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NEAR DENTABILITY AND CONTINUITY OF THE SET-VALUED METRIC GENERALIZED INVERSE IN BANACH SPACES

SHAOQIANG SHANG

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Abstract. In this paper, upper semicontinuity and continuity of the set-valued metric generalized inverse T^{∂} in nearly dentable spaces are investigated using the methods of Banach space geometry. Moreover, it is proved that if X is a nearly dentable space and if C is a closed convex set of X, then C is approximatively compact if and only if $P_C(x)$ is compact for any $x \in X$.

1. Introduction and preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. Let $S(X)$ and $B(X)$ denote the unit sphere and the unit ball of X, respectively. By X^* we denote the dual space of X. Let $N, R,$ and R^+ denote the set of natural numbers, reals, and nonnegative reals, respectively. By $x_n \xrightarrow{w} x$, we denote that $\{x_n\}_{n=1}^{\infty}$ is weakly convergent to x. By $\overline{C}$ we denote the closed hull of C, while $\operatorname{dist}(x, C)$ denotes the distance of x and C, and $B(x, r)$ denotes the closed ball centered at x and of radius $r > 0$. Let $C \subset X$ be a nonempty subset of X. Then the set-valued mapping $P_C : X \to C$,

$$P_C(x) = \big\{ z \in C : \|x - z\| = \operatorname{dist}(x, C) := \inf_{y \in C} \|x - y\| \big\},$$

is said to be the *metric projection operator* from X onto C. A nonempty set C is said to be a *Chebyshev set* if $P_C(x)$ is a singleton. A subspace $L \subset X$ is said to be a *maximal subspace* of X if there exists $x^* \in S(X^*)$ such that

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$L = \{x \in X : x^*(x) = 0\}$. Moreover, if a maximal subspace L is a Chebyshev set, then L is said to be a *Chebyshev maximal subspace*.

It is well known that geometric properties of Banach spaces have also brought great attention to many mathematicians, and such is the case, for example, of differentiability (see [9]), nonsquareness (see [6]), the Mazur intersection property (see [3]), the Banach–Saks property (see [2]), and the approximative compactness (see [1], [4], [8], [11]).

Definition 1.1 (see [8]). A nonempty subset C of X is said to be *approximatively compact* if, for any $\{y_n\}_{n=1}^{\infty} \subset C$ and any $x \in X$ satisfying $\|x - y_n\| \to \inf_{y \in C} \|x - y\|$ as $n \to \infty$, the sequence $\{y_n\}_{n=1}^{\infty}$ has a subsequence converging to an element in C. Also, X is considered approximatively compact if every nonempty closed convex subset of X is approximatively compact.

Definition 1.2. A Banach space X is said to be *nearly dentable* if, for any $f \in S(X^*)$ and any open set $U_{A_f} \supset A_f$, we have $A_f \neq \emptyset$ and $A_f \cap \overline{\mathrm{co}}(B(X) \backslash U_{A_f}) = \emptyset$, where $A_f = \{x : f(x) = \|x\| = 1\}$.

In the present article, we present the history of near dentability and related notions. This notion was introduced in [13] as a property of Banach spaces that guarantees that the metric projection operator is upper-semicontinuous. By the James theorem, it is easy to see that if X is a nearly dentable space, then X is reflexive; Shang, Cui, and Fu in [13] proved that X is approximatively compact if and only if X is a nearly dentable space and if X is a nearly strictly convex space. In 2015, Shang and Cui [12] proved that if X is a nearly dentable space and if H is a hyperplane of X, then H is approximatively compact if and only if $P_H(x)$ is compact for any $x \in X$.

Definition 1.3 (see [1, p. 39]). Set-valued mapping $F : X \to Y$ is said to be *upper-semicontinuous* at x_0 if, for each norm open set W with $F(x_0) \subset W$, there exists a norm neighborhood U of x_0 such that $F(x) \subset W$ for all x in U. Note that F is called *lower-continuous* at x_0 if, for any $y \in F(x_0)$ and any $\{x_n\}_{n=1}^{\infty}$ in X with $x_n \to x_0$, there exists $y_n \in F(x_n)$ such that $y_n \to y$ as $n \to \infty$. Also, F is called *continuous* at x_0 if F is upper-semicontinuous and is lower-continuous at x_0.

Let T be a linear bounded operator from X into Y. Let $D(T)$, $R(T)$, and $N(T)$ denote the domain, range, and null space of T, respectively. If $N(T) \neq \{0\}$ or $R(T) \neq Y$, then the operator equation $Tx = y$ is generally ill posed; that is, there exists $y_0 \in Y$ such that $\|Tx - y_0\| \neq 0$ for any $x \in D(T)$. In applications, one usually looks for the *best approximative solution* (bas) to the equation $Tx = y$. A point $x_0 \in D(T)$ is said to be the best approximative solution to the operator equation $Tx = y$ if

$$\|Tx_0 - y\| = \inf\bigl\{\|Tx - y\| : x \in D(T)\bigr\}$$

and if

$$\|x_0\| = \min\bigl\{\|v\| : v \in D(T), \|Tv - y\| = \inf_{x \in D(T)} \|Tx - y\|\bigr\}.$$

Nashed and Votruba in [10] introduced the concept of the set-valued metric generalized inverse T as follows.

