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# HADAMARD GAP SERIES IN WEIGHTED-TYPE SPACES ON THE UNIT BALL

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Abstract. We give a sufficient and necessary condition for an analytic function $f(z)$ on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ with Hadamard gaps, that is, for $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ where $P_{n_k}(z)$ is a homogeneous polynomial of degree $n_k$ and $n_{k+1}/n_k \geq c > 1$ for all $k \in \mathbb{N}$, to belong to the weighted-type space $H_\mu^\infty$ and the corresponding little weighted-type space $H_{\mu,0}^\infty$ under some condition posed on the weighted funtion $\mu$. We also study the growth rate of those functions in $H_\mu^\infty$.

## 1. Introduction

Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^n$ with $\mathbb{S}$ as its boundary and let $H(\mathbb{B})$ be the collection of all holomorphic functions in $\mathbb{B}$. Here $H^\infty(\mathbb{B})$ denotes the Banach space consisting of all bounded holomorphic functions in $\mathbb{B}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{B}} |f(z)|$.

A positive continuous function $\mu$ on $[0, 1)$ is called *normal* if there exists positive numbers $\alpha$ and $\beta$, $0 < \alpha < \beta$, and $\delta \in (0, 1)$ such that

$$\begin{aligned} &\frac{\mu(r)}{(1-r)^\alpha} \text{ is decreasing on } [\delta, 1), \quad \lim_{r\to 1} \frac{\mu(r)}{(1-r)^\alpha} = 0, \\ &\frac{\mu(r)}{(1-r)^\beta} \text{ is increasing on } [\delta, 1), \quad \lim_{r\to 1} \frac{\mu(r)}{(1-r)^\beta} = \infty \end{aligned} \tag{1.1}$$

(see, e.g., [7]). Note that a normal function $\mu : [0, 1) \to [0, \infty)$ is decreasing in a neighborhood of 1 and satisfies $\lim_{r\to 1^-} \mu(r) = 0$.


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An $f \in H(\mathbb{B})$ is said to belong to the weighted-type space denoted by $H_\mu^\infty = H_\mu^\infty(\mathbb{B})$ if

$$\|f\| = \sup_{z\in\mathbb{B}} \mu\big(|z|\big)\big|f(z)\big| < \infty,$$

where $\mu$ is normal on $[0,1)$. It is well known that $H_\mu^\infty$ is a Banach space with the norm $\|\cdot\|$.

The little weighted-type space, denoted by $H_{\mu,0}^\infty$, is the closed subspace of $H_\mu^\infty$ consisting of those $f \in H_\mu^\infty$ such that $\lim_{|z|\to 1^-} \mu(|z|)|f(z)| = 0$. When $\mu(|z|) = (1-|z|^2)^\alpha, \alpha > 0$, the induced spaces $H_\mu^\infty$ and $H_{\mu,0}^\infty$ are denoted by $H_\alpha^\infty$ and $H_{\alpha,0}^\infty$, respectively.

We say that an $f \in H(\mathbb{B})$ has the *Hadamard gaps* if $f(z) = \sum_{k=0}^\infty P_{n_k}(z)$, where $P_{n_k}$ is a homogeneous polynomial of degree $n_k$ and there exists some $c > 1$,

$$\frac{n_{k+1}}{n_k} \geq c, \quad \forall k \geq 0$$

(see, e.g., [8]).

Hadamard gap series on spaces of holomorphic functions in the unit disk $\mathbb{D}$ or in the unit ball $\mathbb{B}$ have been studied quite well. We refer the readers to the related results in [2], [4], [5], [8], [10]–[14], and the reference therein.

In [12], the authors studied the Hadamard gap series and the growth rate of the functions in $H_\mu^\infty$ in the unit disk. Motivated by [12], the aim of this paper is to study the Hadamard gap series in $H_\mu^\infty$ as well as its little space $H_{\mu,0}^\infty$ on the unit ball. Moreover, as an application of our main result, we characterize the growth rate of those functions in $H_\mu^\infty$.

Throughout this article, for $a, b \in \mathbb{R}$, $a \lesssim b$ ($a \gtrsim b$, respectively) means that there exists a positive number $C$, which is independent of $a$ and $b$, such that $a \leq Cb$ ($a \geq Cb$, respectively). Moreover, if both $a \lesssim b$ and $a \gtrsim b$ hold, then we say that $a \simeq b$.

