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Research Article

The Tensor Product Representation of Polynomials of Weak Type in a DF-Space

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Let E and F be locally convex spaces over $\mathbf{C}$ and let $P(^nE;F)$ be the space of all continuous n-homogeneous polynomials from E to F. We denote by $\bigotimes_{n,s,\pi}E$ the n-fold symmetric tensor product space of E endowed with the projective topology. Then, it is well known that each polynomial $p \in P(^nE;F)$ is represented as an element in the space $L(\bigotimes_{n,s,\pi}E;F)$ of all continuous linear mappings from $\bigotimes_{n,s,\pi}E$ to F. A polynomial $p \in P(^nE;F)$ is said to be *of weak type* if, for every bounded set B of E, $p|_B$ is weakly continuous on B. We denote by $P_w(^nE;F)$ the space of all n-homogeneous polynomials of weak type from E to F. In this paper, in case that E is a DF space, we will give the tensor product representation of the space $P_w(^nE;F)$.

1. Notations and Preliminaries

In this section, we collect some notations, some definitions, and some basic properties of locally convex spaces which we use throughout this paper.

Let E_1 and E_2 be complex vector spaces. Then the pair $\langle E_1, E_2\rangle$ is called a *dual pair* if there exists a bilinear form:

$$(x_1, x_2) \longrightarrow \langle x_1, x_2\rangle \quad ((x_1, x_2) \in E_1 \times E_2) \tag{1}$$

satisfying the following conditions:

(1) If $\langle x_1, x_2\rangle = 0$ for every $x_2 \in E_2$, $x_1 = 0$.

(2) If $\langle x_1, x_2\rangle = 0$ for every $x_1 \in E_1$, $x_2 = 0$.

We denote by $\sigma(E_1, E_2)$ (resp., $\sigma(E_2, E_1)$) the topology on E_1 (resp., E_2) defined by the subset of seminorms:

$$\{|\langle \cdot, x_2\rangle| ; x_2 \in E_2\} \quad (\text{resp. } \{|\langle x_1, \cdot\rangle| ; x_1 \in E_1\}). \tag{2}$$

Let E be a locally convex space. We denote by $cs(E)$ the set of all nontrivial continuous seminorms on E. The topology $\sigma(E, E')$ on E is called the *weak topology* of E and the topology $\sigma(E', E)$ on E' is called the *weak* * *topology* of E'. We denote by $\mathscr{B}(E)$ the family of all bounded subsets of E. We denote by $\|\ \|_B$ the seminorm on E' defined by

$$\|x'\|_B = \sup\{|\langle x, x'\rangle| ; x \in B\}, \tag{3}$$

for every $B \in \mathscr{B}(E)$. The *strong topology* on E' is the topology on E' defined by the set of seminorms $\{\|\ \|_B; B \in \mathscr{B}(E)\}$ on E'. We denote by E'_β the locally convex space E' endowed with the strong topology. We denote by E'' the dual space of E'_β. Let A be a subset of E. We denote by $\overline{A}$ the topological closure of the subset A of $E \subset E''$ for the topology $\sigma(E'', E')$. The *polar set* A° of A is defined by

$$A^\circ = \left\{x' \in E'; \sup_{x\in A} |\langle x, x'\rangle| \leq 1\right\}. \tag{4}$$

We denote by $A^{\circ\circ}$ the bipolar set of A for the dual pair $\langle E'', E'\rangle$. A subset S of E' is said to be *equicontinuous* if there exists a neighborhood V of 0 in E such that $S \subset V^\circ$.

Lemma 1. *Let M be a bounded subset of a locally convex space E. Then the following statements hold:*

(1) $\overline{M} = M^{\circ\circ}$ *if M is absolutely convex.*

(2) $\overline{M}$ *is compact with the topology* $\sigma(E'', E')$.

(3) $E'' = \bigcup_{M\in\mathscr{B}(E)} \overline{M}$.

(4) *Let* A *be an equicontinuous subset of* E'' *for the dual pair* $\langle E'', E'\rangle$. *Then there exists an absolutely convex bounded subset* M *of* E, *such that* $A \subset M^{\circ\circ}$.

