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Research Article

Fractional Integral and Derivative Formulas by Using Marichev-Saigo-Maeda Operators Involving the S-Function

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We establish fractional integral and derivative formulas by using Marichev-Saigo-Maeda operators involving the S-function. The results are expressed in terms of the generalized Gauss hypergeometric functions. Corresponding assertions in terms of Saigo, Erdélyi-Kober, Riemann-Liouville, and Weyl type of fractional integrals and derivatives are presented. Also we develop their composition formula by applying the Beta and Laplace transforms. Further, we point out also their relevance.

1. Introduction and Preliminaries

In recent times, the fractional calculus is the most fast growing subject of mathematical analysis. It is concern with applied mathematics that deals with integrals and derivatives of arbitrary orders. The fractional calculus operator linking diverse special functions has found substantial significance and applications in a variety of subfields of applicable mathematical analysis. Numerous applications of fractional calculus can be found in astrophysics, turbulence, nonlinear biological systems, fluid dynamics, stochastic dynamical system, plasma physics and nonlinear control theory, image processing, and quantum mechanics. Since last four decades, a number of workers like [1–9] so forth have studied, in depth, the properties, applications, and diverse extensions of a range of operators of fractional calculus. A comprehensive account of generalized fractional calculus operators along with their properties and applications can be found in [10–13] and also the research monographs [14, 15] and so forth.

On relation of success of the Saigo operators [16, 17], in their study on various function spaces and their application in the integral equation and differential equations, Saigo and Maeda [18] introduced the following generalized fractional integral and differential operators of any complex order with Appell function $F_3(.)$ in the kernel which is extension of Marichev [19], as follows.

Let $\mu, \mu', \nu, \nu', \delta \in \mathbb{C}$ and $x > 0$, then the generalized fractional calculus formulas (the Marichev-Saigo-Maeda operators) involving the Appell function or Horn's F_3-function are defined by the following equations:

$$\left(I_{0,x}^{\mu,\mu',\nu,\nu',\delta} f\right)(x) = \frac{x^{-\mu}}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1} \cdot t^{-\mu'} F_3\left(\mu, \mu', \nu, \nu', \delta; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, dt, \tag{1}$$

$$= \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{\mu,\mu',\nu+k,\nu',\delta+k} f\right)(x), \quad (\Re(\delta) \le 0;\ k = [-\Re(\delta)+1]); \tag{2}$$

$$\left(I_{x,\infty}^{\mu,\mu',\nu,\nu',\delta} f\right)(x) = \frac{x^{-\mu'}}{\Gamma(\delta)} \int_x^\infty (t-x)^{\delta-1} \cdot t^{-\mu} F_3\left(\mu, \mu', \nu, \nu', \delta; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\, dt, \tag{3}$$

$$= \left(-\frac{d}{dx}\right)^k \left(I_{x,\infty}^{\mu,\mu',\nu,\nu'+k,\delta+k} f\right)(x), \quad (\Re(\delta) \le 0;\ k = [-\Re(\delta)+1]), \tag{4}$$

and

$$\left(D_{0,x}^{\mu,\mu',\nu,\nu',\delta} f\right)(x) = \left(I_{0,x}^{-\mu',-\mu,-\nu'+k,-\nu,-\delta+k} f\right)(x) \tag{5}$$

$$= \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{-\mu',-\mu,-\nu'+k,-\nu,-\delta+k} f\right)(x), \quad (\Re(\delta) > 0;\ k = [\Re(\delta) + 1]); \tag{6}$$

$$\left(D_{x,\infty}^{\mu,\mu',\nu,\nu',\delta} f\right)(x) = \left(I_{x,\infty}^{-\mu',-\mu,-\nu',-\nu,-\delta} f\right)(x) \tag{7}$$

$$= \left(-\frac{d}{dx}\right)^k \left(I_{x,\infty}^{-\mu',-\mu,-\nu',-\nu+k,-\delta+k} f\right)(x), \quad (\Re(\delta) > 0;\ k = [\Re(\delta) + 1]). \tag{8}$$

In (1) and (3), $F_3(.)$ denotes Appell function [20] in two variables defined as

$$F_3\left(\mu, \mu', \nu, \nu', \delta; 1 - \frac{x}{t}, 1 - \frac{t}{x}\right) = \sum_{m.n=0}^{\infty} \frac{(\mu)_m (\mu')_n (\nu)_m (\nu')_n}{(\delta)_{m+n}} \frac{x^m}{m!} \frac{x^n}{n!}, \quad (\max\{|x|, |y|\} < 1). \tag{9}$$

Remark. The Appell function defined in above equation reduces to Gauss hypergeometric function ${}_2F_1$ as given in the following relations:

$$F_3(\mu, \delta - \mu, \nu, \delta - \nu, \delta; x, y) = {}_2F_1(\mu, \nu; \delta; x + y - xy), \tag{10}$$

and

$$F_3\left(\mu, 0, \nu, \nu', \delta; x, y\right) = {}_2F_1(\mu, \nu; \delta; x), \tag{11}$$

and

$$F_3\left(0, \mu', \nu, \nu', \delta; x, y\right) = {}_2F_1\left(\mu', \nu'; \delta; y\right). \tag{12}$$

In view of the above reduction formula as given in (10), the general fractional calculus operators reduce to the Saigo operators [16] defined as follows:

$$\left(I_{0,x}^{\mu,\nu,\delta} f\right)(x) = \frac{x^{-\mu-\nu}}{\Gamma(\delta)} \cdot \int_0^x (x - t)^{\mu-1} {}_2F_1\left(\mu + \nu, -\delta; \mu; 1 - \frac{x}{t}\right) f(t)\, dt, \tag{13}$$

$$= \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{\mu+k,\nu-k,\delta-k} f\right)(x), \quad \Re(\mu) \le 0;\ k = [\Re(-\mu) + 1]; \tag{14}$$

$$\left(I_{x,\infty}^{\mu,\nu,\delta} f\right)(x) = \frac{1}{\Gamma(\delta)} \cdot \int_x^\infty (t - x)^{\mu-1} t^{-\mu-\nu} {}_2F_1\left(\mu + \nu, -\delta; \mu; 1 - \frac{x}{t}\right) f(t)\, dt, \tag{15}$$

$$= \left(-\frac{d}{dx}\right)^k \left(I_{x,\infty}^{\mu-k,\nu-k,\delta} f\right)(x), \quad \Re(\mu) \le 0;\ k = [\Re(-\mu) + 1]; \tag{16}$$

and

$$\left(D_{0,x}^{\mu,\nu,\delta} f\right)(x) = \left(I_{0,x}^{-\mu,-\nu,\mu+\delta} f\right)(x) = \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{-\mu+k,-\nu-k,\mu+\delta-k} f\right)(x) \tag{17}$$

$$\left(D_{x,\infty}^{\mu,\nu,\delta} f\right)(x) = \left(I_{x,\infty}^{-\mu,-\nu,\delta+\mu} f\right)(x) = \left(-\frac{d}{dx}\right)^k \left(I_{x,\infty}^{-\mu+k,-\nu-k,\delta+\mu} f\right)(x). \tag{18}$$

where ${}_2F_1(.)$, a special case of the generalized hypergeometric function, is the Gauss hypergeometric function and the function $f(t)$ is so constrained that the integrals in (13) and (15) converge.

If we take $\nu = 0$ in (13), (15), (17), and (18), we obtain the Erdélyi-Kober fractional integral and derivative operators [11, 21], defined as follows:

$$\left(I_{0,x}^{\mu,\delta} f\right)(x) = \frac{x^{-\mu-\delta}}{\Gamma(\mu)} \int_0^x (x - t)^{\mu-1} t^{\mu-1} f(t)\, dt, \quad \Re(\mu) > 0, \tag{19}$$

$$\left(I_{x,\infty}^{\mu,\delta} f\right)(x) = \frac{x^{\delta}}{\Gamma(\mu)} \int_x^\infty (t - x)^{\mu-1} t^{-\mu-\delta} f(t)\, dt, \quad \Re(\mu) > 0, \tag{20}$$

and

$$\left(D_{0,x}^{\mu,\delta} f\right)(x) = \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{-\mu+k,-\mu,\mu+\delta-k} f\right)(x), \quad \Re(\mu) > 0;\ k = [\Re(\mu) + 1]. \tag{21}$$

$$\left(D_{x,\infty}^{\mu,\delta} f\right)(x) = (-1)^k \left(\frac{d}{dx}\right)^k \left(I_{x,\infty}^{-\mu+k,-\mu,\mu+\delta} f\right)(x), \quad \Re(\mu) > 0;\ k = [\Re(\mu) + 1]. \tag{22}$$

When $\nu = -\mu$, then operators in (13), (15), (17), and (18) give the Riemann-Liouville and the Weyl fractional integral and derivative operators [11, 22] are defined as follows:

$$\left(I_{0,x}^{\mu} f\right)(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x - t)^{\mu-1} f(t)\, dt, \quad \Re(\mu) > 0, \tag{23}$$

$$\left(I_{x,\infty}^{\mu} f\right)(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty (t - x)^{\mu-1} f(t)\, dt, \quad \Re(\mu) > 0, \tag{24}$$

and

$$\left(D_{0,x}^{\mu}f\right)(x)=\left(\frac{d}{dx}\right)^{k}\left(I_{o,x}^{-\mu+k}f\right)(x),\qquad \Re(\mu)>0;\ k=\left[\Re(\mu)+1\right]. \tag{25}$$

$$\left(D_{x,\infty}^{\mu}f\right)(x)=(-1)^{k}\left(\frac{d}{dx}\right)^{k}\left(I_{x,\infty}^{-\mu+k}f\right)(x),\qquad \Re(\mu)>0;\ k=\left[\Re(\mu)+1\right]. \tag{26}$$

Power function formulas of the above discussed fractional operators are required for our present study as given in the following lemmas [16, 18, 23].

