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Research Article

The Dirichlet Problem for Second-Order Divergence Form Elliptic Operators with Variable Coefficients: The Simple Layer Potential Ansatz

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We investigate the Dirichlet problem related to linear elliptic second-order partial differential operators with smooth coefficients in divergence form in bounded connected domains of $\mathbb{R}^m$ ($m \geq 3$) with Lyapunov boundary. In particular, we show how to represent the solution in terms of a simple layer potential. We use an indirect boundary integral method hinging on the theory of reducible operators and the theory of differential forms.

1. Introduction

As remarked in [1, p. 121], elliptic operators with variable coefficients naturally arise in several areas of physics and engineering. In this paper, we study the Dirichlet problem related to a scalar elliptic second-order differential operator with smooth coefficients in divergence form in a bounded simply connected domain of $\mathbb{R}^m$ ($m \geq 3$) with Lyapunov boundary.

This is a classical problem which nowadays can be treated in several ways. In particular, different potential methods have been developed for such operators (see, e.g., [1–6]).

In the present paper, we obtain the solution of the Dirichlet problem by means of a simple layer potential instead of the classical double layer potential (see, e.g., [6, pp. 73–75]). We use an indirect boundary integral method introduced for the first time in [7] for the m-dimensional Laplacian. It requires neither the knowledge of pseudodifferential operators nor the use of hypersingular integrals, but it hinges on the theory of singular integral operators and the theory of differential forms (for details of the method, see, e.g., [8, Section 2]). The method has been also used to treat different boundary value problems in simply connected domains: the Neumann problem for Laplace equation (via a double layer potential), the Dirichlet problem for the Lamé and Stokes systems, the four boundary value problems of the theory of thermoelastic pseudooscillations, the traction problem for Lamé and Stokes systems, the four basic boundary value problems arising in couple-stress elasticity, and the two boundary value problems of the linear theory of viscoelastic materials with voids (see [9, 10] and the references therein). The method can be applied also in multiply connected domains, as shown for the Laplacian, the linearized elastostatics, and the Stokes system (see [11] and the references therein).

The present paper is organized as follows.

In Section 2, after giving preliminary results, we make use of Fichera's construction of a principal fundamental solution [12] and we prove some identities for the related nuclear double form.

Section 3 is devoted to the study of the Dirichlet problem. It contains the main results concerning the reduction of a certain singular integral operator acting in spaces of differential forms and the integral representation of the solution of the Dirichlet problem by means of a simple layer potential.

2. Preliminary Results

Let Ω be a bounded domain (open connected set) of $\mathbb{R}^m$ ($m \geq 3$).

In this paper, we deal with the Dirichlet problem:

$$\begin{aligned} Eu &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \Sigma, \end{aligned} \tag{1}$$

where E is a scalar second-order differential operator (throughout this paper, we use the Einstein summation convention):

$$Eu(x) = \frac{\partial}{\partial x^i}\left(a^{ij}(x)\frac{\partial u(x)}{\partial x^j}\right). \tag{2}$$

We suppose that the coefficients a^{ij} are defined on $\overline{T}$, T being an open ball containing $\overline{\Omega}$, and we assume that they belong to $C^{2,\lambda}(\overline{T})$, $0 < \lambda \leq 1$.

Moreover, assume that $A = (a^{ij})_{i,j=1,\ldots,m}$ is a symmetric contravariant positive-definite tensor. Then, E is a uniform elliptic operator; that is, there exists $c > 0$ such that $a^{ij}(x)\xi_i\xi_j \geq c|\xi|^2$, for every $(\xi_1,\ldots,\xi_m) \in \mathbb{R}^m$ and for any $x \in T$.

For the sake of simplicity, we suppose that the determinant $|A|$ of A is equal to 1.

It is known that to the contravariant tensor A there corresponds a covariant tensor $A^{-1} = (a_{ij})_{i,j=1,\ldots,m}$ such that

$$a^{ij}a_{jh} = \delta^i_h, \quad \text{for every } i,h = 1,\ldots,m, \tag{3}$$

δ^i_h being the Kronecker delta.

A differential form of degree k (in short a k-form) on T is a function defined on T whose values are in the k-covectors space of $\mathbb{R}^m$. A k-form u can be represented as

$$u = \frac{1}{k!}u_{s_1\cdots s_k}\mathrm{d}x^{s_1}\cdots\mathrm{d}x^{s_k} \tag{4}$$

with respect to an admissible coordinate system $(x_1,\ldots,x_m)$, where $u_{s_1\cdots s_k}$ are the components of a skew-symmetric covariant tensor (for details about differential forms, we refer to [13, 14]).

The symbol $C^h_k(T)$ means the space of all k-forms whose components are continuously differentiable up to the order h in a coordinate system of class C^{h+1} (and then in every coordinate system of class C^{h+1}).

If $u \in C^1_k(T)$, the differential of u is a $(k+1)$-form defined as

$$\mathrm{d}u = \frac{1}{k!}\frac{\partial u_{s_1\cdots s_k}}{\partial x^j}\mathrm{d}x^j\mathrm{d}x^{s_1}\cdots\mathrm{d}x^{s_k}. \tag{5}$$

Further, if $u \in C^0_k(T)$, the adjoint of u is the following $(m-k)$-form:

$$\begin{aligned} *u &= \frac{1}{k!(m-k)!}\delta^{1\cdots m}_{j_1\cdots j_k i_{k+1}\cdots i_m} \\ &\quad \cdot a^{s_1j_1}\cdots a^{s_kj_k}u_{s_1\cdots s_k}\mathrm{d}x^{i_{k+1}}\cdots\mathrm{d}x^{i_m}, \end{aligned} \tag{6}$$

where $\delta^{p_1\cdots p_r}_{q_1\cdots q_r}$ is the generalized Kronecker delta ($r \leq m$). We recall that (see, e.g., [15, p. 127])

$$\delta^{j_1\cdots j_s j_{s+1}\cdots j_m}_{h_1\cdots h_s h_{s+1}\cdots h_m}\delta^{h_{s+1}\cdots h_m}_{k_{s+1}\cdots k_m} = (m-s)!\,\delta^{j_1\cdots j_s j_{s+1}\cdots j_m}_{h_1\cdots h_s k_{s+1}\cdots k_m}. \tag{7}$$

We remark that (see, e.g., [13, p. 285])

$$**u = (-1)^{k(m+1)}u. \tag{8}$$

If $u \in C^1_k(T)$, we define the codifferential of u as the following $(k-1)$-form:

$$\delta u = (-1)^{m(k+1)+1} * \mathrm{d} * u. \tag{9}$$

A differential double form $u_{h,k}(x,y)$ of degree h with respect to x and of degree k with respect to y (in short a double (h,k)-form) is represented as

$$\begin{aligned} &u_{h,k}(x,y) \\ &\quad = \frac{1}{h!k!}u_{s_1\cdots s_h j_1\cdots j_k}(x,y)\,\mathrm{d}x^{s_1}\cdots\mathrm{d}x^{s_h}\mathrm{d}y^{j_1}\cdots\mathrm{d}y^{j_k}. \end{aligned} \tag{10}$$

If $h = k$, we denote it briefly by $u_k(x,y)$.

