]) = [\varphi(A), B] + [A, \varphi(B)]$ for all $A, B \in \mathcal{A}$, where $[A, B] = AB - BA$ is the usual Lie product, also called a *commutator*. A Lie derivation φ of $\mathcal{A}$ is standard if it can be decomposed as $\varphi = d + \tau$, where d is a derivation from $\mathcal{A}$ into itself and τ is a linear map from $\mathcal{A}$ into its center vanishing on each commutator. The classical problem, which has been studied for many years, is to find conditions on $\mathcal{A}$ under which each Lie derivation is standard or standard-like. This problem has been investigated for general operator algebras (see [4], [9], and [12]).

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A linear map φ of an algebra $\mathcal{A}$ is called a *local Lie derivation* if, for each $A \in \mathcal{A}$, there exists a Lie derivation φ_A of $\mathcal{A}$ such that $\varphi(A) = \varphi_A(A)$. In [3], Chen et al. proved that each local Lie derivation of $B(X)$ where X is a Banach space of dimension greater than 2 is a Lie derivation. Later, Chen and Lu [2] proved that each local Lie derivation of nest algebras on Hilbert spaces is a Lie derivation. It is quite common to study local derivations in algebras that contain many idempotents in the sense that the linear span of all idempotents is "large." The main novelty of this paper is that we deal with the subalgebra generated by all idempotents instead of the span. Let $\mathcal{M}_2$ be the algebra of 2×2 matrices over $L^\infty[0,1]$. By [8], $\mathcal{M}_2$ is generated by, but not spanned by, its idempotents. In what follows, we denote by $\mathcal{J}(\mathcal{A})$ the subalgebra of $\mathcal{A}$ generated by all idempotents in $\mathcal{A}$. The purpose of the present paper is to study local Lie derivations of a certain class of operator algebras. We also provide an example of an algebra with a nontrivial local Lie derivation.

Let X and Y be Banach spaces over a real or complex field $\mathbb{F}$. By $B(X)$ we denote the algebra of all bounded linear operators on X. Let $\mathcal{A}$ and $\mathcal{B}$ be unital subalgebras of $B(X)$ and $B(Y)$, respectively, and let $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Under the usual matrix operations,

$$\mathcal{T} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} : A \in \mathcal{A}, M \in \mathcal{M}, B \in \mathcal{B} \right\}$$

is an operator algebra with the unit $1 = \left(\begin{smallmatrix} 1_\mathcal{A} & 0 \\ 0 & 1_\mathcal{B} \end{smallmatrix}\right)$, where $1_\mathcal{A}$ and $1_\mathcal{B}$ are the units of the algebras $\mathcal{A}$ and $\mathcal{B}$, respectively.

Let $Z(\mathcal{T})$ be the center of $\mathcal{T}$. It follows from [4] that

$$Z(\mathcal{T}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : AM = MB \text{ for all } M \in \mathcal{M} \right\}.$$

Let us define two natural projections $\pi_\mathcal{A} : \mathcal{T} \to \mathcal{A}$ and $\pi_\mathcal{B} : \mathcal{T} \to \mathcal{B}$ by

$$\pi_\mathcal{A} : \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \mapsto A \qquad \text{and} \qquad \pi_\mathcal{B} : \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \mapsto B.$$

Then $\pi_\mathcal{A}(Z(\mathcal{T})) \subseteq Z(\mathcal{A})$ and $\pi_\mathcal{B}(Z(\mathcal{T})) \subseteq Z(\mathcal{B})$.

Throughout this paper we will use following notation:

$$P_1 = \begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_2 = 1 - P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1_\mathcal{B} \end{pmatrix},$$

and

$$\mathcal{T}_{ij} = P_i \mathcal{T} P_j \quad \text{for } 1 \leq i \leq j \leq 2.$$

It is clear that the algebra $\mathcal{T}$ may be represented as

$$\mathcal{T} = \mathcal{T}_{11} \oplus \mathcal{T}_{12} \oplus \mathcal{T}_{22}.$$

We close this section with a well-known result concerning Lie derivations.

Proposition 1.1 ([4, Theorem 11]). *Let $\mathcal{T} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ with $Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(\mathcal{T}))$ and $Z(\mathcal{B}) = \pi_{\mathcal{B}}(Z(\mathcal{T}))$. Then every Lie derivation $\varphi : \mathcal{T} \to \mathcal{T}$ is standard; that is, φ is the sum of a derivation d and a linear central-valued map τ vanishing on each commutator.*

2. Main results

Our main result reads as follows.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital subalgebras of $B(X)$ and $B(Y)$, respectively, let $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, and let $\mathcal{T} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$. Suppose that*

(1) *$\mathcal{A} = \mathcal{J}(\mathcal{A})$ and $\mathcal{B} = \mathcal{J}(\mathcal{B})$,*
(2) *$Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(\mathcal{T}))$ and $Z(\mathcal{B}) = \pi_{\mathcal{B}}(Z(\mathcal{T}))$.*

Then every local Lie derivation φ from $\mathcal{T}$ into itself is a Lie derivation.

