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Research Article

On Convergence in L-Valued Fuzzy Topological Spaces

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We introduce the concept of L-fuzzy neighborhood systems using complete MV-algebras and present important links with the theory of L-fuzzy topological spaces. We investigate the relationships among the degrees of L-fuzzy r-adherent points (r-convergent, r-cluster, and r-limit, resp.) in an L-fuzzy topological spaces. Also, we investigate the concept of LF-continuous functions and their properties.

1. Introduction

Šostak [1–3] introduced a new definition of L-fuzzy topology as the concept of the degree of the openness of fuzzy set. It is an extension of $I = [0, 1]$-fuzzy topology defined by Chang [4]. It has been developed in many directions [5–11]. The study of neighborhood systems and convergence of nets in Chang fuzzy topology was initiated by Pao-Ming and Ying-Ming [11] and Liu and Luo [12]. In [13] Ying introduced the degree to which a fuzzy point x_t belongs to a fuzzy subset λ by $m(x_t, \lambda) = \min(1, 1 - t + \lambda(x))$ and gave the idea of graded neighborhood on fuzzy topological spaces. This plays an important role in the theory of convergence in Chang fuzzy topology see also [14–18]. Following Ying [13], Demirci [5] introduced the idea of graded neighborhood systems in smooth toplogical spaces [19] (a smooth topology is similar to fuzzy topology as defined by Šostak [1], Hazra and Samanta [6]) in a different approach but restricted himself to the I-valued fuzzy sets.

In this paper, we study the concept of L-fuzzy neighborhood systems and present important links with the theory of L-fuzzy topological spaces and investigate some of their properties. We investigate the relationships among the degrees of L-fuzzy r-adherent points (r-convergent, r-cluster, and r-limit, resp.) nets in an L-fuzzy topological spaces. Also, we give some related examples to illustrate some of the introduced notions. In the end, we characterize LF-continuous functions in terms of some of the various notions introduced in this paper.

2. Preliminaries

Throughout the text we consider $(L, \leq, \wedge, \vee, 0, 1)$ as a completely distributive lattice with 0 and 1, respectively, being the universal upper and lower bound and $L_0 = L - \{0\}$. A lattice L is called order dense if for each $a, b \in L$ such that $a < b$, there exist $c \in L$ such that $a < c < b$. If L is a completely distributive lattice and $x \lhd \bigvee_{i\in\Gamma} y_i$, then there must be $i_0 \in \Gamma$ such that $x \lhd y_{i_0}$, where $x \lhd a$ means $K \subset L$, $a \leq \bigvee K \Rightarrow \exists y \in K$ such that $x \leq y$. If $a \lhd b$ and $c \lhd d$, we always assume $a \wedge c \lhd b \wedge d$ [20] and some properties of $\lhd$ can be found in [12].

A completely distributive lattice $L = (L, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$ (or L, in short) is called a residuated lattice [9, 21–23] if it satisfies the following conditions: for each $x, y, z \in L$,

(R1) $(L, \odot, 1)$ is a commutative monoid,

(R2) if $x \leq y$, then $x \odot z \leq y \odot z$ ($\odot$ is isotone operation),

(R3) (Galois correspondence) $x \leq y \rightarrow z \Leftrightarrow x \odot y \leq z$.

In a residuated lattice L, $x' = x \rightarrow 0$ is called complement of $x \in L$.

A residuated lattice L is called a BL-algebra [9, 21, 23] if it satisfies the following conditions: for each $x, y, z \in L$,

(B1) $x \wedge y = x \odot (x \rightarrow y)$,

(B2) $x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$,

(B3) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

A BL-algebra is called an MV-algebra if $x = x''$, for each $x \in L$.

Lemma 1 (see [9, 21, 23]). *Let L be a complete MV-algebra. For each $x, y, z \in L$, $\{y_i, x_i \mid i \in \Gamma\} \subset L$, one has the following properties:*

(1) $x \odot y \leq x \wedge y \leq x \vee y$,

(2) $x \odot y \leq x, y$,

(3) *If* $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ *and* $z \rightarrow x \leq y \rightarrow x$,

(4) $x \odot y = (x \rightarrow y')'$,

(5) $x \leq y$ *iff* $x' \geq y'$,

(6) $x \rightarrow y = y' \rightarrow x'$,

(7) $\bigwedge_{i\in\Gamma}(x \odot y_i) = x \odot (\bigwedge_{i\in\Gamma} y_i)$,

(8) $\bigvee_{i\in\Gamma}(x \odot y_i) = x \odot (\bigvee_{i\in\Gamma} y_i)$,

(9) $x \rightarrow 1 = 1, 0 \rightarrow x = 1, x \rightarrow x = 1$,

(10) $x \leq y \Leftrightarrow x \rightarrow y = 1$ *and* $1 \rightarrow x = x$,

(11) $x \rightarrow \bigwedge_{i\in\Gamma} y_i = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$,

(12) $(\bigvee_{i\in\Gamma} y_i) \rightarrow x = \bigwedge_{i\in\Gamma}(y_i \rightarrow x)$,

(13) $x \rightarrow \bigvee_{i\in\Gamma} y_i = \bigvee_{i\in\Gamma}(x \rightarrow y_i)$,

(14) $\bigwedge_{i\in\Gamma} y_i \rightarrow x = \bigvee_{i\in\Gamma}(y_i \rightarrow x)$,

(15) $\bigwedge_{i\in\Gamma} y_i' = (\bigvee_{i\in\Gamma} y_i)'$ *and* $\bigvee_{i\in\Gamma} y_i' = (\bigwedge_{i\in\Gamma} y_i)'$.

In this paper, we always assume that L is a complete MV-algebra. Let X be a nonempty set, and the family L^X denotes the set of all L-fuzzy subsets of a given set X. For $\alpha \in L, \lambda \in L^X$, we denote $(\alpha \rightarrow \lambda)$, $(\alpha \odot \lambda)$, and $\alpha_X \in L^X$ as $(\alpha \rightarrow \lambda)(x) = \alpha \rightarrow \lambda(x)$, $(\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$, and $\alpha_X(x) = \alpha$.

A fuzzy point x_t for $t \in L_0$ is an element of L^X such that

$$x_t(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases} \tag{1}$$

The set of all fuzzy points in X is denoted by $\mathrm{Pt}(X)$. For $\lambda \in L^X$ and $x_t \in \mathrm{Pt}(X)$, $x_t \in \lambda$ if and only if $t \leq \lambda(x)$.

Given a mapping $\phi : X \rightarrow Y$, we write $\phi^{\leftarrow}$ for the mapping $L^Y \rightarrow L^X$ defined by $\phi^{\leftarrow}(\mu) = \mu \circ \phi$; we write $\phi^{\rightarrow}$ for the mapping $L^X \rightarrow L^Y$ defined by $\phi^{\rightarrow}(\mu)(y) = \bigvee\{\mu(x) \mid \phi(x) = y\}$ for all $\mu \in L^X, y \in Y$.

For a given set X, define a binary mapping $S(,) : L^X \times L^X \rightarrow L$ as

$$S(\lambda, \mu) = \bigwedge_{x\in X} (\lambda(x) \longrightarrow \mu(x)), \quad \forall(\lambda, \mu) \in L^X \times L^X. \tag{2}$$

For each $\lambda, \mu \in L^X$, $S(\lambda, \mu)$ can be interpreted as the degree to which λ is fuzzy included in μ. It is called the L-fuzzy inclusion order [24].

Lemma 2 (see [24]). *For each $\lambda, \mu, \rho, \mu_i \in L^X$, $i \in \Gamma$ and $e, x_t \in Pt(X)$, the following properties hold:*

(1) $\lambda \leq \mu \Leftrightarrow S(\lambda, \mu) = 1$,

(2) $\lambda \leq \mu \Rightarrow S(\rho, \lambda) \leq S(\rho, \mu)$ *and* $S(\lambda, \rho) \geq S(\mu, \rho)$, *for any* $\rho \in L^X$,

(3) $S(x, \lambda) = \lambda(x)$, *for any* $\lambda \in L^X$,

(4) $S(x_t, \lambda) = 0$ *if and only if* $t = 1$ *and* $\lambda(x) = 0$,

(5) $S(e, \lambda) \wedge S(e, \mu) = S(e, \lambda \wedge \mu)$,

(6) $S(x_t, \bigwedge_{i\in\Gamma}\mu_i) = \bigwedge_{i\in\Gamma} S(x_t, \mu_i)$, *for any* $\{\mu_i\}_{i\in\Gamma} \subset L^X$,

(7) $S(x_t, \bigvee_{i\in\Gamma}\mu_i) = \bigvee_{i\in\Gamma} S(x_t, \mu_i)$, *for any* $\{\mu_i\}_{i\in\Gamma} \subset L^X$.