We proceed to introduce the following double k-form (see [13, p. 204]) defined, for every $x, y \in \overline{T}$, $x \neq y$, as

$$\begin{aligned} \lambda_k(x,y) &= \frac{1}{(k!)^2}L(x,y) \\ &\quad \cdot a_{s_1\cdots s_k i_1\cdots i_k}(y)\,\mathrm{d}x^{s_1}\cdots\mathrm{d}x^{s_k}\mathrm{d}y^{i_1}\cdots\mathrm{d}y^{i_k}, \end{aligned} \tag{11}$$

where, for $k \leq m$,

$$a_{s_1\cdots s_k i_1\cdots i_k} = \begin{vmatrix} a_{s_1i_1} & \cdots & a_{s_1i_k} \\ \vdots & \ddots & \vdots \\ a_{s_ki_1} & \cdots & a_{s_ki_k} \end{vmatrix} = \delta^{l_1\cdots l_k}_{i_1\cdots i_k}a_{s_1l_1}\cdots a_{s_kl_k}, \tag{12}$$

$$\begin{aligned} &L(x,y) \\ &\quad = \frac{1}{(m-2)\,\omega_m}\left[a_{ij}(y)\left(x^i-y^i\right)\left(x^j-y^j\right)\right]^{(2-m)/2} \end{aligned} \tag{13}$$

(ω_m being the hypersurface measure of the unit sphere in $\mathbb{R}^m$) is a parametrix in the sense of Hilbert and E.E. Levi for the operator E. We recall that (if we write $u_{h,k}(x,y) = \mathcal{O}(|x-y|^\alpha)$, $u_{h,k}(x,y)$ being a double (h,k)-form, we mean that all its components are $\mathcal{O}(|x-y|^\alpha)$)

$$L(x,y) = \mathcal{O}\left(|x-y|^{2-m}\right), \tag{14}$$

$$\begin{aligned} \mathrm{d}_xL(x,y) &= \mathcal{O}\left(|x-y|^{1-m}\right), \\ \mathrm{d}_yL(x,y) &= \mathcal{O}\left(|x-y|^{1-m}\right) \end{aligned} \tag{15}$$

(see [13, Section 9]).

The next results provide other properties of L and λ_k.

Lemma 1. *For every $p = 1,\ldots,m$,*

$$\frac{\partial L(x,y)}{\partial y^p} = -\frac{\partial L(x,y)}{\partial x^p} + M(x,y), \qquad x, y \in \overline{T},\ x \neq y, \tag{16}$$

where $M(x,y) = \mathcal{O}(|x-y|^{2-m})$.

Proof. Taking definition (13) into account, we have

$$\begin{aligned}\frac{\partial L(x,y)}{\partial x^p} &= -\frac{1}{2\omega_m}\left[a_{ij}(y)\left(x^i-y^i\right)\left(x^j-y^j\right)\right]^{-m/2}\\ &\cdot\left\{a_{ij}(y)\left[\delta_p^i\left(x^j-y^j\right)+\delta_p^j\left(x^i-y^i\right)\right]\right\}\\ &= -\frac{1}{\omega_m}\left[a_{ij}(y)\left(x^i-y^i\right)\left(x^j-y^j\right)\right]^{-m/2}\\ &\cdot\left[a_{pj}(y)\left(x^j-y^j\right)\right].\end{aligned} \tag{17}$$

On the other hand,

$$\begin{aligned}\frac{\partial L(x,y)}{\partial y^p} &= -\frac{1}{2\omega_m}\left[a_{ij}(y)\left(x^i-y^i\right)\left(x^j-y^j\right)\right]^{-m/2}\\ &\cdot\left[\frac{\partial a_{ij}(y)}{\partial y^p}\left(x^i-y^i\right)\left(x^j-y^j\right)\right]\\ &-\frac{1}{2\omega_m}\left[a_{ij}(y)\left(x^i-y^i\right)\left(x^j-y^j\right)\right]^{-m/2}\\ &\cdot\left\{a_{ij}(y)\left[-\delta_p^i\left(x^j-y^j\right)-\delta_p^j\left(x^i-y^i\right)\right]\right\}\\ &= M(x,y)+\frac{1}{\omega_m}\left[a_{ij}(y)\left(x^i-y^i\right)\left(x^j-y^j\right)\right]^{-m/2}\\ &\cdot\left[a_{pj}(y)\left(x^j-y^j\right)\right] = M(x,y)-\frac{\partial L(x,y)}{\partial x^p}\end{aligned} \tag{18}$$

and this yields the claim. □

The identities proved in the next proposition generalize the ones obtained by Colautti [16, p. 309] for the Laplacian.

Proposition 2. *Let λ_k be the double k-form defined by (11). Then, for every $x \neq y$, the following properties hold:*

$$\begin{aligned}*_x\lambda_k(x,y) &= (-1)^{k(m-k)} *_y\lambda_{m-k}(x,y)\\ &\quad+\tau_{m-k,k}(x,y), \quad k\le m,\end{aligned} \tag{19}$$

where

$$\tau_{m-k,k}(x,y) = \mathcal{O}\left(|x-y|^{3-m}\right), \tag{20}$$

$$\begin{aligned}*_x\mathrm{d}_x\lambda_k(x,y) &= (-1)^{mk+1} *_y\mathrm{d}_y\lambda_{m-k-1}(x,y)\\ &\quad+\gamma_{m-k-1,k}(x,y), \quad k<m,\end{aligned} \tag{21}$$

where $\gamma_{m-k-1,k}(x,y) = \mathcal{O}(|x-y|^{2-m})$; and

$$\delta_x\lambda_{k+1}(x,y) = \mathrm{d}_y\lambda_k(x,y)+\epsilon_{k,k+1}(x,y), \quad k<m, \tag{22}$$

where

$$\epsilon_{k,k+1}(x,y) = \mathcal{O}\left(|x-y|^{2-m}\right). \tag{23}$$

Proof. First, we prove (19). It follows from (12), (3), and (7) that

$$\begin{aligned}*_y\lambda_{m-k}(x,y) &= \frac{(-1)^{k(m-k)}}{k!\,(m-k)!}\\ &\cdot\delta^{1\cdots m}_{p_1\cdots p_k q_1\cdots q_{m-k}} L(x,y)\,\mathrm{d}x^{q_1}\cdots\mathrm{d}x^{q_{m-k}}\mathrm{d}y^{p_1}\cdots\mathrm{d}y^{p_k}.\end{aligned} \tag{24}$$

On the other hand,

$$\begin{aligned}*_x\lambda_k(x,y) &= \frac{1}{(m-k)!\,(k!)^2}\\ &\cdot\delta^{1\cdots m}_{j_1\cdots j_k q_1\cdots q_{m-k}}\left[\left(a^{s_1j_1}\cdots a^{s_kj_k}\right)(x)\right.\\ &\left.-\left(a^{s_1j_1}\cdots a^{s_kj_k}\right)(y)\right]\\ &\cdot L(x,y)\,a_{s_1\cdots s_k p_1\cdots p_k}(y)\,\mathrm{d}x^{q_1}\cdots\mathrm{d}x^{q_{m-k}}\mathrm{d}y^{p_1}\cdots\mathrm{d}y^{p_k}\\ &+\frac{1}{(m-k)!\,(k!)^2}\delta^{1\cdots m}_{j_1\cdots j_k q_1\cdots q_{m-k}}\left(a^{s_1j_1}\cdots a^{s_kj_k}\right)(y)\\ &\cdot L(x,y)\,a_{s_1\cdots s_k p_1\cdots p_k}(y)\,\mathrm{d}x^{q_1}\cdots\mathrm{d}x^{q_{m-k}}\mathrm{d}y^{p_1}\cdots\mathrm{d}y^{p_k}.\end{aligned} \tag{25}$$

From (12), (3), and (7), we have that

$$\begin{aligned}*_x\lambda_k(x,y) &= \frac{1}{(m-k)!\,(k!)^2}\\ &\cdot\delta^{1\cdots m}_{j_1\cdots j_k q_1\cdots q_{m-k}}\left[\left(a^{s_1j_1}\cdots a^{s_kj_k}\right)(x)\right.\\ &\left.-\left(a^{s_1j_1}\cdots a^{s_kj_k}\right)(y)\right]L(x,y)\\ &\cdot a_{s_1\cdots s_k p_1\cdots p_k}(y)\,\mathrm{d}x^{q_1}\cdots\mathrm{d}x^{q_{m-k}}\mathrm{d}y^{p_1}\cdots\mathrm{d}y^{p_k}\\ &+\frac{1}{(m-k)!k!}\\ &\cdot\delta^{1\cdots m}_{p_1\cdots p_k q_1\cdots q_{m-k}} L(x,y)\,\mathrm{d}x^{q_1}\cdots\mathrm{d}x^{q_{m-k}}\mathrm{d}y^{p_1}\cdots\mathrm{d}y^{p_k}\\ &= \tau_{m-k,k}(x,y)+(-1)^{k(m-k)} *_y\lambda_{m-k}(x,y),\end{aligned} \tag{26}$$

where $\tau_{m-k,k}(x,y)$ satisfies (20) on account of

$$\left(a^{s_1j_1}\cdots a^{s_kj_k}\right)(x)-\left(a^{s_1j_1}\cdots a^{s_kj_k}\right)(y) = \mathcal{O}\left(|x-y|\right) \tag{27}$$

and (8).