Lemma 1. *Let $\mu,\mu',\nu,\nu',\delta,\gamma\in\mathbb{C}$, and $x>0$ be such that $\Re(\delta)>0$, then the following formulas hold true:*

$$\left(I_{0,x}^{\mu,\mu',\nu,\nu',\delta}t^{\gamma-1}\right)(x)=\frac{\Gamma(\gamma)\Gamma\left(\gamma+\delta-\mu-\mu'-\nu\right)\Gamma\left(\gamma+\nu'-\mu'\right)}{\Gamma(\gamma+\nu')\Gamma(\gamma+\delta-\mu-\mu')\Gamma(\gamma+\delta-\mu'-\nu)}x^{\gamma+\delta-\mu-\mu'-1}, \tag{27}$$

$$\Re(\gamma)>\max\left\{0,\Re\left(\mu+\mu'+\nu-\delta\right),\Re\left(\mu'-\nu'\right)\right\}.$$

$$\left(I_{x,\infty}^{\mu,\mu',\nu,\nu',\delta}t^{\gamma-1}\right)(x)=\frac{\Gamma(1-\gamma-\nu)\Gamma\left(1-\gamma-\delta+\mu+\mu'\right)\Gamma\left(1-\gamma-\delta+\mu+\nu'\right)}{\Gamma(1-\gamma)\Gamma(1-\gamma-\delta+\mu+\mu'+\nu')\Gamma(1-\gamma+\mu-\nu)}\times x^{\gamma+\delta-\mu-\mu'-1}, \tag{28}$$

$$\Re(\gamma)<1+\min\left\{\Re(-\nu),\Re\left(\mu+\mu'-\delta\right),\Re\left(\mu+\nu'-\delta\right)\right\},$$

and

$$\left(D_{0,x}^{\mu,\mu',\nu,\nu',\delta}t^{\gamma-1}\right)(x)=\frac{\Gamma(\gamma)\Gamma\left(\gamma-\delta+\mu+\mu'+\nu'\right)\Gamma\left(\gamma-\nu+\mu\right)}{\Gamma(\gamma-\nu)\Gamma(\gamma-\delta+\mu+\mu')\Gamma(\gamma-\delta+\mu+\nu')}x^{\gamma-\delta+\mu+\mu'-1}, \tag{29}$$

$$\Re(\gamma)>\max\left\{0,\Re\left(\delta-\mu-\mu'-\nu'\right),\ \Re(\nu-\mu)\right\}.$$

$$\left(D_{x,\infty}^{\mu,\mu',\nu,\nu',\delta}t^{\gamma-1}\right)(x)=\frac{\Gamma\left(1-\gamma+\nu'\right)\Gamma\left(1-\gamma+\delta-\mu-\mu'\right)\Gamma\left(1-\gamma+\delta-\mu'-\nu\right)}{\Gamma(1-\gamma)\Gamma(1-\gamma+\delta-\mu-\mu'-\nu)\Gamma(1-\gamma-\mu'+\nu')}\times x^{\gamma-\delta+\mu+\mu'-1}, \tag{30}$$

$$\Re(\gamma)<1+\min\left\{\Re\left(\nu'\right),\Re\left(\delta-\mu-\mu'\right),\Re\left(\delta-\mu'-\nu\right)\right\}.$$

Lemma 2. *Let $\mu,\nu,\delta,\gamma\in\mathbb{C}$, and $x>0$ be such that $\Re(\mu)>0$, then the following formulas hold true:*

$$\left(I_{0,x}^{\mu,\nu,\delta}t^{\gamma-1}\right)(x)=\frac{\Gamma(\gamma)\Gamma(\gamma+\delta-\nu)}{\Gamma(\gamma-\nu)\Gamma(\gamma+\delta+\mu)}x^{\gamma-\nu-1},\qquad \Re(\gamma)>\max\{0,\Re(\nu-\delta)\}. \tag{31}$$

$$\left(I_{x,\infty}^{\mu,\nu,\delta}t^{\gamma-1}\right)(x)=\frac{\Gamma(1-\gamma+\nu)\Gamma(1-\gamma+\delta)}{\Gamma(1-\gamma)\Gamma(1-\gamma+\delta+\mu+\nu)}x^{\gamma-\nu-1},\qquad \Re(\gamma)<1+\min\{\Re(\nu),\Re(\delta)\}, \tag{32}$$

and

$$\left(D_{0,x}^{\mu,\nu,\delta}t^{\gamma-1}\right)(x)=\frac{\Gamma(\gamma)\Gamma(\gamma+\delta+\mu+\nu)}{\Gamma(\gamma+\delta)\Gamma(\gamma+\nu)}x^{\gamma+\nu-1},\qquad \Re(\gamma)>-\min\{0,\Re(\mu+\nu+\delta)\}. \tag{33}$$

$$\left(D_{x,\infty}^{\mu,\nu,\delta}t^{\gamma-1}\right)(x)=\frac{\Gamma(1-\gamma-\nu)\Gamma(1-\gamma+\mu+\delta)}{\Gamma(1-\gamma)\Gamma(1-\gamma+\delta-\nu)}x^{\gamma+\nu-1},\qquad \Re(\gamma)<1+\min\{\Re(-\nu-n),\Re(\delta+\mu)\},\ n=\Re(\mu)+1. \tag{34}$$

Lemma 3. *Let $\mu,\delta,\gamma\in\mathbb{C}$, and $x>0$ be such that $\Re(\mu)>0$, then the following formulas hold true:*

$$\left(I_{0,x}^{\mu,\delta}t^{\gamma-1}\right)(x)=\frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma+\delta+\mu)}x^{\gamma-1},\qquad \Re(\gamma)>-\Re(\delta). \tag{35}$$

$$\left(I_{x,\infty}^{\mu,\delta}t^{\gamma-1}\right)(x)=\frac{\Gamma(1-\gamma+\delta)}{\Gamma(1-\gamma+\delta+\mu)}x^{\gamma-1},\qquad \Re(\gamma)<1+\Re(\delta), \tag{36}$$

and

$$\left(D_{0,x}^{\mu,\delta}t^{\gamma-1}\right)(x)=\frac{\Gamma(\gamma+\delta+\mu)}{\Gamma(\gamma+\delta)}x^{\gamma-1},\qquad \Re(\gamma)>-\Re(\mu+\delta). \tag{37}$$

$$\left(D_{x,\infty}^{\mu,\delta}t^{\gamma-1}\right)(x)=\frac{\Gamma(1-\gamma+\mu+\delta)}{\Gamma(1-\gamma+\delta)}x^{\gamma-1},\qquad \Re(\gamma)<1+\Re(\mu+\delta)-n. \tag{38}$$

Lemma 4. *Let $\mu,\gamma\in\mathbb{C}$, and $x>0$ be such that $\Re(\mu)>0$, then the following formulas hold true:*

$$\left(I_{0,x}^{\mu}t^{\gamma-1}\right)(x)=\frac{\Gamma(\gamma)}{\Gamma(\gamma+\mu)}x^{\gamma+\mu-1},\quad \Re(\gamma)>0. \tag{39}$$

$$\left(I_{x,\infty}^{\mu}t^{\gamma-1}\right)(x)=\frac{\Gamma(1-\gamma-\mu)}{\Gamma(1-\gamma)}x^{\gamma+\mu-1},\qquad 0<\Re(\gamma)<1-\Re(\gamma), \tag{40}$$

and

$$\left(D_{0,x}^{\mu}t^{\gamma-1}\right)(x)=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\mu)}x^{\gamma-\mu-1},\quad \Re(\gamma)>\Re(\mu)>0. \tag{41}$$

$$\left(D_{x,\infty}^{\mu}t^{\gamma-1}\right)(x)=\frac{\Gamma(1-\gamma+\mu)}{\Gamma(1-\gamma)}x^{\gamma-\mu-1},\quad \Re(\gamma)<1+\Re(\mu)-n. \tag{42}$$

2. S-Function

The S-function is defined by Saxena and Daiya [24] as

$$S_{(p,q)}^{\alpha,\beta,\varepsilon,\tau,k}\left[a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;x\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n(\varepsilon)_{n\tau,k}}{(b_1)_n\cdots(b_q)_n\Gamma_k(n\alpha+\beta)}\frac{x^n}{n!}, \tag{43}$$

$k\in\Re,\alpha,\beta,\varepsilon,\tau\in C,\Re(\alpha)>0,a_i$ $(i=1,2,3,\ldots,p),b_j$ $(i=1,2,3,\ldots,q)$, $\Re(\alpha)>k\Re(\tau)$, and $p<q+1$. The k-Pochhammer symbol and k-gamma function introduced by Diaz and Pariguan [25] are as follows:

$$(\varepsilon)_{n,k}=\begin{cases}\dfrac{\Gamma_k(\varepsilon+nk)}{\Gamma_k(\varepsilon)}, & k\in\Re,\varepsilon\in\mathbb{C}/\{0\}\\ \varepsilon(\varepsilon+k)\cdots(\varepsilon+(n-1)k), & (n\in\mathbb{C},\varepsilon\in\mathbb{C})\end{cases} \tag{44}$$

and the relation with the classical Euler's gamma function is as follows:

$$\Gamma_k(\varepsilon)=k^{\varepsilon/k-1}\Gamma\left(\frac{\varepsilon}{k}\right) \tag{45}$$

where $\varepsilon\in\mathbb{C},k\in\Re$, and $n\in\mathbb{N}$. For further details of k-Pochhammer symbol and k-special functions one can refer to the papers by Romero et al. [26].

Special Cases

(1) When $p=q=0$ in (43), the S-function reduced to generalized k-Mittag-Leffler function, defined by Saxena et al. [27]:

$$S_{(0,0)}^{\alpha,\beta,\varepsilon,\tau,k}[-;-;x]=\sum_{n=0}^{\infty}\frac{(\varepsilon)_{n\tau,k}x^n}{\Gamma_k(n\alpha+\beta)n!} \tag{46}$$

$$=E_{k,\alpha,\beta}^{\varepsilon,\tau}(x),\quad \Re\left(\frac{\alpha}{k}-\tau\right)>p-q. \tag{47}$$

(2) When $k=\tau=1$ the S-function reduced to generalized K-function, defined by Sharma [28]:

$$S_{(p,q)}^{\alpha,\beta,\varepsilon,1,1}\left[a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;x\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n(\varepsilon)_nx^n}{(b_1)_n\cdots(b_q)_n\Gamma(n\alpha+\beta)n!} \tag{48}$$

$$=K_{(p,q)}^{\alpha,\beta,\varepsilon}\left[a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;x\right],\quad \Re(\alpha)>p-q. \tag{49}$$

(3) When $\tau=k=\varepsilon=1$, the S-function reduced to generalized M-series defined by Sharma and Jain [29]:

$$S_{(p,q)}^{\alpha,\beta,1,1,1}\left[a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;x\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n\Gamma(n\alpha+\beta)}\frac{x^n}{n!} \tag{50}$$

$$=M_{(p,q)}^{\alpha,\beta}\left[a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;x\right],\quad \Re(\alpha)>p-q-1. \tag{51}$$

Here, our aim is to establish composition formula of Marichev-Saigo-Maeda fractional integral and derivative operators of the product of S-function. The main formulas obtained here are represented in terms of the generalized Wright function ${}_p\psi_q(z)$ defined for $Z\in\mathbb{C}$, $a_i,b_j\in\mathbb{C}$, and $A_i,B_j\in\Re(A_i,B_j\neq0;i=1,2,\ldots,p;j=1,2,\ldots,q)$ which is given by the series

$${}_p\psi_q(z)={}_p\psi_q\left[\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\middle| z\right]=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+A_1k)\ldots\Gamma(a_p+A_pk)z^k}{\Gamma(b_1+B_1k)\ldots\Gamma(b_p+B_pk)k!}, \tag{52}$$

where $\Gamma(z)$ is the Euler gamma function and the function was introduced by Wright [30] and is known as generalized Wright function. Several theorems are on the asymptotic expansion of ${}_p\psi_q(z)$ for all values of the argument z, under the condition

$$1+\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i\geq0. \tag{53}$$

For detailed study of various properties, generalization and application of Wright function, and generalized Wright function, we refer to, for instance, [30, 31].