To prove Theorem 2.1, we need some lemmas. In the following, φ is a local Lie derivation and, for any $A \in \mathcal{T}$, the symbol φ_A stands for a Lie derivation from $\mathcal{T}$ into itself such that $\varphi(A) = \varphi_A(A)$. It follows from $\mathcal{A} = \mathcal{J}(\mathcal{A})$ and $\mathcal{B} = \mathcal{J}(\mathcal{B})$ that every A_{kk} in $\mathcal{T}_{kk}$ can be written as a linear combination of some elements $A_{kk}^{(i_1)} A_{kk}^{(i_2)} \cdots A_{kk}^{(i_{n_i})}$ $(i = 1, 2, \ldots, m)$, where $A_{kk}^{(i_1)}, A_{kk}^{(i_2)}, \ldots, A_{kk}^{(i_{n_i})}$ are idempotents in $\mathcal{T}_{kk}$ $(k = 1, 2)$.

Lemma 2.2. *For every idempotent $P, Q \in \mathcal{T}$ and $A \in \mathcal{T}$, there exist linear maps $\tau_1, \tau_2, \tau_3, \tau_4 : \mathcal{T} \to Z(\mathcal{T})$ vanishing on each commutator such that*

$$\begin{aligned} \varphi(PAQ) = {} & \varphi(PA)Q + P\varphi(AQ) - P\varphi(A)Q + P^{\perp}\tau_1(PAQ)Q^{\perp} \\ & - P\tau_2(P^{\perp}AQ)Q^{\perp} + P\tau_3(P^{\perp}AQ^{\perp})Q - P^{\perp}\tau_4(PAQ^{\perp})Q, \end{aligned}$$

where $P^{\perp} = 1 - P$ and $Q^{\perp} = 1 - Q$.

Proof. Assumption (2) of Theorem 2.1 and Proposition 1.1 imply that, for every idempotent $P, Q \in \mathcal{T}$ and $A \in \mathcal{T}$, there exist derivations $d_1, d_2, d_3, d_4 : \mathcal{T} \to \mathcal{T}$ and linear maps $\tau_1, \tau_2, \tau_3, \tau_4 : \mathcal{T} \to Z(\mathcal{T})$ vanishing on each commutator such that

$$\varphi(PAQ) = \varphi_{PAQ}(PAQ) = d_1(PAQ) + \tau_1(PAQ), \tag{2.1}$$
$$\varphi(P^{\perp}AQ) = \varphi_{P^{\perp}AQ}(P^{\perp}AQ) = d_2(P^{\perp}AQ) + \tau_2(P^{\perp}AQ), \tag{2.2}$$
$$\varphi(P^{\perp}AQ^{\perp}) = \varphi_{P^{\perp}AQ^{\perp}}(P^{\perp}AQ^{\perp}) = d_3(P^{\perp}AQ^{\perp}) + \tau_3(P^{\perp}AQ^{\perp}), \tag{2.3}$$
$$\varphi(PAQ^{\perp}) = \varphi_{PAQ^{\perp}}(PAQ^{\perp}) = d_4(PAQ^{\perp}) + \tau_4(PAQ^{\perp}). \tag{2.4}$$

It follows from (2.1)–(2.4) that

$$\begin{aligned} P^{\perp}\varphi(PAQ)Q^{\perp} &= P^{\perp}\tau_1(PAQ)Q^{\perp}, \\ P\varphi(P^{\perp}AQ)Q^{\perp} &= P\tau_2(P^{\perp}AQ)Q^{\perp}, \\ P\varphi(P^{\perp}AQ^{\perp})Q &= P\tau_3(P^{\perp}AQ^{\perp})Q, \\ P^{\perp}\varphi(PAQ^{\perp})Q &= P^{\perp}\tau_4(PAQ^{\perp})Q. \end{aligned}$$

Hence

$$\begin{aligned}\varphi(PAQ)Q^{\perp} &= P\varphi(PAQ)Q^{\perp} + P^{\perp}\varphi(PAQ)Q^{\perp}\\ &= P\varphi(AQ)Q^{\perp} - P\varphi(P^{\perp}AQ)Q^{\perp} + P^{\perp}\varphi(PAQ)Q^{\perp}\\ &= P\varphi(AQ)Q^{\perp} + P^{\perp}\tau_1(PAQ)Q^{\perp} - P\tau_2(P^{\perp}AQ)Q^{\perp}\\ &= P\varphi(AQ) - P\varphi(AQ)Q + P^{\perp}\tau_1(PAQ)Q^{\perp} - P\tau_2(P^{\perp}AQ)Q^{\perp}\end{aligned}$$

and

$$\begin{aligned}\varphi(PAQ^{\perp})Q &= P\varphi(PAQ^{\perp})Q + P^{\perp}\varphi(PAQ^{\perp})Q\\ &= P\varphi(AQ^{\perp})Q - P\varphi(P^{\perp}AQ^{\perp})Q + P^{\perp}\varphi(PAQ^{\perp})Q\\ &= P\varphi(AQ^{\perp})Q - P\tau_3(P^{\perp}AQ^{\perp})Q + P^{\perp}\tau_4(PAQ^{\perp})Q.\end{aligned}$$

Thus

$$\begin{aligned}\varphi(PAQ) &= \varphi(PAQ)Q^{\perp} + \varphi(PAQ)Q\\ &= \varphi(PAQ)Q^{\perp} + \varphi(PA)Q - \varphi(PAQ^{\perp})Q\\ &= \varphi(PA)Q + P\varphi(AQ) - P\varphi(A)Q + P^{\perp}\tau_1(PAQ)Q^{\perp}\\ &\quad - P\tau_2(P^{\perp}AQ)Q^{\perp} + P\tau_3(P^{\perp}AQ^{\perp})Q - P^{\perp}\tau_4(PAQ^{\perp})Q,\end{aligned}$$

where we have used $\varphi(AQ) = \varphi(A) - \varphi(AQ^{\perp})$. □

Lemma 2.3. *For any $A_{ij} \in \mathcal{T}_{ij}$ $(1 \leq i \leq j \leq 2)$, we have*

(1) $P_1\varphi(P_1)P_1 + P_2\varphi(P_1)P_2 \in Z(\mathcal{T})$ *and* $\varphi(A_{12}) \in \mathcal{T}_{12}$,
(2) $P_1\varphi(A_{11})P_2 = A_{11}\varphi(P_1)P_2$ *and* $P_1\varphi(A_{22})P_2 = -P_1\varphi(P_1)A_{22}$.