Now we pass to show (21). With calculations analogue to (26), we have that

$$\begin{aligned}
*_x \mathrm{d}_x \lambda_k(x,y) &= \frac{1}{(m-k-1)!\,(k!)^2}\delta^{1\cdots m}_{jj_1\cdots j_k i_{k+2}\cdots i_m} \\
&\cdot \left[\left(a^{sj}a^{s_1 j_1}\cdots a^{s_k j_k}\right)(x)\right. \\
&\left.-\left(a^{sj}a^{s_1 j_1}\cdots a^{s_k j_k}\right)(y)\right]\frac{\partial L(x,y)}{\partial x^s} a_{s_1\cdots s_k i_1\cdots i_k}(y) \\
&\cdot \mathrm{d}x^{i_{k+2}}\cdots \mathrm{d}x^{i_m}\mathrm{d}y^{i_1}\cdots \mathrm{d}y^{i_k} + \frac{1}{(m-k-1)!\,(k!)^2} \\
&\cdot \delta^{1\cdots m}_{jj_1\cdots j_k i_{k+2}\cdots i_m}\left(a^{sj}a^{s_1 j_1}\cdots a^{s_k j_k}\right)(y)\frac{\partial L(x,y)}{\partial x^s} \\
&\cdot a_{s_1\cdots s_k i_1\cdots i_k}(y)\,\mathrm{d}x^{i_{k+2}}\cdots \mathrm{d}x^{i_m}\mathrm{d}y^{i_1}\cdots \mathrm{d}y^{i_k} \\
&= \gamma'_{m-k-1,k}(x,y) + \frac{1}{(m-k-1)!k!} \\
&\cdot \delta^{1\cdots m}_{ji_{k+2}\cdots i_m i_1\cdots i_k} a^{sj}(y)\cdot \frac{\partial L(x,y)}{\partial x^s} \\
&\cdot \mathrm{d}x^{i_{k+2}}\cdots \mathrm{d}x^{i_m}\mathrm{d}y^{i_1}\cdots \mathrm{d}y^{i_k},
\end{aligned} \tag{28}$$

where $\gamma'_{m-k-1,k}(x,y) = \mathcal{O}(|x-y|^{1-m})$ thanks to (15) and (27).

Moreover,

$$\begin{aligned}
*_y \mathrm{d}_y \lambda_{m-k-1}(x,y) &= \frac{1}{k!\,[(m-k-1)!]^2}\delta^{1\cdots m}_{jq_1\cdots q_{m-k-1}i_1\cdots i_k} \\
&\cdot \left(a^{sj}a^{p_1 q_1}\cdots a^{p_{m-k-1}q_{m-k-1}}\right)(y) \\
&\cdot \frac{\partial}{\partial y^s}\left[L(x,y)\,a_{s_1\cdots s_{m-k-1}p_1\cdots p_{m-k-1}}(y)\right] \\
&\cdot \mathrm{d}x^{s_1}\cdots \mathrm{d}x^{s_{m-k-1}}\mathrm{d}y^{i_1}\cdots \mathrm{d}y^{i_k}.
\end{aligned} \tag{29}$$

Arguing again as in (26) and taking Lemma 1 into account, we get

$$\begin{aligned}
*_y \mathrm{d}_y \lambda_{m-k-1}(x,y) &= \frac{(-1)^{k(m-k-1)}}{k!\,(m-k-1)!} \\
&\cdot \delta^{1\cdots m}_{ji_1\cdots i_k s_1\cdots s_{m-k-1}} a^{sj}(y)\frac{\partial L(x,y)}{\partial y^s} \\
&\cdot \mathrm{d}x^{s_1}\cdots \mathrm{d}x^{s_{m-k-1}}\mathrm{d}y^{i_1}\cdots \mathrm{d}y^{i_k} + \frac{1}{k!\,[(m-k-1)!]^2} \\
&\cdot \delta^{1\cdots m}_{jq_1\cdots q_{m-k-1}i_1\cdots i_k}\left(a^{sj}a^{p_1 q_1}\cdots a^{p_{m-k-1}q_{m-k-1}}\right)(y) \\
&\cdot L(x,y)\frac{\partial}{\partial y^s}a_{s_1\cdots s_{m-k-1}p_1\cdots p_{m-k-1}}(y) \\
&\cdot \mathrm{d}x^{s_1}\cdots \mathrm{d}x^{s_{m-k-1}}\mathrm{d}y^{i_1}\cdots \mathrm{d}y^{i_k} = -\frac{(-1)^{mk}}{k!\,(m-k-1)!} \\
&\cdot \delta^{1\cdots m}_{ji_1\cdots i_k s_1\cdots s_{m-k-1}} a^{sj}(y) \\
&\cdot \frac{\partial L(x,y)}{\partial x^s}\mathrm{d}x^{s_1}\cdots \mathrm{d}x^{s_{m-k-1}}\mathrm{d}y^{i_1}\cdots \mathrm{d}y^{i_k} \\
&+ \frac{(-1)^{mk}}{k!\,(m-k-1)!}\delta^{1\cdots m}_{ji_1\cdots i_k s_1\cdots s_{m-k-1}} a^{sj}(y) \\
&\cdot M(x,y)\,\mathrm{d}x^{s_1}\cdots \mathrm{d}x^{s_{m-k-1}}\mathrm{d}y^{i_1}\cdots \mathrm{d}y^{i_k} \\
&+ \gamma''_{m-k-1,k}(x,y) = (-1)^{mk+1} *_x \mathrm{d}_x \lambda_k(x,y) \\
&+ \gamma'''_{m-k-1,k}(x,y) + \gamma''_{m-k-1,k}(x,y),
\end{aligned} \tag{30}$$

where both $\gamma''_{m-k-1,k}(x,y)$ and $\gamma'''_{m-k-1,k}(x,y)$ are $\mathcal{O}(|x-y|^{2-m})$. Then, we obtain the claim by setting

$$\begin{aligned}
\gamma_{m-k-1,k}(x,y) &= (-1)^{mk}\left(\gamma''_{m-k-1,k}(x,y) + \gamma'''_{m-k-1,k}(x,y)\right).
\end{aligned} \tag{31}$$

Finally, we prove (22). Thanks to (9) and (19), we have

$$\begin{aligned}
\delta_x \lambda_{k+1}(x,y) &= (-1)^{m-k} *_x \mathrm{d}_x *_y \lambda_{m-k-1}(x,y) \\
&\quad + (-1)^{mk+1} *_x \mathrm{d}_x \tau_{m-k-1,k+1}(x,y) \\
&= (-1)^{m-k} *_y *_x \mathrm{d}_x \lambda_{m-k-1}(x,y) \\
&\quad + \epsilon'_{k,k+1}(x,y),
\end{aligned} \tag{32}$$

where $\epsilon'_{k,k+1}(x,y) = \mathcal{O}(|x-y|^{2-m})$. Now, by using (21) and (8), we get

$$\begin{aligned}
\delta_x \lambda_{k+1}(x,y) &= (-1)^{m-k-km+1} *_y *_y \mathrm{d}_y \lambda_k(x,y) \\
&\quad + (-1)^{m-k} *_y \gamma_{k,m-k-1}(x,y) \\
&\quad + \epsilon'_{k,k+1}(x,y) \\
&= \mathrm{d}_y \lambda_k(x,y) + \epsilon_{k,k+1}(x,y),
\end{aligned} \tag{33}$$

and hence the claim with

$$\begin{aligned}
\epsilon_{k,k+1}(x,y) &= (-1)^{m-k} *_y \gamma_{k,m-k-1}(x,y) \\
&\quad + \epsilon'_{k,k+1}(x,y).
\end{aligned} \tag{34}$$

□

Proposition 3. *If $u \in C^2_k(T)$, then*

$$(\delta \mathrm{d} + \mathrm{d}\delta)\,u = -Eu + Fu, \tag{35}$$

where F is a linear first-order differential operator whose coefficients depend only on first- and second-order derivatives of entries of the tensor A.