## 2. Hadamard gap series in $H_\mu^\infty$ and $H_{\mu,0}^\infty$

Let $f(z) = \sum_{k=0}^\infty P_k(z)$ be a holomorphic function in $\mathbb{B}$, where $P_k(z)$ is a homogeneous polynomial with degree $k$. For $k \geq 0$, we denote

$$M_k = \sup_{\xi\in\mathbb{S}}\big|P_k(\xi)\big|.$$

We have the following estimations on $M_k$ of a holomorphic function $f \in H_\mu^\infty$ (or $f \in H_{\mu,0}^\infty$, respectively).

**Theorem 2.1.** *Let $\mu$ be a normal function on $[0,1)$. Let $f(z) = \sum_{k=0}^\infty P_k(z), z \in \mathbb{B}$. Then the following statements hold:*

(1) *if $f \in H_\mu^\infty$, then $\sup_{k\geq 0} M_k \mu(1-\frac{1}{k}) < \infty$,*
(2) *if $f \in H_{\mu,0}^\infty$, then $\lim_{k\to\infty} M_k \mu(1-\frac{1}{k}) = 0$.*

*Proof.* (1). Suppose that $f \in H_\mu^\infty$. Fix a $\xi \in \mathbb{S}$, and denote

$$f_\xi(w) = \sum_{k=0}^\infty P_k(\xi)w^k = \sum_{k=0}^\infty P_k(\xi w), \quad w \in \mathbb{D}.$$

Since $f \in H(\mathbb{B})$, it is known that, for a fixed $\xi \in \mathbb{S}$, $f_\xi(w)$ is holomorphic in $\mathbb{D}$ (see, e.g., [6]). Hence, for any $r \in (0,1)$, we have

$$\begin{aligned} M_k = \sup_{\xi\in\mathbb{S}}\big|P_k(\xi)\big| &= \sup_{\xi\in\mathbb{S}}\Big|\frac{1}{2\pi i}\int_{|w|=r}\frac{f_\xi(w)}{w^{k+1}}\,dw\Big| \\ &= \frac{1}{2\pi}\sup_{\xi\in\mathbb{S}}\Big|\int_{|w|=r}\frac{f(\xi w)}{w^{k+1}}\,dw\Big| \\ &\le \frac{1}{2\pi}\sup_{\xi\in\mathbb{S}}\int_{|w|=r}\frac{|f(\xi w)|}{r^{k+1}}|\,dw| \\ &= \frac{1}{2\pi}\sup_{\xi\in\mathbb{S}}\int_{|w|=r}\frac{|f(\xi w)|\mu(|\xi w|)}{r^{k+1}\mu(r)}|\,dw| \le \frac{\|f\|}{r^k\mu(r)}. \end{aligned} \tag{2.1}$$

In (2.1), letting $r = 1-\frac{1}{k}, k \ge 2, k \in \mathbb{N}$, we have

$$M_k \le \frac{\|f\|}{(1-\frac{1}{k})^k\mu(1-\frac{1}{k})}.$$

Thus, for each $k \ge 2$,

$$M_k\mu\Big(1-\frac{1}{k}\Big) \le \frac{\|f\|}{(1-\frac{1}{k})^k} \le 4\|f\|,$$

which implies that

$$\sup_{k\ge 1} M_k\mu\Big(1-\frac{1}{k}\Big) \le \max\big\{\mu(0)M_1, 4\|f\|\big\} < \infty.$$

(2). Suppose $f \in H^\infty_{\mu,0}$; that is, for any $\varepsilon > 0$, there exists a $\delta \in (0,1)$ when $\delta < |z| < 1$, $\mu(|z|)|f(z)| < \varepsilon$. Take $N_0 \in \mathbb{N}$ satisfying $\delta < 1-\frac{1}{k} < 1$ when $k > N_0$. Then for any $k > N_0$ and $r = 1-\frac{1}{k}$, as the proof in the previous part, we have

$$M_k \le \frac{1}{(1-\frac{1}{k})^k\mu(1-\frac{1}{k})}\cdot\sup_{\delta<|z|<1}\mu\big(|z|\big)\big|f(z)\big| < \frac{\varepsilon}{(1-\frac{1}{k})^k\mu(1-\frac{1}{k})},$$

which implies

$$M_k\mu\Big(1-\frac{1}{k}\Big) \le \frac{\varepsilon}{(1-\frac{1}{k})^k} \le 4\varepsilon, \quad k > N_0.$$

