Ann. Funct. Anal. 7 (2016), no. 3, 484–495
http://dx.doi.org/10.1215/20088752-3624877
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

INTERPOLATION OF NONCOMMUTATIVE QUASIMARTINGALE SPACES

CONGBIAN MA* and YOULIANG HOU

Communicated by K. Zhu

Abstract. Let $\widehat{L}_p(\mathcal{M})$ be the space of bounded $L_p(\mathcal{M})$-quasimartingales. We prove that, with equivalent norms, $(\widehat{L}_{p_0}(\mathcal{M}), \widehat{L}_{p_1}(\mathcal{M}))_{\theta,p} = \widehat{L}_p(\mathcal{M})$, where $1 < p_0, p_1 \leq \infty$, $1 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. We also prove that, for $1 < p < q < \infty$, $(\widehat{\mathrm{BMO}}^c(\mathcal{M}), \widehat{\mathcal{H}}^c_p(\mathcal{M}))_{\frac{p}{q},q} = \widehat{\mathcal{H}}^c_q(\mathcal{M})$ and $(\widehat{\mathrm{BMO}}^r(\mathcal{M}), \widehat{\mathcal{H}}^r_p(\mathcal{M}))_{\frac{p}{q},q} = \widehat{\mathcal{H}}^r_q(\mathcal{M})$, where $\widehat{\mathcal{H}}_p(\mathcal{M})$ and $\widehat{\mathrm{BMO}}(\mathcal{M})$ are, respectively, the Hardy space and the bounded mean oscillation space of noncommutative quasimartingales.

1. Introduction

Inspired by quantum mechanics and probability, noncommutative probability has become an independent field of mathematical research. Today, many of the classical martingale inequalities have been transferred to the noncommutative setting. These include, in particular, the Doob maximal inequality, the Burkholder–Gundy inequality, several weak-type $(1,1)$ inequalities, and the Gundy decomposition.

As for the interpolation between the spaces of noncommutative martingales, we recall the formula on real and complex interpolation of noncommutative L_p-spaces. More precisely, for $0 < \theta < 1$ and $1 \leq p_0, p_1 \leq \infty$, we have

$$\big(L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M})\big)_{\theta,p} = L_p(\mathcal{M}) \qquad \text{and} \qquad \big(L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M})\big)_{\theta} = L_p(\mathcal{M}),$$

Received Dec. 2, 2015; Accepted Feb. 3, 2016.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46L53; Secondary 46L52, 60G42.
Keywords. noncommutative quasimartingale, interpolation space, Hardy space.

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. The real interpolation results between bounded mean oscillation (BMO) spaces and L_p-spaces (resp., Hardy spaces) were discussed by M. Musat [3]. In this paper, we study interpolation in the noncommutative quasimartingale setting. We first prove a real interpolation theorem between the spaces $\widehat{L}_p(\mathcal{M})$ of bounded $L_p(\mathcal{M})$-quasimartingales. And then we prove several real interpolation theorems between $\widehat{\mathrm{BMO}}(\mathcal{M})$ and $\widehat{\mathcal{H}}_p(\mathcal{M})$, where $\widehat{\mathrm{BMO}}(\mathcal{M})$ and $\widehat{\mathcal{H}}_p(\mathcal{M})$ are, respectively, the BMO space and the Hardy space of noncommutative quasimartingales.

2. Preliminaries

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H, and let τ be a normal faithful trace on $\mathcal{M}$ with $\tau(1) = 1$. We call $(\mathcal{M}, \tau)$ a *noncommutative probability space*. For $1 \leq p \leq \infty$, let $L_p(\mathcal{M})$ be the associated noncommutative L_p-space. Recall that, for $1 \leq p < \infty$, the norm on $L_p(\mathcal{M})$ is defined by

$$\|x\|_p = \tau\big(|x|^p\big)^{\frac{1}{p}}, \quad x \in L_p(\mathcal{M}),$$

where $|x| = (x^*x)^{\frac{1}{2}}$ is the usual modulus of x. Note that if $p = \infty$, then $L_\infty(\mathcal{M})$ is just $\mathcal{M}$ with the usual operator norm.

The noncommutative column spaces $L_p(\mathcal{M}; l_2^c)$ and the row spaces $L_p(\mathcal{M}; l_2^r)$ were introduced in [4]. For $1 \leq p < \infty$, define $L_p(\mathcal{M}; l_2^c)$ (resp., $L_p(\mathcal{M}; l_2^r)$) as the completion of the family of all finite sequences $x = (x_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ under the norm

$$\|x\|_{L_p(\mathcal{M};l_2^c)} = \Big\|\Big(\sum_n |x_n|^2\Big)^{\frac{1}{2}}\Big\|_p \quad \Big(\text{resp., } \|x\|_{L_p(\mathcal{M};l_2^r)} = \Big\|\Big(\sum_n |x_n^*|^2\Big)^{\frac{1}{2}}\Big\|_p\Big).$$

Let us recall the general setup for noncommutative martingales. Let $(\mathcal{M}_n)_{n\geq 1}$ be an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ such that the union of $\mathcal{M}_n$'s is weak*-dense in $\mathcal{M}$ and $\mathcal{E}_n$ (with $\mathcal{E}_0 = 0$) is the conditional expectation with respect to $\mathcal{M}_n$. A sequence $x = (x_n)_{n\geq 1}$ is said to be *adapted* if $x_n \in L_1(\mathcal{M}_n)$ for all $n \geq 1$ and *predictable* if $x_n \in L_1(\mathcal{M}_{n-1})$. A noncommutative martingale with respect to the filtration $(\mathcal{M}_n)_{n\geq 1}$ is a sequence $x = (x_n)_{n\geq 1}$ in $L_1(\mathcal{M})$ such that

$$\mathcal{E}_n(x_{n+1}) = x_n \quad \text{for all } n \geq 1.$$

If, additionally, $x = (x_n)_{n\geq 1} \subset L_p(\mathcal{M})$ for some $1 \leq p \leq \infty$, we call x an $L_p(\mathcal{M})$-*martingale*. In this case, we set $\|x\|_p = \sup_n \|x_n\|_p$. If $\|x\|_p < \infty$, then x is called a *bounded* $L_p(\mathcal{M})$-*martingale*. We refer to [5] for more information on noncommutative martingales.

We briefly recall some basic notions concerning the real method of interpolation. Let (X_0, X_1) be a compatible couple of quasi-Banach spaces. Its K-functional is defined by

$$K_t(x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$$

for $x \in X_0 + X_1$ and $t > 0$. Let $0 < \theta < 1$ and $0 < q \leq \infty$. Set

$$\|x\|_{\theta,q} = \Big(\int_0^\infty \big[t^{-\theta} K_t(x; X_0, X_1)\big]^q \frac{dt}{t}\Big)^{\frac{1}{q}}.$$

(The usual modification should be made for $q = \infty$.) Then the real interpolation space $(X_0, X_1)_{\theta,q}$ is defined as $(X_0, X_1)_{\theta,q} = \{x \in X_0 + X_1 : \|x\|_{\theta,q} < \infty\}$ equipped with the norm $\|\cdot\|_{\theta,q}$. Another method of interpolation is complex interpolation. We refer to J. Bergh and J. Löfström [1] for more information.