Proof. (1) Since $M \subset M^{\circ\circ}$ and $M^{\circ\circ}$ is $\sigma(E'', E')$-closed, $\overline{M} \subset M^{\circ\circ}$. We shall show $M^{\circ\circ} \subset \overline{M}$. We assume that $x_0'' \notin \overline{M}$. We denote by $E''_{\sigma(E'',E')}$ the locally convex space E'' endowed with the topology $\sigma(E'', E')$. Since $(E''_{\sigma(E'',E')})' = E'$ and $\overline{M}$ is $\sigma(E'', E')$-closed absolutely convex, by Hahn-Banach theorem there exists $x' \in E'$ such that

$$\sup_{x''\in\overline{M}} \left|\left\langle x'', x'\right\rangle\right| \leq 1, \qquad \left|\left\langle x_0'', x'\right\rangle\right| > 1. \tag{5}$$

Thus, it is valid that $x' \in \overline{M}^{\circ}$ and $x_0'' \notin \overline{M}^{\circ\circ}$. Thus we have $\overline{M}^{\circ\circ} \subset \overline{M}$. Since $M^{\circ\circ} \subset \overline{M}^{\circ\circ}$, $M^{\circ\circ} \subset \overline{M}$. Hence, we have $\overline{M} = M^{\circ\circ}$.

(2) We denote by $\Gamma(M)$ the $\sigma(E'', E')$-closed absolutely convex hull of the set M. By the statement (1) we have

$$\Gamma(M) = M^{\circ\circ}. \tag{6}$$

The polar set M° is an absolutely convex neighborhood of 0 in E'_{β}. Therefore, $\Gamma(M)$ is a $\sigma(E'', E')$-closed equicontinuous subset of E''. By Banach-Alaoglu theorem, $\Gamma(M)$ is $\sigma(E'', E')$-compact. Since $\overline{M} \subset \Gamma(M)$, $\overline{M}$ is also $\sigma(E'', E')$-compact.

(3) It is clear that $\bigcup_{M\in\mathscr{B}(E)} \overline{M} \subset E''$. We shall show that $E'' \subset \bigcup_{M\in\mathscr{B}(E)} \overline{M}$. Let x'' be a point of E''. Then, there exists an open neighborhood V of 0 in E'_{β} such that $|\langle x, x''\rangle| \leq 1$ for every $x \in V$. By the definition of the space E'_{β} there exists an absolutely convex bounded subset $M \in \mathscr{B}(E)$ such that $M^{\circ} \subset V$. Thus, by the statement (1) we have

$$x'' \in V^{\circ} \subset M^{\circ\circ} = \overline{M}. \tag{7}$$

Thus, we have $E'' = \bigcup_{M\in\mathscr{B}(E)} \overline{M}$.

(4) Since A is an equicontinuous subset of E'', there exists a neighborhood V of 0 in E'_{β} such that $A \subset V^{\circ}$. Since V is a neighborhood of 0 in E'_{β}, there exists an absolutely convex bounded subset M of E such that $M^{\circ} \subset V$. Thus, we have $A \subset V^{\circ} \subset M^{\circ\circ}$. This completes the proof. □

A filter $\mathscr{F} = \{F_{\alpha}\}$ of a locally convex space E is called a *Cauchy filter* if for every neighborhood U of 0 there exists an $F \in \mathscr{F}$ such that $\{x - y; x, y \in F\} \subset U$. A locally convex space E is said to be *complete* if any Cauchy filter on E converges to a point of E. There exists the smallest complete locally convex space $\widetilde{E}$ containing E as a subspace. The locally convex space $\widetilde{E}$ is called the *completion* of E.