Hence we have $\lim_{k\to\infty} M_k\mu(1-\frac{1}{k}) = 0$. □

**Theorem 2.2.** *Let $\mu$ be a normal function on $[0,1)$. Let $f(z) = \sum_{k=0}^\infty P_{n_k}(z)$ with Hadamard gaps, where $P_{n_k}$ is a homogeneous polynomial of degree $n_k$. Then the following assertions hold:*

(1) *$f \in H^\infty_\mu$ if and only if $\sup_{k\ge1}\mu(1-\frac{1}{n_k})M_{n_k} < \infty$,*

(2) *$f \in H^\infty_{\mu,0}$ if and only if $\lim_{k\to\infty}\mu(1-\frac{1}{n_k})M_{n_k} = 0$.*

*Proof.* By Theorem 2.1, it suffices to show the sufficiency of both statements.

(1). Noting that

$$\big|f(z)\big| = \Big|\sum_{k=0}^\infty P_{n_k}\Big(\frac{z}{|z|}\Big)|z|^{n_k}\Big| \le \sum_{k=0}^\infty M_{n_k}|z|^{n_k} \lesssim \sum_{k=0}^\infty \frac{|z|^{n_k}}{\mu(1-\frac{1}{n_k})},$$

from the proof of [12, Theorem 2.3], we have

$$\frac{|f(z)|}{1-|z|} \lesssim \sum_{m=1}^{\infty}\Big(\sum_{n_k\le m}\frac{1}{\mu(1-\frac{1}{n_k})}\Big)|z|^m \lesssim \sum_{m=1}^{\infty}\frac{|z|^m}{\mu(1-\frac{1}{m})}$$
$$\lesssim \frac{1}{(1-|z|)\mu(|z|)},$$

which implies $f\in H^\infty_\mu$, as desired.

(2). Since $\lim_{k\to\infty}\mu(1-\frac{1}{n_k})M_{n_k}=0$, we have $\sup_{k\ge1}\mu(1-\frac{1}{n_k})M_{n_k}<\infty$. Hence, by part (1), we have $f\in H^\infty_\mu$. For any $\varepsilon>0$, there exists a $N_0\in\mathbb{N}$ satisfying when $m>N_0$

$$M_{n_m}\mu\Big(1-\frac{1}{n_m}\Big)<\varepsilon.$$

For each $m\in\mathbb{N}$, put $f_m(z)=\sum_{k=0}^{m}P_{n_k}(z)$, which clearly belongs to $H^\infty_{\mu,0}$ since it is a polynomial. Hence it suffices to show that $\|f_m-f\|\to0$ as $m\to\infty$. Indeed, for $m>N_0$, we have

$$\big|f_m(z)-f(z)\big| = \Big|\sum_{k=m+1}^{\infty}P_{n_k}(z)\Big| \le \sum_{k=m+1}^{\infty}M_{n_k}|z|^{n_k} \le \varepsilon\sum_{k=m+1}^{\infty}\frac{|z|^{n_k}}{\mu(1-\frac{1}{n_k})}.$$

From this, the result easily follows from the proof of part (1). $\square$

## 3. Growth rate

As an application of Theorem 2.2, in this section, we show the following result.

**Theorem 3.1.** *Let $\mu$ be a normal function on $[0,1)$. Then there exists a positive integer $M=M(n)$ with the following property: there exists $f_i\in H^\infty_\mu, 1\le i\le M$, such that*

$$\sum_{i=1}^{M}\big|f_i(z)\big| \gtrsim \frac{1}{\mu(|z|)},\quad z\in\mathbb{B}.$$

Note that the result in [12, Theorem 2.5] in the unit disk is a particular case of Theorem 3.1 when $n=1$.

*Remark* 3.2. We observe that $M$ cannot be 1. Indeed, assume that there exists a $f\in H^\infty_\mu$ such that

$$\big|f(z)\big| \gtrsim \frac{1}{\mu(|z|)},\quad z\in\mathbb{B}.$$

It implies that $f(z)$ has no zero in $\mathbb{B}$, and it follows that there exists $g\in H(\mathbb{B})$ such that $f=e^g$. Thus

$$\big|f(z)\big| = |e^{g(z)}| = e^{\operatorname{Re} g(z)},$$

which implies that $e^{\operatorname{Re} g(z)}\gtrsim\frac{1}{\mu(|z|)}$, and hence $\operatorname{Re} g(z)\gtrsim\log\frac{1}{\mu(|z|)}$. For each $r\in(0,1)$, integrating on both sides of the above inequality on $r\mathbb{S}=\{z\in\mathbb{B},|z|=r\}$, we have

$$\int_{r\mathbb{S}}\operatorname{Re} g(z)\,d\sigma \gtrsim \int_{r\mathbb{S}}\log\Big(\frac{1}{\mu(|z|)}\Big)\,d\sigma = \log\Big(\frac{1}{\mu(r)}\Big)\cdot\sigma(r\mathbb{S}).$$