In particular,

$$\left(\delta_x \mathrm{d}_x + \mathrm{d}_x \delta_x\right) \lambda_k (x, y) = F_x \left[\lambda_k (x, y)\right], \quad x \neq y. \tag{36}$$

Proof. We begin by observing that

$$\begin{aligned} \mathrm{d}\delta u = {} & (-1)^{m(k+1)+1} \mathrm{d} \frac{1}{(k-1)!k!\,(m-k)!} \delta^{1\cdots m}_{hh_{k+1}\cdots h_m i_2 \cdots i_k} a^{jh} a^{i_{k+1}h_{k+1}} \cdots a^{i_m h_m} \delta^{1\cdots m}_{j_1\cdots j_k i_{k+1}\cdots i_m} a^{s_1 j_1} \cdots a^{s_k j_k} \frac{\partial}{\partial x^j} u_{s_1\cdots s_k} \mathrm{d}x^{i_2} \cdots \mathrm{d}x^{i_k} \\ & + (-1)^{m(k+1)+1} \mathrm{d} \frac{1}{(k-1)!k!\,(m-k)!} \delta^{1\cdots m}_{hh_{k+1}\cdots h_m i_2 \cdots i_k} a^{jh} a^{i_{k+1}h_{k+1}} \cdots a^{i_m h_m} \cdot \delta^{1\cdots m}_{j_1\cdots j_k i_{k+1}\cdots i_m} \frac{\partial}{\partial x^j} \left(a^{s_1 j_1} \cdots a^{s_k j_k}\right) \\ & \cdot u_{s_1\cdots s_k} \mathrm{d}x^{i_2} \cdots \mathrm{d}x^{i_k}. \end{aligned} \tag{37}$$

Since A is symmetric and $|A| = 1$, we get

$$\begin{aligned} & \delta^{1\cdots m}_{j_1\cdots j_k i_{k+1}\cdots i_m} a^{s_1 j_1} \cdots a^{s_k j_k} a^{i_{k+1}h_{k+1}} \cdots a^{i_m h_m} \\ & \quad = \delta^{s_1\cdots s_k h_{k+1}\cdots h_m}_{1\cdots m} \end{aligned} \tag{38}$$

and, keeping in mind (7), we get

$$\begin{aligned} \mathrm{d}\delta u = {} & -\frac{1}{(k-1)!k!} \delta^{s_1\cdots s_k}_{hi_2\cdots i_k} \frac{\partial}{\partial x^{i_1}} \Bigg(a^{jh} \\ & \cdot \frac{\partial u_{s_1\cdots s_k}}{\partial x^j} \Bigg) \mathrm{d}x^{i_1} \mathrm{d}x^{i_2} \cdots \mathrm{d}x^{i_k} \\ & + (-1)^{m(k+1)+1} \frac{1}{(k-1)!k!\,(m-k)!} \\ & \cdot \delta^{1\cdots m}_{hh_{k+1}\cdots h_m i_2\cdots i_k} \delta^{1\cdots m}_{j_1\cdots j_k i_{k+1}\cdots i_m} \\ & \cdot \frac{\partial}{\partial x^{i_1}} \left[a^{jh} a^{i_{k+1}h_{k+1}} \cdots a^{i_m h_m} \frac{\partial}{\partial x^j} \left(a^{s_1 j_1} \cdots a^{s_k j_k}\right) \right. \\ & \left. \cdot u_{s_1\cdots s_k} \right] \mathrm{d}x^{i_1} \mathrm{d}x^{i_2} \cdots \mathrm{d}x^{i_k}. \end{aligned} \tag{39}$$

On the other hand,

$$\begin{aligned} \delta \mathrm{d}u = {} & (-1)^{m(k+2)+1} \frac{1}{(m-k-1)!\,(k!)^2} \\ & \cdot \delta^{1\cdots m}_{qj_{k+2}\cdots j_m i_1\cdots i_k} a^{pq} a^{i_{k+2}j_{k+2}} \cdots a^{i_m j_m} \delta^{1\cdots m}_{hh_1\cdots h_k i_{k+2}\cdots i_m} \\ & \cdot \frac{\partial}{\partial x^p} \left(a^{jh} a^{s_1 h_1} \cdots a^{s_k h_k} \frac{\partial u_{s_1\cdots s_k}}{\partial x^j} \right) \mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_k}. \end{aligned} \tag{40}$$

Then,

$$\begin{aligned} \mathrm{d}\delta u + \delta \mathrm{d}u = {} & -\frac{1}{(k-1)!k!} \delta^{s_1\cdots s_k}_{hi_2\cdots i_k} a^{jh} \frac{\partial^2 u_{s_1\cdots s_k}}{\partial x^{i_1} \partial x^j} \mathrm{d}x^{i_1} \mathrm{d}x^{i_2} \cdots \mathrm{d}x^{i_k} + (-1)^{m(k+2)+1} \frac{1}{(m-k-1)!\,(k!)^2} \delta^{1\cdots m}_{qj_{k+2}\cdots j_m i_1\cdots i_k} \\ & \cdot a^{pq} a^{i_{k+2}j_{k+2}} \cdots a^{i_m j_m} \cdot \delta^{1\cdots m}_{hh_1\cdots h_k i_{k+2}\cdots i_m} a^{jh} a^{s_1 h_1} \cdots a^{s_k h_k} \frac{\partial^2 u_{s_1\cdots s_k}}{\partial x^p \partial x^j} \mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_k} - \frac{1}{(k-1)!k!} \delta^{s_1\cdots s_k}_{hi_2\cdots i_k} \frac{\partial a^{jh}}{\partial x^{i_1}} \frac{\partial u_{s_1\cdots s_k}}{\partial x^j} \\ & \cdot \mathrm{d}x^{i_1} \mathrm{d}x^{i_2} \cdots \mathrm{d}x^{i_k} + (-1)^{m(k+1)+1} \frac{1}{(k-1)!k!\,(m-k)!} \delta^{1\cdots m}_{hh_{k+1}\cdots h_m i_2\cdots i_k} \delta^{1\cdots m}_{j_1\cdots j_k i_{k+1}\cdots i_m} a^{jh} a^{i_{k+1}h_{k+1}} \cdots a^{i_m h_m} \\ & \cdot \frac{\partial}{\partial x^j} \left(a^{s_1 j_1} \cdots a^{s_k j_k}\right) \frac{\partial u_{s_1\cdots s_k}}{\partial x^{i^1}} \mathrm{d}x^{i_1} \mathrm{d}x^{i_2} \cdots \mathrm{d}x^{i_k} + (-1)^{m(k+2)+1} \frac{1}{(m-k-1)!\,(k!)^2} \\ & \cdot \delta^{1\cdots m}_{qj_{k+2}\cdots j_m i_1\cdots i_k} a^{pq} a^{i_{k+2}j_{k+2}} \cdots a^{i_m j_m} \delta^{1\cdots m}_{hh_1\cdots h_k i_{k+2}\cdots i_m} \frac{\partial}{\partial x^p} \left(a^{jh} a^{s_1 h_1} \cdots a^{s_k h_k}\right) \frac{\partial u_{s_1\cdots s_k}}{\partial x^j} \mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_k} + (-1)^{m(k+1)+1} \\ & \cdot \frac{1}{(k-1)!k!\,(m-k)!} \delta^{1\cdots m}_{hh_{k+1}\cdots h_m i_2\cdots i_k} \delta^{1\cdots m}_{j_1\cdots j_k i_{k+1}\cdots i_m} \frac{\partial}{\partial x^{i_1}} \left(a^{jh} a^{i_{k+1}h_{k+1}} \cdots a^{i_m h_m}\right) \frac{\partial}{\partial x^j} \left(a^{s_1 j_1} \cdots a^{s_k j_k}\right) u_{s_1\cdots s_k} \mathrm{d}x^{i_1} \mathrm{d}x^{i_2} \cdots \mathrm{d}x^{i_k} \\ & + (-1)^{m(k+1)+1} \frac{1}{(k-1)!k!\,(m-k)!} \delta^{1\cdots m}_{hh_{k+1}\cdots h_m i_2\cdots i_k} \delta^{1\cdots m}_{j_1\cdots j_k i_{k+1}\cdots i_m} a^{jh} a^{i_{k+1}h_{k+1}} \cdots a^{i_m h_m} \frac{\partial^2}{\partial x^{i_1} \partial x^j} \left(a^{s_1 j_1} \cdots a^{s_k j_k}\right) \\ & \cdot u_{s_1\cdots s_k} \mathrm{d}x^{i_1} \mathrm{d}x^{i_2} \cdots \mathrm{d}x^{i_k} = -\frac{1}{k!} \frac{\partial}{\partial x^p} \left(a^{jp} \frac{\partial u_{s_1\cdots s_k}}{\partial x^j} \right) \mathrm{d}x^{s_1} \cdots \mathrm{d}x^{s_k} + \frac{m! - (m-k)!}{m!k!} \frac{\partial a^{jp}}{\partial x^p} \frac{\partial u_{s_1\cdots s_k}}{\partial x^j} \mathrm{d}x^{s_1} \cdots \mathrm{d}x^{s_k} \\ & - \frac{m! - (m-k)!}{m!\,(k-1)!k!} \delta^{s_1 s_2\cdots s_k}_{hi_2\cdots i_k} \frac{\partial a^{jh}}{\partial x^{i_1}} \frac{\partial u_{s_1\cdots s_k}}{\partial x^j} \mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_k} - \frac{(m-k+1)!}{m!\,(k-1)!^2} \delta^{js_2\cdots s_k}_{hi_2\cdots i_k} \frac{\partial a^{s_1 h}}{\partial x^j} \frac{\partial u_{s_1\cdots s_k}}{\partial x^{i_1}} \mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_k} + \frac{(m-k)!}{m!\,(k-1)!k!} \delta^{js_2\cdots s_k}_{i_1 i_2\cdots i_k} \end{aligned}$$