Definition 1.4 (see [10, p. 834]). Let X, Y be Banach spaces, and let T be a linear operator from X to Y. The set-valued mapping $T^\partial : Y \to X$ defined by

$$T^\partial(y) = \{x_0 \in D(T) : x_0 \text{ is a best approximative solution to } T(x) = y\}$$

for any $y \in D(T^\partial)$ is said to be the (set-valued) metric generalized inverse of T, where

$$D(T^\partial) = \{y \in Y : T(x) = y \text{ has a best approximative solution in } X\}.$$

During the last three decades, the linear generalized inverses of linear operators in Banach spaces and their applications have been investigated by many authors. In 2008, Chen et al. [4] gave some necessary and sufficient conditions for T to have a continuous Moore–Penrose metric generalized inverse in Banach spaces. Also in 2008, Hudzik, Wang, and Zheng [7] gave criteria for the metric generalized inverses of linear operators and their homogeneous selections in terms of Moore–Penrose conditions. Other research on generalized inverses of linear operators can be found in [5], [14]–[16].

In the following, the author proves that if X is a nearly dentable space, then Y is a Banach space, $D(T)$ is a closed subspace of X, and $R(T)$ is a Chebyshev maximal subspace of Y. Then the following statements are equivalent: (1) $P_{R(T)}(y)$ is a compact set for any $x \in T^{-1}(P_{R(T)}(y_0))$; (2) $T^\partial(y_0)$ is a compact set and the set-valued mapping T^∂ is upper-semicontinuous at y_0; (3) $T^\partial(y_0)$ is a compact set, and the set-valued mapping $T^\partial|_{\{\alpha y_0 : \alpha \in R\}}$ is continuous at y_0. Moreover, it is proved that if X is a nearly dentable space, then Y is a Banach space, $D(T)$ is a closed subspace of X, and $R(T)$ is a Chebyshev maximal subspace of Y. Then the following statements are equivalent: (1) $N(T)$ is an approximatively compact subspace of $D(T)$; (2) $T^\partial(y)$ is a compact set for any $y \in Y$, and the set-valued mapping T^∂ is upper-semicontinuous; (3) $T^\partial(y)$ is a compact set for any $y \in Y$, and the set-valued mapping $T^\partial|_{\{\alpha y : \alpha \in R\}}$ is continuous. Finally, the author proves that if X is a nearly dentable space, then the set A_f is compact if and only if x is an H-point for any $x \in A_f$.

2. Main results

In 2015, Shang and Cui [12] studied upper semicontinuity and continuity of the set-valued metric generalized inverse T^∂ in approximatively compact Banach spaces. Moreover, we know that if X is approximatively compact, then X is nearly dentable. It is very natural to ask whether the set-valued metric generalized inverse T^∂ is continuous for a nearly dentable Banach space. In Theorems 2.1 and 2.6, the author answers the question.

Theorem 2.1. *Let X be a nearly dentable space, let Y be a Banach space, let $D(T)$ be a closed subspace of X, and let $R(T)$ be a Chebyshev maximal subspace of Y. Then the following statements are equivalent:*

(1) *$P_{R(T)}(y)$ is a compact set for any $x \in T^{-1}(P_{R(T)}(y_0))$;*

(2) *$T^\partial(y_0)$ is a compact set, and the set-valued mapping T^∂ is upper-semicontinuous at y_0;*
(3) *$T^\partial(y_0)$ is a compact set, and the set-valued mapping $T^\partial|_{\{\alpha y_0:\alpha\in R\}}$ is continuous at y_0.*

In order to prove the theorem, we first give some lemmas.

Lemma 2.2. *Let X be a Banach space, and let H be a Chebyshev maximal subspace of X. Then the metric projector operator P_H is continuous.*

Proof. Since H is a Chebyshev maximal subspace of X, there exists $x_0^* \in S(X^*)$ such that $H = \{x \in X : x_0^*(x) = 0\}$. Let $x_n \to x$ as $n \to \infty$. Pick $z_n \in H$ and $z \in H$ such that $\|x_n - z_n\| = \operatorname{dist}(x_n, H)$ and $\|x - z\| = \operatorname{dist}(x, H)$. Since H is a Chebyshev maximal subspace of X, we have $z_n = P_H(x_n)$ and $z = P_H(x)$. Moreover, there exist $y_n \in S(X)$ and $y \in S(X)$ such that

$$x - z = \alpha y \qquad \text{and} \qquad x_n - z_n = \alpha_n y_n$$

for all $n \in N$, where $\alpha \in R$ and $\{\alpha_n\}_{n=1}^\infty \subset R$. Then it is easy to see that

$$x - z = \frac{x_0^*(x)}{x_0^*(y)}y \qquad \text{and} \qquad x_n - z_n = \frac{x_0^*(x)}{x_0^*(y_n)}y_n. \tag{2.1}$$

Let $\{y_{1,n}\}_{n=1}^\infty \subset S(X)$, and let $x_0^*(y_{1,n}) \to 1$ as $n \to \infty$. Then

$$z_{1,n} = x - \frac{x_0^*(x)}{x_0^*(y_{1,n})}y_{1,n} \in H \qquad \text{and} \qquad \lim_{n\to\infty} \|x - z_{1,n}\| = \big|x_0^*(x)\big|.$$

Therefore, by $\|x - z\| = |x_0^*(x)|(\|y\|/|x_0^*(y)|) \geq |x_0^*(x)|$, we obtain $\|x - z\| = \operatorname{dist}(x, H) = |x_0^*(x)|$. Similarly, we have $\|x_n - z_n\| = \operatorname{dist}(x_n, H) = |x_0^*(x_n)|$ for all $n \in N$. Therefore, by formula (2.1), we obtain $|x_0^*(y)| = \|y\| = 1$ and $|x_0^*(y_n)| = \|y_n\| = 1$ for all $n \in N$. We claim that $y = y_n$ for all $n \in N$. In fact, suppose that $y \neq y_1$. Then

$$z_{1,0} = x - \frac{x_0^*(x)}{x_0^*(y_1)}y_1 \in H \qquad \text{and} \qquad \|x - z_{1,0}\| = \big|x_0^*(x)\big| = \operatorname{dist}(x, H).$$