By the mean value property, we have $\operatorname{Re} g(0) \gtrsim \log(\frac{1}{\mu(r)}), \forall r \in (0,1)$, which is impossible.

Before we formulate the proof of our main result, we need some preliminary results. In the sequel, for $\xi, \zeta \in \mathbb{S}$, denote

$$d(\xi,\zeta) = \left(1 - \left|\langle \xi, \zeta \rangle\right|^2\right)^{\frac{1}{2}}.$$

Then $d$ satisfies the triangle inequality (see, e.g., [1]). Moreover, we write $E_\delta(\zeta)$ for the $d$-ball with radius $\delta \in (0,1)$ and center at $\zeta \in \mathbb{S}$:

$$E_\delta(\zeta) = \left\{\xi \in \mathbb{S} : d(\xi,\zeta) < \delta\right\}.$$

We say that a subset $\Gamma$ of $\mathbb{S}$ is *d-separated* by $\delta > 0$ if $d$-balls with radius $\delta$ and center at points of $\Gamma$ are pairwise disjoint.

We begin with several lemmas which play an important role in the proof of our main result.

**Lemma 3.3** ([3, Lemma 2.2], [9])**.** *For each $a > 0$, there exists a positive integer $M = M_n(a)$ with the following property: if $\delta > 0$, and if $\Gamma \subset \mathbb{S}$ is $d$-separated by $a\delta$, then $\Gamma$ can be decomposed into $\Gamma = \bigcup_{j=1}^M \Gamma_j$ in such a way that each $\Gamma_j$ is $d$-separated by $\delta$.*

**Lemma 3.4** ([3, Lemma 2.3])**.** *Suppose that $\Gamma \subset \mathbb{S}$ is $d$-separated by $\delta$, and let $k$ be a positive integer. If*

$$P(z) = \sum_{\zeta \in \Gamma} \langle z, \zeta \rangle^k, \quad z \in \mathbb{B},$$

*then*

$$\left|P(z)\right| \le 1 + \sum_{m=1}^{\infty} (m+2)^{2n-2} e^{\frac{-m^2\delta^2 k}{2}}.$$

*Proof of Theorem 3.1.* We will prove the theorem by constructing $f_i \in H_\mu^\infty$ satisfying the given property only near the boundary (then, by adding a proper constant, one obtains the given property on all of the unit ball). Since $\mu$ is normal, by the definition of normal function, there exists positive numbers $\alpha, \beta$ with $0 < \alpha < \beta$, and $\delta \in (0,1)$ satisfy (1.1). Take and fix some small positive number $A < 1$ such that

$$\sum_{m=1}^{\infty} (m+2)^{2n-2} e^{\frac{-m^2}{2A^2}} \le \frac{1}{27}. \tag{3.1}$$

Let $M = M_n(\frac{A}{2})$ be a positive integer provided by Lemma 3.3 with $\frac{A}{2}$ in place of $a$. Let $p$ be a sufficiently large positive integer so that

$$1 - \frac{1}{p} \ge \delta,$$

$$\frac{1}{3} \le \left(1 - \frac{1}{p}\right)^p \le \frac{1}{2}, \tag{3.2}$$

$$\frac{1}{p^{\alpha M} - 1} \le \frac{1}{200}, \tag{3.3}$$

and

$$\frac{p^{\beta M} \cdot 2^{-p^{M-0.5}}}{1 - p^{\beta M} \cdot 2^{-(p^{2M-0.5} - p^{M-0.5})}} \leq \frac{1}{200}. \tag{3.4}$$

For each postive integer $j \leq M$, set $\delta_{j,0}$ such that

$$A^2 p^j \delta_{j,0}^2 = 1, \tag{3.5}$$

and inductively choose $\delta_{j,v}$ such that

$$p^M \delta_{j,v}^2 = \delta_{j,v-1}^2, \quad v = 1, 2, \ldots. \tag{3.6}$$

From (3.5) and (3.6), we get

$$A^2 p^{vM+j} \delta_{j,v}^2 = 1. \tag{3.7}$$

For each fixed $j$ and $v$, let $\Gamma^{j,v}$ be a maximal subset of $\mathbb{S}$ subject to the condition that $\Gamma^{j,v}$ is $d$-separated by $A\delta_{j,v}/2$. Then, by Lemma 3.3, write

$$\Gamma^{j,v} = \bigcup_{l=1}^{M} \Gamma_{j,vM+l} \tag{3.8}$$

in such a way that each $\Gamma_{j,vM+l}$ is $d$-separated by $\delta_{j,v}$.