Lemma 3 (see [16]). *Let $f : X \rightarrow Y$ be a mapping. Then the following statement hold:*

(1) $S(\lambda, \mu) \leq S(f^{\rightarrow}(\lambda), f^{\rightarrow}(\mu))$, *for each* $\lambda, \mu \in L^X$

(2) $S(\rho, \nu) \leq S(f^{\leftarrow}(\rho), f^{\leftarrow}(\nu))$, *for each* $\rho, \nu \in L^Y$.

In particular, if the mapping $f : X \rightarrow Y$ is bijective, and then the equalities hold.

Definition 4 (see [1, 9]). A map $\mathcal{T} : L^X \rightarrow L$ is called an *L-fuzzy topology* on X if it satisfies the following conditions:

(LO1) $\mathcal{T}(1_X) = \mathcal{T}(0_X) = 1$,

(LO2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$, for all $\mu_1, \mu_2 \in L^X$,

(LO3) $\mathcal{T}(\bigvee_{i\in\Lambda}\mu_i) \geq \bigwedge_{i\in\Lambda}\mathcal{T}(\mu_i)$, for any $\{\mu_i\}_{i\in\Lambda} \subset L^X$.

The pair $(X, \mathcal{T})$ is called an *L-fuzzy topological space*.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be L-fuzzy topologies on X. We say that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$ ($\mathcal{T}_2$ is *coarser* than $\mathcal{T}_1$), denoted by $\mathcal{T}_2 \leq \mathcal{T}_1$, if $\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(\lambda)$ for all $\lambda \in L^X$. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be L-fuzzy topological space spaces. A map $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ is L-fuzzy continuous (LF-continuous, for short) if $\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(f^{\leftarrow}(\lambda)), \forall \lambda \in L^Y$.

Theorem 5 (see [7, 9]). *Let $(X, \mathcal{T})$ be an L-fuzzy topological space. For each $r \in L_0$ and $\lambda \in L^X$, one defines operators $I_{\mathcal{T}}, C_{\mathcal{T}} : L^X \times L_0 \rightarrow L^X$ as follows:*

$$I_{\mathcal{T}}(\lambda, r) = \bigvee\left\{\rho \in L^X \mid \rho \leq \lambda,\ \mathcal{T}(\rho) \geq r\right\},$$
$$C_{\mathcal{T}}(\lambda, r) = \bigwedge\left\{\nu \in L^X \mid \lambda \leq \nu,\ \mathcal{T}\left(\nu'\right) \geq r\right\}. \tag{3}$$

For each $\lambda, \mu \in L^X$ and $r, s \in L_0$, one has the following properties:

(I1) $\mathcal{I}_{\mathcal{T}}(1_X, r) = 1_X$,

(I2) $\mathcal{I}_{\mathcal{T}}(\lambda, r) \leq \lambda$,

(I3) if $\lambda \leq \mu$ and $r \leq s$, then $\mathcal{I}_{\mathcal{T}}(\lambda, s) \leq \mathcal{I}_{\mathcal{T}}(\mu, r)$,

(I4) $\mathcal{I}_{\mathcal{T}}(\lambda \wedge \mu, r \wedge s) \geq \mathcal{I}_{\mathcal{T}}(\lambda, r) \wedge \mathcal{I}_{\mathcal{T}}(\mu, s)$,

(I5) $\mathcal{I}_{\mathcal{T}}(\mathcal{I}_{\mathcal{T}}(\lambda, r), r) = \mathcal{I}_{\mathcal{T}}(\lambda, r)$,

(I6) $\mathcal{I}_{\mathcal{T}}(\lambda', r) = (C_{\mathcal{T}}(\lambda, r))'$.

Definition 6 (see [12]). Let D be a directed set. A function $T : D \to \mathrm{Pt}(X)$ is called a fuzzy net in X. Let $\lambda \in L^X$, and one says that T is a fuzzy net in λ if $T(n) \in \lambda$ for every $n \in D$.

Definition 7 (see [12, 25]). Let T be a fuzzy net and $\lambda \in L^X$.

(1) T is often in λ if for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $T(n_0) \in \lambda$.

(2) T is finally in λ if there exists $n_0 \in D$ such that for each $n \in D$ with $n \geq n_0$, one has $T(n) \in \lambda$.

Definition 8 (see [12, 25]). Let $T : D \to \mathrm{Pt}(X)$ and $U : E \to \mathrm{Pt}(X)$ be two fuzzy nets. A fuzzy net U is called a subnet of T if there exists a function $N : E \to D$, called by a cofinal selection on T, such that

(1) $U = T \circ N$;

(2) for every $n_0 \in D$, there exists $m_0 \in E$ such that $N(m) \geq n_0$, for $m \geq m_0$.

3. L-Fuzzy Neighborhood Systems

Definition 9. Let $\lambda \in L^X$ and $x_t \in \mathrm{Pt}(X)$. Then the degree to which x_t belongs to λ is

$$S(x_t, \lambda) = \bigwedge_{x \in X} (t \longrightarrow \lambda(x)). \tag{4}$$

Definition 10. Let $(X, \mathcal{T})$ be an L-fuzzy topological space, $\lambda \in L^X$, $e \in \mathrm{Pt}(X)$, and $r \in L_0$. The degree to which λ is a r-neighborhood of e is defined by

$$\left(\mathcal{N}^{\mathcal{T}}\right)_e (\lambda, r) = \bigvee \{S(e, \mu) \mid \mu \leq \lambda,\ r \lhd \mathcal{T}(\mu)\}. \tag{5}$$

A mapping $(\mathcal{N}^{\mathcal{T}})_e : L^X \times L_0 \to L$ is called *the L-fuzzy neighborhood system* of e.

Theorem 11. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and let $(\mathcal{N}^{\mathcal{T}})_e$ be the fuzzy neighborhood system of e. For all $\lambda, \mu \in L^X$ and $r, s \in L_0$, the following properties hold:*

(1) $(\mathcal{N}^{\mathcal{T}})_e(0_X, r) = S(e, 0_X)$ *and* $(\mathcal{N}^{\mathcal{T}})_e(1_X, r) = 1$,

(2) $(\mathcal{N}^{\mathcal{T}})_e(\lambda, r) \leq S(e, \lambda)$,

(3) $(\mathcal{N}^{\mathcal{T}})_e(\lambda, r) \geq (\mathcal{N}^{\mathcal{T}})_e(\lambda, s)$, *if* $r \leq s$,

(4) $(\mathcal{N}^{\mathcal{T}})_e(\lambda, r) \leq (\mathcal{N}^{\mathcal{T}})_e(\mu, r)$, *if* $\lambda \leq \mu$,

(5) $(\mathcal{N}^{\mathcal{T}})_e(\lambda_1, r) \wedge (\mathcal{N}^{\mathcal{T}})_e(\lambda_2, s) \leq (\mathcal{N}^{\mathcal{T}})_e(\lambda_1 \wedge \lambda_2, r \wedge s)$,

(6) $(\mathcal{N}^{\mathcal{T}})_e(\lambda, r) \leq \bigvee\{(\mathcal{N}^{\mathcal{T}})_e(\mu, r) \mid \mu \leq \lambda,\ S(d, \mu) \leq (\mathcal{N}^{\mathcal{T}})_d(\mu, r)\ \forall d \in Pt(X)\}$,

(7) $(\mathcal{N}^{\mathcal{T}})_{x_t}(\lambda, r) = \bigwedge_{x \in X}(t \to (\mathcal{N}^{\mathcal{T}})_{x_1}(\lambda, r))$.

Proof. (1), (3), and (4) are easily proved.