This implies that $z_{1,0} \in P_H(x)$ and $z_{1,0} \neq z$. Hence we obtain that H is not a Chebyshev subspace of X, which is a contradiction. Therefore, by formula (2.1) and $y = y_n$, we obtain

$$z_n = x_n - \frac{x_0^*(x)}{x_0^*(y)}y \to x - \frac{x_0^*(x)}{x_0^*(y)}y = z \quad \text{as } n \to \infty.$$

This implies that the projector operator P_H is continuous. This completes the proof of the lemma. □

Lemma 2.3. *Let X be a nearly dentable Banach space, and let C be a closed convex set of X. Then the following statements are equivalent:*

(1) *the set $P_C(x_0)$ is compact;*
(2) *if $\{y_n\}_{n=1}^\infty \subset C$ and $\|x_0 - y_n\| \to \inf_{y\in C}\|x_0 - y\|$ as $n \to \infty$, then the sequence $\{y_n\}_{n=1}^\infty$ has a subsequence converging to an element in C.*

Proof. (1) $\Rightarrow$ (2). Suppose that $\{y_n\}_{n=1}^{\infty} \subset C$, that $\|x_0 - y_n\| \to \inf_{y \in C} \|x_0 - y\|$ as $n \to \infty$, and that the sequence $\{y_n\}_{n=1}^{\infty}$ does not have a Cauchy subsequence. Moreover, we may assume, without loss of generality, that $x_0 = 0$. For clarity, we will divide the proof into two parts.

Case I. Let $\{y_n\}_{n=1}^{\infty} \cap P_C(0)$ be an infinite set. Since $P_C(0)$ is a compact set, the sequence $\{y_n\}_{n=1}^{\infty}$ has a subsequence converging to an element in C.

Case II. Let $\{y_n\}_{n=1}^{\infty} \cap P_C(0)$ be a finite set. Then we may assume without loss of generality that $\{y_n\}_{n=1}^{\infty} \cap P_C(0) = \emptyset$. We claim that, for any $x \in P_C(0)$, there exists $\varepsilon_x > 0$ such that $\{y_n\}_{n=1}^{\infty} \cap B(x, \varepsilon_x) = \emptyset$. Otherwise, there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ such that $y_{n_k} \to x$ as $k \to \infty$. This implies that the sequence $\{y_n\}_{n=1}^{\infty}$ has a subsequence converging to an element in C, which is a contradiction. Hence

$$\{y_n\}_{n=1}^{\infty} \cap \Big(\bigcup_{x \in P_C(0)} \big\{ z \in X : \|x - z\| < \varepsilon_x \big\} \Big) = \emptyset. \tag{2.2}$$

Moreover, we may assume without loss of generality that $0 \notin C$. Otherwise, it is easy to see that (2) is true. Hence $d = \operatorname{dist}(0, C) > 0$. This implies that

$$\operatorname{int} B(0, d) \cap C = \emptyset \qquad \text{and} \qquad B(0, d) \cap C = P_C(0).$$

Therefore, by the separation theorem, there exists $f \in S(X^*)$ such that

$$d = \sup\big\{ f(x) : x \in \operatorname{int} B(0, d) \big\} \leq \inf\big\{ f(x) : x \in C \big\}.$$

Therefore, by $\overline{\operatorname{int} B(0, d)} = B(0, d)$, we have

$$d = \sup\big\{ f(x) : x \in B(0, d) \big\} \leq \inf\big\{ f(x) : x \in C \big\}.$$

Since $P_C(0) \subset C$, we get $f(x) \geq d$ for any $x \in P_C(0)$. Since $P_C(0) \subset B(0, d)$, we have $f(x) \leq d$ for any $x \in P_C(0)$. This implies that $f(x) = d$ for any $x \in P_C(0)$. Hence $P_C(0) \subset \{x \in B(0, d) : f(x) = d\}$. Moreover, since $B(0, d) \cap C = P_C(0)$, we obtain that $\operatorname{dist}(y, C) > 0$ for any $y \in \{x \in B(0, d) : f(x) = d\} \backslash P_C(0)$. Let $4\varepsilon_y = \operatorname{dist}(y, C)$ for any $y \in \{x \in B(0, d) : f(x) = d\} \backslash P_C(0)$. Then

$$\{y_n\}_{n=1}^{\infty} \cap \big\{ z \in X : \|z - y\| < 2\varepsilon_y \big\} = \emptyset \tag{2.3}$$

for any $y \in \{x \in B(0, d) : f(x) = d\} \backslash P_C(0)$. Hence we define a sequence $\{z_n\}_{n=1}^{\infty}$, where

$$z_n = \frac{nd}{n+1} \cdot \frac{y_n}{\|y_n\|}, \quad n \in N.$$

Then

$$\|z_n - y_n\| = \Big\| \frac{nd}{n+1} \cdot \frac{y_n}{\|y_n\|} - y_n \Big\| = \Big| \frac{nd}{(n+1)\|y_n\|} - 1 \Big| \cdot \|y_n\| \to 0 \tag{2.4}$$

and

$$f(z_n) = f\Big(\frac{nd}{n+1} \cdot \frac{y_n}{\|y_n\|} \Big) = \frac{nd}{n+1} f\Big(\frac{y_n}{\|y_n\|} \Big) < d. \tag{2.5}$$