In this paper, we focus on noncommutative quasimartingales, which are the generalizations of noncommutative martingales and the noncommutative analogues of classical quasimartingales.

Definition 2.1. Let $1 \leq p \leq \infty$. An adapted sequence $x = (x_n)_{n\geq 1}$ in $L_1(\mathcal{M})$ is called a *p-quasimartingale* with respect to $(\mathcal{M}_n)_{n\geq 1}$ if

$$\sum_{n=1}^\infty \big\|\mathcal{E}_{n-1}(dx_n)\big\|_p^p < \infty, \tag{2.1}$$

where $dx_n = x_n - x_{n-1}$ (with $dx_1 = x_1$). If, in addition, $x_n \in L_p(\mathcal{M})$ $(n \geq 1)$, we call x an $L_p(\mathcal{M})$-*quasimartingale*. In this case, we set

$$\|x\|_p := \sup_n \|y_n\|_p + \Big(\sum_{n=1}^\infty \big\|\mathcal{E}_{n-1}(dx_n)\big\|_p^p\Big)^{\frac{1}{p}},$$

where $y_n = \sum_{k=1}^n (dx_k - \mathcal{E}_{k-1}(dx_k))$. If $\|x\|_p < \infty$, then x is called a *bounded* $L_p(\mathcal{M})$-*quasimartingale*. The noncommutative quasimartingale space $\widehat{L}_p(\mathcal{M})$ is defined as the space of all bounded $L_p(\mathcal{M})$-quasimartingales and is equipped with the norm $\|\cdot\|_p$.

Remark 2.2. Another kind of p quasimartingale is defined by replacing (2.1) with

$$\sum_{n=1}^\infty \big\|\mathcal{E}_{n-1}(dx_n)\big\|_p < \infty.$$

The quasimartingale defined in Definition 2.1 is more general, which is more suitable for the study of interpolation theorems.

A basic fact with respect to quasimartingales is that each p quasimartingale can be decomposed as a sum of a martingale and a predicable quasimartingale, which we call *Doob's decomposition*. Doob's decomposition plays a key role in this paper.

Lemma 2.3 (Doob's decomposition)**.** *Let $1 \leq p \leq \infty$, and let $x = (x_n)_{n\geq 1}$ be a p-quasimartingale. Then x can be uniquely decomposed into $x_n = y_n + z_n$ $(n \geq 1)$, where $y = (y_n)_{n\geq 1}$ is a martingale and $z = (z_n)_{n\geq 1}$ is a predictable p-quasimartingale with $z_1 = 0$.*

Proof. We define two sequences $y = (y_n)_{n\geq 1}$ and $z = (z_n)_{n\geq 1}$ by

$$y_n = \sum_{k=1}^{n}\big(dx_k - \mathcal{E}_{k-1}(dx_k)\big) \qquad \text{and} \qquad z_n = \sum_{k=1}^{n}\big(\mathcal{E}_{k-1}(dx_k)\big). \tag{2.2}$$

Then $x_n = y_n + z_n$ is the desired decomposition. The proof is similar to the proof of Lemma 2.2 in [2]. We omit the details. □

The space $F_p(\mathcal{M})$ defined in the following will play an important role later in the paper.

Definition 2.4. For $1 \leq p \leq \infty$, let $F_p(\mathcal{M})$ be the subspace of $l_p(L_p(\mathcal{M}))$ of all sequences $dx = (dx_n)_{n\geq 1}$ such that $x = (x_n)_{n\geq 1}$ is a predictable p-quasimartingale with $x_1 = 0$, equipped with the norm

$$\|dx\|_{F_p(\mathcal{M})} := \|dx\|_{l_p(L_p(\mathcal{M}))} = \Big(\sum_{n=1}^{\infty}\big\|\mathcal{E}_{n-1}(dx_n)\big\|_p^p\Big)^{\frac{1}{p}}.$$

3. Main results

Our first result in this section is concerned with the real interpolation between the spaces $\widehat{L}_p(\mathcal{M})$ of bounded $L_p(\mathcal{M})$-quasimartingales. Later, we use p' to denote the conjugate index of p for $1 \leq p \leq \infty$.

Theorem 3.1. *Let $1 < p_0, p_1 \leq \infty$, $1 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then*

$$\big(\widehat{L}_{p_0}(\mathcal{M}), \widehat{L}_{p_1}(\mathcal{M})\big)_{\theta,p} = \widehat{L}_p(\mathcal{M}) \quad \textit{(with equivalent norms).} \tag{3.1}$$

For the proof of Theorem 3.1, we will need the following lemmas.

Lemma 3.2 (see [1]). *Let (X_0, X_1) be a couple of Banach spaces such that $X_0 \cap X_1$ is dense in both X_0 and X_1. Assume that $1 \leq q < \infty$ and $0 < \theta < 1$. Then*

$$(X_0, X_1)^*_{\theta,q} = (X_0^*, X_1^*)_{\theta,q'} \quad \textit{(with equivalent norms).}$$

Lemma 3.3. *Let $1 \leq p < \infty$. Then $\widehat{L}_p(\mathcal{M})^* = \widehat{L}_{p'}(\mathcal{M})$ with equivalent norms. The duality is given by*

$$(x, u) = \tau(y_\infty v_\infty) + \sum_{n=1}^{\infty}\tau(dz_n\, dw_n), \quad x \in \widehat{L}_p(\mathcal{M}), u \in \widehat{L}_{p'}(\mathcal{M}),$$

where $x_n = y_n + z_n$ and $u_n = v_n + w_n$ are Doob's decompositions of x and u, respectively; y_∞ is the limit of $(y_n)_{n\geq 1}$ in $L_p(\mathcal{M})$; and v_∞ is the limit of $(v_n)_{n\geq 1}$ in $L_{p'}(\mathcal{M})$.