(2) is proved from the following:

$$\begin{aligned} \left(\mathcal{N}^{\mathcal{T}}\right)_e (\lambda, r) &= \bigvee \{S(e, \mu_i) \mid \mu_i \leq \lambda,\ r \lhd \tau(\mu)\} \\ &\leq \bigvee \left\{S\left(e, \bigvee \mu_i\right) \mid \mu_i \leq \lambda,\ r \lhd \tau(\mu)\right\} \\ &\qquad \text{(by Lemma 2 (2))} \\ &\leq \left\{S\left(e, \bigvee \mu_i\right) \mid \bigvee \mu_i \leq \lambda,\ r \leq \tau\left(\bigvee \mu_i\right)\right\} \\ &\leq S(e, \lambda). \end{aligned} \tag{6}$$

In (5) if $a \lhd (\mathcal{N}^{\mathcal{T}})_e(\lambda_1, r) \wedge (\mathcal{N}^{\mathcal{T}})_e(\lambda_2, s)$, then $a \lhd (\mathcal{N}^{\mathcal{T}})_e(\lambda_1, r)$ and $a \lhd (\mathcal{N}^{\mathcal{T}})_e(\lambda_2, s)$, and there exists $\rho_1 \in L^X$ with $\rho_1 \leq \lambda_1$ and $r \lhd \mathcal{T}(\rho_1)$ such that $a \lhd S(e, \rho_1)$. Again, there exists $\rho_1 \in L^X$ with $\rho_2 \leq \lambda_2$ and $r \lhd \mathcal{T}(\rho_2)$ such that $a \lhd S(e, \rho_2)$. So, $\rho_1 \wedge \rho_2 \leq \lambda_1 \wedge \lambda_2, r \wedge s \lhd \mathcal{T}(\rho_1) \wedge \mathcal{T}(\rho_2)$, and $a \leq S(e, \rho_1) \wedge S(e, \rho_2) = S(e, \rho_1 \wedge \rho_2) \leq (\mathcal{N}^{\mathcal{T}})_e(\lambda_1 \wedge \lambda_2, r \wedge s)$. Hence,

$$\left(\mathcal{N}^{\mathcal{T}}\right)_e (\lambda_1 \wedge \lambda_2, r \wedge s) \geq \left(\mathcal{N}^{\mathcal{T}}\right)_e (\lambda_1, r) \wedge \left(\mathcal{N}^{\mathcal{T}}\right)_e (\lambda_2, s). \tag{7}$$

In (6) if $r \lhd \mathcal{T}(\mu)$, then $S(d, \mu) = (\mathcal{N}^{\mathcal{T}})_d(\mu, r)$, for each $d \in \mathrm{Pt}(X)$. It implies

$$\begin{aligned} \left(\mathcal{N}^{\mathcal{T}}\right) (\lambda, r) &= \bigvee \{S(e, \mu) \mid \mu \leq \lambda,\ r \lhd \mathcal{T}(\mu)\} \\ &= \bigvee \Big\{\left(\mathcal{N}^{\mathcal{T}}\right)_e (\mu, r) \mid \mu \leq \lambda, \\ &\qquad S(d, \mu) = \left(\mathcal{N}^{\mathcal{T}}\right)_d (\mu, r), \\ &\qquad \forall d \in \mathrm{Pt}(X)\Big\} \\ &\leq \bigvee \Big\{\left(\mathcal{N}^{\mathcal{T}}\right)_e (\mu, r) \mid \mu \leq \lambda, \\ &\qquad S(d, \mu) \leq \left(\mathcal{N}^{\mathcal{T}}\right)_d (\mu, r), \\ &\qquad \forall d \in \mathrm{Pt}(X)\Big\}. \end{aligned} \tag{8}$$

(7) is proved from

$$\begin{aligned} \left(\mathcal{N}^{\mathcal{T}}\right)_{x_t} (\lambda, r) &= \bigvee \{S(x_t, \mu) \mid \mu \leq \lambda,\ \mathcal{T}(\mu) \geq r\} \\ &= \bigvee \left\{\bigwedge_{x \in X} (t \longrightarrow \mu(x)) \mid \mu \leq \lambda,\ \mathcal{T}(\mu) \geq r\right\} \end{aligned}$$

$$= \bigwedge_{x\in X} \left\{t \longrightarrow \bigvee \{\mu(x) \mid \mu \le \lambda,\ \mathcal{T}(\mu) \ge r\}\right\}$$
$$\text{(by Lemma 2 (7))}$$
$$= \bigwedge_{x\in X} \left(t \longrightarrow \left(\mathcal{N}^{\mathcal{T}}\right)_{x_1}(\lambda, r)\right). \tag{9}$$

□

Theorem 12. *Let X be a nonempty set. Let for each $e \in Pt(X)$, and $\mathcal{N}_e : L^X \times L_0 \to L$ satisfying the above conditions (1)–(5). Define $\mathcal{T}_{\mathcal{N}} : L^X \to L$ by*

$$\mathcal{T}_{\mathcal{N}}(\lambda) = \bigvee \{r \in L_0 \mid S(e,\lambda) = \mathcal{N}_e(\lambda, r),\ \forall e \in Pt(X)\}. \tag{10}$$

Then one has the following:

(a) *$\mathcal{T}_{\mathcal{N}}$ is an L-fuzzy topology on X;*

(b) *if $(\mathcal{N}^{\mathcal{T}})_e$ is the L-fuzzy neighborhood system of e induced by $(X, \mathcal{T})$, then $\mathcal{T}_{\mathcal{N}^{\mathcal{T}}} = \mathcal{T}$;*

(c) *if $\mathcal{N}_e$'s satisfy the conditions (6) and (7), then*

$$\mathcal{T}_{\mathcal{N}}(\lambda) = \bigvee \{r \in L_0 S(x,\lambda) = \mathcal{N}_x(\lambda, r),\ \forall x \in X\}; \tag{11}$$

(d) *$\mathcal{N}_{\mathcal{T}_{\mathcal{N}}} = \mathcal{N}$.*

Proof. (a) (LO1) It is easily proved from Theorem 11(1).

(LO2) It is proved from the following:

$$\begin{aligned}
&\mathcal{T}_{\mathcal{N}}(\lambda_1) \wedge \mathcal{T}_{\mathcal{N}}(\lambda_2) \\
&\quad= \left(\bigvee \{r \in L_0 \mid S(e,\lambda_1) = \mathcal{N}_e(\lambda_1, r)\}\right) \\
&\qquad \wedge \left(\bigvee \{s \in L_0 \mid S(e,\lambda_2) = \mathcal{N}_e(\lambda_2, s)\}\right) \\
&\quad= \bigvee \{r \wedge s \in L_0 \mid S(e,\lambda_1) \wedge S(e,\lambda_2) \\
&\qquad\qquad = \mathcal{N}_e(\lambda_1, r) \wedge \mathcal{N}_e(\lambda_2, s)\} \\
&\quad\le \bigvee \{r \wedge s \in L_0 \mid S(e,\lambda_1) \wedge S(e,\lambda_2) \\
&\qquad\qquad \le \mathcal{N}_e(\lambda_1 \wedge \lambda_2, r \wedge s)\} \\
&\quad\le \bigvee \{r \wedge s \in L_0 \mid S(e,\lambda_1 \wedge \lambda_2) \le \mathcal{N}_e(\lambda_1 \wedge \lambda_2, r \wedge s)\} \\
&\qquad\qquad \text{(by Lemma 2 (5))} \\
&\quad\le \mathcal{T}_{\mathcal{N}}(\lambda_1 \wedge \lambda_2).
\end{aligned} \tag{12}$$

(LO3) If $a \lhd \bigwedge_{i\in\Gamma} \mathcal{T}_{\mathcal{N}}(\lambda_i)$, then $a \lhd \mathcal{T}_{\mathcal{N}}(\lambda_i)$ for each $i \in \Gamma$, and note that

$$\mathcal{T}_{\mathcal{N}}(\lambda_i) = \bigvee \{r_i \in L_0 \mid S(e,\lambda_i) = \mathcal{N}_e(\lambda_i, r_i),\ \forall e \in \mathrm{Pt}(X)\}, \tag{13}$$

so there exists $r_i \in L_0$, with $S(e,\lambda_i) = \mathcal{N}_e(\lambda_i, r_i)$ such that $a \lhd r_i$. Put $r = \bigwedge_{i\in\Gamma} r_i$, and then $a \le r$. By Theorem 11, we have

$$S(e,\lambda_i) \le \mathcal{N}_e(\lambda_i, r_i) \le \mathcal{N}_e(\lambda_i, r) \le S(e,\lambda_i). \tag{14}$$

It implies $S(e,\lambda_i) = \mathcal{N}_e(\lambda_i, r)$. Furthermore, by Lemma 2(7), we have

$$\begin{aligned}
&S\left(e, \bigvee_{i\in\Gamma} \lambda_i\right) \\
&\quad= \bigvee_{i\in\Gamma} S(e,\lambda_i) = \bigvee_{i\in\Gamma} \mathcal{N}_e(\lambda_i, r_i) \\
&\quad\le \bigvee_{i\in\Gamma} \mathcal{N}_e(\lambda_i, r) \le \mathcal{N}_e\left(\bigvee_{i\in\Gamma} \lambda_i, r\right) \le S\left(e, \bigvee_{i\in\Gamma} \lambda_i\right).
\end{aligned} \tag{15}$$

So $\mathcal{N}_e(\bigvee_{i\in\Gamma} \lambda_i, r) = S(e, \bigvee_{i\in\Gamma} \lambda_i)$. Hence, $\mathcal{T}_{\mathcal{N}}(\bigvee_{i\in\Gamma} \lambda_i) \ge r \ge a$. Therefore, $\mathcal{T}_{\mathcal{N}}(\bigvee_{i\in\Gamma} \lambda_i) \ge \bigwedge_{i\in\Gamma} \lambda_i(\lambda_i)$.