$$\cdot \frac{\partial a^{s_1 h}}{\partial x^h} \frac{\partial u_{s_1\cdots s_k}}{\partial x^j} \mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_k} - \frac{(m-k)!}{m!\,(k-1)\,!^2} \delta^{js_2\cdots s_k}_{hi_2\cdots i_k} \frac{\partial a^{s_1 h}}{\partial x^{i_1}} \frac{\partial u_{s_1\cdots s_k}}{\partial x^j} \mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_k} - \frac{(m-k+1)!}{m!\,(k-1)!} \delta^{js_2\cdots s_k}_{hi_2\cdots i_k} \frac{\partial^2 a^{s_1 h}}{\partial x^{i_1} \partial x^j} u_{s_1\cdots s_k} \mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_k}$$
$$= -\frac{1}{k!} E u_{s_1\cdots s_k} \mathrm{d}x^{s_1} \cdots \mathrm{d}x^{s_k} + \frac{1}{k!} F u_{s_1\cdots s_k} \mathrm{d}x^{s_1} \cdots \mathrm{d}x^{s_k} \tag{41}$$

and this proves (35). Finally, (36) follows from (35). □

Finally, following Fichera we employ the parametrix L to construct a principal fundamental solution of the differential operator E (see [12]).

Lemma 4. *There exists $\zeta(x,y)$ such that the function*

$$S(x,y) = L(x,y) + \zeta(x,y), \quad x \in \overline{T},\ y \in T, \tag{42}$$

is a principal fundamental solution of E. In particular, we have

$$\begin{aligned} E_x S(x,y) &= 0, \quad x \in T,\ y \in T,\ x \neq y, \\ S(x,y) &= 0, \quad x \in \partial T,\ y \in T, \\ S(x,y) &= \mathcal{O}\left(|x-y|^{2-m}\right). \end{aligned} \tag{43}$$

Moreover, for every $x \neq y$,

$$\mathrm{d}_x \zeta(x,y) = \mathcal{O}\left(|x-y|^{2-m-\gamma}\right) \tag{44}$$

for some $0 < \gamma \leq 1$.

Proof. The existence of $\zeta(x,y)$ can be obtained as the solution of a certain integral equation (see [12, §2]). In [12] properties (43) are proved and $\zeta(x,y)$ is written as

$$\zeta(x,y) = G(x,y) + \int_T L(x,w)\, R(w,y)\, \mathrm{d}w, \tag{45}$$

where G is a smooth function on T and, for some $0 < \gamma \leq 1$,

$$R(w,y) = \mathcal{O}\left(|w-y|^{1-m-\gamma}\right). \tag{46}$$

Then, by (15),

$$\begin{aligned} \mathrm{d}_x \zeta(x,y) &= \mathrm{d}_x G(x,y) + \int_T \mathrm{d}_x L(x,w)\, R(w,y)\, \mathrm{d}w \\ &= \mathcal{O}\left(|x-y|^{2-m-\gamma}\right). \end{aligned} \tag{47}$$

□

3. The Dirichlet Problem

In this section, we suppose that the domain Ω is such that $\mathbb{R}^m \setminus \overline{\Omega}$ is connected and such that its boundary $\Sigma = \partial\Omega$ is a Lyapunov surface (i.e., $\Sigma \in C^{1,\lambda}$, $0 < \lambda \leq 1$).

By $n(y) = (n_1(y), \dots, n_m(y))$, we denote the outwards unit normal vector at the point $y \in \Sigma$ and by $\nu(y) = (\nu_1(y), \dots, \nu_m(y))$ we denote the conormal vector at the point $y \in \Sigma$ associated with the operator E and defined as $\nu_i(y) = a^{ij}(y) n_j(y)$ $(i = 1, \dots, m)$. By $\partial u/\partial\nu$ we denote the conormal derivative

$$\frac{\partial u}{\partial \nu} = a^{ij} n_j \frac{\partial u}{\partial y^i}. \tag{48}$$

As usual, the symbols $L^p(\Sigma)$ and $W^{1,p}(\Sigma)$ $(1 < p < +\infty)$ stand for the classical Lebesgue and Sobolev spaces, respectively.

By $L^p_k(\Sigma)$, we denote the space of all k-forms whose components are L^p real-valued functions in a coordinate system of class C^1 (and then in every coordinate system of class C^1).

We will look for the solution of the Dirichlet problem for the operator E in the domain Ω in the form of a simple layer potential. To this end, we introduce the space $\mathcal{S}^p$.

Definition 5. The function u belongs to $\mathcal{S}^p$ if and only if there exists $\varphi \in L^p(\Sigma)$ such that it can be represented by means of a simple layer potential; that is,

$$u(x) = \int_\Sigma \varphi(y)\, S(x,y)\, \mathrm{d}\sigma_y, \quad x \in \Omega. \tag{49}$$

Specifically our aim is to give an existence and uniqueness theorem for the Dirichlet problem

$$\begin{aligned} &u \in \mathcal{S}^p, \\ &Eu = 0 \quad \text{in } \Omega, \\ &u = f \quad \text{on } \Sigma,\ f \in W^{1,p}(\Sigma). \end{aligned} \tag{50}$$

First, we prove the following formula.