Proof. Let $u = (u_n)_{n\geq 1} \in \widehat{L}_{p'}(\mathcal{M})$. We define a linear functional on $\widehat{L}_p(\mathcal{M})$ by

$$\ell_u(x) = \tau(y_\infty v_\infty) + \sum_{n=1}^{\infty}\tau(dz_n\, dw_n), \quad x = (x_n)_{n\geq 1} \in \widehat{L}_p(\mathcal{M}),$$

where $x_n = y_n + z_n$ and $u_n = v_n + w_n$ are Doob's decompositions of x and u, respectively; y_∞ is the limit of $(y_n)_{n\geq 1}$ in $L_p(\mathcal{M})$; and v_∞ is the limit of $(v_n)_{n\geq 1}$ in $L_{p'}(\mathcal{M})$. Then, by Hölder's inequality, we have

$$\begin{aligned}\left|\ell_u(x)\right| &\leq \|y_\infty\|_p \|v_\infty\|_{p'} + \Big(\sum_{n=1}^\infty \|dz_n\|_p^p\Big)^{\frac{1}{p}} \Big(\sum_{n=1}^\infty \|dw_n\|_{p'}^{p'}\Big)^{\frac{1}{p'}} \\ &\leq \Big(\sup_n \|y_n\|_p + \Big(\sum_{n=1}^\infty \|dz_n\|_p^p\Big)^{\frac{1}{p}}\Big)\Big(\sup_n \|v_n\|_{p'} + \Big(\sum_{n=1}^\infty \|dw_n\|_{p'}^{p'}\Big)^{\frac{1}{p'}}\Big) \\ &= \|x\|_p \|u\|_{p'}.\end{aligned}$$

Thus $\ell_u(x)$ is continuous on $\widehat{L}_p(\mathcal{M})$ and $\|\ell_u\| \leq \|u\|_{p'}$.

We pass to the converse inclusion. Let $\ell \in \widehat{L}_p(\mathcal{M})^*$, and let ℓ_1 be the restriction of ℓ on $L_p(\mathcal{M})$. Then there exists an element $v \in L_{p'}(\mathcal{M})$ with $\|v\|_{p'} \leq \|\ell\|$ such that

$$\ell_1(a) = \tau(av), \quad a \in L_p(\mathcal{M}). \tag{3.2}$$

On the other hand, we define a functional on $F_p(\mathcal{M})$ by

$$\ell_2(db) = \ell(b), \quad db = (db_n)_{n\geq 1} \in F_p(\mathcal{M}).$$

Then ℓ_2 is a continuous linear functional on $F_p(\mathcal{M})$ and $\|\ell_2\| \leq \|\ell\|$ since

$$\left|\ell_2(db)\right| \leq \|\ell\| \|b\|_{\widehat{L}_p(\mathcal{M})} = \|\ell\| \|db\|_{l_p(L_p(\mathcal{M}))} = \|\ell\| \|db\|_{F_p(\mathcal{M})}.$$

By the Hahn–Banach theorem, ℓ_2 extends to a functional on $l_p(L_p(\mathcal{M}))$. Since $(l_p(L_p(\mathcal{M})))^* = l_{p'}(L_{p'}(\mathcal{M}))$, by the representation theorem there exists a sequence $w' = (w'_n)_{n\geq 1} \in l_{p'}(L_{p'}(\mathcal{M}))$ such that

$$\ell_2(s) = \sum_{n=1}^\infty \tau(w'_n s_n) \quad \big(s = (s_n)_{n\geq 1} \in l_p\big(L_p(\mathcal{M})\big)\big) \tag{3.3}$$

and $\|w'\|_{l_{p'}(L_{p'}(\mathcal{M}))} \leq \|\ell_2\|$. Set $w_1 = 0$ and $w_n = \sum_{k=1}^n \mathcal{E}_{k-1}(w'_k)$ $(n \geq 2)$. For any $db = (db_n)_{n\geq 1} \in F_p(\mathcal{M})$, noting that $db = (db_n)_{n\geq 1}$ is predictable, it follows from (3.3) that

$$\ell_2(db) = \sum_{n=1}^\infty \tau\big(\mathcal{E}_{n-1}(w'_n db_n)\big) = \sum_{n=1}^\infty \tau(dw_n\, db_n). \tag{3.4}$$

It is easy to see that $w = (w_n)_{n\geq 1}$ is a predictable p'-quasimartingale with $w_1 = 0$ and

$$\Big(\sum_{n=1}^\infty \big\|\mathcal{E}_{n-1}(dw_n)\big\|_{p'}^{p'}\Big)^{\frac{1}{p'}} \leq \Big(\sum_{n=1}^\infty \|w'_n\|_{p'}^{p'}\Big)^{\frac{1}{p'}} = \|w'\|_{l_{p'}(L_{p'}(\mathcal{M}))} \leq \|\ell_2\|.$$

Set $u_n = v_n + w_n$ $(n \geq 1)$, where $v_n = \mathcal{E}_n(v)$ $(n \geq 1)$. Then $u = (u_n)_{n\geq 1} \in \widehat{L}_{p'}(\mathcal{M})$ and

$$\|u\|_{\widehat{L}_{p'}(\mathcal{M})} = \|v\|_{p'} + \Big(\sum_{n=1}^\infty \big\|\mathcal{E}_{n-1}(dw_n)\big\|_{p'}^{p'}\Big)^{\frac{1}{p'}} \leq 2\|\ell\|.$$

For any $x = (x_n)_{n\geq 1} \in \widehat{L}_p(\mathcal{M})$, let $x_n = y_n + z_n$ $(n \geq 1)$ be its Doob's decomposition. Noting that $y = (y_n)_{n\geq 1}$ is a bounded $L_p(\mathcal{M})$-martingale and that $dz = (dz_n)_{n\geq 1} \in F_p(\mathcal{M})$, it follows from (3.2) and (3.4) that $\ell(x) = \ell(y) + \ell(z) = \tau(y_\infty v_\infty) + \sum_{n=1}^{\infty} \tau(dw_n\, dz_n)$. ☐

Our last lemma concerns the real interpolation of $F_p(\mathcal{M})$.

Lemma 3.4. *Let $1 \leq p_0, p_1 \leq \infty$, $1 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then*

$$\big(F_{p_0}(\mathcal{M}), F_{p_1}(\mathcal{M})\big)_{\theta,p} = F_p(\mathcal{M}) \quad \textit{(with equivalent norms).} \tag{3.5}$$

Proof. Notice that $F_p(\mathcal{M})$ is 1-complemented in $l_p(L_p(\mathcal{M}))$ via the projection

$$P : \begin{cases} l_p(L_p(\mathcal{M})) \longrightarrow F_p(\mathcal{M}), \\ (a_n)_{n\geq 1} \longrightarrow (\mathcal{E}_{n-1}(a_n))_{n\geq 1}. \end{cases}$$

The fact that $l_p(L_p(\mathcal{M}))$ forms an interpolation scale with respect to the real interpolation yields the required result. ☐

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1.