For each $i, j = 1, 2, \ldots, M$ and $v \geq 0$, set

$$P_{i,vM+j}(z) = \sum_{\xi \in \Gamma_{j,vM+\tau^i(j)}} \langle z, \xi \rangle^{p^{vM+j}},$$

where $\tau^i$ is the $i$th iteration of the permutation $\tau$ on $\{1, 2, \ldots, M\}$ defined by

$$\tau(j) = \begin{cases} j+1, & j < M; \\ 1, & j = M. \end{cases}$$

By (3.7), Lemma 3.4, and (3.1), we get that

$$\begin{aligned} \left|P_{i,vM+j}(z)\right| &\leq 1 + \sum_{m=1}^{\infty} (m+2)^{2n-2} e^{\frac{-m^2 \delta_{j,v}^2 p^{vM+j}}{2}} \\ &\leq 1 + \sum_{m=1}^{\infty} (m+2)^{2n-2} e^{-\frac{m^2}{2A^2}} \leq 2, \quad z \in \mathbb{B} \end{aligned} \tag{3.9}$$

for all $i, j = 1, 2, \ldots, M$ and $v \geq 0$.

Define

$$g_{i,j}(z) = \sum_{v=0}^{\infty} \frac{P_{i,vM+j}(z)}{\mu(1 - \frac{1}{p^{vM+j}})}, \quad z \in \mathbb{B}.$$

By Theorem 2.2, it is clear that, for each $i, j \in \{1, 2, \ldots, M\}$, $g_{i,j} \in H_\mu^\infty$.

We will show that, for every $v \geq 0, 1 \leq j \leq M$, and $z \in \mathbb{B}$ with

$$1 - \frac{1}{p^{vM+j}} \leq |z| \leq 1 - \frac{1}{p^{vM+j+\frac{1}{2}}}, \tag{3.10}$$

there exists an $i \in \{1, 2, \ldots, M\}$ such that $|g_{i,j}(z)| \geq \frac{C}{\mu(|z|)}$, where $C$ is some constant independent of the choice of $i$, $j$, and $z$.

Fix $v, j$, and $z$ for which (3.10) holds. Let $z = |z|\eta$, where $\eta \in \mathbb{S}$. Since $d$-balls with radius $A\delta_{j,v}$ and centers at points of $\Gamma^{j,v}$ cover $\mathbb{S}$ by maximality, there exists some $\zeta \in \Gamma^{j,v}$ such that $\eta \in E_{A\delta_{j,v}}(\zeta)$. Note that $\zeta \in \Gamma_{j,vM+l}$ for some $1 \le l \le M$ by (3.8), and hence $\zeta \in \Gamma_{j,vM+\tau^i(j)}$ for some $1 \le i \le M$.

We now estimate $|g_{i,j}(z)|$. By (3.9),

$$
\begin{aligned}
\big|g_{i,j}(z)\big| &= \Big|\sum_{k=0}^{\infty} \frac{P_{i,kM+j}(z)}{\mu(1-\frac{1}{p^{kM+j}})}\Big| \\
&\ge \Big|\frac{P_{i,vM+j}(z)}{\mu(1-\frac{1}{p^{vM+j}})}\Big| - \Big|\sum_{k\neq v} \frac{P_{i,kM+j}(z)}{\mu(1-\frac{1}{p^{kM+j}})}\Big| \\
&= \frac{|z|^{p^{vM+j}}|P_{i,vM+j}(\eta)|}{\mu(1-\frac{1}{p^{vM+j}})} - \Big|\sum_{k\neq v} \frac{|z|^{kM+j}P_{i,kM+j}(\eta)}{\mu(1-\frac{1}{p^{kM+j}})}\Big| \\
&\ge \frac{|z|^{p^{vM+j}}|P_{i,vM+j}(\eta)|}{\mu(1-\frac{1}{p^{vM+j}})} - 2\sum_{k=0}^{v-1} \frac{|z|^{p^{kM+j}}}{\mu(1-\frac{1}{p^{kM+j}})} \\
&\quad - 2\sum_{k=v+1}^{\infty} \frac{|z|^{p^{kM+j}}}{\mu(1-\frac{1}{p^{kM+j}})} \\
&= I_1 - I_2 - I_3,
\end{aligned}
$$