(b) If $a \lhd \mathcal{T}_{\mathcal{N}}(\lambda)$, then there exists $r_0 \in L_0$ with $S(e,\lambda) = \mathcal{N}_e(\lambda, r_0)$ such that $r_0 \lhd \mathcal{T}(\lambda)$. Since

$$S(e,\lambda) = \mathcal{N}_e(\lambda, r_0) = \bigvee \{S(e,\mu_i) \mid \mu_i \le \lambda,\ r_0 \lhd \mathcal{T}(\mu_i)\}, \tag{16}$$

then, for each $x_1 \in \mathrm{Pt}(X)$,

$$\begin{aligned}
\lambda(x) &= S(x_1, \lambda) \\
&= \bigvee \{S(x_1, \mu_i) \mid \mu_i \le \lambda,\ r_0 \lhd \mathcal{T}(\mu_i)\} \\
&= S\left(x_1, \bigvee_{i\in\Gamma} \mu_i\right) = \bigvee_{i\in\Gamma} \mu_i(x).
\end{aligned} \tag{17}$$

Thus, $\lambda = \bigvee \mu_i$. So $\mathcal{T}(\lambda) \ge r_0 \ge a$. Hence, $\mathcal{T}_{\mathcal{N}}(\lambda) \le \mathcal{T}(\lambda)$. We can easily obtain $\mathcal{T}_{\mathcal{N}}(\lambda) \ge \mathcal{T}(\lambda)$.

(c) We only show that $S(x_t, \lambda) = \mathcal{N}_{x_t}(\lambda, r), \forall x_t \in \mathrm{Pt}(X)$ if and only if $S(x,\lambda) = \lambda(x) = \mathcal{N}_x(\lambda, r), \forall x \in X$.

(⇒) It is trivial.

(⇐) From condition (7),

$$\begin{aligned}
\mathcal{N}_{x_t}(\lambda, r) &= \bigwedge_{x\in X} \left(t \longrightarrow \mathcal{N}_{x_1}(\lambda, r)\right) \\
&= \bigwedge_{x\in X} (t \longrightarrow S(x_1, \lambda)) \\
&= \bigwedge_{x\in X} (t \longrightarrow \lambda(x)) \\
&= S(x_t, \lambda).
\end{aligned} \tag{18}$$

(d) From the proof of Theorem 11(6), we easily obtain $\mathcal{N}_{\mathcal{T}_{\mathcal{N}}} \ge \mathcal{N}$.

If $a \lhd (\mathcal{N}_{\mathcal{T}_{\mathcal{N}}})_e(\lambda, r) = \bigvee\{S(e,\mu) \mid \mu \le \lambda, r \lhd \mathcal{T}_{\mathcal{N}}(\mu)\}$, there exists μ_0 with $\mu_0 \le \lambda$, $r \lhd \mathcal{T}_{\mathcal{N}}(\mu_0)$ such that $a \lhd S(e,\mu_0)$. Note that

$$\mathcal{T}_{\mathcal{N}}(\mu_0) = \bigvee \{t \in L_0 \mid S(e,\mu_0) = \mathcal{N}_e(\mu_0, t),\ \forall e \in \mathrm{Pt}(X)\}, \tag{19}$$

and there exists $t_0 \in L_0$ with $S(e,\mu_0) = \mathcal{N}_e(\mu_0, t_0)$ such that $r \lhd t_0$ (thus $r \le t_0$). So $a \lhd \mathcal{N}_e(\mu_0,t_0) \le \mathcal{N}_e(\mu_0,r) \le \mathcal{N}_e(\lambda,r)$. Therefore, $\mathcal{N}_{\mathcal{T}_{\mathcal{N}}} \le \mathcal{N}$. □

By Theorem 12, we have the following corollary.

Corollary 13. *The set of all L-fuzzy topologies on X and the set of all L-fuzzy neighborhood systems on X are in one to one correspondence.*

Example 14. Let $L = [0,1]$, $X = \{a,b\}$ be a set, $x \to y = \min(1-x+y,1)$, and let $\mu \in L^X$ be defined as follows:

$$\mu(a) = 0.3, \qquad \mu(b) = 0.4. \tag{20}$$

We define an L-fuzzy topology on X as

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } 1_X, \\ \dfrac{1}{2}, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

From Definition 10, $\mathcal{N}_{a_1}, \mathcal{N}_{b_2} : L^X \times L_0 \to L$ as follows:

$$\begin{gathered} \mathcal{N}_{a_1}(\lambda,r) = \begin{cases} 1, & \text{if } \lambda = 1_X, \quad r \in L_0, \\ 0.3, & \text{if } 1_X \ne \lambda \ge \mu, \quad 0 < r \le \dfrac{1}{2}, \\ 0, & \text{otherwise,} \end{cases} \\ \mathcal{N}_{b_1}(\lambda,r) = \begin{cases} 1, & \text{if } \lambda = 1_X, \quad r \in L_0, \\ 0.4, & \text{if } 1_X \ne \lambda \ge \mu, \quad 0 < r \le \dfrac{1}{2}, \\ 0, & \text{otherwise.} \end{cases} \end{gathered} \tag{22}$$

From Theorem 12(c), we have

$$\mathcal{T}_{\mathcal{N}}(\lambda) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } 1_X, \\ \dfrac{1}{2}, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

4. R-Convergence

Definition 15. Let $(X,\mathcal{T})$ be an L-fuzzy topological space, $\lambda \in L^X$, $e \in \mathrm{Pt}(X)$, and $r \in L_0$. The degree to which a fuzzy net T in X is r-convergent to e and T is r-cluster to e are defined, respectively, as follows:

$$\begin{gathered} \mathrm{Con}_e(T,r) = \bigwedge \left\{ \mathcal{N}'_e(\lambda,r) \mid T \text{ is often in } \lambda' \right\}, \\ \mathrm{Cl}_e(T,r) = \bigwedge \left\{ \mathcal{N}'_e(\lambda,r) \mid T \text{ is finally in } \lambda' \right\}. \end{gathered} \tag{24}$$

Definition 16. Let $(X,\mathcal{T})$ be be an L-fuzzy topological space, $\lambda \in L^X$, $e \in \mathrm{Pt}(X)$, and $r \in L_0$. The degree to which e is r-adherent point of e is defined by

$$\mathrm{Ad}_e(\lambda,r) = \mathcal{N}'_e(\lambda',r). \tag{25}$$

Proposition 17. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space. For each $\lambda \in L^X$, $e, x_t \in Pt(X)$ and $r \in L_0$, one has*

(1) $S(e, I_{\mathcal{T}}(\lambda,r)) = \mathcal{N}_e(\lambda,r)$,
(2) $S(e, C'_{\mathcal{T}}(\lambda,r)) = Ad'_e(\lambda,r)$,
(3) $Ad_{x_t}(\lambda,r) = \bigvee_{x\in X}(t \odot Ad_x(\lambda,r))$.