Therefore, by formulas (2.4) and (2.5), we have $z_n \notin \{x \in B(0, d) : f(x) = d\}$ and $z_n \in B(0, d)$. Noticing formula (2.4), we then get, for any $y \in \{x \in B(0, d) :$

$f(x) = d\}\backslash P_C(0)$, that there exists $n_y \in N$ such that $\|z_n - y_n\| < \varepsilon_y/4$ whenever $n > n_y$. Therefore, by formula (2.3), we have

$$\|z_n - y\| \geq \|y - y_n\| - \|y_n - z_n\| \geq \varepsilon_y - \frac{\varepsilon_y}{4} = \frac{3}{4}\varepsilon_y \tag{2.6}$$

whenever $n > n_y$. Since $z_n \notin \{x \in B(0,d) : f(x) = d\}$ and $z_n \in B(0,d)$, it holds that

$$\min\big\{\|z_1 - y\|, \|z_2 - y\|, \ldots, \|z_{n_y} - y\|\big\} > 0 \tag{2.7}$$

for any $y \in \{x \in B(0,d) : f(x) = d\}\backslash P_C(0)$. Let

$$\eta_y = \min\Big\{\frac{3}{4}\varepsilon_y, \min\big\{\|z_1 - y\|, \|z_2 - y\|, \ldots, \|z_{n_y} - y\|\big\}\Big\}$$

for any $y \in \{x \in B(0,d) : f(x) = d\}\backslash P_C(0)$. Then $\eta_y > 0$. Therefore, by formulas (2.6) and (2.7), we have

$$\{z_n\}_{n=1}^{\infty} \cap \big\{z \in X : \|y - z\| < \eta_y\big\} = \emptyset \tag{2.8}$$

for any $y \in \{x \in B(0,d) : f(x) = d\}\backslash P_C(0)$. Moreover, by formulas (2.2) and (2.3), for any $x \in P_C(0)$, there exists $n_x \in N$ such that $\|z_n - y_n\| < \varepsilon_x/4$ whenever $n > n_x$. Therefore, by formula (2.2), we have

$$\|z_n - x\| \geq \|x - y_n\| - \|y_n - z_n\| \geq \varepsilon_x - \frac{1}{4}\varepsilon_x = \frac{3}{4}\varepsilon_x$$

for any $x \in P_C(0)$. Moreover, by $z_n \notin \{x \in B(0,d) : f(x) = d\}$,

$$\min\big\{\|z_1 - x\|, \|z_2 - x\|, \ldots, \|z_{n_x} - x\|\big\} > 0$$

for any $x \in P_C(0)$. Let

$$\eta_x = \min\Big\{\frac{3}{4}\varepsilon_x, \min\big\{\|z_1 - x\|, \|z_2 - x\|, \ldots, \|z_{n_x} - x\|\big\}\Big\}$$

for any $x \in P_C(0)$. Then $\eta_x > 0$. This implies that

$$\{z_n\}_{n=1}^{\infty} \cap \big\{z \in X : \|x - z\| < \eta_x\big\} = \emptyset \tag{2.9}$$

for any $x \in P_C(0)$. Let

$$\begin{aligned} V_f = &\Big(\bigcup_{y \in \{x \in B(0,d) : f(x) = d\}\backslash P_C(0)} \big\{z \in X : \|y - z\| < \eta_y\big\}\Big) \\ &\cup \Big(\bigcup_{x \in P_C(0)} \big\{z \in X : \|x - z\| < \eta_x\big\}\Big). \end{aligned}$$

Then, by formulas (2.8) and (2.9), we have $\{z_n\}_{n=1}^{\infty} \cap V_f = \emptyset$.

Since $\|0 - y_n\| \to \inf_{y \in C} \|0 - y\|$ as $n \to \infty$, it holds that the sequence $\{y_n\}_{n=1}^{\infty}$ is a bounded sequence. Since X is a nearly dentable space, X is a reflexive space. Hence we may assume without loss of generality that $y_n \to^w y_0$ as $n \to \infty$. Since C is a closed convex set, we have that C is a weakly closed convex set. Hence

$y_0 \in C$. Moreover, by the Hahn–Banach theorem, there exists $f_0 \in S(X^*)$ such that $f_0(y_0) = \|y_0\|$. Hence

$$\operatorname{dist}(0, C) = \lim_{n\to\infty} \|0 - y_n\| \geq \lim_{n\to\infty} f_0(y_n - 0) = f_0(y_0) = \|0 - y_0\| \geq \operatorname{dist}(0, C).$$

This implies that $y_0 \in P_C(0)$. Since $y_n \to^w y_0$ and $\|y_n - z_n\| \to 0$ as $n \to \infty$, we obtain $z_n \to^w y_0$ as $n \to \infty$. Since $\{z_n\}_{n=1}^\infty \subset B(0, d)$ and $\{z_n\}_{n=1}^\infty \cap V_f = \emptyset$, we have

$$\left\{\frac{1}{d} z_n\right\}_{n=1}^\infty \subset \frac{1}{d} B(0, d) = B(X) \qquad \text{and} \qquad \left\{\frac{1}{d} z_n\right\}_{n=1}^\infty \cap \frac{1}{d} V_f = \left\{\frac{1}{d} z_n\right\}_{n=1}^\infty \cap U_{A_f}.$$

This implies that $\{z_n/d\}_{n=1}^\infty \subset B(X) \backslash U_{A_f}$. Moreover, it is easy to see that $U_{A_f} \supset A_f$. Since X is a nearly dentable space, we get

$$A_f \cap \overline{\operatorname{co}}\big(B(X) \backslash U_{A_f}\big) = \emptyset.$$