For $A_i=B_i=1$, (52) reduces to the generalized hypergeometric function ${}_pF_q$ (see [32]).

$${}_pF_q\left[\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\middle| z\right]=\frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}{}_p\psi_q\left[\begin{matrix}(a_i,1)_{1,p}\\(b_j,1)_{1,q}\end{matrix}\middle| z\right]. \tag{54}$$

3. Approach to Fractional Calculus

Throughout this paper, we assume that $\mu,\mu',\nu,\nu',\delta,\gamma,\alpha_i,\beta_i,\varepsilon_i,\tau_i\in\mathbb{C},k_i\in\Re,p_i,q_i\in\mathbb{N}$, and $x>0$, such that $\Re(\xi)>0$,

$\Re(\alpha_i) > k_i \Re(\tau_i)$, and $p_i < q_i + 1$. Further, let the constants satisfy the condition $a_i, b_j \in \mathbb{C}$, and $A_i, B_j \in \Re$ ($A_i, B_j \neq 0$; $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, q$), such that condition (53) is also satisfied.

3.1. Left- and Right-Sided Generalized Fractional Integration of S-Function. In this section, we establish image formulas for the product of S-function involving left- and right-sided operators of Marichev-Saigo-Meada fractional integral operators (1) and (3), respectively, in terms of the generalized Wright function. These formulas are given by the following theorems.

Theorem 5. *Let $\Re(\delta) > 0$, $\Re(\gamma) > \max\{0, \Re(\mu + \mu' + \nu - \delta), \Re(\mu' - \nu')\}$, then the generalized fractional integration $I_{0,x}^{\mu,\mu',\nu,\nu',\delta}$ of the product of S-function is given by*

$$\left(I_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r} S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t\right]\right\}\right)(x) = x^{\gamma+\delta-\mu-\mu'-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}\,{}_{(p+1)r+3}\Psi_{(q+1)r+3}$$
$$\cdot\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(\gamma,r),(\gamma+\delta-\mu-\mu'-\nu,r),(\gamma-\mu'+\nu',r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(\gamma+\nu',r),(\gamma+\delta-\mu-\mu',r),(\gamma+\delta-\mu'-\nu,r)\end{matrix}\middle| k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}x^r\right]. \quad (55)$$

Proof. On using (43), writing the function in the series form, the left-hand side of (55) leads to

$$\left(I_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r} S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t\right]\right\}\right)(x) \quad (56)$$
$$= \left(I_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\cdots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\cdots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{t^n}{n!}\right)^r\right\}\right)(x). \quad (57)$$

By interchanging the order of integration and summation, we reduce the right side of (57) to

$$= \prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\cdots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\cdots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{1}{n!}\right)^r$$
$$\cdot\left(I_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma+nr-1}\right\}\right)(x). \quad (58)$$

By applying Lemma 1 (see (27)) in (58), we get

$$= \prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\cdots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\cdots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{1}{n!}\right)^r \quad (59)$$
$$\times\frac{\Gamma(\gamma+nr)\Gamma(\gamma+\delta-\mu-\mu'-\nu+nr)\Gamma(\gamma+\nu'-\mu'+nr)}{\Gamma(\gamma+\nu'+nr)\Gamma(\gamma+\delta-\mu-\mu'+nr)\Gamma(\gamma+\delta-\mu'-\nu+nr)}$$
$$\cdot x^{\gamma+nr+\delta-\mu-\mu'-1}. \quad (60)$$

By applying (44) and (45) in (60) we get

$$= x^{\gamma+\delta-\mu-\mu'-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma(a_{1i}+n)\cdots\Gamma(a_{pi}+n)\Gamma(\varepsilon_i/k_i+n\tau_i)}{\Gamma(b_{1i}+n)\cdots\Gamma(b_{qi}+n)\Gamma(n\alpha_i/k_i+\beta_i/k_i)}\left(\frac{1}{n!}\right)^r \quad (61)$$
$$\times\frac{\Gamma(\gamma+nr)\Gamma(\gamma+\delta-\mu-\mu'-\nu+nr)\Gamma(\gamma+\nu'-\mu'+nr)}{\Gamma(\gamma+\nu'+nr)\Gamma(\gamma+\delta-\mu-\mu'+nr)\Gamma(\gamma+\delta-\mu'-\nu+nr)}(k_i)^{n(\tau_i-\alpha_i/k_i)}x^{nr} \quad (62)$$

Interpreting the right-hand side of the above equation, in view of definition (52), we arrive at result (55). □

Theorem 6. *Let $\Re(\delta) > 0$, $\Re(\gamma) < 1+\min\{\Re(-\nu), \Re(\mu+\mu'-\delta), \Re(\mu+\nu'-\delta)\}$, then the generalized fractional integration $I_{x,\infty}^{\mu,\mu',\nu,\nu',\delta}$ of the product of S-function is given by*

$$\left(I_{x,\infty}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r} S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t^{-1}\right]\right\}\right)(x) = x^{\gamma+\delta-\mu-\mu'-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}\,{}_{(p+1)r+3}\Psi_{(q+1)r+3}$$
$$\cdot\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(1-\gamma-\nu,r),(1-\gamma-\delta+\mu+\mu',r),(1-\gamma-\delta+\mu+\nu',r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(1-\gamma,r),(1-\gamma-\delta+\mu+\mu'+\nu',r),(1-\gamma+\mu-\nu,r)\end{matrix}\middle|\frac{k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}}{x^r}\right]. \quad (63)$$

Proof. The proof of Theorem 6 is a similar manner of Theorem 5. □

3.1.1. Special Cases. Now, we present some special cases of Theorems 5 and 6 given as follows.

If we put $\mu = \mu + \nu$, $\mu' = \nu' = 0$, $\nu = -\delta$, $\delta = \mu$, then we obtain the relationship

$$\left(I_{0,x}^{\mu,\mu',\nu,\nu',\delta}\right)(x) = \left(I_{0,x}^{\mu,\nu,\delta}f\right)(x) \quad (64)$$

and

$$\left(I_{x,\infty}^{\mu,\mu',\nu,\nu',\delta}\right)(x) = \left(I_{x,\infty}^{\mu,\nu,\delta}f\right)(x) \quad (65)$$

which are defined in (13) and (15) as Saigo fractional integral operator.

Corollary 7. *Let $\Re(\mu) > 0$, $\Re(\gamma) > \max\{0, \Re(\nu-\delta)\}$, then the generalized fractional integration $I_{0,x}^{\mu,\nu,\delta}$ of the product of S-function is given by*

$$\left(I_{0,x}^{\mu,\nu,\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t\right]\right\}\right)(x)=x^{\gamma-\nu-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\,\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\,\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$
$$\times\ {}_{(p+1)r+2}\Psi_{(q+1)r+2}\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(\gamma,r),(\gamma+\delta-\nu,r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(\gamma-\nu,r),(\gamma+\delta+\mu,r)\end{matrix}\middle|k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}x^r\right].\tag{66}$$

Corollary 8. *Let $\Re(\mu) > 0$, $\Re(\gamma) < 1+\min\{\Re(\nu), \Re(\delta)\}$, then the generalized fractional integration $I_{x,\infty}^{\mu,\nu,\delta}$ of the product of S-function is given by*

$$\left(I_{x,\infty}^{\mu,\nu,\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t^{-1}\right]\right\}\right)(x)=x^{\gamma-\nu-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\,\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\,\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$
$$\times\ {}_{(p+1)r+2}\Psi_{(q+1)r+2}\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(1-\gamma+\nu,r),(1-\gamma+\delta,r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(1-\gamma,r),(1-\gamma+\delta+\mu+\nu,r)\end{matrix}\middle|\frac{k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}}{x^r}\right].\tag{67}$$

Further, if we set $\nu = 0$, in Corollaries 7 and 8, then Saigo fractional integrals reduce to the following Erdélyi-Kober type fractional integral operators.

Corollary 9. *Let $\Re(\mu) > 0$, $\Re(\gamma) > -\Re(\delta)$, then the generalized fractional integration $I_{0,x}^{\mu,\delta}$ of the product of S-function is given by*

$$\left(I_{0,x}^{\mu,\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t\right]\right\}\right)(x)=x^{\gamma-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\,\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\,\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$
$$\times\ {}_{(p+1)r+1}\Psi_{(q+1)r+1}\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(\gamma+\delta,r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(\gamma+\delta+\mu,r)\end{matrix}\middle|k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}x^r\right].\tag{68}$$

Corollary 10. *Let $\Re(\mu) > 0$, $\Re(\gamma) < 1+\Re(\delta)$, then the generalized fractional integration $I_{x,\infty}^{\mu,\delta}$ of the product of S-function is given by*

$$\left(I_{x,\infty}^{\mu,\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t^{-1}\right]\right\}\right)(x)=x^{\gamma-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\,\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\,\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$
$$\times\ {}_{(p+1)r+1}\Psi_{(q+1)r+1}\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(1-\gamma+\delta,r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(1-\gamma+\delta+\mu,r)\end{matrix}\middle|\frac{k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}}{x^r}\right].\tag{69}$$

Further, if we set $\nu = -\mu$, in Corollaries 9 and 10, then Saigo fractional integrals reduce to the Riemann-Liouville and the Weyl type fractional integral operators as the following results.