where

$$
I_1 = \frac{|z|^{p^{vM+j}}|P_{i,vM+j}(\eta)|}{\mu(1-\frac{1}{p^{vM+j}})}, \qquad I_2 = 2\sum_{k=0}^{v-1} \frac{|z|^{p^{kM+j}}}{\mu(1-\frac{1}{p^{kM+j}})},
$$

and

$$
I_3 = 2\sum_{k=v+1}^{\infty} \frac{|z|^{p^{kM+j}}}{\mu(1-\frac{1}{p^{kM+j}})}.
$$

Now we estimate $I_1, I_2$, and $I_3$, respectively.

• *Estimation of $I_1$.*

By (3.2) and (3.10), we obtain

$$
|z|^{p^{vM+j}} \ge \Big(1 - \frac{1}{p^{vM+j}}\Big)^{p^{vM+j}} \ge \frac{1}{3},
$$

and therefore

$$
\begin{aligned}
I_1 &\ge \frac{|P_{i,vM+j}(\eta)|}{3\mu(1-\frac{1}{p^{vM+j}})} \\
&\ge \frac{\big(|\langle \eta, \zeta\rangle|^{p^{vM+j}} - \sum_{\xi\in\Gamma_{j,vM+\tau^i(j)},\xi\neq\zeta} |\langle \eta, \xi\rangle|^{p^{vM+j}}\big)}{3\mu(1-\frac{1}{p^{vM+J}})} \\
&\qquad \text{(by the proof of [3, Theorem 2.1])} \\
&\ge \frac{2}{27\mu(1-\frac{1}{p^{vM+J}})}.
\end{aligned}
$$

• *Estimation of $I_2$.*

By the definition of normal function, we have for each $s \in \mathbb{N}$

$$\frac{(1-(1-\frac{1}{p^{sM+j}}))^{\alpha}}{(1-(1-\frac{1}{p^{(s+1)M+j}}))^{\alpha}} \le \frac{\mu(1-\frac{1}{p^{sM+j}})}{\mu(1-\frac{1}{p^{(s+1)M+j}})} \le \frac{(1-(1-\frac{1}{p^{sM+j}}))^{\beta}}{(1-(1-\frac{1}{p^{(s+1)M+j}}))^{\beta}};$$

that is,

$$1 < p^{M\alpha} \le \frac{\mu(1-\frac{1}{p^{sM+j}})}{\mu(1-\frac{1}{p^{(s+1)M+j}})} \le p^{M\beta}. \tag{3.11}$$

Combining this with (3.3), we have

$$\begin{aligned}
I_2 &\le 2\sum_{k=0}^{v-1} \frac{1}{\mu(1-\frac{1}{p^{kM+j}})} \\
&= \frac{2}{\mu(1-\frac{1}{p^{vM+j}})} \sum_{k=0}^{v-1} \Big[ \frac{\mu(1-\frac{1}{p^{vM+j}})}{\mu(1-\frac{1}{p^{(v-1)M+j}})} \frac{\mu(1-\frac{1}{p^{(v-1)M+j}})}{\mu(1-\frac{1}{p^{(v-2)M+j}})} \cdots \\
&\quad \times \frac{\mu(1-\frac{1}{p^{(k+1)M+j}})}{\mu(1-\frac{1}{p^{kM+j}})} \Big] \\
&\le \frac{2}{\mu(1-\frac{1}{p^{vM+j}})} \sum_{k=0}^{v-1} \frac{1}{p^{\alpha M(v-k)}} \le \frac{2}{\mu(1-\frac{1}{p^{vM+j}})} \cdot \frac{1}{p^{\alpha M}-1} \\
&\le \frac{1}{100\mu(1-\frac{1}{p^{vM+j}})}.
\end{aligned}$$

• *Estimation of $I_3$.*
Note that, by (3.2) and (3.10), we have

$$|z|^{p^{vM+j}} \le \Big(1-\frac{1}{p^{vM+j+\frac{1}{2}}}\Big)^{p^{vM+j+\frac{1}{2}}\cdot p^{-\frac{1}{2}}} \le \Big(\frac{1}{2}\Big)^{p^{-\frac{1}{2}}}. \tag{3.12}$$