Since X is a reflexive space, A_f and $\overline{\operatorname{co}}(B(X) \backslash U_{A_f})$ are weakly compact. Therefore, by the separation theorem, there exist $g \in S(X^*)$ and $r > 0$ such that

$$\inf\big\{g(x) : x \in A_f\big\} - r \geq \sup\big\{g(x) : x \in \overline{\operatorname{co}}\big(B(X) \backslash U_{A_f}\big)\big\}. \tag{2.10}$$

Since $\{z_n/d\}_{n=1}^\infty \subset B(X) \backslash U_{A_f}$ and $y_0/d \in A_f$, by formula (2.10), it holds that

$$g\Big(\frac{1}{d} y_0\Big) - r \geq \sup\Big\{g(x) : x \in \left\{\frac{1}{d} z_n\right\}_{n=1}^\infty\Big\}.$$

Then $g(y_0) - r/d \geq g(z_n)$ for all $n \in N$, which contradicts $z_n \to^w y_0$ as $n \to \infty$. Hence the sequence $\{y_n\}_{n=1}^\infty$ has a Cauchy subsequence. Then it is easy to see that the sequence $\{y_n\}_{n=1}^\infty$ has a subsequence converging to an element in C.

(2) $\Rightarrow$ (1) is obvious. This completes the proof of the lemma. □

Proof of Theorem 2.1. (1) $\Rightarrow$ (2). Let $y_0 \in Y$ and $P_{R(T)}(x)$ be compact for any $x \in T^{-1}(P_{R(T)}(y_0))$. We next will prove that T^∂ is upper-semicontinuous at y_0; that is, for any $\{y_n\}_{n=1}^\infty \subset Y$, $y_n \to y_0 \in Y$, and any norm open set W with $T^\partial(y_0) \subset W$, there exists a natural number N_0 such that $T^\partial(y_n) \subset W$ whenever $n > N_0$. Otherwise, we may assume that there exists $x_n \in T^\partial(y_n)$ such that $\{x_n\}_{n=1}^\infty \cap W = \emptyset$. Since $R(T)$ is a Chebyshev maximal subspace of Y, by Lemma 2.2 it holds that the metric projector operator $P_{R(T)}$ is continuous. Therefore, by $y_n \to y_0$, we get $P_{R(T)}(y_n) \to P_{R(T)}(y_0)$ as $n \to \infty$. Noticing that $Tx_n = P_{R(T)}(y_n)$, we obtain that $Tx_n \to P_{R(T)}(y_0)$ as $n \to \infty$. Since T is a bounded linear operator, it holds that $N(T)$ is a closed subspace of $D(T)$. Put

$$\overline{T} : D(T)/N(T) \to R(T), \qquad \overline{T}[x] = Tx,$$

where $[x] \in D(T)/N(T)$ and $x \in D(T)$. It is easy to see that $R(\overline{T}) = R(T)$. Moreover, $\overline{R(T)} = R(T)$. In fact, suppose that $\overline{R(T)} \neq R(T)$. Then there exists $y' \in \overline{R(T)}$ such that $y' \notin R(T)$. It is easy to see that $\{y \in R(T) : \|y' - y\| = \operatorname{dist}(y', R(T))\} = \emptyset$. This implies that $R(T)$ is not a Chebyshev subspace of Y, which is a contradiction. Since $\overline{R(T)} = R(T)$, we get that $R(T)$ is a Banach space. It is, moreover, clear that $\overline{T}$ is a bounded linear operator and that $N(\overline{T}) = \{0\}$. This implies that the bounded linear operator $\overline{T}$ is both injective and surjective.

Therefore, by the inverse operator theorem, $\overline{T}^{-1}$ is a bounded linear operator. Hence

$$[x_n] = \overline{T}^{-1}\big(P_{R(T)}(y_n)\big) \to \overline{T}^{-1}\big(P_{R(T)}(y_0)\big) = [x_0] \quad \text{as } n \to \infty. \tag{2.11}$$

This implies that $\|[x_n]\| \to \|[x]\|$ as $n \to \infty$. Noticing that

$$x_n \in T^{\partial}(y_n), \quad \big\|[x_n]\big\| = \inf_{z \in N(T)} \|x_n + z\|, \qquad \overline{T}[x_n] = T(x_n + z) = P_{R(T)}(y_n),$$

$$x_0 \in T^{\partial}(y_0), \quad \big\|[x_0]\big\| = \inf_{z \in N(T)} \|x_0 + z\|, \qquad \overline{T}[x_0] = T(x_0 + z) = P_{R(T)}(y_0),$$

it is easy to see that $\|[x_n]\| = \|x_n\|$ and $\|[x_0]\| = \|x_0\|$. Since $\|[x_n]\| \to \|[x_0]\|$, $\|[x_n]\| = \|x_n\|$, and $\|[x_0]\| = \|x_0\|$, we have $\|x_n\| \to \|x_0\|$ as $n \to \infty$. We will derive a contradiction for each of the following two cases.

Case I. Let $x_0 = 0$. Then, by formula (2.11), we have $[x_n] \to [x_0] = 0$ as $n \to \infty$. This implies that $\|[x_n]\| \to 0$ as $n \to \infty$. Hence $\|x_n\| \to 0$ as $n \to \infty$. Since $x_0 = 0$, we have $0 \in T^{\partial}(y) \subset W$. Moreover, by $\|x_n\| \to 0$, we have $x_n \to 0$ as $n \to \infty$, which contradicts $\{x_n\}_{n=1}^{\infty} \cap W = \emptyset$.