Proposition 6. *For any $u \in W^{1,p}(\Sigma)$,*

$$\begin{aligned} &\frac{\partial}{\partial \nu_z} \left(\int_\Sigma u(x) \frac{\partial}{\partial \nu_x} L(z,x)\, \mathrm{d}\sigma_x \right) \mathrm{d}\sigma_z \\ &= \mathrm{d}_z \int_\Sigma \mathrm{d}u(x) \wedge \lambda_{m-2}(z,x) \\ &\quad + \int_\Sigma u(x) \wedge F_z\left[\lambda_{m-1}(z,x)\right] \\ &\quad - \int_\Sigma u(x) \wedge \eta_{m-1}(z,x), \quad z \in \Sigma, \end{aligned} \tag{51}$$

where F is the linear first-order differential operator considered in Proposition 3 and

$$\eta_{m-1}(z,x) = \mathcal{O}\left(|z-x|^{1-m}\right). \tag{52}$$

Proof. Set, for every $z \notin \Sigma$,

$$\begin{aligned} U(z) &= \int_\Sigma du(x) \wedge \lambda_{m-2}(z,x), \\ V(z) &= \int_\Sigma u(x) \wedge d_z[\lambda_{m-1}(z,x)]. \end{aligned} \tag{53}$$

On account of (22) and (36), we get

$$\begin{aligned} dU(z) &= -\int_\Sigma u(x) \wedge d_z d_x [\lambda_{m-2}(z,x)] \\ &= -\int_\Sigma u(x) \wedge d_z \delta_z [\lambda_{m-1}(z,x)] \\ &\quad + \int_\Sigma u(x) \wedge d_z [\epsilon_{m-2,m-1}(z,x)] \\ &= \int_\Sigma u(x) \wedge \delta_z d_z [\lambda_{m-1}(z,x)] \\ &\quad - \int_\Sigma u(x) \wedge F_z [\lambda_{m-1}(z,x)] \\ &\quad + \int_\Sigma u(x) \wedge d_z [\epsilon_{m-2,m-1}(z,x)] \\ &= \delta V(z) - \int_\Sigma u(x) \wedge F_z [\lambda_{m-1}(z,x)] \\ &\quad + \int_\Sigma u(x) \wedge \eta_{m-1}(z,x) \end{aligned} \tag{54}$$

and (52) follows from (23).

On the other hand, if $A_{\hat{i}}^{\hat{j}}$ is the minor of A^{-1} obtained deleting the ith row and the jth column, for $z \in \Omega$, $x \in \Sigma$ we get

$$\begin{aligned} d_z[\lambda_{m-1}(z,x)] &= d_z L(z,x) \\ &\cdot \left|A_i^j(x)\right| dz^1 \cdots \hat{\imath} \cdots dz^m dx^1 \cdots \hat{\jmath} \cdots dx^m \\ &= \frac{\partial L(z,x)}{\partial z^i} (-1)^{i-j} \\ &\cdot \left|A_i^j(x)\right| dz^1 \cdots dz^m (-1)^{j-1} dx^1 \cdots \hat{\jmath} \cdots dx^m \\ &= \frac{\partial L(z,x)}{\partial z^i} a^{ij}(x) n_j(x) d\sigma_x dz^1 \cdots dz^m = -a^{ij}(x) \\ &\cdot n_j(x) \frac{\partial L(z,x)}{\partial x^i} d\sigma_x dz^1 \cdots dz^m \\ &= -\frac{\partial L(z,x)}{\partial \nu_x} d\sigma_x dz^1 \cdots dz^m. \end{aligned} \tag{55}$$

Therefore,

$$\begin{aligned} V(z) &= -\int_\Sigma u(x) \frac{\partial L(z,x)}{\partial \nu_x} d\sigma_x dz^1 \cdots dz^m \\ &= V_0(z) dz^1 \cdots dz^m, \\ \delta V(z) &= (-1)^{m(m+1)+1} * d * V(z) = - * dV_0(z) \\ &= - * \frac{\partial V_0(z)}{\partial z^j} dz^j \\ &= -\frac{1}{(m-1)!} \delta^{1\cdots m}_{hk_2\cdots k_m} a^{jh}(z) \frac{\partial V_0(z)}{\partial z^j} dz^{k_2} \cdots dz^{k_m} \\ &= -a^{jh}(z) \frac{\partial V_0(z)}{\partial z^j} (-1)^{h-1} dz^1 \cdots \hat{h} \cdots dz^m \\ &= -a^{jh}(z) n_h(z) \frac{\partial V_0(z)}{\partial z^j} d\sigma_z = -\frac{\partial V_0(z)}{\partial \nu_z} d\sigma_z. \end{aligned} \tag{56}$$

Then, if $z \in \Sigma$,

$$\lim_{z' \to z} \delta V(z') = -\frac{\partial V_0(z)}{\partial \nu_z} d\sigma_z \tag{57}$$

and this concludes the proof. □

Remark 7. We note that (51) generalizes the following identity (see [7] [8, Proposition 2.2]):

$$\begin{aligned} &\frac{\partial}{\partial n_z} \left(\int_\Sigma u(x) \frac{\partial}{\partial n_x} s(z,x) d\sigma_x \right) d\sigma_z \\ &\quad = d_z \int_\Sigma du(x) \wedge s_{m-2}(z,x), \quad u \in W^{1,p}(\Sigma), \end{aligned} \tag{58}$$

where $s(z,x)$ and $s_k(z,x)$ denote the fundamental solution for Laplace equation and the double k-form associated with $s(z,x)$, respectively.

We recall that if B and $\widetilde{B}$ are two Banach spaces and $C : B \to \widetilde{B}$ is a continuous linear operator, we say that C can be reduced on the left if there exists a continuous linear operator $C' : \widetilde{B} \to B$ such that $C'C = I + K$, where I stands for the identity operator on B and $K : B \to B$ is compact. One of the main properties of such operators is that equation $C\alpha = \beta$ has a solution if and only if $\langle \gamma, \beta \rangle = 0$ for any γ such that $C^* \gamma = 0$, C^* being the adjoint of C (see [17, 18]).

Theorem 8. *Let $\widetilde{J} : L^p(\Sigma) \to L^p_1(\Sigma)$ be the singular integral operator defined as*

$$\begin{aligned} \widetilde{J}\varphi(x) &= \int_\Sigma \varphi(y) d_x [S(x,y)] d\sigma_y, \\ &\qquad \varphi \in L^p(\Sigma), \ x \in \Sigma. \end{aligned} \tag{59}$$

Then, $\widetilde{J}$ can be reduced on the left by the operator $J' : L^p_1(\Sigma) \to L^p(\Sigma)$:

$$J'\psi(z) = \underset{\Sigma}{*} \int_\Sigma \psi(x) \wedge d_z [\lambda_{m-2}(z,x)], \quad z \in \Sigma, \tag{60}$$

where the symbol $\underset{\Sigma}{*}$ *means that if* $w = w_0 \mathrm{d}\sigma$ *is an* $(m-1)$*-form on* Σ, *then* $\underset{\Sigma}{*} w = w_0$.

Proof. We start with the observation that

$$\begin{aligned} \tilde{J}\varphi(x) &= \int_\Sigma \varphi(y)\, \mathrm{d}_x [L(x,y)]\, \mathrm{d}\sigma_y \\ &\quad + \int_\Sigma \varphi(y)\, \mathrm{d}_x [\zeta(x,y)]\, \mathrm{d}\sigma_y \\ &= J\varphi(x) + Z\varphi(x) \end{aligned} \tag{61}$$

and then

$$J'\tilde{J}\varphi = J'J\varphi + J'Z\varphi. \tag{62}$$

The operator $J'Z$ is compact because of (44). Concerning $J'J$, keeping in mind Proposition 6 and setting $u(x) = \int_\Sigma \varphi(y) L(x,y) \mathrm{d}\sigma_y$, we get

$$\begin{aligned} &J'J\varphi(z) \\ &\quad = \underset{\Sigma}{*} \int_\Sigma \int_\Sigma \varphi(y)\, \mathrm{d}_x [L(x,y)]\, \mathrm{d}\sigma_y \wedge \mathrm{d}_z [\lambda_{m-2}(z,x)] \\ &\quad = \underset{\Sigma}{*} \int_\Sigma \mathrm{d}u(x) \wedge \mathrm{d}_z [\lambda_{m-2}(z,x)] \\ &\quad = \frac{\partial}{\partial \nu_z} \int_\Sigma u(x) \frac{\partial}{\partial \nu_x} L(z,x)\, \mathrm{d}\sigma_x \\ &\qquad - \underset{\Sigma}{*} \int_\Sigma u(x) \wedge F_z [\lambda_{m-1}(z,x)] \\ &\qquad + \underset{\Sigma}{*} \int_\Sigma u(x) \wedge \eta_{m-1}(z,x) \\ &\quad = \frac{\partial}{\partial \nu_z} \int_\Sigma u(x) \frac{\partial}{\partial \nu_x} L(z,x)\, \mathrm{d}\sigma_x + Q\varphi(z). \end{aligned} \tag{63}$$

Since $F_z[\lambda_{m-1}(z,x)] = \mathcal{O}(|z-x|^{1-m})$ and in view of (52), Q is a compact operator from $L^p(\Sigma)$ into itself.