Proof. (1) From Lemma 2(7), we have

$$\begin{aligned} S(e, I_{\mathcal{T}}(\lambda,r)) &= S\left(e, \bigvee \{\mu_i \mid \mu_i \le \lambda,\ \mathcal{T}(\mu_i) \ge r\}\right) \\ &= \bigvee \{S(e,\mu_i) \mid \mu_i \le \lambda,\ \mathcal{T}(\mu_i) \ge r\} \\ &= \mathcal{N}_e(\lambda,r). \end{aligned} \tag{26}$$

(2) From Theorem 5, we have

$$\begin{aligned} S\left(e, C'_{\mathcal{T}}(\lambda,r)\right) &= S\left(e, I_{\mathcal{T}}(\lambda',r)\right) \\ &= \mathcal{N}_e(\lambda',r) \quad (\text{by } (1)) \\ &= \mathrm{Ad}'_e(\lambda,r). \end{aligned} \tag{27}$$

(3) From Theorem 11(7), we have

$$\begin{aligned} \mathrm{Ad}_{x_t}(\lambda,r) &= \mathcal{N}'_{x_t}(\lambda',r) \\ &= \left(\bigwedge_{x\in X} \left(t \longrightarrow \mathcal{N}_{x_t}(\lambda',r)\right)\right)' \\ &= \bigvee_{x\in X} \left(t \longrightarrow \mathcal{N}_{x_t}(\lambda',r)\right)' \\ &= \bigvee_{x\in X} \left(t \odot \mathcal{N}'_{x_1}(\lambda',r)\right) \\ &\qquad (\text{by Lemma } 2(4)) \\ &= \bigvee_{x\in X} \left(t \odot \mathrm{Ad}_x(\lambda,r)\right). \end{aligned} \tag{28}$$

□

Theorem 18. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space. Let $T : D \to Pt(X)$ be fuzzy net and let $U : E \to Pt(X)$ be a subnet of S. For $r, s \in L_0$, the following properties hold:*

(1) *if $r_1 \le r_2$, $Con_e(T,r_1) \le Con_e(T,r_2)$, and $Cl_e(T,r_1) \le Cl_e(T,r_2)$,*
(2) *$Con_e(T,r) \le Cl_e(T,r)$,*
(3) *$Cl_e(U,r) \le Cl_e(T,r)$,*
(4) *$Con_e(T,r) \le Con_e(U,r)$,*
(5) *$Con_{x_t}(T,r) = \bigvee_{x\in X}(t \odot Con_x(T,r))$, and $Cl_{x_t}(T,r) = \bigvee_{x\in X}(t \odot Cl_x(T,r))$.*

Proof. (1) is easily proved.

In (2) if T is finally in λ', T is often in λ'. Hence

$$\begin{aligned} \mathrm{Con}_e(T,r) &= \bigwedge \left\{ \mathcal{N}'_e(\lambda,r) \mid T \text{ is often in } \lambda' \right\} \\ &\le \bigwedge \left\{ \mathcal{N}'_e(\lambda,r) \mid T \text{ is finally in } \lambda' \right\} \\ &= \mathrm{Cl}_e(T,r). \end{aligned} \tag{29}$$

In (3) if T is finally in λ', U is finally in λ'. Hence

$$\begin{aligned}\mathrm{Cl}_e(U,r) &= \bigwedge\left\{\mathcal{N}'_e(\lambda,r) \mid U \text{ is finally in } \lambda'\right\}\\ &\leq \bigwedge\left\{\mathcal{N}'_e(\lambda,r) \mid T \text{ is finally in } \lambda'\right\}\\ &= \mathrm{Cl}_e(T,r).\end{aligned} \tag{30}$$

In (4) let U be often in λ'. We will show that T is often in λ'. Let $n \in D$. Since $U : E \to \mathrm{Pt}(X)$ is a subnet of T, there exists a cofinal selection $N : E \to D$. For each $n \in D$, there exists $m \in E$ such that $N(k) \geq n$ for $k \geq m$. Since U is often in λ', for $m \in E$, there exists $m_0 \in E$ such that $m_0 \geq m$ for $U(m_0) \in \lambda'$. Put $n_0 = N(m_0)$. Then $n_0 \geq n$ and $T(n_0) = T(N(m_0)) = T(n_0) \in \lambda'$. Thus, U is often in λ'. Hence

$$\begin{aligned}\mathrm{Con}_e(T,r) &= \bigwedge\left\{\mathcal{N}'_e(\lambda,r) \mid T \text{ is often in } \lambda'\right\}\\ &\leq \bigwedge\left\{\mathcal{N}'_e(\lambda,r) \mid U \text{ is often in } \lambda'\right\}\\ &= \mathrm{Con}_e(U,r).\end{aligned} \tag{31}$$

In (5) one has

$$\begin{aligned}\mathrm{Con}_{x_t}(T,r) &= \bigwedge\left\{\mathcal{N}'_{x_t}(\lambda,r) \mid T \text{ is often in } \lambda'\right\}\\ &= \bigwedge\left\{\left(\bigwedge_{x\in X}\left(t \longrightarrow \mathcal{N}_{x_1}(\lambda,r)\right)\right)' \mid T \text{ is finally in } \lambda'\right\}\\ &\qquad (\text{by Theorem } 11\,(7))\\ &= \bigvee_{x\in X}\bigwedge\left\{\left(t \longrightarrow \mathcal{N}_{x_1}(\lambda,r)\right)' \mid T \text{ is finally in } \lambda'\right\}\\ &= \bigvee_{x\in X}\bigwedge\left\{t \odot \mathcal{N}'_{x_1}(\lambda,r) \mid T \text{ is finally in } \lambda'\right\}\\ &\qquad (\text{by Lemma } 1\,(4))\\ &= \bigvee_{x\in X}\left(t \odot \bigwedge\left\{\mathcal{N}'_{x_1}(\lambda,r) \mid T \text{ is finally in } \lambda'\right\}\right)\\ &= \bigvee_{x\in X}\left(t \odot \mathrm{Con}_x(T,r)\right).\end{aligned} \tag{32}$$

The other case is the same. □

Proposition 19. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space, let T be a fuzzy net, $e \in Pt(X)$, and $r \in L_0$. Then one has*

$$\begin{aligned}Ad_e(\lambda,r) &= \bigvee\left\{Con_e(T,r) \mid T \text{ is a fuzzy net in } \lambda\right\}\\ &= \bigvee\left\{Cl_e(T,r) \mid T \text{ is a fuzzy net in } \lambda\right\}.\end{aligned} \tag{33}$$

Proof. Since T is finally in λ, T is often in λ. We easily show that

$$\begin{aligned}\mathrm{Ad}_e(\lambda,r) &= \mathcal{N}'_e(\lambda',r)\\ &\geq \bigvee\left\{\mathrm{Cl}_e(T,r) \mid T \text{ is a fuzzy net in } \lambda\right\}\\ &\geq \bigvee\left\{\mathrm{Con}_e(T,r) \mid T \text{ is a fuzzy net in } \lambda\right\}.\end{aligned} \tag{34}$$

We only show that

$$\mathrm{Ad}_e(\lambda,r) \leq \bigvee\left\{\mathrm{Con}_e(T,r) \mid T \text{ is a fuzzy net in } \lambda\right\}. \tag{35}$$

Let $\mathrm{Ad}_e(\lambda,r) = t$. If $t > 0$, then $\mathcal{N}'_e(\lambda',r) = t$. Put $D = \{\mu \in L^X \mid \mathcal{N}_e(\mu,r) > t'\}$. Define a relation on D by

$$\mu_1 \preceq \mu_2 \quad \text{iff } \mu_1 \geq \mu_2,\ \forall \mu_1,\mu_2 \in D. \tag{36}$$

For each $\mu_1,\mu_2 \in D$, since by Theorem 11(5),

$$\mathcal{N}_e(\mu_1\wedge\mu_2,r) \geq \mathcal{N}_e(\mu_1,r)\wedge\mathcal{N}_e(\mu_2,r) > t'. \tag{37}$$

Hence, $\mu_1\wedge\mu_2 \in D$ and $\mu_1,\mu_2 \preceq \mu_1\wedge\mu_2$. Thus, $(D,\preceq)$ is a directed set. For each $\mu \in D$, that is, $\mathcal{N}_e(\mu,r) > t'$, we have $\mu \not\leq \lambda'$; that is, there exists $x \in X$ such that $\lambda(x) > \mu'(x)$. Thus, we can define a fuzzy net $T_0 : D \to \mathrm{Pt}(X)$ by $T_0(\mu) = x_{\lambda(x)}$ where $T_0(\mu) \in \lambda$ and $\lambda(x) = T_0(\mu)(x) > \mu'(x)$.