2. The Extension of Polynomial Mappings of Weak Type

In this section, we will give basic properties of polynomial mappings on locally convex spaces and discuss the extension of weak type on locally convex spaces. For more detailed properties of polynomials on locally convex spaces, see Dineen [1, 2] and Mujica [3]. Let E and F be locally convex spaces and let n be a positive integer. We denote by $L_a({}^nE; F)$ the space of all n-linear mappings from the product space E^n of n-copies of E into F and denote by $L_{aw}({}^nE; F)$ the space of all n-linear mappings, which are $\sigma(E, E')$-continuous on bounded subsets of E^n, from the product space E^n into F. A mapping $p : E \to F$ is called an *n-homogeneous polynomial* from E into F if there exists an n-linear mapping u from E into F such that

$$P(x) = u(x, \dots, x), \tag{8}$$

for every $x \in E$. If p is an n-homogeneous polynomial from E into F, there exists uniquely a symmetric n-linear mapping u. We denote by $P_a({}^nE; F)$ the space of all n-homogeneous polynomials from E into F. We denote by $P({}^nE; F)$ (resp., $L({}^nE; F)$) the space of all continuous n-homogeneous polynomials from E (resp., all continuous n-linear mappings from E^n) into F. We denote by $P_{aw}({}^nE; F)$ the space of all $\sigma(E, E')$-continuous polynomials on each bounded subset of E. We set

$$P_w({}^nE; F) = P({}^nE; F) \cap P_{aw}({}^nE; F), \qquad L_w({}^nE; F) = L({}^nE; F) \cap L_{aw}({}^nE; F). \tag{9}$$

A polynomial belonging to $P_w({}^nE; F)$ is said to be *of weak type*.

Lemma 2. *Let* E *and* F *be locally convex spaces and let* u *be an* n*-linear mapping belonging to* $L_w({}^nE; F)$. *Let* $A_1, \dots, A_n$ *be absolutely convex bounded subsets of* E. *Let* a_i *be any point of* $\sigma(E'', E')$*-closure* $\overline{A}_i$ *of* A_i *for each* i *with* $2 \leq i \leq n$. *We denote by* $\mathscr{N}_{w^*}(E'')(0)$ *the system of all* $\sigma(E'', E')$*-neighborhoods of* 0 *in* E''. *Then, for any* $\alpha \in cs(F)$ *there exists* $V \in \mathscr{N}_{w^*}(E'')(0)$ *such that*

$$\alpha(u(x_1, \dots, x_n)) < 1, \tag{10}$$

for every $(x_1, \dots, x_n) \in (V \cap A_1) \times ((a_2 + V) \cap A_2) \times \cdots \times ((a_n + V) \cap A_n)$.

Proof. We shall prove this lemma by induction on n. Let $n = 1$. Then, the conclusion of this lemma is true since $0 \in A_1$ and $\sigma(E, E')$ is the induced topology of the topology $\sigma(E'', E')$ onto E.

We suppose that the conclusion of this lemma is true for all mappings belonging to $L_w({}^{n-1}E; F)$. And we assume that for $u \in L_w({}^nE; F)$, the conclusion is not true. Then, there exists $\alpha \in cs(F)$ such that for every $V \in \mathscr{N}_{w^*}(E'')(0)$ there is a point:

$$(x_{1V}, x_{2V}, \dots, x_{nV}) \in (V \cap A_1) \times ((a_2 + V) \cap A_2) \times \cdots \times ((a_n + V) \cap A_n), \tag{11}$$

satisfying

$$\alpha\left(u\left(x_{1V}, x_{2V}, \ldots, x_{nV}\right)\right) \geq 1. \tag{12}$$

Since u is $\sigma(E,E')$-continuous at $(0,0,\ldots,0)$ on each absolutely convex bounded subset of E^n and $\sigma(E,E')$ is the induced topology of $\sigma(E'',E')$ onto E, there exists $W \in \mathcal{N}_{w^*}(E'')(0)$ such that

$$\alpha\left(u\left(x_1, x_2, \ldots, x_n\right)\right) \leq \frac{1}{4} \tag{13}$$

for every $x_i \in 2(W \cap A_i)$ with $1 \leq i \leq n$. We choose a decreasing sequence $W \supset W(1) \supset W(2) \supset \cdots \supset W(n-1)$ of elements of $\mathcal{N}_{w*}(E'')(0)$ by the process of $(n-1)$ steps as follows.