In view of the Stokes formula for u and on account of known properties of potentials (see, e.g., [6, p. 35]), we get

$$\begin{aligned} &\frac{\partial}{\partial \nu_z} \int_\Sigma u(x) \frac{\partial}{\partial \nu_x} L(z,x)\, \mathrm{d}\sigma_x \\ &\quad = \frac{\partial}{\partial \nu_z} \left[u(z) + \int_\Sigma \frac{\partial}{\partial \nu_x} u(x) L(z,x)\, \mathrm{d}\sigma_x \right] \\ &\quad = \left(1 - \frac{1}{2}\right) \frac{\partial}{\partial \nu_z} u(z) + \int_\Sigma \frac{\partial}{\partial \nu_x} u(x) \frac{\partial}{\partial \nu_z} L(z,x)\, \mathrm{d}\sigma_x \\ &\quad = \frac{1}{2} \left(-\frac{1}{2}\varphi(z) + \int_\Sigma \varphi(y) \frac{\partial}{\partial \nu_z} L(z,y)\, \mathrm{d}\sigma_y \right) \\ &\qquad + \int_\Sigma \left[-\frac{1}{2}\varphi(x) + \int_\Sigma \varphi(y) \frac{\partial}{\partial \nu_x} L(x,y)\, \mathrm{d}\sigma_y \right] \frac{\partial}{\partial \nu_z} L(z,x)\, \mathrm{d}\sigma_x \\ &\quad = -\frac{1}{4}\varphi(z) + \int_\Sigma \varphi(y)\, \mathrm{d}\sigma_y \int_\Sigma \frac{\partial}{\partial \nu_x} L(x,y) \frac{\partial}{\partial \nu_z} L(z,x)\, \mathrm{d}\sigma_x. \end{aligned} \tag{64}$$

Then,

$$\begin{aligned} &J'J\varphi(z) \\ &\quad = -\frac{1}{4}\varphi(z) \\ &\qquad + \int_\Sigma \varphi(y)\, \mathrm{d}\sigma_y \int_\Sigma \frac{\partial}{\partial \nu_x} L(x,y) \frac{\partial}{\partial \nu_z} L(z,x)\, \mathrm{d}\sigma_x \\ &\qquad + Q\varphi(z) = -\frac{1}{4}\varphi(z) + K^2\varphi(z) + Q\varphi(z). \end{aligned} \tag{65}$$

Since $\partial/\partial\nu_x L(x,y) = \mathcal{O}(|x-y|^{1-m+\lambda})$, K is a compact operator.

Thus,

$$J'\tilde{J}\varphi = J'J\varphi + J'Z\varphi = -\frac{1}{4}\varphi + \left(K^2 + Q + J'Z\right)\varphi \tag{66}$$

is a Fredholm operator and the assertion is proved. □

Theorem 9. *Given* $\omega \in L_1^p(\Sigma)$, *there exists a solution of the singular integral equation*

$$\tilde{J}\varphi(x) = \omega(x), \quad \varphi \in L^p(\Sigma), \ x \in \Sigma \tag{67}$$

if and only if

$$\int_\Sigma \gamma \wedge \omega = 0 \tag{68}$$

for every weakly closed form $\gamma \in L_{m-2}^q(\Sigma)(1/p + 1/q = 1)$.

Proof. Denote by $\tilde{J}^* : L_{m-2}^q(\Sigma) \to L^q(\Sigma)$ the adjoint of $\tilde{J}$; that is,

$$\tilde{J}^*\gamma(x) = \int_\Sigma \gamma(y) \wedge \mathrm{d}_y [S(x,y)], \quad x \in \Sigma. \tag{69}$$

From Theorem 8, it follows that operator $\tilde{J}$ can be reduced on the left; therefore, (67) admits a solution $\varphi \in L^p(\Sigma)$ if and only if

$$\int_\Sigma \gamma \wedge \omega = 0, \quad \forall \gamma \in L_{m-2}^q(\Sigma), \ \tilde{J}^*\gamma = 0. \tag{70}$$

On the other hand, $\tilde{J}^*\gamma = 0$ if and only if γ is a weakly closed form; that is,

$$\int_\Sigma \gamma \wedge \mathrm{d}g = 0, \quad \forall g \in C^\infty(\mathbb{R}^m). \tag{71}$$

In fact, if

$$\int_\Sigma \gamma(y) \wedge \mathrm{d}_y [S(x,y)] = 0, \quad \text{a.e. } x \in \Sigma, \tag{72}$$

we have

$$\int_\Sigma p(x)\, \mathrm{d}\sigma_x \int_\Sigma \gamma(y) \wedge \mathrm{d}_y [S(x,y)] = 0, \qquad \forall p \in C^\lambda(\Sigma) \tag{73}$$

and then

$$0=\int_\Sigma \gamma(y)\wedge \mathrm{d}_y\int_\Sigma p(x)\,S(x,y)\,\mathrm{d}\sigma_x=\int_\Sigma \gamma\wedge \mathrm{d}u \qquad (74)$$

for any smooth solution u of $Eu=0$ in $\overline{\Omega}$. Therefore, we have

$$\int_\Sigma \gamma(y)\wedge \mathrm{d}_y\left[S(x,y)\right]=0,\quad \forall x\in T\setminus\overline{\Omega}. \qquad (75)$$

Let us consider

$$z(x)=\int_\Sigma \gamma(y)\wedge \mathrm{d}_y\left[S(x,y)\right],\quad x\in T. \qquad (76)$$

If $v\in C^\infty(T)$ and $\eta\in C^1(\overline{\Omega})\cap C^2(\Omega)$ are such that $E\eta=Ev$ in Ω and $\eta=0$ on Σ, we have

$$\begin{aligned}\int_\Omega zEv\,\mathrm{d}x&=\int_\Omega zE\eta\,\mathrm{d}x\\&=\int_\Omega E\eta(x)\,\mathrm{d}x\int_\Sigma\gamma(y)\wedge \mathrm{d}_y\left[S(x,y)\right]\\&=\int_\Sigma\gamma(y)\wedge \mathrm{d}_y\int_\Omega E\eta(x)\,S(x,y)\,\mathrm{d}x.\end{aligned} \qquad (77)$$

From the Green formulas we have

$$\int_\Omega S(x,y)\,E\eta(x)\,\mathrm{d}x=\int_\Sigma S(x,y)\frac{\partial\eta}{\partial\nu}(x)\,\mathrm{d}\sigma_x,\qquad y\in\Sigma. \qquad (78)$$

In view of (72), we find

$$\begin{aligned}\int_\Omega zEv\,\mathrm{d}x&=\int_\Sigma\gamma(y)\wedge \mathrm{d}_y\int_\Omega E\eta(x)\,S(x,y)\,\mathrm{d}x\\&=\int_\Sigma\gamma(y)\wedge \mathrm{d}_y\int_\Sigma\frac{\partial\eta}{\partial\nu}(x)\,S(x,y)\,\mathrm{d}\sigma_x\\&=\int_\Sigma\frac{\partial\eta}{\partial\nu}(x)\,\mathrm{d}\sigma_x\int_\Sigma\gamma(y)\wedge \mathrm{d}_y\left[S(x,y)\right]\\&=0.\end{aligned} \qquad (79)$$

We have proved that $z=0$ on Σ, $z=0$ in Ω, and then $z=0$ in T. Therefore,

$$\begin{aligned}0=\int_T zE\varphi\,\mathrm{d}x&=\int_T E\varphi\,\mathrm{d}x\int_\Sigma\gamma(y)\wedge \mathrm{d}_y\left[S(x,y)\right]\\&=\int_\Sigma\gamma(y)\wedge \mathrm{d}_y\int_T E\varphi(x)\,S(x,y)\,\mathrm{d}x\\&=\int_\Sigma\gamma(y)\wedge \mathrm{d}\varphi(y),\end{aligned} \qquad (80)$$

for any $\varphi\in\mathring{C}^\infty(T)$. This implies (71) and the theorem is proved. □

Lemma 10. *For every $f\in W^{1,p}(\Sigma)$, there exists a solution of the boundary value problem*

$$\begin{gathered}w\in\mathcal{S}^p,\\ Ew=0\quad in\ \Omega,\\ \mathrm{d}w=\mathrm{d}f\quad on\ \Sigma.\end{gathered} \qquad (81)$$

Its solution w is a simple layer potential (49) whose density φ solves $\widetilde{J}\varphi=\mathrm{d}f$ (see (59)).