We will show that if $\mu \in D$, then T_0 is not often in μ'. Suppose that T_0 is often in μ'. For $\mu \in D$, there exists $\rho \in D$ such that $\mu \preceq \rho$ such that

$$T_0(\rho) = y_{\lambda(y)} \in \mu', \tag{38}$$

and $\lambda(y) = T_0(\rho)(y) > \rho'(y)$. Since $\mu \preceq \rho$ implies $\mu \geq \rho$, it implies

$$\lambda(y) \leq \mu'(y) \leq \rho'(y), \tag{39}$$

It is contradiction for the definition of T_0. Thus, if T_0 is often in μ', then $\mu \notin D$; that is, $\mathcal{N}_e(\mu,r) \leq t'$. Therefore,

$$\begin{aligned}&\bigvee\left\{\mathrm{Con}_e(T,r) \mid T \text{ is a fuzzy net in } \lambda\right\}\\ &\quad\geq \mathrm{Con}_e(T,r)\\ &\quad= \bigwedge\left\{\mathcal{N}'_e(\mu,r) \mid T_0 \text{ is often in } \mu'\right\}\\ &\quad\geq t = \mathrm{Ad}_e(\lambda,r).\end{aligned} \tag{40}$$

□

Theorem 20. *Let $(X,\mathcal{T})$ be L-fuzzy topological space and let $T,U : D \to Pt(X)$ be fuzzy nets such that $T(n)\vee U(n), T(n)\wedge U(n) \in Pt(X)$ for each $n \in D$. Define fuzzy nets $T\vee U, T\wedge U : D \to Pt(X)$ by, for each $n \in D$,*

$$\begin{aligned}(T\vee U)(n) &= T(n)\vee U(n),\\ (T\wedge U)(n) &= T(n)\wedge U(n).\end{aligned} \tag{41}$$

For each $r \in L_0$, the following properties hold:

(1) *if $T(n) \leq U(n)$ for all $n \in D$, then*

$$Cl_e(T,r) \leq Cl_e(U,r), \qquad Con_e(T,r) \leq Con_e(U,r), \quad (42)$$

(2) $Cl_e(T \wedge U, r) \leq Cl_e(T,r) \wedge Cl_e(U,r)$,

(3) $Con_e(T \vee U, r) \geq Con_e(T,r) \vee Con_e(U,r)$,

(4) $Con_e(T \wedge U, r) \leq Con_e(T,r) \wedge Con_e(U,r)$,

(5) *if L is order dense, then $Cl_e(T \vee U, r) = Cl_e(T,r) \vee Cl_e(U,r)$.*

Proof. In (1) let U be finally (often) in λ. Then let T be finally (often) in λ, respectively. Thus it is trivial. (2), (3), and (4) are easily proved.

In (5) since $T \leq T \vee U$ and $U \leq T \vee U$, by (1), we have

$$\mathrm{Cl}_e(T \vee U, r) \geq \mathrm{Cl}_e(T,r) \vee \mathrm{Cl}_e(U,r). \quad (43)$$

Suppose that $\mathrm{Cl}_e(T \vee U, r) \ngeq \mathrm{Cl}_e(T,r) \vee \mathrm{Cl}_e(U,r)$. Since L is order dense, then there exist $t \in L_0$ and a fuzzy point $e \in \mathrm{Pt}(X)$ such that

$$\mathrm{Cl}_e(T \vee U, r) > t > \mathrm{Cl}_e(T,r) \vee \mathrm{Cl}_e(U,r). \quad (44)$$

Since $\mathrm{Cl}_e(T,r) < t$ and $\mathrm{Cl}_e(U,r) < t$, by the definition Cl_e, there exist $\lambda, \mu \in L^X$ such that T and U are finally in λ' and μ', respectively, with

$$\mathrm{Cl}_e(T,r) \vee \mathrm{Cl}_e(U,r) \leq \mathcal{N}'_e(\lambda,r) \vee \mathcal{N}'_e(\mu,r) < t. \quad (45)$$

Since T is finally in λ', there exists $n_1 \in D$ such that $T(n) \in \lambda'$ for every $n \in D$ with $n \geq n_1$. Since U is finally in μ', there exists $n_2 \in D$ such that $T(n) \in \mu'$ for every $n \in D$ with $n \geq n_2$. Let $n_3 \in D$ such that $n_3 \geq n_1$ and $n_3 \geq n_2$. For $n \geq n_3$, we have

$$(T \vee U)(n) \leq \lambda' \vee \mu' = (\lambda \wedge \mu)'. \quad (46)$$

Thus, $(T \vee U)$ is finally in $(\lambda \wedge \mu)'$. It implies

$$\begin{aligned} \mathrm{Cl}_e(T \vee U, r) &\leq \mathcal{N}'_e(\lambda \wedge \mu, r) \\ &\leq \mathcal{N}'_e(\lambda,r) \vee \mathcal{N}'_e(\mu,r) < t. \end{aligned} \quad (47)$$

It is a contradiction. Hence, we have

$$\mathrm{Cl}_e(T \vee U, r) \leq \mathrm{Cl}_e(T,r) \vee \mathrm{Cl}_e(U,r). \quad (48)$$

□

Example 21. Let $(L = [0,1], \rightarrow)$ be defined as Example 14. Let $X = \{a, b\}$ be a set and $\mu \in I^X$ as follows:

$$\mu(x) = 0.3, \qquad \mu(y) = 0.4. \quad (49)$$

We define L-fuzzy topology $\mathcal{T} : I^X \rightarrow I$ as follows:

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } 1_X, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases} \quad (50)$$

(1) In general, $\mathrm{Cl}_e(T \wedge U, r) \neq \mathrm{Cl}_e(T,r) \wedge \mathrm{Cl}_e(U,r)$.

Let N be a natural numbers. Define fuzzy nets $T, U : N \rightarrow \mathrm{Pt}(X)$ by

$$\begin{aligned} T(n) &= x_{a_n}, \quad a_n = 0.8 + (-1)^n 0.2. \\ U(n) &= x_{b_n}, \quad b_n = 0.8 + (-1)^{n+1} 0.2. \end{aligned} \quad (51)$$

From Theorem 20, $(T \wedge U)(n) = x_{0.6}$ is a fuzzy net. Let $e = x_{0.3}$. From Definition 15, we have for $0 < r \leq 1/2$,

$$\mathrm{Cl}_e(x_{0.6}, r) = 1 - \mathcal{N}_e(\mu, r) = 1 - m(x_{0.3}, \mu) = 0. \quad (52)$$

Since T or U is finally in 1_X,

$$\mathrm{Cl}_e(T,r) = 1 - \mathcal{N}_e(0_X, r) = 1 - m(x_{0.3}, 0_X) = 0.3. \quad (53)$$

Similarly, $\mathrm{Cl}_e(U,r) = 0.3$. For $0 < r \leq 1/2$,

$$0 = \mathrm{Cl}_e(T \wedge U, r) \neq \mathrm{Cl}_e(T,r) \wedge \mathrm{Cl}_e(U,r) = 0.3. \quad (54)$$

(2) In general, $\mathrm{Con}_e(T \vee U, r) \neq \mathrm{Con}_e(T,r) \vee \mathrm{Con}_e(U,r)$.

Define fuzzy nets $T, U : N \rightarrow \mathrm{Pt}(X)$ by

$$\begin{aligned} T(n) &= x_{a_n}, \quad a_n = 0.6 + (-1)^n 0.2. \\ U(n) &= x_{b_n}, \quad b_n = 0.6 + (-1)^{n+1} 0.2. \end{aligned} \quad (55)$$

From Theorem 20, $(T \vee U)(n) = x_{0.8}$ is a fuzzy net. Let $e = x_{0.3}$. For all $r \in I_0$,

$$\mathrm{Ad}_e(x_{0.8}, r) = 1 - \mathcal{N}_e(0_X, r) = 1 - m(x_{0.3}, 0_X) = 0.3. \quad (56)$$

Since T or U is often in μ', for $0 < r \leq 1/2$,

$$\mathrm{Cl}_e(T,r) = 1 - \mathcal{N}_e(\mu, r) = 1 - m(x_{0.3}, \mu) = 0. \quad (57)$$

Similarly, $\mathrm{Cl}_e(U,r) = 0$. For $0 < r \leq 1/2$

$$0.3 = \mathrm{Con}_e(T \vee U, r) > (\mathrm{Con}_e(T,r) \vee \mathrm{Con}_e(U,r)) = 0. \quad (58)$$

5. Fuzzy r-Limit Nets and LF-Continuous Mappings

Definition 22. Let $(X, \mathcal{T})$ be an L-fuzzy toplogical space. Let $T : D \rightarrow \mathrm{Pt}(X)$ be fuzzy net in X, $e \in \mathrm{Pt}(X)$, and $r \in L_0$. Then the degree to which T is r-limit to e is defined, denoted by $\lim_e(T,r) = t$, if $\mathrm{Cl}_e(T,r) = \mathrm{Con}_e(T,r) = t$.