Corollary 11. *Let $\Re(\mu) > 0$, $\Re(\gamma) > 0$, then the generalized fractional integration $I^{\mu}_{0,x}$ of the product of S-function is given by*

$$\left(I^{\mu}_{0,x}\left\{t^{\gamma-1}\prod_{i=1}^{r}S^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}_{(p_i,q_i)}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t\right]\right\}\right)(x) = x^{\gamma+\mu-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$
$$\times\,{}_{(p+1)r+1}\Psi_{(q+1)r+1}\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(\gamma,r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(\gamma+\mu,r)\end{matrix}\middle| k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}x^r\right]. \tag{70}$$

Corollary 12. *Let $1-\Re(\gamma) > \Re(\mu) > 0$, then the generalized fractional integration $I^{\mu}_{x,\infty}$ of the product of S-function is given by*

$$\left(I^{\mu}_{x,\infty}\left\{t^{\gamma-1}\prod_{i=1}^{r}S^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}_{(p_i,q_i)}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t^{-1}\right]\right\}\right)(x) = x^{\gamma+\mu-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$
$$\times\,{}_{(p+1)r+1}\Psi_{(q+1)r+1}\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(1-\gamma-\mu,r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(1-\gamma,r)\end{matrix}\middle| \frac{k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}}{x^r}\right]. \tag{71}$$

If we put $r=1$, then the results in (55), (63), and (66) to (71) reduce to the following form.

Corollary 13. *Let $\Re(\delta) > 0$, $\Re(\gamma) > \max\{0,\Re(\mu+\mu'+\nu-\delta),\Re(\mu'-\nu')\}$, then the generalized fractional integration $I^{\mu,\mu',\nu,\nu',\delta}_{0,x}$ of the S-function is given by*

$$\left(I^{\mu,\mu',\nu,\nu',\delta}_{0,x}\left\{t^{\gamma-1}S^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}_{(p_i,q_i)}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t\right]\right\}\right)(x) = x^{\gamma+\delta-\mu-\mu'-1}$$
$$\cdot\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)}\,{}_{p+4}\Psi_{q+4}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),\left(\frac{\varepsilon}{k},\tau\right),(\gamma,1),(\gamma+\delta-\mu-\mu'-\nu,1),(\gamma-\mu'+\nu',1)\\(b_1,1)\ldots(b_q,1),\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(\gamma+\nu',1),(\gamma+\delta-\mu-\mu',1),(\gamma+\delta-\mu'-\nu,1)\end{matrix}\middle| k^{\tau-\alpha/k}x\right]. \tag{72}$$

Corollary 14. *Let $\Re(\delta) > 0$, $\Re(\gamma) < 1+\min\{\Re(-\nu),\Re(\mu+\mu'-\delta),\Re(\mu+\nu'-\delta)\}$, then the generalized fractional integration $I^{\mu,\mu',\nu,\nu',\delta}_{x,\infty}$ of the S-function is given by*

$$\left(I^{\mu,\mu',\nu,\nu',\delta}_{x,\infty}\left\{t^{\gamma-1}S^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}_{(p_i,q_i)}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t^{-1}\right]\right\}\right)(x) = x^{\gamma+\delta-\mu-\mu'-1}$$
$$\cdot\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)}\,{}_{p+4}\Psi_{q+4}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),\left(\frac{\varepsilon}{k},\tau\right),(1-\gamma-\nu,1),(1-\gamma-\delta+\mu+\mu',1),(1-\gamma-\delta+\mu+\nu',1)\\(b_1,1)\ldots(b_q,1),\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(1-\gamma,1),(1-\gamma-\delta+\mu+\mu'+\nu',1),(1-\gamma+\mu-\nu,1)\end{matrix}\middle| \frac{k^{\tau-\alpha/k}}{x}\right]. \tag{73}$$

Corollary 15. *Let $\Re(\mu) > 0$, $\Re(\gamma) > \max\{0, \Re(\nu-\delta)\}$, then the generalized fractional integration $I_{0,x}^{\mu,\nu,\delta}$ of the S-function is given by*

$$\left(I_{0,x}^{\mu,\nu,\delta}\left\{t^{\gamma-1}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t\right]\right\}\right)(x) = x^{\gamma-\nu-1}\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)} \times {}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),\left(\frac{\varepsilon}{k},\tau\right),(\gamma,1),(\gamma+\delta-\nu,1)\\(b_1,1)\ldots(b_q,1),\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(\gamma-\nu,1),(\gamma+\delta+\mu,1)\end{matrix}\middle| k^{\tau-\alpha/k}x\right]. \tag{74}$$

Corollary 16. *Let $\Re(\mu) > 0$, $\Re(\gamma) < 1 + \min\{\Re(\nu), \Re(\delta)\}$, then the generalized fractional integration $I_{x,\infty}^{\mu,\nu,\delta}$ of the S-function is given by*

$$\begin{aligned}&\left(I_{x,\infty}^{\mu,\nu,\delta}\left\{t^{\gamma-1}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t^{-1}\right]\right\}\right)(x)\\&= x^{\gamma-\nu-1}\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)}\\&\times {}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),\left(\frac{\varepsilon}{k},\tau\right),(1-\gamma+\nu,1),(1-\gamma+\delta,1)\\(b_1,1)\ldots(b_q,1),\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(1-\gamma,1),(1-\gamma+\delta+\mu+\nu,1)\end{matrix}\middle| \frac{k^{\tau-\alpha/k}}{x}\right].\end{aligned} \tag{75}$$

Corollary 17. *Let $\Re(\mu) > 0$, $\Re(\gamma) > -\Re(\delta)$, then the generalized fractional integration $I_{0,x}^{\mu,\delta}$ of the S-function is given by*

$$\left(I_{0,x}^{\mu,\delta}\left\{t^{\gamma-1}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t\right]\right\}\right)(x) = x^{\gamma-1}\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)} \times {}_{p+2}\Psi_{q+2}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),\left(\frac{\varepsilon}{k},\tau\right),(\gamma+\delta,1)\\(b_1,1)\ldots(b_q,1),\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(\gamma+\delta+\mu,1)\end{matrix}\middle| k^{\tau-\alpha/k}x\right]. \tag{76}$$

Corollary 18. *Let $\Re(\mu) > 0$, $\Re(\gamma) < 1 + \Re(\delta)$, then the generalized fractional integration $I_{x,\infty}^{\mu,\delta}$ of the S-function is given by*

$$\left(I_{x,\infty}^{\mu,\delta}\left\{t^{\gamma-1}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t^{-1}\right]\right\}\right)(x) = x^{\gamma-1}\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)} \times {}_{p+2}\Psi_{q+2}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),\left(\frac{\varepsilon}{k},\tau\right),(1-\gamma+\delta,1)\\(b_1,1)\ldots(b_q,1),\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(1-\gamma+\delta+\mu,1)\end{matrix}\middle| \frac{k^{\tau-\alpha/k}}{x}\right]. \tag{77}$$

Corollary 19. *Let $\Re(\mu) > 0$, $\Re(\gamma) > 0$, then the generalized fractional integration $I^{\mu}_{0,x}$ of the S-function is given by*

$$\left(I^{\mu}_{0,x}\left\{t^{\gamma-1}S^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}_{(p_i,q_i)}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t\right]\right\}\right)(x) = x^{\gamma-1}\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)} \times {}_{p+2}\psi_{q+2}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),\left(\dfrac{\varepsilon}{k},\tau\right),(\gamma,1)\\(b_1,1)\ldots(b_q,1),\left(\dfrac{\beta}{k},\dfrac{\alpha}{k}\right),(\gamma+\mu,1)\end{matrix}\,\middle|\,k^{\tau-\alpha/k}x\right]. \tag{78}$$

Corollary 20. *Let $1-\Re(\gamma) > \Re(\mu) > 0$, then the generalized fractional integration $I^{\mu}_{x,\infty}$ of the S-function is given by*

$$\left(I^{\mu}_{x,\infty}\left\{t^{\gamma-1}S^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}_{(p_i,q_i)}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t^{-1}\right]\right\}\right)(x) = x^{\gamma+\mu-1}\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)} \times {}_{p+2}\psi_{q+2}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),\left(\dfrac{\varepsilon}{k},\tau\right),(1-\gamma-\mu,1)\\(b_1,1)\ldots(b_q,1),\left(\dfrac{\beta}{k},\dfrac{\alpha}{k}\right),(1-\gamma,1)\end{matrix}\,\middle|\,\frac{k^{\tau-\alpha/k}}{x}\right]. \tag{79}$$

3.2. Left- and Right-Sided Generalized Fractional Differentiation of S-Function. In this section, we establish image formulas for the product of S-function involving left- and right-sided operators of Marichev-Saigo-Meada fractional derivative operators (6) and (8), respectively, in terms of the generalized Wright function. These formulas are given by the following theorems.

Theorem 21. *Let $\Re(\delta) > 0$, $\Re(\gamma) > \max\{0,\Re(\delta-\mu-\mu'-\nu'),\Re(\nu-\mu)\}$, then the generalized fractional differentiation $D^{\mu,\mu',\nu,\nu',\delta}_{0,x}$ of the product of S-function is given by*

$$\left(D^{\mu,\mu',\nu,\nu',\delta}_{0,x}\left\{t^{\gamma-1}\prod_{i=1}^{r}S^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}_{(p_i,q_i)}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t\right]\right\}\right)(x) = x^{\gamma-\delta+\mu+\mu'-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}\,{}_{(p+1)r+3}\psi_{(q+1)r+3}$$
$$\cdot\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\dfrac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\dfrac{\varepsilon_r}{k_r},\tau_r\right),(\gamma,r),(\gamma-\delta+\mu+\mu'+\nu',r),(\gamma-\nu+\mu,r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\dfrac{\beta_1}{k_1},\dfrac{\alpha_1}{k_1}\right),\ldots,\left(\dfrac{\beta_r}{k_r},\dfrac{\alpha_r}{k_r}\right),(\gamma-\nu,r),(\gamma-\delta+\mu+\mu',r),(\gamma-\delta+\mu+\nu',r)\end{matrix}\,\middle|\,k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}x^r\right]. \tag{80}$$

Proof. Using (43) we can rewrite left side of (80) as follows:

$$\left(D^{\mu,\mu',\nu,\nu',\delta}_{0,x}\left\{t^{\gamma-1}\prod_{i=1}^{r}S^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}_{(p_i,q_i)}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t\right]\right\}\right)(x) \tag{81}$$

$$=\left(D^{\mu,\mu',\nu,\nu',\delta}_{0,x}\left\{t^{\gamma-1}\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\ldots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\ldots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{t^n}{n!}\right)^r\right\}\right)(x). \tag{82}$$

By interchanging the order of differentiation and summation, we reduce the right side of (82) to

$$=\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\ldots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\ldots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{1}{n!}\right)^r \cdot\left(D^{\mu,\mu',\nu,\nu',\delta}_{0,x}\left\{t^{\gamma+nr-1}\right\}\right)(x). \tag{83}$$

By applying Lemma 1 (see (29)) in (83), we obtain

$$=\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\ldots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\ldots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{1}{n!}\right)^r \tag{84}$$

$$\times\frac{\Gamma(\gamma+nr)\Gamma(\gamma-\delta+\mu+\mu'+\nu'+nr)\Gamma(\gamma-\nu+\mu+nr)}{\Gamma(\gamma-\nu+nr)\Gamma(\gamma-\delta+\mu+\mu'+nr)\Gamma(\gamma-\delta+\mu+\nu'+nr)} \cdot x^{\gamma+nr-\delta+\mu+\mu'-1}. \tag{85}$$