Proof. (1) For any $A_{12} \in \mathcal{T}_{12}$, we have

$$\begin{aligned}\varphi_{P_1}(A_{12}) &= \varphi_{P_1}\big([P_1, A_{12}]\big)\\ &= \big[\varphi(P_1), A_{12}\big] + \big[P_1, \varphi_{P_1}(A_{12})\big]\\ &= \varphi(P_1)A_{12} - A_{12}\varphi(P_1) + P_1\varphi_{P_1}(A_{12})P_2.\end{aligned}$$

Left-multiplying by P_1 and right-multiplying by P_2, this implies that $P_1\varphi(P_1)A_{12} = A_{12}\varphi(P_1)P_2$, and so

$$P_1\varphi(P_1)P_1 + P_2\varphi(P_1)P_2 \in Z(\mathcal{T}).$$

It follows from $A_{12} = [P_1, A_{12}]$ that

$$\begin{aligned}\varphi(A_{12}) &= \varphi_{A_{12}}\big([P_1, A_{12}]\big)\\ &= \big[\varphi_{A_{12}}(P_1), A_{12}\big] + \big[P_1, \varphi(A_{12})\big]\\ &= \varphi_{A_{12}}(P_1)A_{12} - A_{12}\varphi_{A_{12}}(P_1) + P_1\varphi(A_{12})P_2.\end{aligned}$$

Multiplying the above equality from both sides by P_1 and P_2, respectively, we have $P_1\varphi(A_{12})P_1 = P_2\varphi(A_{12})P_2 = 0$. Hence $\varphi(A_{12}) = P_1\varphi(A_{12})P_2 \in \mathcal{T}_{12}$.

(2) Let $B_{11} \in \mathcal{T}_{11}, A_{12} \in \mathcal{T}_{12}$. Taking $P = A_{11}^{(1)}$, $A = B_{11}$, and $Q = P_1$ in Lemma 2.2, we have from $PAQ^\perp = P^\perp AQ^\perp = 0$ that

$$\begin{aligned}\varphi(A_{11}^{(1)}B_{11}) &= \varphi(A_{11}^{(1)}B_{11})P_1 + A_{11}^{(1)}\varphi(B_{11}) - A_{11}^{(1)}\varphi(B_{11})P_1 \\ &\quad + (1 - A_{11}^{(1)})\tau_1(A_{11}^{(1)}B_{11})P_2 - A_{11}^{(1)}\tau_2(B_{11} - A_{11}^{(1)}B_{11})P_2 \\ &= \varphi(A_{11}^{(1)}B_{11})P_1 + A_{11}^{(1)}\varphi(B_{11})P_2 + \tau_1(A_{11}^{(1)}B_{11})P_2.\end{aligned}$$

This implies that

$$P_1\varphi(A_{11}^{(1)}B_{11})P_2 = A_{11}^{(1)}\varphi(B_{11})P_2.$$

In particular,

$$P_1\varphi(A_{11}^{(1)})P_2 = A_{11}^{(1)}\varphi(P_1)P_2.$$

By the above two equations, then

$$\begin{aligned}P_1\varphi(A_{11}^{(1)}A_{11}^{(2)}\cdots A_{11}^{(n)})P_2 &= A_{11}^{(1)}\varphi(A_{11}^{(2)}\cdots A_{11}^{(n)})P_2 \\ &= A_{11}^{(1)}A_{11}^{(2)}\cdots A_{11}^{(n-1)}\varphi(A_{11}^{(n)})P_2 \\ &= A_{11}^{(1)}A_{11}^{(2)}\cdots A_{11}^{(n)}\varphi(P_1)P_2\end{aligned}$$

for any idempotents $A_{11}^{(1)}, A_{11}^{(2)}, \ldots, A_{11}^{(n)} \in \mathcal{T}_{11}$. It follows from $\mathcal{A} = \mathcal{J}(\mathcal{A})$ that $P_1\varphi(A_{11})P_2 = A_{11}\varphi(P_1)P_2$ for all $A_{11} \in \mathcal{T}_{11}$. Similarly, we can obtain from Lemma 2.2 and the fact $\varphi(1) \in Z(\mathcal{T})$ that

$$P_1\varphi(A_{22})P_2 = P_1\varphi(P_2)A_{22} = -P_1\varphi(P_1)A_{22}$$

for all $A_{22} \in \mathcal{T}_{22}$. □

Next we define a linear map $\delta : \mathcal{T} \to \mathcal{T}$ by $\delta(A) = \varphi(A) - [A, P_1\varphi(P_1)P_2]$. Then δ is also a local Lie derivation, and by Lemma 2.3, $\delta(P_1) \in Z(\mathcal{T})$ and

$$\delta(\mathcal{T}_{12}) \subseteq \mathcal{T}_{12}, \qquad \delta(\mathcal{T}_{ii}) \subseteq \mathcal{T}_{11} \oplus \mathcal{T}_{22} \quad (i = 1, 2). \tag{2.5}$$