Case 1: $1 < p_0, p_1 < \infty$. Let $x \in (\widehat{L}_{p_0}(\mathcal{M}), \widehat{L}_{p_1}(\mathcal{M}))_{\theta,p}$, and let $x = x^0 + x^1$ be a decomposition with $x^0 \in \widehat{L}_{p_0}(\mathcal{M})$ and $x^1 \in \widehat{L}_{p_1}(\mathcal{M})$. Let $x_n^k = y_n^k + z_n^k$ $(n \geq 1)$ be the Doob's decomposition of x^k $(k = 0, 1)$. Then $y^k = (y_n^k)_{n\geq 1}$ is a bounded $L_{p_k}(\mathcal{M})$-martingale and $dz^k = (dz_n^k)_{n\geq 1} \in F_{p_k}(\mathcal{M})$. Let $y = y^0 + y^1$ and $z = z^0 + z^1$. Then

$$\begin{aligned} &K_t\big(y; L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M})\big) + K_t\big(dz; F_{p_0}(\mathcal{M}), F_{p_1}(\mathcal{M})\big) \\ &\quad \leq \|y^0\|_{L_{p_0}(\mathcal{M})} + t\|y^1\|_{L_{p_1}(\mathcal{M})} + \|dz^0\|_{F_{p_0}(\mathcal{M})} + t\|dz^1\|_{F_{p_1}(\mathcal{M})} \\ &\quad = \|x^0\|_{\widehat{L}_{p_0}(\mathcal{M})} + t\|x^1\|_{\widehat{L}_{p_1}(\mathcal{M})}. \end{aligned}$$

Taking the infimum over all decompositions of x, we get

$$K_t\big(y; L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M})\big) + K_t\big(dz; F_{p_0}(\mathcal{M}), F_{p_1}(\mathcal{M})\big) \leq K_t\big(x; \widehat{L}_{p_0}(\mathcal{M}), \widehat{L}_{p_1}(\mathcal{M})\big).$$

By the equality $\|x\|_{(X_0,X_1)_{\theta,p}} = (\int_0^\infty [t^{-\theta} K_t(x; X_0, X_1)]^p \frac{dt}{t})^{\frac{1}{p}}$, we get that

$$\|y\|_{(L_{p_0}(\mathcal{M}),L_{p_1}(\mathcal{M}))_{\theta,p}} + \|dz\|_{(F_{p_0}(\mathcal{M}),F_{p_1}(\mathcal{M}))_{\theta,p}} \leq 2^{1-\frac{1}{p}} \|x\|_{(\widehat{L}_{p_0}(\mathcal{M}),\widehat{L}_{p_1}(\mathcal{M}))_{\theta,p}}.$$

Noting that $(L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M}))_{\theta,p} = L_p(\mathcal{M})$ and by Lemma 3.4, we obtain that

$$\|y\|_p + \|dz\|_{F_p(\mathcal{M})} \leq 2^{1-\frac{1}{p}} \|x\|_{(\widehat{L}_{p_0}(\mathcal{M}),\widehat{L}_{p_1}(\mathcal{M}))_{\theta,p}}.$$

This means that $\|x\|_{\widehat{L}_p(\mathcal{M})} \leq 2^{1-\frac{1}{p}} \|x\|_{(\widehat{L}_{p_0}(\mathcal{M}),\widehat{L}_{p_1}(\mathcal{M}))_{\theta,p}}$ and that

$$\big(\widehat{L}_{p_0}(\mathcal{M}), \widehat{L}_{p_1}(\mathcal{M})\big)_{\theta,p} \subset \widehat{L}_p(\mathcal{M}). \tag{3.6}$$

Using (3.6) and Lemma 3.2, we have that

$$\widehat{L}_{p'}(\mathcal{M})^* \subset \big(\widehat{L}_{p_0'}(\mathcal{M}), \widehat{L}_{p_1'}(\mathcal{M})\big)^*_{\theta,p'} = \big(\widehat{L}_{p_0'}(\mathcal{M})^*, \widehat{L}_{p_1'}(\mathcal{M})^*\big)_{\theta,p}.$$

By Lemma 3.3, we have that

$$\widehat{L}_p(\mathcal{M}) \subset \big(\widehat{L}_{p_0}(\mathcal{M}), \widehat{L}_{p_1}(\mathcal{M})\big)_{\theta,p}. \tag{3.7}$$

Putting (3.6) and (3.7) together, we obtain that

$$\big(\widehat{L}_{p_0}(\mathcal{M}), \widehat{L}_{p_1}(\mathcal{M})\big)_{\theta,p} = \widehat{L}_p(\mathcal{M}).$$

Case 2: $1 < p_0 < p_1 = \infty$. Notice that (3.6) holds when $p_1 = 1$ or $p_1 = \infty$. Then, by Lemma 3.2,

$$\widehat{L}_{p'}(\mathcal{M})^* \subset \big(\widehat{L}_{p_0'}(\mathcal{M}), \widehat{L}_1(\mathcal{M})\big)^*_{\theta,p'} = \big(\widehat{L}_{p_0'}(\mathcal{M})^*, \widehat{L}_1(\mathcal{M})^*\big)_{\theta,p}.$$

Using Lemma 3.3, we have that

$$\widehat{L}_p(\mathcal{M}) \subset \big(\widehat{L}_{p_0}(\mathcal{M}), \widehat{L}_\infty(\mathcal{M})\big)_{\theta,p}.$$

Therefore,

$$\widehat{L}_p(\mathcal{M}) = \big(\widehat{L}_{p_0}(\mathcal{M}), \widehat{L}_\infty(\mathcal{M})\big)_{\theta,p}. \qquad \square$$

By the connection between real and complex interpolation, we obtain a result for complex interpolation.

Corollary 3.5. *Let $1 < p_0 < p_1 < \infty$, $0 < \eta < 1$, and $\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$. Then the following holds with equivalent norms:*

$$\big(\widehat{L}_{p_0}(\mathcal{M}), \widehat{L}_{p_1}(\mathcal{M})\big)_\eta = \widehat{L}_p(\mathcal{M}).$$

Proof. Take p_2 such that $1 < p_2 < p_0$, and let $\theta_0 = 1 - \frac{p_2}{p_0}$, $\theta_1 = 1 - \frac{p_2}{p_1}$, and $\theta = 1 - \frac{p_2}{p}$. Then $0 < \theta_0 < \theta_1 < 1$ and $\theta = (1-\eta)\theta_0 + \eta\theta_1$. By Theorem 4.7.2 of [1] on the connection between real and complex interpolation, we obtain

$$\Big(\big(\widehat{L}_{p_2}(\mathcal{M}), \widehat{L}_\infty(\mathcal{M})\big)_{\theta_0,p_0}, \big(\widehat{L}_{p_2}(\mathcal{M}), \widehat{L}_\infty(\mathcal{M})\big)_{\theta_1,p_1}\Big)_\eta = \big(\widehat{L}_{p_2}(\mathcal{M}), \widehat{L}_\infty(\mathcal{M})\big)_{\theta,p}.$$

Notice that $\frac{1-\theta_0}{p_2} = \frac{1}{p_0}$, $\frac{1-\theta_1}{p_2} = \frac{1}{p_1}$, and $\frac{1-\theta}{p_2} = \frac{1}{p}$. Using Theorem 3.1, we get that $(\widehat{L}_{p_0}(\mathcal{M}), \widehat{L}_{p_1}(\mathcal{M}))_\eta = \widehat{L}_p(\mathcal{M})$. $\square$

The second part of this section is concerned with real interpolation between the spaces $\widehat{\mathrm{BMO}}(\mathcal{M})$ and the Hardy spaces $\widehat{\mathcal{H}}_p(\mathcal{M})$ of noncommutative quasimartingales. First, we introduce the Hardy spaces of noncommutative quasimartingales.