By applying (44) and (45) in (85), we get

$$=x^{\gamma-\delta+\mu+\mu'-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma(a_{1i}+n)\ldots\Gamma(a_{pi}+n)\Gamma(\varepsilon_i/k_i+n\tau_i)}{\Gamma(b_{1i}+n)\ldots\Gamma(b_{qi}+n)\Gamma(n\alpha_i/k_i+\beta_i/k_i)}\left(\frac{1}{n!}\right)^r \tag{86}$$

$$\times \frac{\Gamma(\gamma+nr)\Gamma(\gamma-\delta+\mu+\mu'+\nu'+nr)\Gamma(\gamma-\nu+\mu+nr)}{\Gamma(\gamma-\nu+nr)\Gamma(\gamma-\delta+\mu+\mu'+nr)\Gamma(\gamma-\delta+\mu+\nu'+nr)}(k_i)^{n(\tau_i-\alpha_i/k_i)}x^{nr}. \tag{87}$$

Interpreting the right-hand side of the above equation, in view of definition (52), we arrive at result (80). □

Theorem 22. *Let $\Re(\delta) > 0$, $\Re(\gamma) < 1 + \min\{\Re(\nu'), \Re(\delta - \mu - \mu'), \Re(\delta - \mu' - \nu)\}$, then the generalized fractional differentiation $D_{x,\infty}^{\mu,\mu',\nu,\nu',\delta}$ of the product of S-function is given by*

$$\left(D_{x,\infty}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t^{-1}\right]\right\}\right)(x) = x^{\gamma-\delta+\mu+\mu'-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}\,{}_{(p+1)r+3}\Psi_{(q+1)r+3}$$
$$\cdot\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(1-\gamma+\nu',r),(1-\gamma+\delta-\mu-\mu',r),(1-\gamma+\delta-\mu'-\nu,r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(1-\gamma,r),(1-\gamma+\delta-\mu-\mu'-\nu,r),(1-\gamma-\mu'+\nu',r)\end{matrix}\middle|\frac{k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}}{x^r}\right]. \tag{88}$$

Proof. The proof of Theorem 22 is a similar manner of Theorem 21. □

3.2.1. Special Cases. Now, we present some special cases of Theorems 21 and 22 given as follows.

If we put $\mu = \mu+\nu, \mu' = \nu' = 0, \nu = -\delta, \delta = \mu$, then we obtain the relationship

$$\left(D_{0,x}^{\mu,\mu',\nu,\nu',\delta}\right)(x) = \left(D_{0,x}^{\mu,\nu,\delta}f\right)(x) \tag{89}$$

and

$$\left(D_{x,\infty}^{\mu,\mu',\nu,\nu',\delta}\right)(x) = \left(D_{x,\infty}^{\mu,\nu,\delta}f\right)(x) \tag{90}$$

which are defined in (17) and (18) as Saigo fractional derivative operator.

Corollary 23. *Let $\Re(\mu) > 0$, $\Re(\gamma) > -\min\{0, \Re(\mu+\nu+\delta)\}$, then the generalized fractional differentiation $D_{0,x}^{\mu,\nu,\delta}$ of the product of S-function is given by*

$$\left(D_{0,x}^{\mu,\nu,\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t\right]\right\}\right)(x) = x^{\gamma+\nu-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$
$$\times\,{}_{(p+1)r+2}\Psi_{(q+1)r+2}\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(\gamma,r),(\gamma+\delta+\mu+\nu,r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(\gamma+\nu,r),(\gamma+\delta,r)\end{matrix}\middle|k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}x^r\right]. \tag{91}$$

Corollary 24. *Let $\Re(\mu) > 0$, $\Re(\gamma) < 1 + \min\{\Re(-\nu), \Re(\delta+\mu)\}$, then the generalized fractional differentiation $D_{x,\infty}^{\mu,\nu,\delta}$ of the product of S-function is given by*

$$\left(D_{x,\infty}^{\mu,\nu,\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t^{-1}\right]\right\}\right)(x) = x^{\gamma+\nu-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$
$$\times\,{}_{(p+1)r+2}\Psi_{(q+1)r+2}\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(1-\gamma-\nu,r),(1-\gamma+\mu+\delta,r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(1-\gamma,r),(1-\gamma+\delta-\nu,r)\end{matrix}\middle|\frac{k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}}{x^r}\right]. \tag{92}$$

Further, if we set $\nu = 0$, Corollaries 23 and 24 and then Saigo fractional differential formulas reduce to the following Erdélyi-Kober type fractional differential formulas.

Corollary 25. *Let $\Re(\mu) > 0$, $\Re(\gamma) > -\Re(\delta+\mu)$, then the generalized fractional differentiation $D_{0,x}^{\mu,\delta}$ of the product of S-function is given by*

$$\left(D_{0,x}^{\mu,\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};t\right]\right\}\right)(x) = x^{\gamma-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$

$$\times {}_{(p+1)r+1}\Psi_{(q+1)r+1}\left[\begin{matrix}(a_{11},1)\dots(a_{1r},1),\dots,(a_{p1},1)\dots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\dots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(\gamma+\delta+\mu,r)\\(b_{11},1)\dots(b_{1r},1),\dots,(b_{q1},1)\dots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\dots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(\gamma+\delta,r)\end{matrix}\middle| k_1^{\tau_1-\alpha_1/k_1}\dots k_r^{\tau_r-\alpha_r/k_r}x^r\right]. \quad (93)$$

Corollary 26. *Let $\Re(\mu)>0$, $\Re(\gamma)<1+\Re(\delta+\mu)$, then the generalized fractional differentiation $D_{x,\infty}^{\mu,\delta}$ of the product of S-function is given by*

$$\left(D_{x,\infty}^{\mu,\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\dots,a_{pi};b_{1i},\dots,b_{qi};t^{-1}\right]\right\}\right)(x)=x^{\gamma-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\dots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\dots\Gamma(a_{pi})}$$
$$\times {}_{(p+1)r+1}\Psi_{(q+1)r+1}\left[\begin{matrix}(a_{11},1)\dots(a_{1r},1),\dots,(a_{p1},1)\dots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\dots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(1-\gamma+\mu+\delta,r)\\(b_{11},1)\dots(b_{1r},1),\dots,(b_{q1},1)\dots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\dots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(1-\gamma+\delta,r)\end{matrix}\middle| \frac{k_1^{\tau_1-\alpha_1/k_1}\dots k_r^{\tau_r-\alpha_r/k_r}}{x^r}\right]. \quad (94)$$

Further, if we set $\nu=-\mu$, in Corollaries 9 and 10 then Saigo fractional derivative formulas reduce to the Riemann-Liouville and the Weyl type fractional derivative formulas as the following results.

Corollary 27. *Let $\Re(\mu)>0$, $\Re(\gamma)>0$, and then the generalized fractional differentiation $D_{0,x}^{\mu}$ of the product of S-function is given by*

$$\left(D_{0,x}^{\mu}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\dots,a_{pi};b_{1i},\dots,b_{qi};t\right]\right\}\right)(x)=x^{\gamma-\mu-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\dots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\dots\Gamma(a_{pi})}$$
$$\times {}_{(p+1)r+1}\Psi_{(q+1)r+1}\left[\begin{matrix}(a_{11},1)\dots(a_{1r},1),\dots,(a_{p1},1)\dots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\dots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(\gamma,r)\\(b_{11},1)\dots(b_{1r},1),\dots,(b_{q1},1)\dots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\dots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(\gamma-\mu,r)\end{matrix}\middle| k_1^{\tau_1-\alpha_1/k_1}\dots k_r^{\tau_r-\alpha_r/k_r}x^r\right]. \quad (95)$$

Corollary 28. *Let $\Re(\gamma)<1+\Re(\mu)$, then the generalized fractional differentiation $D_{x,\infty}^{\mu}$ of the product of S-function is given by*

$$\left(D_{x,\infty}^{\mu}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\dots,a_{pi};b_{1i},\dots,b_{qi};t^{-1}\right]\right\}\right)(x)=x^{\gamma-\mu-1}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\dots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\dots\Gamma(a_{pi})}$$
$$\times {}_{(p+1)r+1}\Psi_{(q+1)r+1}\left[\begin{matrix}(a_{11},1)\dots(a_{1r},1),\dots,(a_{p1},1)\dots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\dots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(1-\gamma+\mu,r)\\(b_{11},1)\dots(b_{1r},1),\dots,(b_{q1},1)\dots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\dots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(1-\gamma,r)\end{matrix}\middle| \frac{k_1^{\tau_1-\alpha_1/k_1}\dots k_r^{\tau_r-\alpha_r/k_r}}{x^r}\right]. \quad (96)$$

If we put $r=1$, then the results in (80), (88), and (91) to (96) reduce to the following form.

Corollary 29. *Let $\Re(\delta)>0$, $\Re(\gamma)>\max\{0,\Re(\delta-\mu-\mu'-\nu'),\Re(\nu-\mu)\}$, then the generalized fractional differentiation $D_{0,x}^{\mu,\mu',\nu,\nu',\delta}$ of the S-function is given by*

$$\left(D_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}S_{(p,q)}^{\alpha,\beta,\varepsilon,\tau,k}\left[a_1,\dots,a_p;b_1,\dots,b_q;t\right]\right\}\right)(x)=x^{\gamma-\delta+\mu+\mu'-1}$$
$$\cdot\frac{(k)^{1-\beta/k}\Gamma(b_1)\dots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\dots\Gamma(a_p)}\,{}_{p+4}\Psi_{q+4}\left[\begin{matrix}(a_1,1)\dots(a_p,1),\left(\frac{\varepsilon}{k},\tau\right),(\gamma,1),(\gamma-\delta+\mu+\mu'+\nu',1),(\gamma-\nu+\mu,1)\\(b_1,1)\dots(b_q,1),\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(\gamma-\nu,1),(\gamma-\delta+\mu+\mu',1),(\gamma-\delta+\mu+\nu',1)\end{matrix}\middle| k^{\tau-\alpha/k}x\right]. \quad (97)$$