Remark 2.4. It is easy to verify that, for each derivation $d : \mathcal{T} \to \mathcal{T}$, we have

$$d(\mathcal{T}_{12}) \subseteq \mathcal{T}_{12}, \qquad d(\mathcal{T}_{ii}) \subseteq \mathcal{T}_{ii} \oplus \mathcal{T}_{12} \quad (i = 1, 2). \tag{2.6}$$

Lemma 2.5. *We have the following.*

(1) $\delta([A_{11}, A_{12}]) = [\delta(A_{11}), A_{12}] + [A_{11}, \delta(A_{12})]$ *for all* $A_{11} \in \mathcal{T}_{11}$ *and* $A_{12} \in \mathcal{T}_{12}$,
(2) $\delta([A_{12}, A_{22}]) = [\delta(A_{12}), A_{22}] + [A_{12}, \delta(A_{22})]$ *for all* $A_{12} \in \mathcal{T}_{12}$ *and* $A_{22} \in \mathcal{T}_{22}$.

Proof. (1) To prove this statement, we only need to prove that

$$\begin{aligned}&\delta\big([A_{11}^{(1)}A_{11}^{(2)}\cdots A_{11}^{(n)}, A_{12}]\big) \\ &\quad = \big[\delta(A_{11}^{(1)}A_{11}^{(2)}\cdots A_{11}^{(n)}), A_{12}\big] + \big[A_{11}^{(1)}A_{11}^{(2)}\cdots A_{11}^{(n)}, \delta(A_{12})\big]\end{aligned} \tag{2.7}$$

for any idempotents $A_{11}^{(1)}, A_{11}^{(2)}, \ldots, A_{11}^{(n)} \in \mathcal{T}_{11}$ and $A_{12} \in \mathcal{T}_{12}$.

Let $B_{11} \in \mathcal{T}_{11}$, and let $A_{12} \in \mathcal{T}_{12}$. Taking $P = A_{11}^{(1)}$, $A = B_{11}$, and $Q = P_1 + A_{12}$ in (2.2) and Lemma 2.2, it follows from the facts $P^{\perp}AQ^{\perp}$ and $PAQ^{\perp}$ can be written as commutators that $\tau_3(P^{\perp}AQ^{\perp}) = \tau_4(PAQ^{\perp}) = 0$. Then we can get

$$\begin{aligned}
&\delta(B_{11} + B_{11}A_{12} - A_{11}^{(1)}B_{11} - A_{11}^{(1)}B_{11}A_{12}) \\
&\quad = d_2(B_{11} + B_{11}A_{12} - A_{11}^{(1)}B_{11} - A_{11}^{(1)}B_{11}A_{12}) \\
&\qquad + \tau_2(B_{11} + B_{11}A_{12} - A_{11}^{(1)}B_{11} - A_{11}^{(1)}B_{11}A_{12}) \\
&\quad = d_2(B_{11} + B_{11}A_{12} - A_{11}^{(1)}B_{11} - A_{11}^{(1)}B_{11}A_{12}) \\
&\qquad + \tau_2(B_{11} - A_{11}^{(1)}B_{11})
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
&\delta(A_{11}^{(1)}B_{11} + A_{11}^{(1)}B_{11}A_{12}) \\
&\quad = \delta(A_{11}^{(1)}B_{11})(P_1 + A_{12}) + A_{11}^{(1)}\delta(B_{11} + B_{11}A_{12}) - A_{11}^{(1)}\delta(B_{11})(P_1 + A_{12}) \\
&\qquad + (1 - A_{11}^{(1)})\tau_1(A_{11}^{(1)}B_{11} + A_{11}^{(1)}B_{11}A_{12})(P_2 - A_{12}) \\
&\qquad - A_{11}^{(1)}\tau_2(B_{11} + B_{11}A_{12} - A_{11}^{(1)}B_{11} - A_{11}^{(1)}B_{11}A_{12})(P_2 - A_{12}) \\
&\quad = \delta(A_{11}^{(1)}B_{11})(P_1 + A_{12}) + A_{11}^{(1)}\delta(B_{11} + B_{11}A_{12}) - A_{11}^{(1)}\delta(B_{11})(P_1 + A_{12}) \\
&\qquad + (1 - A_{11}^{(1)})\tau_1(A_{11}^{(1)}B_{11})(P_2 - A_{12}) - A_{11}^{(1)}\tau_2(B_{11} - A_{11}^{(1)}B_{11})(P_2 - A_{12}) \\
&\quad = \delta(A_{11}^{(1)}B_{11})P_1 + \delta(A_{11}^{(1)}B_{11})A_{12} + A_{11}^{(1)}\delta(B_{11}A_{12}) - A_{11}^{(1)}\delta(B_{11})A_{12} \\
&\qquad + \tau_1(A_{11}^{(1)}B_{11})P_2 - A_{12}\tau_1(A_{11}^{(1)}B_{11}) + A_{11}^{(1)}A_{12}\tau_1(A_{11}^{(1)}B_{11}) \\
&\qquad + A_{11}^{(1)}A_{12}\tau_2(B_{11} - A_{11}^{(1)}B_{11}),
\end{aligned} \tag{2.9}$$

where we have used (2.5) in the third equality. It follows from (2.5), (2.6), and (2.8) that

$$P_2\delta(B_{11} - A_{11}^{(1)}B_{11})P_2 = \tau_2(B_{11} - A_{11}^{(1)}B_{11})P_2. \tag{2.10}$$