Hence, by (3.4), (3.11), and (3.12), we have

$$\begin{aligned}
I_3 &= \frac{2|z|^{p^{(v+1)M+j}}}{\mu(1-\frac{1}{p^{vM+j}})} \cdot \sum_{k=v+1}^{\infty} \Big[ \frac{\mu(1-\frac{1}{p^{vM+j}})}{\mu(1-\frac{1}{p^{kM+j}})} |z|^{(p^{kM+j}-p^{(v+1)M+j})} \Big] \\
&= \frac{2|z|^{p^{(v+1)M+j}}}{\mu(1-\frac{1}{p^{vM+j}})} \cdot \sum_{k=v+1}^{\infty} \Big[ \frac{\mu(1-\frac{1}{p^{vM+j}})}{\mu(1-\frac{1}{p^{(v+1)M+j}})} \cdots \frac{\mu(1-\frac{1}{p^{(k-1)M+j}})}{\mu(1-\frac{1}{p^{kM+j}})} \\
&\quad \times |z|^{(p^{kM+j}-p^{(v+1)M+j})} \Big] \\
&\le \frac{2|z|^{p^{(v+1)M+j}}}{\mu(1-\frac{1}{p^{vM+j}})} \cdot \sum_{k=v+1}^{\infty} [p^{(\beta M)(k-v)} |z|^{(p^{kM+j}-p^{(v+1)M+j})}] \\
&= \frac{2|z|^{p^{(v+1)M+j}}}{\mu(1-\frac{1}{p^{vM+j}})} \cdot \sum_{k=v+1}^{\infty} [p^{\beta M} p^{(\beta M)(k-v-1)} |z|^{p^j(p^{kM}-p^{(v+1)M})}] \\
&\quad \text{(Let } s=k-v-1.)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2|z|^{p^{(v+1)M+j}}}{\mu(1-\frac{1}{p^{vM+j}})} \cdot \sum_{s=0}^{\infty} \big[p^{\beta M} p^{\beta M s} |z|^{p^{j+(v+1)M}(p^{sM}-1)}\big] \\
&\quad \big(\text{by } p^{sM-1} \geq s(p^M-1), \text{ where } s \text{ and } M \text{ are two positive integers}\big) \\
&\leq \frac{2|z|^{p^{(v+1)M+j}}}{\mu(1-\frac{1}{p^{vM+j}})} \cdot \sum_{s=0}^{\infty} \big[p^{\beta M} p^{\beta M s} |z|^{p^{j+(v+1)M}(p^{M}-1)s}\big] \\
&= \frac{2|z|^{p^{(v+1)M+j}}}{\mu(1-\frac{1}{p^{vM+j}})} \cdot \sum_{s=0}^{\infty} \Big[p^{\beta M} \big(p^{\beta M} |z|^{(p^{(v+2)M+j}-p^{(v+1)M+j})}\big)^s\Big] \\
&= \frac{2}{\mu(1-\frac{1}{p^{vM+j}})} \cdot \frac{p^{\beta M}(|z|^{p^{vM+j}})^{p^M}}{1-p^{\beta M}|z|^{p^{vM+j}(p^{2M}-p^M)}} \\
&\leq \frac{2}{\mu(1-\frac{1}{p^{vM+j}})} \cdot \frac{p^{\beta M}\cdot 2^{-p^{M-0.5}}}{1-p^{\beta M}\cdot 2^{-(p^{2M-0.5}-p^{M-0.5})}} \leq \frac{1}{100\mu(1-\frac{1}{p^{vM+j}})}.
\end{aligned}
$$

Combining all the estimates for $I_1, I_2$, and $I_3$, we get

$$
\begin{aligned}
\big|g_{i,j}(z)\big| &\geq I_1 - I_2 - I_3 \geq \frac{1}{\mu(1-\frac{1}{p^{vM+j}})}\Big(\frac{2}{27}-\frac{1}{100}-\frac{1}{100}\Big) \\
&> \frac{1}{20\mu(1-\frac{1}{p^{vM+j}})} = \frac{1}{20\mu(1-\frac{1}{p^{vM+j+\frac{1}{2}}})} \cdot \frac{\mu(1-\frac{1}{p^{vM+j+\frac{1}{2}}})}{\mu(1-\frac{1}{p^{vM+j}})} \\
&\geq \frac{1}{20p^{\frac{\beta}{2}}\mu(1-\frac{1}{p^{vM+j+\frac{1}{2}}})} \geq \frac{1}{20p^{\frac{\beta}{2}}\mu(|z|)}.
\end{aligned}
$$