Corollary 30. *Let $\Re(\delta) > 0$, $\Re(\gamma) < 1 + \min\{\Re(\nu'), \Re(\delta - \mu - \mu'), \Re(\delta - \mu' - \nu)\}$, then the generalized fractional differentiation $D_{x,\infty}^{\mu,\mu',\nu,\nu',\delta}$ of the S-function is given by*

$$\begin{aligned}&\left(D_{x,\infty}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t^{-1}\right]\right\}\right)(x) = x^{\gamma-\delta+\mu+\mu'-1}\\&\cdot\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)}\,{}_{p+4}\Psi_{q+4}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),\left(\frac{\varepsilon}{k},\tau\right),(1-\gamma+\nu',1),(1-\gamma+\delta-\mu-\mu',1),(1-\gamma+\delta-\mu'-\nu,1)\\(b_1,1)\ldots(b_q,1),\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(1-\gamma,1),(1-\gamma+\delta-\mu-\mu'-\nu,1),(1-\gamma-\mu'+\nu',1)\end{matrix}\;\middle|\;\frac{k^{\tau-\alpha/k}}{x}\right].\end{aligned}\tag{98}$$

Corollary 31. *Let $\Re(\mu) > 0$, $\Re(\gamma) > -\min\{0, \Re(\mu+\nu+\delta)\}$, then the generalized fractional differentiation $D_{0,x}^{\mu,\nu,\delta}$ of the S-function is given by*

$$\begin{aligned}&\left(D_{0,x}^{\mu,\nu,\delta}\left\{t^{\gamma-1}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t\right]\right\}\right)(x) = x^{\gamma+\nu-1}\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)}\\&\times\,{}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),\left(\frac{\varepsilon}{k},\tau\right),(\gamma,1),(\gamma+\delta+\mu+\nu,1)\\(b_1,1)\ldots(b_q,1),\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(\gamma+\nu,1),(\gamma+\delta,1)\end{matrix}\;\middle|\;k^{\tau-\alpha/k}x\right].\end{aligned}\tag{99}$$

Corollary 32. *Let $\Re(\mu) > 0$, $\Re(\gamma) < 1 + \min\{\Re(-\nu), \Re(\delta+\mu)\}$, then the generalized fractional differentiation $D_{x,\infty}^{\mu,\nu,\delta}$ of the S-function is given by*

$$\begin{aligned}&\left(D_{x,\infty}^{\mu,\nu,\delta}\left\{t^{\gamma-1}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t^{-1}\right]\right\}\right)(x) = x^{\gamma+\nu-1}\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)}\\&\times\,{}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),\left(\frac{\varepsilon}{k},\tau\right),(1-\gamma-\nu,1),(1-\gamma+\mu+\delta,1)\\(b_1,1)\ldots(b_q,1),\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(1-\gamma,1),(1-\gamma+\delta-\nu,1)\end{matrix}\;\middle|\;\frac{k^{\tau-\alpha/k}}{x}\right].\end{aligned}\tag{100}$$

Corollary 33. *Let $\Re(\mu) > 0$, $\Re(\gamma) > -\Re(\delta+\mu)$, then the generalized fractional differentiation $D_{0,x}^{\mu,\delta}$ of the S-function is given by*

$$\begin{aligned}&\left(D_{0,x}^{\mu,\delta}\left\{t^{\gamma-1}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t\right]\right\}\right)(x) = x^{\gamma-1}\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)}\\&\times\,{}_{p+2}\Psi_{q+2}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),\left(\frac{\varepsilon}{k},\tau\right),(\gamma+\delta+\mu,1)\\(b_1,1)\ldots(b_q,1),\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(\gamma+\delta,1)\end{matrix}\;\middle|\;k^{\tau-\alpha/k}x\right].\end{aligned}\tag{101}$$

Corollary 34. *Let $\Re(\mu) > 0$, $\Re(\gamma) < 1 + \Re(\delta + \mu)$, then the generalized fractional differentiation $D_{x,\infty}^{\mu,\delta}$ of the S-function is given by*

$$\left(D_{x,\infty}^{\mu,\delta}\left\{t^{\gamma-1}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t^{-1}\right]\right\}\right)(x) = x^{\gamma-1}\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)} \times {}_{p+2}\psi_{q+2}\left[\begin{array}{c} (a_1,1)\ldots(a_p,1),\left(\frac{\varepsilon}{k},\tau\right),(1-\gamma+\mu+\delta,1) \\ (b_1,1)\ldots(b_q,1),\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(1-\gamma+\delta,1)\end{array}\middle|\frac{k^{\tau-\alpha/k}}{x}\right]. \tag{102}$$

Corollary 35. *Let $\Re(\mu) > 0$, $\Re(\gamma) > 0$, and then the generalized fractional differentiation $D_{0,x}^{\mu}$ of the S-function is given by*

$$\left(D_{0,x}^{\mu}\left\{t^{\gamma-1}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t\right]\right\}\right)(x) = x^{\gamma-\mu-1}\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)} \times {}_{p+2}\psi_{q+2}\left[\begin{array}{c} (a_1,1)\ldots(a_p,1),\left(\frac{\varepsilon}{k},\tau\right),(\gamma,1) \\ (b_1,1)\ldots(b_q,1),\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(\gamma-\mu,1)\end{array}\middle|k^{\tau-\alpha/k}x\right] \tag{103}$$

Corollary 36. *Let $\Re(\gamma) < 1 + \Re(\mu)$, then the generalized fractional differentiation $D_{x,\infty}^{\mu}$ of the S-function is given by*

$$\left(D_{x,\infty}^{\mu}\left\{t^{\gamma-1}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_1,\ldots,a_p;b_1,\ldots,b_q;t^{-1}\right]\right\}\right)(x) = x^{\gamma-\mu-1}\frac{(k)^{1-\beta/k}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(\varepsilon/k)\Gamma(a_1)\ldots\Gamma(a_p)} \times {}_{p+2}\psi_{q+2}\left[\begin{array}{c} (a_1,1)\ldots(a_p,1),\left(\frac{\varepsilon}{k},\tau\right),(1-\gamma+\mu,1) \\ (b_1,1)\ldots(b_q,1),\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(1-\gamma,1)\end{array}\middle|\frac{k^{\tau-\alpha/k}}{x}\right]. \tag{104}$$

4. Integral Transform Formulas of the Product of S-Function

In this section, we establish some theorems involving the results obtained in previous sections pertaining with the integral transform as like the Beta transform and the Laplace transform.

4.1. Beta Transform. The Beta transform [33] of the function $f(z)$ is defined as

$$B(f(z);g.h) = \int_0^1 z^{g-1}(1-z)^{h-1} f(z)\,dz,$$

$$g,h \in \mathbb{C},\ \Re(g) > 0,\ \Re(h) > 0. \tag{105}$$

Theorem 37. *Let $\mu,\mu',\nu,\nu',\delta,\gamma,\alpha_i,\beta_i,\varepsilon_i,\tau_i \in \mathbb{C}$, $k_i \in \Re$, $p_i,q_i \in \mathbb{N}$, $\Re(\alpha_i) > k_i\Re(\tau_i)$, $x > 0$, and $p_i < q_i + 1$, where $(i = 1,\ldots,r)$, such that $\Re(\delta) > 0$, $\Re(\gamma) > \max\{0,\Re(\mu+\mu'+\nu-\delta),\Re(\mu'-\nu')\}$, then the following fractional integral holds true:*

$$B\left(\left(I_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots a_{pi};b_{1i},\ldots b_{qi};tz\right]\right\}\right)(x):g,h\right) = x^{\gamma+\delta-\mu-\mu'-1}\Gamma(h)\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$

$$\times {}_{(p+1)r+4}\Psi_{(q+1)r+4}\left[\begin{matrix}(a_{11},1)\dots(a_{1r},1),\dots,(a_{p1},1)\dots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\dots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(\gamma,r),(\gamma+\delta-\mu-\mu'-\nu,r),(\gamma-\mu'+\nu',r),(g,r)\\(b_{11},1)\dots(b_{1r},1),\dots,(b_{q1},1)\dots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\dots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(\gamma+\nu',r),(\gamma+\delta-\mu-\mu',r),(\gamma+\delta-\mu'-\nu,r),(g+h,r)\end{matrix}\middle| k_1^{\tau_1-\alpha_1/k_1}\dots k_r^{\tau_r-\alpha_r/k_r}x^r\right]. \tag{106}$$

Proof. On using (105), the left-hand side of (106) leads to

$$B\left(\left(I_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\dots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\dots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{(tz)^n}{n!}\right)^r\right\}\right)(x),g,h\right). \tag{107}$$

Using definition of Beta transform right side of (107) becomes

$$= \int_0^{\infty} z^{g-1}(1-z)^{h-1} \tag{108}$$

$$\times\left(I_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\dots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\dots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{(tz)^n}{n!}\right)^r\right\}\right)dz. \tag{109}$$

By interchanging the order of integration and summation we reduce the right side of (109) to

$$= \prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\dots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\dots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{1}{n!}\right)^r \tag{110}$$

$$\times\left(I_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma+nr-1}\right\}\right)(x)\int_0^{\infty} z^{g+nr-1}(1-z)^{h-1}\,dz. \tag{111}$$

By applying definition of Beta transform in (111), we get

$$= x^{\gamma+\delta-\mu-\mu'-1}\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})\dots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\dots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{1}{n!}\right)^r \tag{112}$$

$$\times\frac{\Gamma(\gamma+nr)\Gamma(\gamma+\delta-\mu-\mu'-\nu+nr)\Gamma(\gamma+\nu'-\mu'+nr)\Gamma(g+nr)\Gamma(h)}{\Gamma(\gamma+\nu'+nr)\Gamma(\gamma+\delta-\mu-\mu'+nr)\Gamma(\gamma+\delta-\mu'-\nu+nr)\Gamma(g+h+nr)}x^{nr}, \tag{113}$$

and using (44) and (45) in (113) we get

$$= x^{\gamma+\delta-\mu-\mu'-1}\Gamma(h)\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\dots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\dots\Gamma(a_{pi})}\times\sum_{n=0}^{\infty}\frac{\Gamma(a_{1i}+n)\dots\Gamma(a_{pi}+n)\Gamma(\varepsilon_i/k_i+n\tau_i)}{\Gamma(b_{1i}+n)\dots\Gamma(b_{qi}+n)\Gamma(n\alpha_i/k_i+\beta_i/k_i)}\left(\frac{1}{n!}\right)^r\frac{\Gamma(\gamma+nr)}{\Gamma(\gamma+\nu'+nr)} \tag{114}$$

$$\times\frac{\Gamma(\gamma+\delta-\mu-\mu'-\nu+nr)\Gamma(\gamma+\nu'-\mu'+nr)\Gamma(g+nr)}{\Gamma(\gamma+\delta-\mu-\mu'+nr)\Gamma(\gamma+\delta-\mu'-\nu+nr)\Gamma(g+h+nr)}(k_i)^{n(\tau_i-\alpha_i/k_i)}x^{nr}. \tag{115}$$