It follows from (2.5) and (2.9) that

$$P_2\delta(A_{11}^{(1)}B_{11})P_2 = \tau_1(A_{11}^{(1)}B_{11})P_2. \tag{2.11}$$

By (2.10) and (2.11), then $A_{12}\tau_1(A_{11}^{(1)}B_{11}) = A_{12}\delta(A_{11}^{(1)}B_{11})$ and

$$\begin{aligned}
A_{11}^{(1)}A_{12}\tau_2(B_{11} - A_{11}^{(1)}B_{11}) &= A_{11}^{(1)}A_{12}\delta(B_{11} - A_{11}^{(1)}B_{11}) \\
&= A_{11}^{(1)}A_{12}\delta(B_{11}) - A_{11}^{(1)}A_{12}\delta(A_{11}^{(1)}B_{11}) \\
&= A_{11}^{(1)}A_{12}\delta(B_{11}) - A_{11}^{(1)}A_{12}\tau_1(A_{11}^{(1)}B_{11}).
\end{aligned}$$

This together with (2.9) gives us that

$$\begin{aligned}
&\delta(A_{11}^{(1)}B_{11} + A_{11}^{(1)}B_{11}A_{12}) \\
&\quad = \delta(A_{11}^{(1)}B_{11})P_1 + \delta(A_{11}^{(1)}B_{11})A_{12} + A_{11}^{(1)}\delta(B_{11}A_{12}) - A_{11}^{(1)}\delta(B_{11})A_{12} \\
&\qquad + P_2\delta(A_{11}^{(1)}B_{11})P_2 - A_{12}\delta(A_{11}^{(1)}B_{11}) + A_{11}^{(1)}A_{12}\delta(B_{11}).
\end{aligned} \tag{2.12}$$

It follows from $\delta(\mathcal{T}_{11}) \subseteq \mathcal{T}_{11} \oplus \mathcal{T}_{22}$ that $\delta(A_{11}^{(1)}B_{11}) = \delta(A_{11}^{(1)}B_{11})P_1 + P_2\delta(A_{11}^{(1)}B_{11})P_2$, and so, by (2.12),

$$\begin{aligned}\delta(A_{11}^{(1)}B_{11}A_{12}) = \delta(A_{11}^{(1)}B_{11})A_{12} + A_{11}^{(1)}\delta(B_{11}A_{12}) - A_{11}^{(1)}\delta(B_{11})A_{12} \\ - A_{12}\delta(A_{11}^{(1)}B_{11}) + A_{11}^{(1)}A_{12}\delta(B_{11}).\end{aligned} \tag{2.13}$$

Taking $B_{11} = P_1$ in (2.13), we have from $\delta(\mathcal{T}_{12}) \subseteq \mathcal{T}_{12}$, $\mathcal{T}_{12}\mathcal{T}_{11} = 0$ and $\delta(P_1) \in Z(\mathcal{T})$ that

$$\delta\big([A_{11}^{(1)}, A_{12}]\big) = \big[\delta(A_{11}^{(1)}), A_{12}\big] + \big[A_{11}^{(1)}, \delta(A_{12})\big].$$

This shows that (2.7) is true for $n = 1$. One can verify that (2.7) follows easily by induction based on (2.13). Similarly, we can show that statement (2) is valid. $\square$

Lemma 2.6. *We have the following.*

(1) $\delta([A_{11}, B_{11}]) = [\delta(A_{11}), B_{11}] + [A_{11}, \delta(B_{11})]$ *for all* $A_{11}, B_{11} \in \mathcal{T}_{11}$,
(2) $\delta([A_{22}, B_{22}]) = [\delta(A_{22}), B_{22}] + [A_{22}, \delta(B_{22})]$ *for all* $A_{22}, B_{22} \in \mathcal{T}_{22}$.

Proof. Let $A_{11}, B_{11} \in \mathcal{T}_{11}$. The assumption (2) of Theorem 2.1 and Proposition 1.1 imply that there exist a derivation $d : \mathcal{T} \to \mathcal{T}$and a linear map $\tau : \mathcal{T} \to Z(\mathcal{T})$ vanishing on each commutator such that

$$\begin{aligned}\delta\big([A_{11}, B_{11}]\big) &= \delta_{[A_{11},B_{11}]}\big([A_{11}, B_{11}]\big) \\ &= d\big([A_{11}, B_{11}]\big) + \tau\big([A_{11}, B_{11}]\big) \\ &= d\big([A_{11}, B_{11}]\big).\end{aligned}$$

This and the facts $\delta(\mathcal{T}_{11}) \subseteq \mathcal{T}_{11} \oplus \mathcal{T}_{22}$ and $d(\mathcal{T}_{11}) \subseteq \mathcal{T}_{11} \oplus \mathcal{T}_{12}$ imply that $\delta([A_{11}, B_{11}]) \in \mathcal{T}_{11}$.