At the first step, we consider the $(n-1)$-linear mapping:

$$\left(z_1, \ldots, z_{n-1}\right) \longrightarrow u\left(z_1, \ldots, z_{n-1}, x_{nW}\right), \tag{14}$$

belonging to $L_w({}^{n-1}E;F)$. By the assumption of induction, there exists $W(1) \in \mathcal{N}_{w^*}(E'')(0)$ with $W(1) \subset W$ such that

$$\alpha\left(u\left(z_1, \ldots, z_{n-1}, x_{nW}\right)\right) \leq \frac{1}{2^n} \tag{15}$$

for every $(z_1,\ldots,z_{n-1}) \in (W(1) \cap A_1) \times ((a_2 + W(1)) \cap A_2) \times \cdots \times ((a_{n-1} + W(1)) \cap A_{n-1})$.

At the second step, we consider the $(n-1)$-linear mapping

$$\left(z_1, \ldots, z_{n-2}, z_n\right) \longrightarrow u\left(z_1, \ldots, z_{n-2}, x_{n-1W(1)}, z_n\right) \tag{16}$$

belonging to $L_w({}^{n-1}E;F)$. By the assumption of induction, there exists $W(2) \in \mathcal{N}_{w^*}(E'')(0)$ with $W(2) \subset W(1)$ such that

$$\alpha\left(u\left(z_1, \ldots, z_{n-2}, x_{n-1W(1)}, z_n\right)\right) \leq \frac{1}{2^n} \tag{17}$$

for every point $(z_1,\ldots,z_{n-2},z_n)$ of

$$\begin{aligned}&(W(2) \cap A_1) \times ((a_2 + W(2)) \cap A_2) \\ &\quad \times \cdots \times ((a_{n-2} + W(2)) \cap A_{n-2}) \times ((a_n + W(2)) \cap A_n).\end{aligned} \tag{18}$$

Repeating this process, at the $(n-1)$th step, we consider the $(n-1)$-linear mapping:

$$\left(z_1, z_3, \ldots, z_n\right) \longrightarrow u\left(z_1, x_{2W(n-2)}, z_3, \ldots, z_n\right), \tag{19}$$

belonging to $L_w({}^{n-1}E;F)$. By the assumption of induction, there exists $W(n-1) \in \mathcal{N}_{w^*}(E'')(0)$ with $W(n-1) \subset W(n-2)$ such that

$$\alpha\left(u\left(z_1, x_{2W(n-2)}, z_3, \ldots, z_n\right)\right) \leq \frac{1}{2^n} \tag{20}$$

for every $(z_1, z_3 \ldots, z_n) \in (W(n-1) \cap A_1) \times ((a_3 + W(n-1)) \cap A_3) \times \cdots \times ((a_n + W(n-1)) \cap A_n)$. Then we have

$$\begin{aligned}&\alpha\left(u\left(x_{1W(n-1)}, x_{2W(n-1)} - x_{2W(n-2)}, \ldots, x_{n-1W(n-1)}\right.\right.\\ &\qquad\left.\left.-x_{n-1W(1)}, x_{nW(n-1)} - x_{nW}\right)\right)\\ &\quad\geq \alpha\left(u\left(x_{1W(n-1)}, x_{2W(n-1)}, \ldots, x_{nW(n-1)}\right)\right)\\ &\qquad- \sum_{(k_i)\in K} \alpha\left(u\left(x_{1W(n-1)}, x_{2W(k_2)}, \ldots, x_{nW(k_n)}\right)\right)\\ &\quad\geq 1 - \left(2^{n-1} - 1\right) \times \frac{1}{2^n} \geq \frac{1}{2},\end{aligned} \tag{21}$$

where $K = \{(k_2,\ldots,k_n); k_i \in \{n-1, n-i\}, i = 2,\ldots,n\} \setminus \{(n-1,\ldots,n-1)\}$ and $W(0) = W$.