Interpreting the right-hand side of the above equation, in view of definition (52), we arrive at result (106). □

Theorem 38. *Let* $\mu,\mu',\nu,\nu',\delta,\gamma,\alpha_i,\beta_i,\varepsilon_i,\tau_i \in \mathbb{C}$, $k_i \in \mathfrak{R}$, $p_i,q_i \in \mathbb{N}$, $\mathfrak{R}(\alpha_i) > k_i\mathfrak{R}(\tau_i)$, $x > 0$, *and* $p_i < q_i + 1$, *where* $(i = 1,\dots,r)$, *such that* $\mathfrak{R}(\delta) > 0$, $\mathfrak{R}(\gamma) < 1 + \min\{\mathfrak{R}(-\nu),\mathfrak{R}(\mu+\mu'-\delta),\mathfrak{R}(\mu+\nu'-\delta)\}$, *then the following fractional integral holds true:*

$$B\left(\left(I_{x,\infty}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i}\cdots a_{pi};b_{1i}\cdots b_{qi};zt^{-1}\right]\right\}\right)(x):g,h\right) = x^{\gamma+\delta-\mu-\mu'-1}\Gamma(h)\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\dots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\dots\Gamma(a_{pi})}$$

$$\times {}_{(p+1)r+4}\Psi_{(q+1)r+4}\left[\begin{matrix}(a_{11},1)\dots(a_{1r},1),\dots,(a_{p1},1)\dots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\dots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(1-\gamma-\nu,r),(1-\gamma-\delta+\mu+\mu',r),(1-\gamma-\delta+\mu+\nu',r),(g,r)\\(b_{11},1)\dots(b_{1r},1),\dots,(b_{q1},1)\dots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\dots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(1-\gamma,r),(1-\gamma-\delta+\mu+\mu'+\nu',r),(1-\gamma+\mu-\nu,r),(g+h,r)\end{matrix}\middle|\frac{k_1^{\tau_1-\alpha_1/k_1}\dots k_r^{\tau_r-\alpha_r/k_r}}{x^r}\right]. \tag{116}$$

Proof. The proof of Theorem 38 is a similar manner of Theorem 37. □

Theorem 39. *Let* $\mu,\mu',\nu,\nu',\delta,\gamma,\alpha_i,\beta_i,\varepsilon_i,\tau_i \in \mathbb{C}$, $k_i \in \mathfrak{R}$, $p_i,q_i \in \mathbb{N}$, $\mathfrak{R}(\alpha_i) > k_i\mathfrak{R}(\tau_i)$, $x > 0$, *and*

$p_i < q_i + 1$, *where* $(i = 1, \ldots, r)$, *such that* $\Re(\delta) > 0$, $\Re(\gamma) > \max\{0, \Re(\delta - \mu - \mu' - \nu'), \Re(\nu - \mu)\}$, *then the following fractional derivative formula holds true:*

$$\begin{aligned} &B\left(\left(D_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r} S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots.a_{pi};b_{1i},\ldots,b_{qi};tz\right]\right\}\right)(x):g,h\right) = x^{\gamma-\delta+\mu+\mu'-1}\Gamma(h)\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})} \\ &\times {}_{(p+1)r+4}\Psi_{(q+1)r+4}\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(\gamma,r),(\gamma-\delta+\mu+\mu'+\nu',r),(\gamma-\nu+\mu,r),(g,r) \\ (b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(\gamma-\nu,r),(\gamma-\delta+\mu+\mu',r),(\gamma-\delta+\mu+\nu',r),(g+h,r)\end{matrix}\middle|k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}x^r\right]. \end{aligned} \tag{117}$$

Proof. On using (105), the left-hand side of (117) leads to

$$B\left(\left(D_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\ldots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\ldots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{(tz)^n}{n!}\right)^r\right\}\right)(x):g,h\right). \tag{118}$$

Using definition of Beta transform right side of (118) becomes

$$= \int_0^\infty z^{g-1}(1-z)^{h-1} \tag{119}$$

$$\times\left(D_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\ldots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\ldots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{(tz)^n}{n!}\right)^r\right\}\right)dz. \tag{120}$$

By interchanging the order of integration and summation we reduce the right side of (120) to

$$= \prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\ldots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\ldots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{1}{n!}\right)^r \tag{121}$$

$$\times\left(D_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma+nr-1}\right\}\right)(x)\int_0^\infty z^{g+nr-1}(1-z)^{h-1}dz. \tag{122}$$

By applying definition of Beta transform in (122), we get

$$= x^{\gamma-\delta+\mu+\mu'-1}\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})\ldots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\ldots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{1}{n!}\right)^r \tag{123}$$

$$\times\frac{\Gamma(\gamma+nr)\Gamma(\gamma-\delta+\mu+\mu'+\nu'+nr)\Gamma(\gamma-\nu+\mu+nr)\Gamma(g+nr)\Gamma(h)}{\Gamma(\gamma-\nu+nr)\Gamma(\gamma-\delta+\mu+\mu'+nr)\Gamma(\gamma-\delta+\mu+\nu'+nr)\Gamma(g+h+nr)}x^{nr}, \tag{124}$$

and using (44) and (45) in (124) we get

$$\begin{aligned} &= x^{\gamma-\delta+\mu+\mu'-1}\Gamma(h)\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}\times\sum_{n=0}^{\infty}\frac{\Gamma(a_{1i}+n)\ldots\Gamma(a_{pi}+n)\Gamma(\varepsilon_i/k_i+n\tau_i)}{\Gamma(b_{1i}+n)\ldots\Gamma(b_{qi}+n)\Gamma(n\alpha_i/k_i+\beta_i/k_i)}\left(\frac{1}{n!}\right)^r \\ &\times\frac{\Gamma(\gamma+nr)\Gamma(\gamma-\delta+\mu+\mu'+\nu'+nr)\Gamma(\gamma-\nu+\mu+nr)\Gamma(g+nr)}{\Gamma(\gamma-\nu+nr)\Gamma(\gamma-\delta+\mu+\mu'+nr)\Gamma(\gamma-\delta+\mu+\nu'+nr)\Gamma(g+h+nr)}\times(k_i)^{n(\tau_i-\alpha_i/k_i)}x^{nr}. \end{aligned} \tag{125}$$

Interpreting the right-hand side of the above equation, in view of definition (52), we arrive at result (117). □

Theorem 40. *Let* $\mu, \mu', \nu, \nu', \delta, \gamma, \alpha_i, \beta_i, \varepsilon_i, \tau_i \in \mathbb{C}$, $k_i \in \Re$, $p_i, q_i \in \mathbb{N}$, $\Re(\alpha_i) > k_i\Re(\tau_i)$, $x > 0$, *and* $p_i < q_i + 1$, *where* $(i = 1, \ldots, r)$, *such that* $\Re(\delta) > 0$, $\Re(\gamma) < 1 + \min\{\Re(\nu'), \Re(\delta - \mu - \mu'), \Re(\delta - \mu' - \nu)\}$, *then the following fractional derivative formula holds true:*

$$\begin{aligned} &B\left(\left(D_{x,\infty}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r} S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};zt^{-1}\right]\right\}\right)(x):g,h\right) = x^{\gamma-\delta+\mu+\mu'-1}\Gamma(h)\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})} \\ &\times {}_{(p+1)r+4}\Psi_{(q+1)r+4}\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(1-\gamma+\nu',r),(1-\gamma+\delta-\mu-\mu',r),(1-\gamma+\delta-\mu'-\nu,r),(g,r) \\ (b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(1-\gamma,r),(1-\gamma+\delta-\mu-\mu'-\nu,r),(1-\gamma-\mu'+\nu',r),(g+h,r)\end{matrix}\middle|\frac{k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}}{x^r}\right]. \end{aligned} \tag{126}$$

Proof. The proof of Theorem 40 is a similar manner of Theorem 39. □

4.2. Laplace Transform. The Laplace transform [33, 34] of $f(z)$ is defined as

$$L(f(z)) = \int_0^{\infty} e^{-sz} f(z)\,dz, \quad \Re(s) > 0. \tag{127}$$

Theorem 41. *Let $\mu, \mu', \nu, \nu', \delta, \gamma, \alpha_i, \beta_i, \varepsilon_i, \tau_i \in \mathbb{C}$, $k_i \in \Re$, $p_i, q_i \in \mathbb{N}$, $\Re(\alpha_i) > k_i \Re(\tau_i)$, $x > 0$, and $p_i < q_i + 1$, where $(i = 1, \ldots, r)$, such that $\Re(\delta) > 0$, $\Re(\gamma) > \max\{0, \Re(\mu + \mu' + \nu - \delta), \Re(\mu' - \nu')\}$, then the following fractional integral holds true:*

$$\begin{aligned} & L\left(z^{g-1}\left(I_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r} S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};tz\right]\right\}\right)(x)\right) = \frac{x^{\gamma+\delta-\mu-\mu'-1}}{s^g}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})} \\ & \times {}_{(p+1)r+4}\Psi_{(q+1)r+3}\left[\begin{matrix} (a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(\gamma,r),(\gamma+\delta-\mu-\mu'-\nu,r),(\gamma-\mu'+\nu',r),(g,r) \\ (b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(\gamma+\nu',r),(\gamma+\delta-\mu-\mu',r),(\gamma+\delta-\mu'-\nu,r) \end{matrix}\middle| k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}\left(\frac{x}{s}\right)^r\right]. \end{aligned} \tag{128}$$

Proof. In order to prove (128), we use (127) as

$$\begin{aligned} & L\left(z^{g-1}\left(I_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r} S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};tz\right]\right\}\right)(x)\right) \\ & = \int_0^{\infty} e^{-sz}\left(I_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\ldots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\ldots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{(zt)^n}{n!}\right)^r\right\}\right)dz, \end{aligned} \tag{129}$$

$$\begin{aligned} & = x^{\gamma-\delta+\mu+\mu'-1}\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\ldots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\ldots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{1}{n!}\right)^r \\ & \times \frac{\Gamma(\gamma+nr)\Gamma(\gamma+\delta-\mu-\mu'-\nu+nr)\Gamma(\gamma+\nu'-\mu'+nr)}{\Gamma(\gamma+\nu'+nr)\Gamma(\gamma+\delta-\mu-\mu'+nr)\Gamma(\gamma+\delta-\mu'-\nu+nr)}x^{nr} \\ & \times \int_0^{\infty} e^{-sz}z^{g+nr-1}dz. \end{aligned} \tag{130}$$