For any $C_{12} \in \mathcal{T}_{12}$, we have from Lemma 2.5 that

$$\begin{aligned}&\delta\big(\big[[A_{11}, B_{11}], C_{12}\big]\big) \\ &= \delta\big([A_{11}, B_{11}C_{12}]\big) - \delta\big([B_{11}, A_{11}C_{12}]\big) \\ &= \big[\delta(A_{11}), B_{11}C_{12}\big] + \big[A_{11}, \delta(B_{11}C_{12})\big] \\ &\quad - \big[\delta(B_{11}), A_{11}C_{12}\big] - \big[B_{11}, \delta(A_{11}C_{12})\big] \\ &= \big[\delta(A_{11}), B_{11}C_{12}\big] + \big[A_{11}, \big[\delta(B_{11}), C_{12}\big] + \big[B_{11}, \delta(C_{12})\big]\big] \\ &\quad - \big[\delta(B_{11}), A_{11}C_{12}\big] - \big[B_{11}, \big[\delta(A_{11}), C_{12}\big] + \big[A_{11}, \delta(C_{12})\big]\big] \\ &= \big[\delta(A_{11}), B_{11}\big]C_{12} + \big[A_{11}, \delta(B_{11})\big]C_{12} + [A_{11}, B_{11}]\delta(C_{12}),\end{aligned}$$

where we have used (2.5) in the fourth equality. On the other hand, we have from $\delta([A_{11}, B_{11}]) \in \mathcal{T}_{11}$ and $\delta(C_{12}) \in \mathcal{T}_{12}$ that

$$\begin{aligned}\delta\big(\big[[A_{11}, B_{11}], C_{12}\big]\big) &= \big[\delta\big([A_{11}, B_{11}]\big), C_{12}\big] + \big[[A_{11}, B_{11}], \delta(C_{12})\big] \\ &= \delta\big([A_{11}, B_{11}]\big)C_{12} + [A_{11}, B_{11}]\delta(C_{12}).\end{aligned}$$

Comparing the above two equalities, we have

$$\big\{\delta\big([A_{11}, B_{11}]\big) - \big[\delta(A_{11}), B_{11}\big] - \big[A_{11}, \delta(B_{11})\big]\big\}C_{12} = 0$$

for any $C_{12} \in \mathcal{T}_{12}$. Since $\mathcal{T}_{12}$ is a faithful left $\mathcal{T}_{11}$-module, we get

$$\delta\big([A_{11}, B_{11}]\big) = \big[\delta(A_{11}), B_{11}\big] + \big[A_{11}, \delta(B_{11})\big]$$

for all $A_{11}, B_{11} \in \mathcal{T}_{11}$. Similarly, we can show that statement (2) is valid. □

Proof of Theorem 2.1. Let $A, B \in \mathcal{T}$. Then

$$A = A_{11} + A_{12} + A_{22}, \qquad B = B_{11} + B_{12} + B_{22}$$

for some $A_{ij}, B_{ij} \in \mathcal{T}_{ij}$. It follows from (2.11) that

$$P_2\delta(\mathcal{T}_{11})P_2 \subseteq Z(\mathcal{T})P_2 \qquad \text{and} \qquad P_1\delta(\mathcal{T}_{22})P_1 \subseteq Z(\mathcal{T})P_1.$$

This implies that $[\delta(A_{ii}), B_{jj}] = 0$ for all $A_{ii} \in \mathcal{T}_{ii}$ and that $B_{jj} \in \mathcal{T}_{jj}$ $(1 \leq i \neq j \leq 2)$. Hence we have from $\delta(\mathcal{T}_{12}) \subseteq \mathcal{T}_{12}$ that

$$\begin{aligned}
&\big[\delta(A), B\big] + \big[A, \delta(B)\big] \\
&\quad = \big[\delta(A_{11} + A_{12} + A_{22}), B_{11} + B_{12} + B_{22}\big] \\
&\qquad + \big[A_{11} + A_{12} + A_{22}, \delta(B_{11} + B_{12} + B_{22})\big] \\
&\quad = \big[\delta(A_{11}), B_{11}\big] + \big[A_{11}, \delta(B_{11})\big] + \big[\delta(A_{11}), B_{12}\big] + \big[A_{11}, \delta(B_{12})\big] \\
&\qquad + \big[\delta(A_{12}), B_{11}\big] + \big[A_{12}, \delta(B_{11})\big] + \big[\delta(A_{12}), B_{22}\big] + \big[A_{12}, \delta(B_{22})\big] \\
&\qquad + \big[\delta(A_{22}), B_{12}\big] + \big[A_{22}, \delta(B_{12})\big] + \big[\delta(A_{22}), B_{22}\big] + \big[A_{22}, \delta(B_{22})\big].
\end{aligned}$$

On the other hand, it follows from Lemmas 2.5 and 2.6 that

$$\begin{aligned}
\delta\big([A, B]\big) &= \delta\big([A_{11}, B_{11}]\big) + \delta\big([A_{11}, B_{12}]\big) + \delta\big([A_{12}, B_{11}]\big) \\
&\quad + \delta\big([A_{12}, B_{22}]\big) + \delta\big([A_{22}, B_{12}]\big) + \delta\big([A_{22}, B_{22}]\big) \\
&= \big[\delta(A_{11}), B_{11}\big] + \big[A_{11}, \delta(B_{11})\big] + \big[\delta(A_{11}), B_{12}\big] + \big[A_{11}, \delta(B_{12})\big] \\
&\quad + \big[\delta(A_{12}), B_{11}\big] + \big[A_{12}, \delta(B_{11})\big] + \big[\delta(A_{12}), B_{22}\big] + \big[A_{12}, \delta(B_{22})\big] \\
&\quad + \big[\delta(A_{22}), B_{12}\big] + \big[A_{22}, \delta(B_{12})\big] + \big[\delta(A_{22}), B_{22}\big] + \big[A_{22}, \delta(B_{22})\big].
\end{aligned}$$