If we set

$$\begin{aligned}\left(y_1, y_2, \ldots, y_n\right) = &\left(x_{1W(n-1)}, x_{2W(n-1)} - x_{2W(n-2)}, \ldots,\right.\\ &\left.x_{nW(n-1)} - x_{nW}\right),\end{aligned} \tag{22}$$

we have

$$\begin{aligned}&\alpha\left(u\left(y_1, \ldots, y_n\right)\right) \geq \frac{1}{2},\\ &\left(y_1, y_2, \ldots, y_n\right) \in \left(W \cap A_1\right) \times \left\{2\left(W \cap A_2\right)\right\}\\ &\qquad \times \cdots \times \left\{2\left(W \cap A_n\right)\right\}.\end{aligned} \tag{23}$$

This is a contradiction by (13). This completes the proof. □

Lemma 3. *Let E be a locally convex space, let F be a complete locally convex space and let u be an n-linear mapping belonging to $L_w({}^nE;F)$. Let $A_1,\ldots,A_n$ be absolutely convex bounded subsets of E. Then there exists a $\sigma(E'',E')$-continuous mapping $\tilde{u}_{\overline{A}_1\times\cdots\times\overline{A}_n}$ from $\overline{A}_1 \times \cdots \times \overline{A}_n$ into F such that $\tilde{u}_{\overline{A}_1\times\cdots\times\overline{A}_n} = u$ on $A_1 \times \cdots \times A_n$.*

Proof. Let a_i be any point of the $\sigma(E'',E')$-closure $\overline{A}_i$ of A_i for each i with $1 \leq i \leq n$. At first, we shall show that a filter of F

$$\begin{aligned}\mathcal{F}\left(a_1, \ldots, a_n\right) = &\left\{u\left(\left(a_1 + V\right) \cap A_1, \ldots, \left(a_n + V\right) \cap A_n\right);\right.\\ &\left. V \in \mathcal{N}_{w^*}\left(E''\right)(0)\right\}\end{aligned} \tag{24}$$

is a Cauchy filter. Let α be an arbitrary continuous seminorm of F. By Lemma 2 there exists $V \in \mathcal{N}_{w^*}(E'')(0)$ such that

$$\alpha\left(u\left(x_1, x_2, \ldots, x_n\right)\right) < \frac{1}{n}, \tag{25}$$

for every point $(x_1,x_2,\ldots,x_n)$ of

$$\bigcup_{i=1}^{n}\left(\prod_{j=1}^{n}\left(\left(\gamma_{ij}a_j + 2V\right) \cap 2A_j\right)\right), \tag{26}$$

where $\gamma_{ij} = 1$ $(i \neq j)$ and $\gamma_{ij} = 0$ $(i = j)$. Then, we have

$$\begin{aligned}
&\alpha\left(u\left(x_1, x_2, \ldots, x_n\right) - u\left(y_1, y_2, \ldots, y_n\right)\right) \\
&\quad \leq \alpha\left(u\left(x_1 - y_1, x_2, \ldots, x_n\right)\right) \\
&\qquad + \alpha\left(u\left(y_1, x_2 - y_2, x_3, \ldots, x_n\right)\right) \\
&\qquad + \alpha\left(u\left(y_1, y_2, x_3 - y_3, x_4, \ldots, x_n\right)\right) \\
&\qquad + \cdots + \alpha\left(u\left(y_1, y_2, \ldots, y_{n-1}, x_n - y_n\right)\right) \\
&\quad \leq n \times \frac{1}{n} = 1,
\end{aligned} \tag{27}$$

for every $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \prod_{i=1}^{n}((a_i + V) \cap A_i)$. Thus, the filter $\mathscr{F}(a_1, \ldots, a_n)$ is a Cauchy filter. Since F is complete and Hausdorff, there exists uniquely the limit point of the filter $\mathscr{F}(a_1, \ldots, a_n)$ for every $(a_1, \ldots, a_n) \in \overline{A}_1 \times \cdots \times \overline{A}_n$. We denote by $\tilde{u}_{\overline{A}_1 \times \cdots \times \overline{A}_n}(a_1, \ldots, a_n)$ the limit point of the filter $\mathscr{F}(a_1, \ldots, a_n)$ for every $(a_1, \ldots, a_n) \in \overline{A}_1 \times \cdots \times \overline{A}_n$. Then $\tilde{u}_{\overline{A}_1 \times \cdots \times \overline{A}_n}$ defines $\sigma(E'', E')$-continuous mapping from $\overline{A}_1 \times \cdots \times \overline{A}_n$ into F with

$$\tilde{u}_{\overline{A}_1 \times \cdots \times \overline{A}_n} = u \quad \text{on } A_1 \times \cdots \times A_n. \tag{28}$$