By interchanging the order of integration and summation and little simplification, we have

$$= \frac{x^{\gamma+\delta-\mu-\mu'-1}}{s^g}\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\ldots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\ldots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{1}{n!}\right)^r \tag{131}$$

$$\times\frac{\Gamma(\gamma+nr)\Gamma(\gamma+\delta-\mu-\mu'-\nu+nr)\Gamma(\gamma+\nu'-\mu'+nr)\Gamma(g+nr)}{\Gamma(\gamma+\nu'+nr)\Gamma(\gamma+\delta-\mu-\mu'+nr)\Gamma(\gamma+\delta-\mu'-\nu+nr)}\left(\frac{x}{s}\right)^{nr}. \tag{132}$$

By applying (44) and (45) in (132) we get

$$\begin{aligned} & = \frac{x^{\gamma+\delta-\mu-\mu'-1}}{s^g}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}\times\sum_{n=0}^{\infty}\frac{\Gamma(a_{1i}+n)\ldots\Gamma(a_{pi}+n)\Gamma(\varepsilon_i/k_i+n\tau_i)}{\Gamma(b_{1i}+n)\ldots\Gamma(b_{qi}+n)\Gamma(n\alpha_i/k_i+\beta_i/k_i)}\left(\frac{1}{n!}\right)^r \\ & \times\frac{\Gamma(\gamma+nr)\Gamma(\gamma+\delta-\mu-\mu'-\nu+nr)\Gamma(\gamma+\nu'-\mu'+nr)\Gamma(g+nr)}{\Gamma(\gamma+\nu'+nr)\Gamma(\gamma+\delta-\mu-\mu'+nr)\Gamma(\gamma+\delta-\mu'-\nu+nr)} \end{aligned} \tag{133}$$

$$\times(k_i)^{n(\tau_i-\alpha_i/k_i)}\left(\frac{x}{s}\right)^{nr}. \tag{134}$$

Interpreting the right-hand side of the above equation, in view of definition (52), we arrive at result (128). □

Theorem 42. *Let $\mu, \mu', \nu, \nu', \delta, \gamma, \alpha_i, \beta_i, \varepsilon_i, \tau_i \in \mathbb{C}$, $k_i \in \Re$, $p_i, q_i \in \mathbb{N}$, $\Re(\alpha_i) > k_i\Re(\tau_i)$, $x > 0$, and $p_i < q_i + 1$, where*

$(i = 1, \ldots, r)$, *such that* $\Re(\delta) > 0$, $\Re(\gamma) < 1 + \min\{\Re(-\nu), \Re(\mu + \mu' - \delta), \Re(\mu + \nu' - \delta)\}$, *then the following fractional integral holds true:*

$$L\left(z^{g-1}\left(I_{x,\infty}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};zt^{-1}\right]\right\}\right)(x)\right) = \frac{x^{\gamma+\delta-\mu-\mu'-1}}{s^g}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$
$$\times {}_{(p+1)r+4}\Psi_{(q+1)r+3}\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(\gamma,r),(\gamma+\delta-\mu-\mu'-\nu,r),(\gamma-\mu'+\nu',r),(g,r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(\gamma+\nu',r),(\gamma+\delta-\mu-\mu',r),(\gamma+\delta-\mu'-\nu,r)\end{matrix}\middle| k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}\left(\frac{1}{xs}\right)^r\right]. \quad (135)$$

Proof. The proof of Theorem 42 is a similar manner of Theorem 41. □

Theorem 43. *Let* $\mu, \mu', \nu, \nu', \delta, \gamma, \alpha_i, \beta_i, \varepsilon_i, \tau_i \in \mathbb{C}$, $k_i \in \Re$, $p_i, q_i \in \mathbb{N}$, $\Re(\alpha_i) > k_i\Re(\tau_i)$, $x > 0$, *and* $p_i < q_i + 1$, *where* $(i = 1, \ldots, r)$, *such that* $\Re(\delta) > 0\Re(\gamma) > \max\{0, \Re(\delta-\mu-\mu'-\nu'), \Re(\nu-\mu)\}$, *then the following fractional derivative formula holds true:*

$$L\left(z^{g-1}\left(D_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};tz\right]\right\}\right)(x)\right) = \frac{x^{\gamma-\delta+\mu+\mu'-1}}{s^g}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$
$$\times {}_{(p+1)r+4}\Psi_{(q+1)r+3}\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(\gamma,r),(\gamma-\delta+\mu+\mu'+\nu',r),(\gamma-\nu+\mu,r),(g,r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(\gamma-\nu,r),(\gamma-\delta+\mu+\mu',r),(\gamma-\delta+\mu+\nu',r)\end{matrix}\middle| k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}\left(\frac{x}{s}\right)r\right]. \quad (136)$$

Proof. In order to prove (136), we use (127) as

$$L\left(z^{g-1}\left(D_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};tz\right]\right\}\right)(x)\right) \quad (137)$$

$$= \int_0^\infty e^{-sz}\left(D_{0,x}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\ldots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\ldots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{(zt)^n}{n!}\right)^r\right\}\right)dz. \quad (138)$$

By interchanging the order of integration and summation, we reduce the right side of (138) to

$$= x^{\gamma-\delta+\mu+\mu'-1}\prod_{i=1}^{r}\sum_{n=0}^{\infty}\frac{(a_{1i})_n\ldots(a_{pi})_n(\varepsilon_i)_{n\tau_i,k_i}}{(b_{1i})_n\ldots(b_{qi})_n\Gamma_{k_i}(n\alpha_i+\beta_i)}\left(\frac{1}{n!}\right)^r$$
$$\times\frac{\Gamma(\gamma+nr)\Gamma(\gamma-\delta+\mu+\mu'+\nu'+nr)\Gamma(\gamma-\nu+\mu+nr)}{\Gamma(\gamma-\nu+nr)\Gamma(\gamma-\delta+\mu+\mu'+nr)\Gamma(\gamma-\delta+\mu+\nu'+nr)}x^{nr} \quad (139)$$
$$\times\int_0^\infty e^{-sz}z^{g+nr-1}dz. \quad (140)$$

After a little simplification and using (44) and (45), we obtain

$$= \frac{x^{\gamma+\delta-\mu-\mu'-1}}{s^g}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}\times\sum_{n=0}^{\infty}\frac{\Gamma(a_{1i}+n)\ldots\Gamma(a_{pi}+n)\Gamma(\varepsilon_i/k_i+n\tau_i)}{\Gamma(b_{1i}+n)\ldots\Gamma(b_{qi}+n)\Gamma(n\alpha_i/k_i+\beta_i/k_i)}\left(\frac{1}{n!}\right)^r$$
$$\times\frac{\Gamma(\gamma+nr)\Gamma(\gamma-\delta+\mu+\mu'+\nu'+nr)\Gamma(\gamma-\nu+\mu+nr)\Gamma(g+nr)}{\Gamma(\gamma-\nu+nr)\Gamma(\gamma-\delta+\mu+\mu'+nr)\Gamma(\gamma-\delta+\mu+\nu'+nr)} \quad (141)$$
$$\times(k_i)^{n(\tau_i-\alpha_i/k_i)}\left(\frac{x}{s}\right)^{nr}. \quad (142)$$

Interpreting the right-hand side of the above equation, in view of definition (52), we arrive at result (136). □

Theorem 44. *Let* $\mu, \mu', \nu, \nu', \delta, \gamma, \alpha_i, \beta_i, \varepsilon_i, \tau_i \in \mathbb{C}, k_i \in \Re, p_i, q_i \in \mathbb{N}, \Re(\alpha_i) > k_i(\tau_i), x > 0$, *and* $p_i < q_i + 1$, *where*

$(i = 1, \ldots, r)$, *such that* $\Re(\delta) > 0$ $\Re(\gamma) < 1 + \min\{\Re(\nu'), \Re(\delta - \mu - \mu'), \Re(\delta - \mu' - \nu)\}$, *then the following fractional derivative formula holds true:*

$$L\left(z^{g-1}\left(D_{x,\infty}^{\mu,\mu',\nu,\nu',\delta}\left\{t^{\gamma-1}\prod_{i=1}^{r}S_{(p_i,q_i)}^{\alpha_i,\beta_i,\varepsilon_i,\tau_i,k_i}\left[a_{1i},\ldots,a_{pi};b_{1i},\ldots,b_{qi};zt^{-1}\right]\right\}\right)(x)\right) = \frac{x^{\gamma-\delta+\mu+\mu'-1}}{s^g}\prod_{i=1}^{r}\frac{(k_i)^{1-\beta_i/k_i}\Gamma(b_{1i})\ldots\Gamma(b_{qi})}{\Gamma(\varepsilon_i/k_i)\Gamma(a_{1i})\ldots\Gamma(a_{pi})}$$
$$\times {}_{(p+1)r+4}\Psi_{(q+1)r+3}\left[\begin{matrix}(a_{11},1)\ldots(a_{1r},1),\ldots,(a_{p1},1)\ldots(a_{pr},1),\left(\frac{\varepsilon_1}{k_1},\tau_1\right),\ldots,\left(\frac{\varepsilon_r}{k_r},\tau_r\right),(g,r),(1-\gamma+\nu',r),(1-\gamma+\delta-\mu-\mu',r),(1-\gamma+\delta-\mu'-\nu,r)\\(b_{11},1)\ldots(b_{1r},1),\ldots,(b_{q1},1)\ldots(b_{qr},1),\left(\frac{\beta_1}{k_1},\frac{\alpha_1}{k_1}\right),\ldots,\left(\frac{\beta_r}{k_r},\frac{\alpha_r}{k_r}\right),(1-\gamma,r),(1-\gamma+\delta-\mu-\mu'-\nu,r),(1-\gamma-\mu'+\nu',r)\end{matrix}\middle| k_1^{\tau_1-\alpha_1/k_1}\ldots k_r^{\tau_r-\alpha_r/k_r}\left(\frac{1}{xs}\right)^r\right]. \tag{143}$$

5. Consequence Results and Concluding Remarks

Marichev-Saigo-Maeda fractional integral and derivative operators have advantage that they generalize the R-L, Weyl, Erdélyi-Kober, and Saigo's fractional integral and derivative operators; therefore, many authors called this a general operator. So, we conclude this paper by emphasizing that many other interesting image formulas can be derived as the specific cases of our leading results (Theorems 5, 6, 21, and 22), involving familiar fractional integral and derivative operators as above said. Further, the S-function defined in (43) possesses the lead that a number of k-Mittag-Leffler functions, K-function, M-series, and Mittag-Leffler function happen to be the particular cases of this function. Some special cases of fractional calculus involved as above said function have been explored in the literature by a numeral of authors ([35–41]) with different arguments. Therefore, results presented in this paper are easily converted in terms of a comparable type of novel interesting integrals with diverse arguments after various suitable parametric replacements.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare to have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the present investigation. All authors read and approved the final manuscript.

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