Definition 3.6. Let $1 \le p < \infty$.

(1) The column Hardy space $\widehat{\mathcal{H}}_p^c(\mathcal{M})$ of noncommutative quasimartingales is defined as the space of all $L_p(\mathcal{M})$-quasimartingales $x = (x_n)_{n\ge 1}$ such that $(dx)_{n\ge1} \in L_p(\mathcal{M}; l_2^c)$, equipped with the norm

$$\|x\|_{\widehat{\mathcal{H}}_p^c(\mathcal{M})} = \Big\|\Big(\sum_{n=1}^\infty |dy_n|^2\Big)^{\frac12}\Big\|_p + \Big(\sum_{n=1}^\infty \big\|\mathcal{E}_{n-1}(dx_n)\big\|_p^p\Big)^{\frac1p},$$

where $y_n = \sum_{k=1}^n (dx_k - \mathcal{E}_{k-1}(dx_k))$. Similarly, the row space $\widehat{\mathcal{H}}_p^r(\mathcal{M})$ is defined as the space of all $L_p(\mathcal{M})$-quasimartingales $x = (x_n)_{n\ge1}$ such that $x^* \in \widehat{\mathcal{H}}_p^c(\mathcal{M})$, equipped with the norm $\|x\|_{\widehat{\mathcal{H}}_p^r(\mathcal{M})} = \|x^*\|_{\widehat{\mathcal{H}}_p^c(\mathcal{M})}$.

(2) The space $\widehat{\mathcal{H}}_p(\mathcal{M})$ is defined as the sum space $\widehat{\mathcal{H}}_p(\mathcal{M}) = \widehat{\mathcal{H}}_p^c(\mathcal{M}) + \widehat{\mathcal{H}}_p^r(\mathcal{M})$ for $1 \leq p < 2$ and the intersection space $\widehat{\mathcal{H}}_p(\mathcal{M}) = \widehat{\mathcal{H}}_p^c(\mathcal{M}) \cap \widehat{\mathcal{H}}_p^r(\mathcal{M})$ for $2 \leq p < \infty$.

Now we turn to the definition of the BMO space of quasimartingales $\widehat{\mathrm{BMO}}(\mathcal{M})$. Recall that the dual space of $\mathcal{H}_1(\mathcal{M})$ is $\mathrm{BMO}(\mathcal{M})$, which is defined in [4] as the intersection space

$$\mathrm{BMO}(\mathcal{M}) = \mathrm{BMO}^c(\mathcal{M}) \cap \mathrm{BMO}^r(\mathcal{M}),$$

where

$$\mathrm{BMO}^c(\mathcal{M}) = \Big\{x \in L_2(\mathcal{M}) : \|x\|_{\mathrm{BMO}^c(\mathcal{M})} = \sup_n \Big\|\mathcal{E}_n\big(\big|x - \mathcal{E}_{n-1}(x)\big|^2\big)\Big\|_\infty^{1/2}\Big\},$$
$$\mathrm{BMO}^r(\mathcal{M}) = \Big\{x \in L_2(\mathcal{M}) : \|x\|_{\mathrm{BMO}^r(\mathcal{M})} = \|x^*\|_{\mathrm{BMO}^c(\mathcal{M})}\Big\}.$$

This leads us to consider the spaces defined in the following.

Definition 3.7. We define $\widehat{\mathrm{BMO}}^c(\mathcal{M})$ as the space of all adaptable sequences $x = (x_n)_{n\geq 1}$ which can be decomposed as $x = y + z$ such that $y \in \mathrm{BMO}^c(\mathcal{M})$ and $dz \in F_\infty(\mathcal{M})$, equipped with the norm

$$\|x\|_{\widehat{\mathrm{BMO}}^c(\mathcal{M})} = \|y\|_{\mathrm{BMO}^c(\mathcal{M})} + \sup_n \|dz_n\|_\infty. \tag{3.8}$$

Similarly, the space $\widehat{\mathrm{BMO}}^r(\mathcal{M})$ is defined as the space of all adaptable sequences $x = (x_n)_{n\geq 1}$ such that $x^* = (x_n^*)_{n\geq 1} \in \widehat{\mathrm{BMO}}^c(\mathcal{M})$, equipped with the norm $\|x\|_{\widehat{\mathrm{BMO}}^r(\mathcal{M})} = \|x^*\|_{\widehat{\mathrm{BMO}}^c(\mathcal{M})}$. We define $\widehat{\mathrm{BMO}}(\mathcal{M})$ as the intersection space of $\widehat{\mathrm{BMO}}^c(\mathcal{M})$ and $\widehat{\mathrm{BMO}}^r(\mathcal{M})$.

We are ready to state the following result.

Theorem 3.8. *Let $1 < p < q < \infty$. Then, with equivalent norms,*

$$\big(\widehat{\mathrm{BMO}}^c(\mathcal{M}), \widehat{\mathcal{H}}_p^c(\mathcal{M})\big)_{\frac{p}{q},q} = \widehat{\mathcal{H}}_q^c(\mathcal{M})$$

and

$$\big(\widehat{\mathrm{BMO}}^r(\mathcal{M}), \widehat{\mathcal{H}}_p^r(\mathcal{M})\big)_{\frac{p}{q},q} = \widehat{\mathcal{H}}_q^r(\mathcal{M}).$$

The following lemma is a key step toward the proof of Theorem 3.8.

Lemma 3.9.