Case II. Let $x_0 \neq 0$. Pick $x' \in T^{-1}(P_{R(T)}(y_0))$. Then, by the definition of the set-valued metric generalized inverse, there exists $\pi_{N(T)}(x') \in P_{N(T)}(x')$ such that $x_0 = x' - \pi_{N(T)}(x')$. Since $P_{R(T)}(y_n) \to P_{R(T)}(y_0)$, by the definition of the quotient space, there exists $x_{1n} \in T^{-1}(P_{R(T)}(y_n))$ such that $\|[x'] - [x_{1n}]\| \to 0$ as $n \to \infty$. Hence we may assume without loss of generality that $\|x' - x_{1n}\| \to 0$ as $n \to \infty$. Moreover, by the definition of the set-valued metric generalized inverse, there exists $\pi_{N(T)}(x_{1n}) \in P_{N(T)}(x_{1n})$ such that $x_n = x_{1n} - \pi_{N(T)}(x_{1n})$. Therefore, by $\|x_{1n} - \pi_{N(T)}(x_{1n})\| = \|x_n\| \to \|x_0\|$ and $\|x' - x_{1n}\| \to 0$ as $n \to \infty$, we have

$$\begin{aligned}
\big\|x' - \pi_{N(T)}(x')\big\| &\leq \lim \inf_{n\to\infty} \big\|x' - \pi_{N(T)}(x_{1n})\big\| \\
&\leq \lim \sup_{n\to\infty} \big\|x' - \pi_{N(T)}(x_{1n})\big\| \\
&\leq \lim \sup_{n\to\infty} \big[\|x' - x_{1n}\| + \big\|x_{1n} - \pi_{N(T)}(x_{1n})\big\|\big] \\
&\leq \|x_0\| = \big\|x' - \pi_{N(T)}(x')\big\|.
\end{aligned}$$

This implies that $\|x' - \pi_{N(T)}(x_{1n})\| \to \|x' - \pi_{N(T)}(x')\|$ as $n \to \infty$. Since the set $P_{N(T)}(x')$ is compact, by Lemma 2.3, the sequence $\{\pi_{N(T)}(x_{1n})\}_{n=1}^{\infty}$ has a Cauchy subsequence. Hence we may assume without loss of generality that $\pi_{N(T)}(x_{1n}) \to x \in N(T)$ as $n \to \infty$. Then $\|x' - x\| = \|x' - \pi_{N(T)}(x')\|$. This implies that $x' - x \in T^{\partial}(y_0)$. Since $x_n = x_{1n} - \pi_{N(T)}(x_{1n}) \to x' - x$ as $n \to \infty$, there exists $n_0 \in N$ such that $x_n \in W$ whenever $n > n_0$, which is a contradiction. Hence the set-valued mapping T^{∂} is upper-semicontinuous at y_0.

Pick $x \in T^{-1}(P_{R(T)}(y_0))$. Then, by the definition of the set-valued metric generalized inverse, we have $T^{\partial}(y_0) = x - P_{N(T)}(x)$. Since $P_{N(T)}(x)$ is compact, $T^{\partial}(y_0)$ is a compact set.

(2) $\Rightarrow$ (3). Suppose that $T^{\partial}(y_0)$ is a compact set, let the set-valued mapping T^{∂} be upper-semicontinuous at y_0, and let $y_n \to y_0$, where $\{y_n\}_{n=1}^{\infty} \subset \{\alpha y_0 : \alpha \in R\}$. Then there exists $\{\alpha_n\}_{n=1}^{\infty} \subset R$ such that $y_n = \alpha_n y_0$. If $y_0 = 0$, then it is easy

to see that (3) is true. If $y_0 \neq 0$, then we may assume without loss of generality that $\alpha_n \neq 0$. Hence, for any $y \in R(T)$, we have

$$\|y_n - y\| = \Big\|y_n - \alpha_n\Big(\frac{1}{\alpha_n}y\Big)\Big\| = \alpha_n\Big\|y_0 - \Big(\frac{1}{\alpha_n}y\Big)\Big\| \geq \alpha_n \operatorname{dist}\big(y_0, R(T)\big).$$

Moreover, since $R(T)$ is a Chebyshev subspace of Y, we get

$$\big\|y_n - \alpha_n P_{R(T)}(y_0)\big\| = \alpha_n\big\|y_0 - P_{R(T)}(y_0)\big\| = \alpha_n \operatorname{dist}\big(y_0, R(T)\big).$$

This implies that

$$\big\|y_n - \alpha_n P_{R(T)}(y_0)\big\| = \alpha_n \operatorname{dist}\big(y_0, R(T)\big) = \operatorname{dist}\big(y_n, R(T)\big). \tag{2.12}$$

Since $R(T)$ is a Chebyshev subspace of Y, by formula (2.12), we have

$$\big\{P_{R(T)}(y_n)\big\}_{n=1}^{\infty} \subset \big\{\alpha P_{R(T)}(y_0) : \alpha \in R\big\}. \tag{2.13}$$