In summary, we have

$$
\sum_{i=1}^{M}\sum_{j=1}^{M}\big|g_{i,j}(z)\big| \geq \frac{1}{20p^{\frac{\beta}{2}}\mu(|z|)} \tag{3.13}
$$

for all $z$ such that $1-\frac{1}{p^k} \leq |z| \leq 1-\frac{1}{p^{k+\frac{1}{2}}}, k = 1, 2, \ldots$.

Next, pick a sequence of positive integers $q_k$ such that $0 \leq q_k - p^{k+\frac{1}{2}} < 1$, and, for each $1 \leq j \leq M$, pick a sequence of positive numbers $\varepsilon_{j,v}$ such that $A^2 q_{vM+j}\varepsilon_{j,v}^2 = 1$.

Choose a sequence of subsets $\Psi_{j,v}$ of $\mathbb{S}$ with the following property: for each nonnegative interger $v$, the set $\bigcup_{l=1}^{M}\Psi_{j,vM+l}$ is a maximal subset of $\mathbb{S}$ which is $d$-separated by $A\varepsilon_{j,v}/2$, and each $\Psi_{j,vM+l}$ is $d$-separated by $\varepsilon_{j,v}$.

For each $i, j = 1, 2, \ldots, M$ and $v \geq 0$, set

$$
Q_{i,vM+j}(z) = \sum_{\xi\in\Psi_{j,vM+\tau^i(j)}} \langle z, \xi\rangle^{q_{vM+j}},
$$

and define

$$
h_{i,j}(z) = \sum_{v=0}^{\infty} \frac{Q_{i,vM+j}(z)}{\mu(1-\frac{1}{q_{vM+j}})}.
$$

Then $h_{i,j}$ is in the Hadamard gap since, for each $v \geq 0$,

$$\frac{q_{vM+j}}{q_{(v-1)M+j}} \geq \frac{p^{vM+\frac{1}{2}}}{p^{(v-1)M+\frac{1}{2}}+1} \geq \frac{p^M}{2} > 1.$$

Moreover, the homogeneous polynomials $Q_{i,vM+j}$ are uniformally bounded by 2 as before. Hence each $h_{i,j}$ belongs to $H_\mu^\infty$ by Theorem 2.2, and an easy modification of the previous arguments yields that, for each $v \geq 0, 1 \leq j \leq M$, and $z \in \mathbb{B}$ satisfying

$$1 - \frac{1}{p^{vM+j+\frac{1}{2}}} \leq |z| \leq 1 - \frac{1}{p^{vM+j+1}},$$

there exists an index $i \in \{1, 2, \ldots, M\}$ such that

$$\left|h_{i,j}(z)\right| \geq \frac{C_p}{\mu(|z|)},$$

where $C_p > 0$.

Hence

$$\sum_{i=1}^{M}\sum_{j=1}^{M}\left|h_{i,j}(z)\right| \geq \frac{C_p}{\mu(|z|)} \tag{3.14}$$

for all $z$ such that $1 - \frac{1}{p^{k+\frac{1}{2}}} \leq |z| \leq 1 - \frac{1}{p^{k+1}}, k = 1, 2, \ldots$.

Consequently, we finally have

$$\sum_{i=1}^{M}\sum_{j=1}^{M}\left(\left|g_{i,j}(z)\right| + \left|h_{i,j}(z)\right|\right) \geq \frac{C}{\mu(|z|)}$$

for all $z \in \mathbb{B}$ sufficiently close to the boundary and for some constant $C$. Therefore, the proof is complete. □

As a corollary, we get the following description of the growth rate on the space $H_\alpha^\infty$ $(\alpha > 0)$ by taking $\mu(|z|) = (1 - |z|^2)^\alpha$ in Theorem 3.1.

**Corollary 3.5.** *There exists some positive integer $M$ and a sequence of functions $f_i \in H_\alpha^\infty, 1 \leq i \leq M$ such that*

$$\sum_{i=1}^{M}\left|f_i(z)\right| \gtrsim \frac{1}{(1-|z|^2)^\alpha}, \quad z \in \mathbb{B}.$$

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