(1) *Let $1 < p < \infty$. Then $\widehat{\mathcal{H}}_p^c(\mathcal{M})^* = \widehat{\mathcal{H}}_{p'}^c(\mathcal{M})$ and $\widehat{\mathcal{H}}_p^r(\mathcal{M})^* = \widehat{\mathcal{H}}_{p'}^r(\mathcal{M})$ with equivalent norms.*

(2) $\widehat{\mathcal{H}}_1^c(\mathcal{M})^* = \widehat{\mathrm{BMO}}^c(\mathcal{M})$ *and* $\widehat{\mathcal{H}}_1^r(\mathcal{M})^* = \widehat{\mathrm{BMO}}^r(\mathcal{M})$ *with equivalent norms.*

Proof. (1) Let $u = (u_n)_{n\geq 1} \in \widehat{\mathcal{H}}_{p'}^c(\mathcal{M})$, and let $u_n = v_n + w_n$ $(n \geq 1)$ be its Doob's decomposition. Define a linear functional on $\widehat{\mathcal{H}}_p^c(\mathcal{M})$ by

$$\ell_u(x) = \sum_{n=1}^\infty \tau(dv_n^*\, dy_n) + \sum_{n=1}^\infty \tau(dw_n^*\, dz_n) \quad \big(x \in \widehat{\mathcal{H}}_p^c(\mathcal{M})\big),$$

where $x_n = y_n + z_n$ $(n \geq 1)$ is the Doob's decomposition of x. Notice that the series $\sum_n dv_n^* \, dy_n$ converges in $L_1(\mathcal{M})$ and that

$$\Big\|\sum_{n=1}^{\infty} dv_n^* \, dy_n\Big\|_1 \leq \Big\|\Big(\sum_{n=1}^{\infty} |dv_n|^2\Big)^{\frac{1}{2}}\Big\|_{p'}\Big\|\Big(\sum_{n} |dy_n|^2\Big)^{\frac{1}{2}}\Big\|_{p}.$$

It follows that the series $\sum_n \tau(dv_n^* \, dy_n)$ converges and that

$$\Big|\sum_{n=1}^{\infty} \tau(dv_n^* \, dy_n)\Big| \leq \Big\|\sum_{n=1}^{\infty} dv_n^* \, dy_n\Big\|_1 \leq \Big\|\Big(\sum_{n=1}^{\infty} |dv_n|^2\Big)^{\frac{1}{2}}\Big\|_{p'}\Big\|\Big(\sum_{n=1}^{\infty} |dy_n|^2\Big)^{\frac{1}{2}}\Big\|_{p}.$$

Then, by Hölder's inequality,

$$\begin{aligned}
\big|\ell_u(x)\big| &\leq \Big\|\Big(\sum_{n=1}^{\infty} |dv_n|^2\Big)^{\frac{1}{2}}\Big\|_{p'}\Big\|\Big(\sum_{n=1}^{\infty} |dy_n|^2\Big)^{\frac{1}{2}}\Big\|_{p} + \Big(\sum_{n=1}^{\infty} \|dw_n\|_{p'}^{p'}\Big)^{\frac{1}{p'}}\Big(\sum_{n=1}^{\infty} \|dz_n\|_{p}^{p}\Big)^{\frac{1}{p}} \\
&\leq \bigg(\Big\|\Big(\sum_{n=1}^{\infty} |dv_n|^2\Big)^{\frac{1}{2}}\Big\|_{p'} + \Big(\sum_{n=1}^{\infty} \|dw_n\|_{p'}^{p'}\Big)^{\frac{1}{p'}}\bigg) \\
&\quad \times \bigg(\Big\|\Big(\sum_{n=1}^{\infty} |dy_n|^2\Big)^{\frac{1}{2}}\Big\|_{p} + \Big(\sum_{n=1}^{\infty} \|dz_n\|_{p}^{p}\Big)^{\frac{1}{p}}\bigg) \\
&= \|u\|_{\widehat{\mathcal{H}}_{p'}^c(\mathcal{M})}\|x\|_{\widehat{\mathcal{H}}_{p}^c(\mathcal{M})}.
\end{aligned}$$

Thus ℓ_u is continuous on $\widehat{\mathcal{H}}_p^c(\mathcal{M})$ and $\|\ell_u\| \leq \|u\|_{\widehat{\mathcal{H}}_{p'}^c(\mathcal{M})}$.

We pass to the converse inclusion. Let $\ell \in \widehat{\mathcal{H}}_p^c(\mathcal{M})^*$. First, we restrict ℓ on the subspace $\mathcal{H}_p^c(\mathcal{M})$. Since $\mathcal{H}_p^c(\mathcal{M})^* = \mathcal{H}_{p'}^c(\mathcal{M})$, there exists a sequence $v = (v_n)_{n\geq 1} \in \mathcal{H}_{p'}^c(\mathcal{M})$ such that

$$\ell(a) = \sum_{n=1}^{\infty} \tau(dv_n^* \, da_n) \quad \big(a = (a_n)_{n\geq 1} \in \mathcal{H}_p^c(\mathcal{M})\big). \tag{3.9}$$

Imitating the proof of Theorem 3.1, there exists a predictable quasimartingale $w = (w_n)_{n\geq 1}$ in $\widehat{L}_{p'}(\mathcal{M})$ with $w_1 = 0$ such that

$$\ell(b) = \sum_{n=1}^{\infty} \tau(dw_n^* \, db_n) \tag{3.10}$$

for any $db = (db_n)_{n\geq 1} \in F_p(\mathcal{M})$ and $(\sum_{n=1}^{\infty} \|dw_n\|_{p'}^{p'})^{\frac{1}{p'}} \leq \|\ell\|$. Set $u_n = v_n + w_n$ $(n \geq 1)$. Then $u = (u_n)_{n\geq 1} \in \widehat{\mathcal{H}}_{p'}^c(\mathcal{M})$ and

$$\|u\|_{\widehat{\mathcal{H}}_{p'}^c(\mathcal{M})} = \|v\|_{\mathcal{H}_{p'}^c(\mathcal{M})} + \Big(\sum_{n=1}^{\infty} \|dw_n\|_{p'}^{p'}\Big)^{\frac{1}{p'}} \leq 2\|\ell\|.$$

For any $x = (x_n)_{n\geq 1} \in \widehat{\mathcal{H}}_p^c(\mathcal{M})$, let $x_n = y_n + z_n$ $(n \geq 1)$ be the Doob's decomposition of x. It follows from (3.9) and (3.10) that

$$\ell(x) = \ell(y) + \ell(z) = \sum_{n=1}^{\infty} \tau(dv_n^* \, dy_n) + \sum_{n=1}^{\infty} \tau(dw_n^* \, dz_n).$$

Therefore, this proves that $\widehat{\mathcal{H}}_p^c(\mathcal{M})^* = \widehat{\mathcal{H}}_{p'}^c(\mathcal{M})$. Passing to adjoint, we obtain the identity $\widehat{\mathcal{H}}_p^r(\mathcal{M})^* = \widehat{\mathcal{H}}_{p'}^r(\mathcal{M})$.