Pick $x_0 \in T^{-1}(P_{R(T)}(y_0))$. Then $Tx_0 = P_{R(T)}(y_0)$. This implies that $T(\alpha_n x_0) = \alpha_n P_{R(T)}(y_0)$. Therefore, by formula (2.12), we have $\alpha_n x_0 \in T^{-1}(P_{R(T)}(y_n))$. Therefore, by the definition of the set-valued metric generalized inverse, we obtain that

$$T^{\partial}(y_0) = x_0 - P_{N(T)}(x_0) \qquad \text{and} \qquad T^{\partial}(y_n) = \alpha_n x_0 - P_{N(T)}(\alpha_n x_0) \tag{2.14}$$

for all $n \in N$. Let us define a subspace

$$X_0 = \big\{\alpha x_0 + y : \alpha \in R, y \in N(T)\big\}$$

of X. Then X_0 is a closed subspace of X, and $N(T)$ is a maximal subspace of X_0. Hence there exists $f_0 \in S((X_0)^*)$ such that

$$N(T) = \big\{x \in X_0 : f_0(x) = 0\big\}.$$

Moreover, by formula (2.14), it holds that $T^{\partial}(y_0) \subset X_0$ and $T^{\partial}(y_n) \subset X_0$ for all $n \in N$. From the proof of Lemma 2.2, we get

$$P_{N(T)}(x_0) = \Big\{x_0 - \frac{f_0(x_0)}{f_0(x)}x \in X_0 : f_0(x) = \|x\| = 1\Big\}$$

and

$$P_{N(T)}(\alpha_n x_0) = \Big\{\alpha_n x_0 - \frac{f_0(\alpha_n x_0)}{f_0(x)}x \in X_0 : f_0(x) = \|x\| = 1\Big\}.$$

Then

$$\begin{aligned} T^{\partial}(y_0) &= x_0 - P_{N(T)}(x_0) \\ &= x_0 - \Big\{x_0 - \frac{f_0(x_0)}{f_0(x)}x \in X_0 : f_0(x) = \|x\| = 1\Big\} \\ &= \Big\{\frac{f_0(x_0)}{f_0(x)}x \in X_0 : f_0(x) = \|x\| = 1\Big\}. \end{aligned}$$

Similarly, we have

$$T^{\partial}(y_n) = T^{\partial}(\alpha_n y_0) = \Big\{\frac{f_0(\alpha_n x_0)}{f_0(x)}x \in X_0 : f_0(x) = \|x\| = 1\Big\}$$

for all $n \in N$. Since $y_n = \alpha_n y_0$ and $y_n \to y_0$, by $y_0 \neq 0$, we have $\alpha_n \to 1$ as $n \to \infty$. Hence, for any $(f_0(x_0)/f_0(z))z \in T^\partial(y_0)$, we have

$$\lim_{n\to\infty} \frac{f_0(\alpha_n x_0)}{f_0(z)} z = \frac{f_0(x_0)}{f_0(z)} z \qquad \text{and} \qquad \frac{f_0(\alpha_n x_0)}{f_0(z)} z \in T^\partial(\alpha_n y_0) = T^\partial(y_n).$$

This implies that the set-valued mapping $T^\partial|_{\{\alpha y_0:\alpha\in R\}}$ is lower-semicontinuous at y_0. Since the set-valued mapping $T^\partial|_{\{\alpha y_0:\alpha\in R\}}$ is upper-semicontinuous at y_0, we obtain that the set-valued mapping $T^\partial|_{\{\alpha y_0:\alpha\in R\}}$ is continuous at y_0.

(3) $\Rightarrow$ (1). By the definition of the set-valued metric generalized inverse, we have $T^\partial(y_0) = x - P_{N(T)}(x)$ for any $x \in T^{-1}(P_{R(T)}(y_0))$. Since $T^\partial(y_0)$ is compact, by $T^\partial(y_0) = x - P_{N(T)}(x)$, it holds that the set $P_{N(T)}(x)$ is a compact set. This completes the proof. □

Theorem 2.4. *Let X be a nearly dentable space, let Y be a Banach space, let $D(T)$ be a closed subspace of X, and let $R(T)$ be a Chebyshev maximal subspace of Y. Then the following statements are equivalent:*

(1) *$N(T)$ is an approximatively compact subspace of $D(T)$;*
(2) *$T^\partial(y)$ is a compact set, and the set-valued mapping T^∂ is upper-semicontinuous for any $y \in Y$;*
(3) *$T^\partial(y)$ is a compact set, and the set-valued mapping $T^\partial|_{\{\alpha y:\alpha\in R\}}$ is continuous for any $y \in Y$.*

Proof. By Theorem 2.1, it is easy to see that (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) is true. We next will prove that (3) $\Rightarrow$ (1) is true. Let $x \in D(T)$. Then $Tx \in R(T)$. Therefore, by the definition of the set-valued metric generalized inverse, we have $T^\partial(Tx) = x - P_{N(T)}(x)$. Since $T^\partial(y_0)$ is a compact set, by $T^\partial(y_0) = x - P_{N(T)}(x)$, we obtain that $P_{N(T)}(x)$ is a compact set. Therefore, by Lemma 2.3, it holds that if $\{y_n\}_{n=1}^\infty \subset N(T)$ and if $\|x - y_n\| \to \inf_{y\in N(T)} \|x - y\|$ as $n \to \infty$, then the sequence $\{y_n\}_{n=1}^\infty$ has a subsequence converging to an element in $N(T)$. This implies that $N(T)$ is an approximatively compact subspace of $D(T)$. This completes the proof. □

Theorem 2.5. *Let X be a nearly dentable space, and let C be a closed convex set of X. Then the following statements are equivalent:*

(1) *$P_C(x)$ is compact for any $x \in X$,*
(2) *C is approximatively compact.*

Proof. By Lemma 2.3, it is easy to see that Theorem 2.5 is true. This completes the proof. □

Theorem 2.6. *Let X and Y be nearly dentable spaces, let $D(T)$ be a closed subspace of X, and let $R(T)$ be a Chebyshev subspace of Y. Then the following statements are equivalent:*

(1) *$N(T)$ is an approximatively compact subspace of $D(T)$, and $R(T)$ is an approximatively compact subspace of Y;*
(2) *$T^\partial(y)$ is a compact set, and the set-valued mapping T^∂ is upper-semicontinuous for any $y \in Y$;*

(3) *$T^\partial(y)$ is a compact set, and the set-valued mapping $T^\partial|_{\{\alpha y:\alpha\in R\}}$ is continuous for any $y \in Y$.*

Proof. (1) $\Rightarrow$ (2). Since $R(T)$ is an approximatively compact subspace of Y and is a Chebyshev subspace of Y, we obtain that the metric projector operator $P_{R(T)}$ is continuous. Therefore, by the proof of Theorem 2.1, we obtain that (1) $\Rightarrow$ (2) is true.