Then $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$ for all $A, B \in \mathcal{T}$; that is, δ is a Lie derivation. By the definition of δ, we have $\varphi(A) = \delta(A) + [A, P_1\varphi(P_1)P_2]$ for all $A \in \mathcal{T}$. Hence φ is a Lie derivation as required. □

Let $\mathcal{N}$ be a von Neumann algebra acting on a separable Hilbert space H. A nest β in $\mathcal{N}$ is a totally operator-ordered family of projections in $\mathcal{N}$, which is closed in the strong operator topology, and which include 0 and I. The nest subalgebras of $\mathcal{N}$ associated to a nest β, denoted by $\mathrm{Alg}_{\mathcal{N}}\beta$, is the set

$$\mathrm{Alg}_{\mathcal{N}}\beta = \{T \in \mathcal{N} : PTP = TP \text{ for all } P \in \beta\}.$$

Corollary 2.7. *Let β be a nontrivial finite nest in a factor von Neumann algebra $\mathcal{N}$. Then every local Lie derivation of $\mathrm{Alg}_{\mathcal{N}}\beta$ is a Lie derivation.*

Proof. Let $P \in \beta$ be a nontrivial projection. Write $\mathcal{N}_1 = P\mathcal{N}|_{PH}$, $\mathcal{N}_2 = P^\perp\mathcal{N}|_{P^\perp H}$. Let $\beta_1 = \{QP : Q \in \beta\}$, and let $\beta_2 = \{QP^\perp : Q \in \beta\}$. Then β_1 and β_2 are nests in factor von Neumann algebras $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. Let

$\mathcal{M} = P\mathcal{N}|_{P^\perp H}$. Then $\mathcal{M}$ is a faithful $(\mathrm{Alg}_{\mathcal{N}_1}\beta_1, \mathrm{Alg}_{\mathcal{N}_2}\beta_2)$-bimodule. The nest subalgebra $\mathrm{Alg}_{\mathcal{N}}\beta$ can be represented as

$$\begin{pmatrix} \mathrm{Alg}_{\mathcal{N}_1}\beta_1 & \mathcal{M} \\ 0 & \mathrm{Alg}_{\mathcal{N}_2}\beta_2 \end{pmatrix}.$$

Since $Z(\mathrm{Alg}_{\mathcal{N}}\beta) = \mathbb{C}I$, $Z(\mathrm{Alg}_{\mathcal{N}_1}\beta_1) = \mathbb{C}P$, and $Z(\mathrm{Alg}_{\mathcal{N}_2}\beta_2) = \mathbb{C}P^\perp$, the second condition of Theorem 2.1 is satisfied. The first condition is also satisfied because β_1 and β_2 are finite. It follows from Theorem 2.1 that every local Lie derivation of $\mathrm{Alg}_{\mathcal{N}}\beta$ is a Lie derivation. □

Let $\mathcal{A}$ and $\mathcal{B}$ be norm-closed unital subalgebras of $B(H)$ and $B(K)$, respectively. In [5], Gilfeather and Smith defined an operator algebra analog $\mathcal{A}\sharp\mathcal{B}$, which is called the *join of $\mathcal{A}$ and $\mathcal{B}$*, as a subalgebra of $B(H \oplus K)$ of the form

$$\begin{pmatrix} \mathcal{A} & 0 \\ B(H,K) & \mathcal{B} \end{pmatrix}.$$

Corollary 2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be factor von Neumann algebras of $B(H)$ and $B(K)$, respectively. Then every local Lie derivation of $\mathcal{A}\sharp\mathcal{B}$ is a Lie derivation.*

Proof. It is clear that $\mathcal{A}$ and $\mathcal{B}$ are generated by their idempotents. Since $Z(\mathcal{A}) = \mathbb{C}I_H$, $Z(\mathcal{B}) = \mathbb{C}I_K$, and $Z(\mathcal{A}\sharp\mathcal{B}) = \mathbb{C}I_{H\oplus K}$, the second condition of Theorem 2.1 is satisfied. It follows from Theorem 2.1 that every local Lie derivation of $\mathcal{A}\sharp\mathcal{B}$ is a Lie derivation. □

Let $M_{n\times k}(\mathbb{F})$ be the set of all $n \times k$ matrices over $\mathbb{F}$. For $n \geq 2$ and $m \leq n$, the block upper triangular matrix algebra $T_n^{\bar{k}}(\mathbb{F})$ is a subalgebra of $M_n(\mathbb{F})$ of the form

$$\begin{pmatrix} M_{k_1}(\mathbb{F}) & M_{k_1\times k_2}(\mathbb{F}) & \cdots & M_{k_1\times k_m}(\mathbb{F}) \\ 0 & M_{k_2}(\mathbb{F}) & \cdots & M_{k_2\times k_m}(\mathbb{F}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{k_m}(\mathbb{F}) \end{pmatrix},$$

where $\bar{k} = (k_1, k_2, \ldots, k_m) \in \mathbb{N}^m$ is an ordered m-vector of positive integers such that $k_1 + k_2 + \cdots + k_m = n$.