(2) Let $u = (u_n)_{n\geq 1} \in \widehat{\mathrm{BMO}}^c(\mathcal{M})$, and let $u_n = v_n + w_n$ $(n \geq 1)$ be its decomposition. Define a linear functional on $\widehat{\mathcal{H}}_1^c(\mathcal{M})$ by

$$\ell_u(x) = \sum_{n=1}^{\infty} \tau(dv_n^* \, dy_n) + \sum_{n=1}^{\infty} \tau(dw_n^* \, dz_n) \quad \big(x \in \widehat{\mathcal{H}}_1^c(\mathcal{M})\big),$$

where $x_n = y_n + z_n$ $(n \geq 1)$ is the Doob's decomposition of x. Notice that

$$\Big|\sum_{n=1}^{\infty} \tau(dv_n^* \, dy_n)\Big| \leq \sqrt{2}\|y\|_{\mathcal{H}_1^c(\mathcal{M})}\|v\|_{\mathrm{BMO}^c(\mathcal{M})}$$

(see [4, Appendix]). Putting the preceding inequalities together, we obtain that

$$\begin{aligned}
\big|\ell_u(x)\big| &\leq \sqrt{2}\|y\|_{\mathcal{H}_1^c(\mathcal{M})}\|v\|_{\mathrm{BMO}^c(\mathcal{M})} + \sup_n \|dw_n\|_\infty \sum_{n=1}^{\infty} \|dz_n\|_1 \\
&\leq \sqrt{2}\Big(\|y\|_{\mathcal{H}_1^c(\mathcal{M})} + \sum_{n=1}^{\infty} \|dz_n\|_1\Big)\big(\|v\|_{\mathrm{BMO}^c(\mathcal{M})} + \sup_n \|dw_n\|_\infty\big) \\
&= \sqrt{2}\|x\|_{\widehat{\mathcal{H}}_1^c(\mathcal{M})}\|u\|_{\widehat{\mathrm{BMO}}^c(\mathcal{M})}.
\end{aligned}$$

Thus $\ell_u \in \widehat{\mathcal{H}}_1^c(\mathcal{M})^*$ and $\|\ell_u\| \leq \sqrt{2}\|u\|_{\widehat{\mathrm{BMO}}^c(\mathcal{M})}$.

We pass to the converse inclusion. Let $\ell \in \widehat{\mathcal{H}}_1^c(\mathcal{M})^*$. First, we restrict ℓ on the subspace $\mathcal{H}_1^c(\mathcal{M})$. Since $\mathcal{H}_1^c(\mathcal{M})^* = \mathrm{BMO}^c(\mathcal{M})$, there exists a martingale $v = (v_n)_{n\geq 1} \in \mathrm{BMO}^c(\mathcal{M})$ such that

$$\ell(s) = \sum_{n=1}^{\infty} \tau(dv_n^* \, ds_n) \quad \big(s = (s_n)_{n\geq 1} \in \mathcal{H}_1^c(\mathcal{M})\big) \tag{3.11}$$

and $\|v\|_{\mathrm{BMO}^c(\mathcal{M})} \leq \|\ell\|$. Similarly to the proof of (i), there exists $u = (u_n)_{n\geq 1} \in \widehat{\mathrm{BMO}}^c(\mathcal{M})$ and $\|u\|_{\widehat{\mathrm{BMO}}^c(\mathcal{M})} \leq 2\|\ell\|$ such that

$$\ell(x) = \sum_{n=1}^{\infty} \tau(dv_n^* \, dy_n) + \sum_{n=1}^{\infty} \tau(dw_n^* \, dz_n) \quad \big(x \in \widehat{\mathcal{H}}_1^c(\mathcal{M})\big).$$

Therefore, this proves that $\widetilde{\mathcal{H}}_1^c(\mathcal{M})^* = \widehat{\mathrm{BMO}}^c(\mathcal{M})$. Passing to adjoint, we obtain the identity $\widetilde{\mathcal{H}}_1^r(\mathcal{M})^* = \widehat{\mathrm{BMO}}^r(\mathcal{M})$. □

Proof of Theorem 3.8. Let $x \in (\widehat{\mathrm{BMO}}^c(\mathcal{M}), \widehat{\mathcal{H}}_p^c(\mathcal{M}))_{\frac{p}{q},q}$, and let $x = x^0 + x^1$ be a decomposition with $x^0 \in \widehat{\mathrm{BMO}}^c(\mathcal{M})$ and $x^1 \in \widehat{\mathcal{H}}_p^c(\mathcal{M})$. Let $x_n^0 = y_n^0 + z_n^0$ $(n \geq 1)$ be the decomposition as in (3.8). Then $y^0 = (y_n^0)_{n\geq 1} \in \mathrm{BMO}^c(\mathcal{M})$ and $dz^0 = (dz_n^0)_{n\geq 1} \in F_\infty(\mathcal{M})$. Let $x_n^1 = y_n^1 + z_n^1$ $(n \geq 1)$ be the Doob's decomposition of x^1. Then $y^1=(y^1_n)_{n\geq 1}\in \mathcal{H}^c_p(\mathcal{M})$ and $dz^1=(dz^1_n)_{n\geq 1}\in F_p(\mathcal{M})$. Let $y=y^0+y^1$ and $z=z^0+z^1$. Then

$$\begin{aligned}&K_t\big(y;\mathrm{BMO}^c(\mathcal{M}),\mathcal{H}^c_p(\mathcal{M})\big)+K_t\big(dz;F_\infty(\mathcal{M}),F_p(\mathcal{M})\big)\\&\quad\leq \|y^0\|_{\mathrm{BMO}^c(\mathcal{M})}+t\|y^1\|_{\mathcal{H}^c_p(\mathcal{M})}+\|dz^0\|_{F_\infty(\mathcal{M})}+t\|dz^1\|_{F_p(\mathcal{M})}\\&\quad=\|x^0\|_{\widehat{\mathrm{BMO}}^c(\mathcal{M})}+t\|x^1\|_{\widehat{\mathcal{H}}^c_p(\mathcal{M})}.\end{aligned}$$

Taking the infimum over all decompositions of x, we get

$$\begin{aligned}&K_t\big(y;\mathrm{BMO}^c(\mathcal{M}),\mathcal{H}^c_p(\mathcal{M})\big)+K_t\big(dz;F_\infty(\mathcal{M}),F_p(\mathcal{M})\big)\\&\quad\leq K_t\big(x;\widehat{\mathrm{BMO}}^c(\mathcal{M}),\widehat{\mathcal{H}}^c_p(\mathcal{M})\big).\end{aligned}$$

By the definition of $\|x\|_{(A_0,A_1)_{\frac{p}{q},q}}$, we get that

$$\|y\|_{(\mathrm{BMO}^c(\mathcal{M}),\mathcal{H}^c_p(\mathcal{M}))_{\frac{p}{q},q}}+\|dz\|_{(F_\infty(\mathcal{M}),F_p(\mathcal{M}))_{\frac{p}{q},q}\leq 2^{1-\frac{1}{p}}\|x\|_{(\widehat{\mathrm{BMO}}^c(\mathcal{M}),\widehat{\mathcal{H}}^c_p(\mathcal{M}))_{\frac{p}{q},q}}}.$$