(2) $\Rightarrow$ (3). From the proof of Theorem 2.1, (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Since $R(T)$ is a Chebyshev subspace of Y, we obtain that, for any $y \in Y$, $P_{R(T)}(y)$ is a compact set. Therefore, by Theorem 2.5, we obtain that $R(T)$ is an approximatively compact subspace of Y. Therefore, by the proof of Theorem 2.1, (3) $\Rightarrow$ (1) is obvious. This completes the proof. □

Finally, we will discuss the relationship between near dentability and other geometric properties.

Definition 2.7 (see [4, p. 294]). A point $x \in S(X)$ is said to be an H-point if, for $\{x_n\}_{n=1}^\infty \subset S(X)$ and $x_n \to^w x$ as $n \to \infty$, we have $x_n \to x$ as $n \to \infty$. Moreover, if the set of all H-points is equal to $S(X)$, then X is said to have the H-property.

Theorem 2.8. *Let X be a nearly dentable space. Then the set A_f is compact if and only if x is an H-point for any $x \in A_f$.*

Proof. $\Longleftarrow$ Since X is a nearly dentable space, we obtain that X is reflexive. Hence, for any $\{x_n\}_{n=1}^\infty \subset A_f$, there exist $x' \in B(X)$ and a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $x_{n_k} \to^w x'$ as $k \to \infty$. It is easy to see that $x' \in A_f$. Then x' is an H-point. Hence $x_{n_k} \to x'$ as $k \to \infty$. This implies that the set A_f is compact.

$\Longrightarrow$ Pick $x_0 \in A_f$ and $x_n \to^w x_0$ as $n \to \infty$, where $\{x_n\}_{n=1}^\infty \subset S(X)$. Then $f(x_0) = 1$. For clarity, we will divide the proof into two parts.

Case I. Let $\{x_n\}_{n=1}^\infty \cap A_f$ be an infinite set. Since the set A_f is compact, there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $\{x_{n_k}\}_{k=1}^\infty$ is a Cauchy sequence. Therefore, by $x_n \to^w x_0$, we have $x_{n_k} \to x_0$ as $k \to \infty$. Then $x_n \to x_0$ as $n \to \infty$. Otherwise, there exist $\varepsilon_0 > 0$ and a subsequence $\{x_{n_l}\}_{l=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $\|x_{n_l} - x_0\| \geq \varepsilon_0$. From the previous proof, there exists a subsequence $\{x_{n_i}\}_{i=1}^\infty$ of $\{x_{n_l}\}_{l=1}^\infty$ such that $x_{n_i} \to x_0$ as $i \to \infty$, which is a contradiction.

Case II. Let $\{x_n\}_{n=1}^\infty \cap A_f$ be a finite set. Then we may assume without loss of generality that $\{x_n\}_{n=1}^\infty \cap A_f = \emptyset$. We next will prove that there exists $x' \in A_f$ such that x' is an accumulation point of $\{x_n\}_{n=1}^\infty$. In fact, suppose that x is not an accumulation point of $\{x_n\}_{n=1}^\infty$ for any $x \in A_f$. Then there exists $\varepsilon_x > 0$ such that $\{x_n\}_{n=1}^\infty \cap \{y \in X : \|y - x\| < \varepsilon_x\} = \emptyset$ for any $x \in A_f$. Put

$$U_{A_f} = \bigcup_{x \in A_f} \{y \in X : \|y - x\| < \varepsilon_x\}.$$

Then it is easy to see that $U_{A_f} \supset A_f$ and $\{x_n\}_{n=1}^\infty \subset B(X) \backslash U_{A_f}$. Since X is a nearly dentable space, we have

$$A_f \cap \overline{\mathrm{co}}\big(B(X) \backslash U_{A_f}\big) = \emptyset.$$

Since X is a nearly dentable space, we obtain that X is reflexive. Hence A_f and $\overline{\text{co}}(B(X)\backslash U_{A_f})$ are weakly compact. Therefore, by the separation theorem, there exist $g \in S(X^*)$ and $r > 0$ such that

$$\inf\{g(x) : x \in A_f\} - r \geq \sup\{g(x) : x \in \overline{\text{co}}\big(B(X)\backslash U_{A_f}\big)\}. \tag{2.15}$$

Since $x_0 \in A_f$, by formula (2.15), we have $g(x_0) - r \geq g(x_n)$ for all $n \in N$, which is a contradiction. Hence there exists $x' \in A_f$ such that x' is an accumulation point of $\{x_n\}_{n=1}^{\infty}$. Then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \to x'$ as $k \to \infty$. Therefore, by $x_n \to^w x_0$, it holds that $x_0 = x'$. Then $x_{n_k} \to x_0$ as $k \to \infty$. Therefore, by the proof of Case I, it is easy to see that $x_n \to x_0$ as $n \to \infty$. Hence x_0 is an H-point. This completes the proof. □

Corollary 2.9. *Let X be a nearly dentable space. Then the following statements are equivalent:*

(1) *the set A_f is compact for any $f \in S(X^*)$,*
(2) *X has the H-property.*

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Department of Mathematics, Northeast Forestry University, Harbin 150040, P. R. China.
E-mail address: sqshang@163.com