Theorem 23. *Let $(X, \mathcal{T})$ be L-fuzzy topological space and let $T, U : D \rightarrow Pt(X)$ be fuzzy nets such that $T(n) \vee U(n) \in Pt(X)$ for each $n \in D$. If L is order dense, $Cl_e(T,r) = Con_e(T,r)$, and $Cl_e(U,r) = Con_e(U,r)$, then*

$$\lim_e(T \vee U, r) = \lim_e(T,r) \vee \lim_e(U,r). \quad (59)$$

Proof. From Theorem 20, $T \vee U$ is a fuzzy net. We easily proved it from the following:

$$\begin{aligned}
&\mathrm{Cl}_e(T \vee U, r)\\
&\quad = \mathrm{Cl}_e(T, r) \vee \mathrm{Cl}_e(U, r) \quad (\text{by Theorem } 20\,(2))\\
&(\text{since } \mathrm{Cl}_e(T, r) = \mathrm{Con}_e(T, r),\ \mathrm{Cl}_e(U, r) = \mathrm{Con}_e(U, r))\\
&\quad = \mathrm{Con}_e(T, r) \vee \mathrm{Con}_e(U, r)\\
&\quad \le \mathrm{Con}_e(T \vee U, r) \quad (\text{by Theorem } 20\,(4))\\
&\quad \le \mathrm{Cl}_e(T \vee U, r) \quad (\text{by Theorem } 20\,(2)).
\end{aligned} \tag{60}$$

□

Theorem 24. *Let $(X, \mathcal{T})$ be L-fuzzy topological space. Let T be a fuzzy net and $\mathscr{H} = \{U \mid U$ is a subnet of $T\}$. Then, if L is an order dense, the following statements hold:*

(1) $Con_e(T, r) = \bigwedge_{T \in \mathscr{H}} Cl_e(U, r)$;
(2) $Cl_e(T, r) = \bigvee_{T \in \mathscr{H}} Con_e(U, r)$.

Proof. (1) For each $U \in \mathscr{H}$, by Theorem 18, we have

$$\mathrm{Con}_e(T, r) \le \mathrm{Con}_e(U, r) \le \mathrm{Cl}_e(U, r) \le \mathrm{Cl}_e(T, r). \tag{61}$$

Hence

$$\mathrm{Con}_e(T, r) \le \bigwedge_{U \in \mathscr{H}} \mathrm{Cl}_e(U, r). \tag{62}$$

Suppose

$$\mathrm{Con}_e(T, r) \ngeq \bigwedge_{U \in \mathscr{H}} \mathrm{Cl}_e(U, r). \tag{63}$$

Then there exist $x_p \in \mathrm{Pt}(X)$ and $t \in L_0$ such that

$$\mathrm{Con}_{x_p}(T, r) < t < \bigwedge_{U \in \mathscr{H}} \mathrm{Cl}_{x_p}(U, r). \tag{64}$$

Since $\mathrm{Con}_{x_p}(T, r) < t$, there exists $\mu \in L^X$ with T is often in μ' such that

$$\mathrm{Con}_{x_p}(T, r) \le \mathscr{N}'_{x_p}(\mu, r) < \bigwedge_{U \in \mathscr{H}} \mathrm{Cl}_{x_p}(U, r). \tag{65}$$

Since T is often in μ', for each $n \in D$ there exists $N(n) \in D$ with $N(n) \ge n$ and $T(N(n)) \in \mu'$. Hence there exists a cofinal selection $N : E \to D$ such that $U = T \circ N$. Thus U is a subnet of T and U is finally in μ'. It is a contradiction.

(2) From (1), we have

$$\bigvee_{U \in \mathscr{H}} \mathrm{Con}_e(U, r) \le \mathrm{Cl}_e(T, r). \tag{66}$$

Conversely, let $\mathrm{Cl}_e(T, r) = t > 0$. Then $\mathscr{N}_e(\lambda, r) \le t'$, for T is finally in λ'. Let $F = \{\mu \mid \mathscr{N}_e(\mu, r) > t'\}$. Define a relation on $E = D \times F$ by

$$(m, \mu_1) \le (n, \mu_2) \quad \text{iff } m \le n,\ \mu_1 \ge \mu_2. \tag{67}$$

Then $(E, \le)$ is a directed set. If $\mu \in F$, then T is not finally in μ'. For each $(n, \mu) \in E$, there exists $N(n, \mu) \in D$ with $N(n, \mu) \ge n$ such that $T(N(n, \mu)) \nleq \lambda'$. So, we can define $N : E \to D$. For each $n_0 \in D$ and $\mu_0 \in F$, there exists $N(n_0, \mu_0) \in D$ with $N(n_0, \mu_0) \ge n_0$ such that $T(N(n_0, \mu_0)) \nleq \mu_0'$. Hence for every $(n, \mu) \ge (n_0, \mu_0)$, since $n \ge n_0$, we have $N(n, \mu) \ge n \ge n_0$. Therefore N is a cofinal selection on T. So $U = T \circ N$ is a fuzzy subnet of T and U is finally to every member of F. If U is often in λ', then U is not finally of λ; that is, $\lambda \notin F$. Thus

$$\bigvee_{U \in \mathscr{H}} \mathrm{Con}_e(T, r) = \bigwedge \left\{\mathscr{N}'_e(\lambda, r) \mid U \text{ is often in } \lambda'\right\} \ge t. \tag{68}$$

Since t is arbitrary, we complete the proof. □

Theorem 25. *Let L be an order dense, let $(X, \mathcal{T})$ be L-fuzzy topological space, and let T be a fuzzy net. If every subnet U of T has a subnet K of U such that $\lim_e(K, r) = t$, then $\lim_e(T, r) = t$.*

Proof. Let $\mathscr{H} = \{U \mid U$ is a subnet of $T\}$. For each $U \in \mathscr{H}$, since U has a subnet K with $\lim_{\mathcal{T}}(K, r) = t$, by Theorem 18(4), we have

$$\mathrm{Con}_e(U, r) \le \mathrm{Con}_e(K, r) = \mathrm{Cl}_e(K, r) = t. \tag{69}$$

Hence, by Theorem 24(2),

$$\mathrm{Cl}_e(T, r) = \bigvee_{U \in \mathscr{H}} \mathrm{Con}_e(U, r) \le t. \tag{70}$$

Conversely, by Theorem 18(2),

$$t = \mathrm{Con}_e(K, r) = \mathrm{Cl}_e(K, r) \le \mathrm{Cl}_e(U, r). \tag{71}$$

Hence, by Theorem 24(1),

$$t \le \bigwedge_{U \in \mathscr{H}} \mathrm{Cl}_e(U, r) = \mathrm{Con}_e(T, r). \tag{72}$$

Hence, $\mathrm{Cl}_e(T, r) \le \mathrm{Con}_e(T, r)$. Since $\mathrm{Con}_e(T, r) \le \mathrm{Cl}_e(T, r)$ from Theorem 18(2), $\mathrm{Cl}_e(T, r) = \mathrm{Con}_e(T, r)$; that is, $\lim_e(T, r) = t$. □

Example 26. Let $(L = [0, 1], \to)$ be defined as in Example 21. Let N be a natural number set. Define a fuzzy net $T : N \to \mathrm{Pt}(X)$ by

$$T(n) = x_{a_n}, \quad a_n = 0.6 + (-1)^n 0.2. \tag{73}$$

Let $e = x_{0.3}$. Since T is often in μ', for $0 < r \le 1/2$,

$$\mathrm{Con}_e(T, r) = 1 - \mathscr{N}_e(\mu, r) = 1 - m(x_{0.3}, \mu) = 0. \tag{74}$$

Since T is finally in 1_X, for each $r \in I_0$,

$$\mathrm{Cl}_e(T, r) = 1 - \mathscr{N}_e(0_X, r) = 1 - m(x_{0.3}, 0_X) = 0.3. \tag{75}$$

Thus, since $\mathrm{Con}_e(T, r) \ne \mathrm{Cl}_e(T, r)$ for $0 < r \le 1/2$, $\lim_e(T, r)$ does not exists.

Since $\mathrm{Con}_e(T, r) = \mathrm{Cl}_e(T, r) = 0.3$ for $1/2 < r \le 1$, $\lim_e(T, r) = 0.3$.