Corollary 2.9. *Every local Lie derivation of a block upper triangular matrix algebra $T_n^{\bar{k}}(\mathbb{F})$ is a Lie derivation.*

Proof. The block upper triangular matrix algebra $T_n^{\bar{k}}(\mathbb{F})$ can be represented as

$$\begin{pmatrix} T_l^{\bar{k_1}}(\mathbb{F}) & M_{l\times(n-l)}(\mathbb{F}) \\ 0 & T_{n-l}^{\bar{k_2}}(\mathbb{F}) \end{pmatrix},$$

where $1 \leq l < m$ and $\bar{k_1} \in \mathbb{N}^l$, $\bar{k_2} \in \mathbb{N}^{n-l}$. It is clear that $T_l^{\bar{k_1}}(\mathbb{F})$ and $T_{n-l}^{\bar{k_2}}(\mathbb{F})$ satisfy the condition (1) of Theorem 2.1. Since $Z(T_n^{\bar{k}}(\mathbb{F})) = \mathbb{F}I_n$, $Z(T_l^{\bar{k_1}}(\mathbb{F})) = \mathbb{F}I_l$, and $Z(T_{n-l}^{\bar{k_2}}(\mathbb{F})) = \mathbb{F}I_{n-l}$, the condition (2) of Theorem 2.1 is satisfied. □

Corollary 2.10. *Let $\mathcal{A}$ be a unital algebra over $\mathbb{F}$. Then every local Lie derivation of the algebra $\mathcal{T} = \mathrm{Tri}(M_n(\mathcal{A}), M_{n\times k}(\mathcal{A}), M_k(\mathcal{A}))$ is a Lie derivation for $n, k \geq 2$.*

Proof. For $n, k \geq 2$, it follows from the result of [1] that $M_n(\mathcal{A})$ and $M_k(\mathcal{A})$ are generated by their idempotents. On the other hand, we have $Z(\mathcal{T}) = Z(\mathcal{A})I_{n+k}$, $Z(M_n(\mathcal{A})) = Z(\mathcal{A})I_n$, and $Z(M_k(\mathcal{A})) = Z(\mathcal{A})I_k$. Hence, by Theorem 2.1, every local Lie derivation of $\mathcal{T}$ is a Lie derivation. □

We denote by $\{E_{ij}\}$ the standard matrix units of $M_3(\mathbb{F})$. The following example shows that the conditions (1) and (2) of Theorem 2.1 cannot both be dropped.

Example 2.11. Let $\mathcal{A} = \text{span}\{E_{11} + E_{22}, E_{12}\}$, let $\mathcal{B} = \text{span}\{E_{33}\}$, let $\mathcal{M} = \text{span}\{E_{13}, E_{23}\}$, and let $\mathcal{T} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$. Then there exists a local Lie derivation of $\mathcal{T}$ which is not a Lie derivation.

Proof. It is clear that $\mathcal{A} \neq \mathcal{J}(\mathcal{A})$ and $Z(\mathcal{A}) \neq \pi_{\mathcal{A}}(Z(\mathcal{T}))$. It can be shown that a linear map $\varphi : \mathcal{T} \to \mathcal{T}$ is a Lie derivation if there exist scalars $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$ such that

$$\varphi(I) = 0, \qquad \varphi(E_{12}) = \lambda_1 E_{12}, \qquad \varphi(E_{23}) = \lambda_2 E_{13} + \lambda_3 E_{23},$$

and

$$\varphi(E_{13}) = (\lambda_1 + \lambda_3)E_{13}.$$

We define a linear map $\Phi : \mathcal{T} \to \mathcal{T}$ by

$$\Phi(A) = (2a_{13} - a_{23})E_{13} + a_{12}E_{12}$$

for each $A = (a_{ij}) \in \mathcal{T}$. It is easy to check that

$$\Phi(I) = 0, \qquad \Phi(E_{12}) = E_{12}, \qquad \Phi(E_{23}) = -E_{13}, \qquad \text{and} \qquad \Phi(E_{13}) = 2E_{13}.$$

If $a_{23} \neq 0$, then let φ_1 be the linear map of $\mathcal{T}$ with $\varphi_1(I) = 0, \varphi_1(E_{12}) = E_{12}$, $\varphi_1(E_{23}) = (a_{23}^{-1}a_{13} - 1)E_{13}$, and $\varphi_1(E_{13}) = E_{13}$. Then φ_1 is a Lie derivation of $\mathcal{T}$. It follows from the definition of Φ that $\Phi(A) = \varphi_1(A)$. If $a_{23} = 0$, then let φ_2 be a linear map with $\varphi_2(I) = 0, \varphi_2(E_{12}) = E_{12}$, $\varphi_2(E_{23}) = E_{23}$, and $\varphi_2(E_{13}) = E_{13}$. Then φ_2 is a Lie derivation of $\mathcal{T}$. By the definition of Φ, we have $\Phi(A) = \varphi_2(A)$. Therefore, Φ is a local Lie derivation. Let $A = E_{12}$ and $B = E_{12} + E_{23}$. Then $\Phi([A, B]) = \Phi(E_{13}) = 2E_{13}$ and $[\Phi(A), B] + [A, \Phi(B)] = [E_{12}, E_{12} + E_{23}] + [E_{12}, E_{12} - E_{13}] = E_{13}$. Hence

$$\Phi\big([A, B]\big) \neq \big[\Phi(A), B\big] + \big[A, \Phi(B)\big).$$

We conclude that Φ is a local Lie derivation, which is not a Lie derivation. □

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