Proof. Consider the following singular integral equation:

$$\int_\Sigma\varphi(y)\,\mathrm{d}_x\left[S(x,y)\right]\mathrm{d}\sigma_y=\mathrm{d}f(x),\quad x\in\Sigma, \qquad (82)$$

in which the unknown is $\varphi\in L^p(\Sigma)$ and the datum is $\mathrm{d}f\in L^p_1(\Sigma)$. With conditions (68) being satisfied, in view of Theorem 9 there exists a solution φ of (82). □

Lemma 11. *Let $\mathcal{A}$ be the eigenspace of the Fredholm integral equation*

$$-\frac{1}{2}\psi(x)+\int_\Sigma\psi(y)\frac{\partial}{\partial\nu_x}S(x,y)\,\mathrm{d}\sigma_y=0,\qquad a.e.\ x\in\Sigma. \qquad (83)$$

The dimension of $\mathcal{A}$ is 1.

Proof. The Fredholm equation (83) has the same number of linearly independent solutions of the following equation:

$$-\frac{1}{2}\gamma(x)+\int_\Sigma\gamma(y)\frac{\partial}{\partial\nu_y}S(x,y)\,\mathrm{d}\sigma_y=0,\quad \text{a.e. } x\in\Sigma. \qquad (84)$$

Obviously, the constant functions are eigensolutions of (84). We want to show that they are the only ones. Let γ_1 and γ_2 be two linearly independent eigensolutions of (84) and set

$$u_i(x)=\int_\Sigma\gamma_i(y)\,S(x,y)\,\mathrm{d}\sigma_y,\quad i=1,2. \qquad (85)$$

We note that γ_1 and γ_2 are Hölder continuous functions. With potentials u_i being smooth solutions of the problem

$$\begin{gathered}Eu=0\quad\text{in }\Omega,\\ \frac{\partial u}{\partial\nu^+}=0\quad\text{on }\Sigma,\end{gathered} \qquad (86)$$

we get $u_i=\alpha_i$ in Ω. Choose $(c_1,c_2)\neq(0,0)$ such that $c_1\alpha_1+c_2\alpha_2=0$ and set

$$u(x)=\int_\Sigma\left(c_1\gamma_1(y)+c_2\gamma_2(y)\right)S(x,y)\,\mathrm{d}\sigma_y. \qquad (87)$$

Since $u=c_1\alpha_1+c_2\alpha_2=0$ in Ω, u satisfies the following boundary value problem:

$$\begin{gathered}Eu=0\quad\text{in }T\setminus\overline{\Omega},\\ u=0\quad\text{on }\Sigma,\\ u=0\quad\text{on }\partial T.\end{gathered} \qquad (88)$$

By Green's formula, $u=0$ in $T\setminus\Omega$ and therefore $u=0$ in T. This implies $c_1\gamma_1+c_2\gamma_2=0$, which is a contradiction. □

Lemma 12. *Given $c \in \mathbb{R}$, there exists a solution of the following boundary value problem:*

$$\begin{gathered} v \in \mathcal{S}^p, \\ Ev = 0 \quad in\ \Omega, \\ v = c \quad on\ \Sigma. \end{gathered} \tag{89}$$

It is given by

$$v(x) = c \int_\Sigma \psi_0(y) S(x, y)\, d\sigma_y, \quad x \in \Omega, \tag{90}$$

where ψ_0 is the unique element of $\mathscr{A}$ such that

$$\int_\Sigma \psi_0(y) S(x, y)\, d\sigma_y = 1, \quad \forall x \in \overline{\Omega}. \tag{91}$$

Proof. Let $\psi \in \mathscr{A}$, $\psi \neq 0$. Setting

$$P\psi(x) = \int_\Sigma \psi(y) S(x, y)\, d\sigma_y \tag{92}$$

we have that $P\psi = c$ in Ω. As in Lemma 11, this implies that if $c = 0$, we have that $\psi = 0$. Then, $c \neq 0$. Function $\psi_0 = (1/c)\psi$ satisfies (91) and v given by (90) is solution of (89). □

Theorem 13. *The Dirichlet problem (50) has a unique solution for every $f \in W^{1,p}(\Sigma)$. In particular, the density φ of u can be written as $\varphi = \varphi_0 + \psi$, where φ_0 solves the singular integral system*

$$\int_\Sigma \varphi_0(y)\, d_x [S(x, y)]\, d\sigma_y = df(x), \quad a.e.\ x \in \Sigma \tag{93}$$

and $\psi \in \mathscr{A}$.

Proof. Let w be a solution of the boundary value problem (81). Since $dw = df$ on Σ, $w = f - c$ on Σ for some $c \in \mathbb{R}$. Function $u = w + v$, where v is given by (90), solves problem (50).

Consider now two solutions of the same problem (50):

$$\begin{aligned} u(x) &= \int_\Sigma \varphi(y) S(x, y)\, d\sigma_y, \\ u'(x) &= \int_\Sigma \varphi'(y) S(x, y)\, d\sigma_y. \end{aligned} \tag{94}$$

Therefore, the potential

$$v(x) = \int_\Sigma \psi(y) S(x, y)\, d\sigma_y, \tag{95}$$

where $\psi = \varphi - \varphi'$, solves the problem

$$\begin{gathered} v \in \mathcal{S}^p, \\ Ev = 0 \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \Sigma. \end{gathered} \tag{96}$$

Since

$$\int_\Sigma \psi(y)\, d_x [S(x, y)]\, d\sigma_y = 0 \quad \text{on } \Sigma, \tag{97}$$

we have $J'\tilde{J}\psi = 0$ (see (66)). By standard arguments, ψ is Hölder continuous and then $v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. The weak maximum principle (see, e.g., [19, p. 32]) shows that $v = 0$ in Ω; that is, $u = u'$. □

We end this section by observing that when we study the Dirichlet problem (50), we need to solve the singular integral equation $\tilde{J}\varphi = df$, $\varphi \in L^p(\Sigma)$. We have proved that this equation can be reduced to a Fredholm one by means of the operator J'. This reduction is not an equivalent reduction in the usual sense (see, e.g., [18, pp. 19-20]); that is, it is not true that $\mathcal{N}(J') = \{0\}$, $\mathcal{N}(J')$ being the kernel of the operator J'. However, if the condition

$$\mathcal{N}(J'\tilde{J}) = \mathcal{N}(\tilde{J}) \tag{98}$$

is true, J' still provides equivalence in a certain sense. In fact, we have the following lemma.

Lemma 14. *If condition (98) holds, the singular integral equation (82) is equivalent to the Fredholm equation $J'\tilde{J}\varphi = J'df$.*

Proof. Condition (98) implies that if g is such that there exists a solution φ of the equation $\tilde{J}\varphi = g$, then this equation is satisfied if and only if $J'\tilde{J}\varphi = J'g$. Since the equation $\tilde{J}\varphi = df$ is solvable (see Lemma 10), we have that $\tilde{J}\varphi = df$ if and only if $J'\tilde{J}\varphi = J'df$. □

Condition (98) is satisfied, for example, if the differential operator E has constant coefficients. This can be proved as in [20, Remark 1, p. 1045], replacing the Laplacian and the normal derivative by E and the conormal derivative, respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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