Theorem 27. *Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be L-fuzzy topological spaces. For every fuzzy net T in X, $x_t \in Pt(X)$, $r \in L_0$, and $\lambda \in L^X$, the following statements are equivalent:*

(1) $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ *is LF-continuous;*

(2) $\mathcal{N}_{f^{\to}(e)}(\mu, r) \leq \bigvee\{\mathcal{N}_e(\lambda, r) \mid f^{\to}(\lambda) \leq \mu\}$;

(3) $Cl_e(T, r) \leq Cl_{f^{\to}(e)}(f \circ T, r)$;

(4) $Con_e(T, r) \leq Con_{f^{\to}(e)}(f \circ T, r)$;

(5) $f^{\to}(C_{\mathcal{T}_1}(\lambda, r)) \leq C_{\mathcal{T}_2}(f^{\to}(\lambda), r)$;

(6) $C_{\mathcal{T}_1}(f^{\leftarrow}(\mu), r)) \leq f^{\leftarrow}(C_{\mathcal{T}_2}(\mu), r)$;

(7) $f^{\leftarrow}(I_{\mathcal{T}_2}(\mu, r)) \leq I_{\mathcal{T}_1}(f^{\leftarrow}(\mu), r)$.

Proof. (1) $\Rightarrow$ (2) For any $\rho \in L^Y$ such that $\mathcal{T}_2(\rho) \geq r$ and $\rho \leq \mu$. Since f is LF-continuous, then $\mathcal{T}_1(f^{\leftarrow}(\rho)) \geq \mathcal{T}_2(\rho) \geq r$, and we have by Lemma 3(2)

$$\begin{aligned} & S(f^{\to}(e), \rho) \\ & \quad \leq S(e, f^{\leftarrow}(\rho)) \quad (e = x_t, f^{\to}(e) = f(x)_t) \\ & \quad = \mathcal{N}_e(f^{\leftarrow}(\rho), r) \quad (\mathcal{T}_1(f^{\to}(f^{\leftarrow})(\rho)) \geq r) \\ & \quad \leq \bigvee\{\mathcal{N}_e(\lambda, r) \mid f^{\to}(\lambda) \leq \mu\} \\ & \qquad\qquad (f^{\to}(f^{\leftarrow}(\rho)) \leq \rho \leq \mu). \end{aligned} \tag{76}$$

Thus, $\mathcal{N}_{f^{\to}(e)}(\mu, r) \leq \bigvee\{\mathcal{N}_e(\lambda, r) \mid f^{\to}(\lambda) \leq \mu\}$.

(2) $\Rightarrow$ (3) If $f^{\to}(\lambda) \leq \mu$ and $f \circ T$ is finally in μ', there exists $n_0 \in D$ such that, for all $n \geq n_0$, $f(T(n)) \in \mu'$. Let $T(n) = x_t$. Then

$$t \leq \mu'(f(x)) \leq (f(\lambda))'(f(x)) \leq \lambda'(x). \tag{77}$$

It implies $T(n) \in \lambda'$. Therefore, T is finally in λ'. One has

$$\begin{aligned} \mathrm{Cl}_e(T, r) &= \bigwedge\left\{\mathcal{N}'_e(\lambda, r) \mid T \text{ is finally in } \lambda'\right\} \\ &\leq \bigwedge\left\{\mathcal{N}'_e(\lambda, r) \mid \exists\mu,\ f^{\to}(\lambda) \leq \mu, \right. \\ &\qquad\qquad \left. f \circ T \text{ is finally in } \mu'\right\} \\ &= \bigwedge\left\{\bigvee\{\mathcal{N}_e(\lambda, r) \mid f^{\to}(\lambda) \leq \mu\}', \right. \\ &\qquad\qquad \left. f \circ T \text{ is finally in } \mu'\right\} \\ &\leq \bigwedge\left\{\mathcal{N}'_{f^{\to}(e)}(\mu, r), f \circ T \text{ is finally in } \mu'\right\} \\ &= \mathrm{Cl}_e(f \circ T, r) \quad (\text{by } (2)). \end{aligned} \tag{78}$$

(3) $\Rightarrow$ (4) Every subnet $U : E \to \mathrm{Pt}(Y)$ of $f(T)$, and there exists a cofinal selection $N : E \to D$ such that $U = f(T) \circ N = f \circ (T \circ N)$. Put $K = T \circ N$. Then K is a subnet of T. We can prove it from the following:

$$\begin{aligned} \mathrm{Con}_e(T, r) &\leq \mathrm{Con}_e(K, r) \quad (\text{by Theorem 18 (5)}) \\ &\leq \mathrm{Cl}_e(K, r) \quad (\text{by Theorem 18 (2)}) \\ &\leq \mathrm{Cl}_{f^{\to}(e)}(f \circ K, r) \quad (\text{by } (3)) \\ &= \mathrm{Cl}_{f^{\to}(e)}(f \circ (T \circ N), r) \\ &= \mathrm{Cl}_{f^{\to}(e)}(U, r). \end{aligned} \tag{79}$$

From Theorem 18(2), we have $\mathrm{Con}_e(T, r) \leq \mathrm{Con}_{f^{\to}(e)}(f \circ T, r)$.

(4) $\Rightarrow$ (5) From Theorem 5 and Proposition 17(2),

$$S(x_1, C'_{\mathcal{T}_1}(\lambda, r)) = C'_{\mathcal{T}_1}(\lambda, r)(x) = \mathrm{Ad}'_x(\lambda, r). \tag{80}$$

It implies

$$C_{\mathcal{T}_1}(\lambda, r)(x) = \mathrm{Ad}_x(\lambda, r). \tag{81}$$

Thus, we have

$$\begin{aligned} & f^{\to}\left(C_{\mathcal{T}_1}(\lambda, r)\right)(y) \\ & \quad = \bigvee\left\{C_{\mathcal{T}_1}(\lambda, r)(x) \mid f(x) = y\right\} \\ & \quad = \bigvee\{\mathrm{Ad}_x(\lambda, r) \mid f(x) = y\} \quad (\text{by } (81)) \\ & \quad = \bigvee_{f(x)=y} \bigvee\{\mathrm{Con}_x(T, r) \mid T \text{ is fuzzy net in } \lambda\} \\ & \qquad\qquad (\text{by Proposition 19}) \\ & \quad \leq \bigvee_{f(x)=y} \bigvee\left\{\mathrm{Con}_y(f \circ T, r) \mid T \text{ is fuzzy net in } \lambda\right\} \\ & \qquad\qquad (\text{by } (4)) \\ & \quad = \bigvee\left\{\mathrm{Con}_y(f \circ T, r) \mid T \text{ is fuzzy net in } \lambda\right\} \\ & \quad \leq \bigvee\left\{\mathrm{Con}_y(T, r) \mid T \text{ is fuzzy net in } f^{\to}(\lambda)\right\} \\ & \quad = \mathrm{Ad}_y(f^{\to}(\lambda), r) \quad (\text{by Proposition 19}) \\ & \quad = C_{\mathcal{T}_2}(f^{\to}(\lambda), r)(y) \quad (\text{by } (81)). \end{aligned} \tag{82}$$

(5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (7) are easily proved.

(7) $\Rightarrow$ (1) We will show that $\mathcal{T}_1(f^{\leftarrow}(\mu)) \geq \mathcal{T}_2(\mu)$, for all $\mu \in L^Y$.

Let $\mathcal{T}_2(\mu) = 0$. It is trivial.

Let $\mathcal{T}_2(\mu) = r > 0$. Since $\mathcal{T}_N = \mathcal{T}_2$ from Theorem 12(b), we have for all $y \in Y$,

$$S(y, \mu) = \mathcal{N}_y(\mu, r). \tag{83}$$

It implies, for all $x \in X$,

$$S(f(x), \mu) = S(x, f^{\leftarrow}(\mu)) = \mathcal{N}_{f(x)}(\mu, r). \tag{84}$$

Since $f^{\leftarrow}(I_{\mathcal{T}_2}(\mu, r)) = f^{\leftarrow}(\mu)$,

$$\begin{aligned} &S\left(x, f^{-1}(\mu)\right) \\ &\quad = S\left(x, f^{\leftarrow}\left(I_{\mathcal{T}_2}(\mu, r)\right)\right) \\ &\left(\text{since } f^{\leftarrow}\left(I_{\mathcal{T}_2}(\mu, r)\right) \leq I_{\mathcal{T}_1}\left(f^{\leftarrow}(\mu), r\right)\right) \\ &\quad \leq S\left(x, I_{\mathcal{T}_1}\left(f^{\leftarrow}(\mu), r\right)\right) \\ &\quad = \mathcal{N}_x\left(f^{\leftarrow}(\mu), r\right) \quad (\text{by Proposition } 17\,(1)). \end{aligned} \tag{85}$$

Thus, by Theorem 11(2), we have

$$S\left(x, f^{\leftarrow}(\mu)\right) = \mathcal{N}_x\left(f^{\leftarrow}(\mu), r\right). \tag{86}$$

Hence, $\mathcal{T}_1(f^{\leftarrow}(\mu)) \geq r$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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