By the equality $(\mathrm{BMO}^c(\mathcal{M}),\mathcal{H}^c_p(\mathcal{M}))_{\frac{p}{q},q}=\mathcal{H}^c_q(\mathcal{M})$ and Lemma 3.4, we obtain

$$\|y\|_{\mathcal{H}^c_q(\mathcal{M})}+\|dz\|_{F_q(\mathcal{M})}\leq 2^{1-\frac{1}{p}}\|x\|_{(\widehat{\mathrm{BMO}}^c(\mathcal{M}),\widehat{\mathcal{H}}^c_p(\mathcal{M}))_{\frac{p}{q},q}}.$$

This means that $\|x\|_{\widehat{\mathcal{H}}^c_q(\mathcal{M})}\leq 2^{1-\frac{1}{p}}\|x\|_{(\widehat{\mathrm{BMO}}^c(\mathcal{M}),\widehat{\mathcal{H}}^c_p(\mathcal{M}))_{\frac{p}{q},q}}$ and

$$\big(\widehat{\mathrm{BMO}}^c(\mathcal{M}),\widehat{\mathcal{H}}^c_p(\mathcal{M})\big)_{\frac{p}{q},q}\subset \widehat{\mathcal{H}}^c_q(\mathcal{M}). \tag{3.12}$$

Replacing $\widehat{L}_p(\mathcal{M})$ (resp., $\widehat{L}_{p_k}(\mathcal{M})$ $(k=0,1)$) with $\widehat{\mathcal{H}}^c_p(\mathcal{M})$ (resp., $\widehat{\mathcal{H}}_{p_k}(\mathcal{M})$ $(k=0,1)$) and $L_p(\mathcal{M})$ (resp., $\widehat{L}_{p_k}(\mathcal{M})$ $(k=0,1)$) with $\mathcal{H}^c_p(\mathcal{M})$ (resp., $\widehat{\mathcal{H}}_{p_k}(\mathcal{M})$ $(k=0,1)$) in the proof of (3.6), we have that, for $1\leq p_0,p_1<\infty$, and $1<\theta<1$,

$$\big(\widehat{\mathcal{H}}^c_{p_0}(\mathcal{M}),\widehat{\mathcal{H}}^c_{p_1}(\mathcal{M})\big)_{\theta,p}\subset \widehat{\mathcal{H}}^c_p(\mathcal{M}),$$

where $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$. Then, by Lemma 3.2,

$$\widehat{\mathcal{H}}^c_{q'}(\mathcal{M})^*\subset \big(\widehat{\mathcal{H}}^c_1(\mathcal{M}),\widehat{\mathcal{H}}^c_{p'}(\mathcal{M})\big)^*_{\frac{p}{q},q'}=\big(\widehat{\mathcal{H}}^c_1(\mathcal{M})^*,\widehat{\mathcal{H}}^c_{p'}(\mathcal{M})^*\big)_{\frac{p}{q},q}.$$

By Lemma 3.9, we have that

$$\widehat{\mathcal{H}}^c_q(\mathcal{M})\subset \big(\widehat{\mathrm{BMO}}^c(\mathcal{M}),\widehat{\mathcal{H}}^c_p(\mathcal{M})\big)_{\frac{p}{q},q}. \tag{3.13}$$

Putting (3.12) and (3.13) together, we obtain

$$\big(\widehat{\mathrm{BMO}}^c(\mathcal{M}),\widehat{\mathcal{H}}^c_p(\mathcal{M})\big)_{\frac{p}{q},q}=\widehat{\mathcal{H}}^c_q(\mathcal{M}).$$

Similarly, we have that

$$\big(\widehat{\mathrm{BMO}}^r(\mathcal{M}),\widehat{\mathcal{H}}^r_p(\mathcal{M})\big)_{\frac{p}{q},q}=\widehat{\mathcal{H}}^r_q(\mathcal{M}). \qquad \square$$

Lemma 3.10. *Let $1 < p < \infty$. Then there exist two positive constants α_p and β_p depending only on p such that, for each quasimartingale x, it holds that*

$$\alpha_p{}^{-1}\|x\|_p \leq \|x\|_{\widehat{\mathcal{H}}_p(\mathcal{M})} \leq \beta_p\|x\|_p.$$

Proof. Let x be a quasimartingale, and let $x_n = y_n + z_n$ $(n \geq 1)$ be the Doob's decomposition of x. Then $\|x\|_p = \|y\|_p + (\sum_{n=1}^{\infty} \|dz_n\|_p^p)^{\frac{1}{p}}$ and $\|x\|_{\widehat{\mathcal{H}}_p(\mathcal{M})} = \|y\|_{\mathcal{H}_p(\mathcal{M})} + (\sum_{n=1}^{\infty} \|dz_n\|_p^p)^{\frac{1}{p}}$. The desired inequality follows from the Burkholder–Gundy inequalities for noncommutative martingales. □

Corollary 3.11. *Let $1 < p_0, p_1 < \infty$, $1 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then, with equivalent norms,*

$$\big(\widehat{\mathcal{H}}_{p_0}(\mathcal{M}), \widehat{\mathcal{H}}_{p_1}(\mathcal{M})\big)_{\theta,p} = \widehat{\mathcal{H}}_p(\mathcal{M}).$$

Proof. Since $\widehat{L}_p(\mathcal{M}) = \widehat{\mathcal{H}}_p(\mathcal{M})$ by Lemma 3.9, the desired result comes from (3.1). □

Corollary 3.12. *Let $2 \leq p < q < \infty$. Then, with equivalent norms,*

$$\big(\widehat{\mathrm{BMO}}(\mathcal{M}), \widehat{L}_p(\mathcal{M})\big)_{\frac{p}{q},q} = \widehat{L}_q(\mathcal{M}).$$

Proof. Using Lemma 3.10 and Theorem 3.8, we obtain, for $2 \leq p < q < \infty$,

$$\begin{aligned}\big(\widehat{\mathrm{BMO}}(\mathcal{M}), \widehat{L}_p(\mathcal{M})\big)_{\frac{p}{q},q} &= \big(\widehat{\mathrm{BMO}}(\mathcal{M}), \widehat{\mathcal{H}}_p(\mathcal{M})\big)_{\frac{p}{q},q} \subset \widehat{\mathcal{H}}_q^c(\mathcal{M}) \cap \widehat{\mathcal{H}}_q^r(\mathcal{M})\\ &= \widehat{L}_q(\mathcal{M}).\end{aligned}$$

By Theorem 3.1, we get that

$$\widehat{L}_q(\mathcal{M}) = \big(\widehat{L}_\infty(\mathcal{M}), \widehat{L}_p(\mathcal{M})\big)_{\frac{p}{q},q} \subset \big(\widehat{\mathrm{BMO}}(\mathcal{M}), \widehat{L}_p(\mathcal{M})\big)_{\frac{p}{q},q}.$$ □

Acknowledgment. This research is supported by the National Science Foundation of China (NSFC) (grant numbers 11271293 and 11471251).

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School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.
E-mail address: congbianm@whu.edu.cn; ylhou323@whu.edu.cn