This completes the proof. □

Lemma 4. *Let E be a locally convex space, let F be a complete locally convex space, and let u be an n-linear mapping belonging to $L_w({}^nE; F)$. Then, there exists $\tilde{u} \in L_a({}^nE''; F)$ with $\tilde{u} \mid E^n = u$ such that $\tilde{u}$ is $\sigma(E'', E')$-continuous on $\overline{A}_1 \times \cdots \times \overline{A}_n$ for all absolutely convex bounded subsets $A_1, \ldots, A_n$ of E.*

Proof. By Lemma 3 for all absolutely convex bounded subsets $A_1, \ldots, A_n$ of E, there exists a $\sigma(E'', E')$-continuous mapping $\tilde{u}_{\overline{A}_1 \times \cdots \times \overline{A}_n}$ from $\overline{A}_1 \times \cdots \times \overline{A}_n$ to F with $\tilde{u}_{\overline{A}_1 \times \cdots \times \overline{A}_n} = u$ on $A_1 \times \cdots \times A_n$. If $A_1, \ldots, A_n, B_1, \ldots, B_n$ are absolutely convex bounded subsets of E with $\overline{A}_i \cap \overline{B}_i \neq \emptyset$ for $1 \leq i \leq n$, we have

$$\tilde{u}_{\overline{A}_1 \times \cdots \times \overline{A}_n} = \tilde{u}_{\overline{B}_1 \times \cdots \times \overline{B}_n} \tag{29}$$

on $(\overline{A}_1 \times \cdots \times \overline{A}_n) \cap (\overline{B}_1 \times \cdots \times \overline{B}_n)$. Thus, by Lemma 1, we can define an n-linear mapping $\tilde{u}$ from E''^n into F by setting $\tilde{u} = \tilde{u}_{\overline{A}_1 \times \cdots \times \overline{A}_n}$ on $\overline{A}_1 \times \cdots \times \overline{A}_n$ for all absolutely convex bounded subsets $A_1, \ldots, A_n$ of E. Then, the mapping $\tilde{u}$ satisfies all required conditions of this lemma. This completes the proof. □

The following theorem is proved by Aron et al. [4], González and Gutiérrez [5], Honda et al. [6].

Theorem 5. *Let E be a locally convex space and let F be a complete locally convex space. Let $p \in P({}^nE; F)$. Then, the following statements are equivalent.*

(1) $p \in P_w({}^nE; F)$.

(2) *For each absolutely convex bounded subset M of E, p can be extended $\sigma(E'', E')$-continuously to $M^{\circ\circ}$, where $M^{\circ\circ}$ is the bipolar set of M for the dual pair $\langle E'', E' \rangle$.*

(3) *There exists $\tilde{p} \in P_a({}^nE''; F)$ such that $\tilde{p}$ is $\sigma(E'', E')$-continuous on each equicontinuous subset of E'' and $\tilde{p} \mid E = p$.*

(4) *p is weakly uniformly continuous on every bounded subset of E.*

Proof. We shall show that (1) implies (3). There exists a symmetric n-linear mapping u from E^n into F such that $p(x) = u(x, \ldots, x)$ for every $x \in E$. By the polarization formula, we have $u \in L_w({}^nE, F)$. By Lemma 4 there exists $\tilde{u} \in L_a({}^nE''; F)$ with $\tilde{u} \mid E^n = u$ such that $\tilde{u}$ is $\sigma(E'', E')$-continuous on $\overline{A}_1 \times \cdots \times \overline{A}_n$ for all absolutely convex bounded subsets $A_1, \ldots, A_n$ of E. We define $\tilde{p} \in P_a({}^nE''; F)$ by $\tilde{p}(x) = \tilde{u}(x, \ldots, x)$ for every $x \in E''$. By Lemma 1, $\tilde{p}$ satisfies all required conditions of the statement (3).

(3) implies (2) since $M^{\circ\circ}$ is equicontinuous for every absolutely convex bounded subset M of E.

We shall show that (2) implies (4). Let A be a bounded subset of E. We denote by M the absolutely convex hull of A in E. Then, M is a bounded subset of E, and $A \subset M$. By statement (2), there exists a $\sigma(E'', E')$-continuous mapping $\tilde{p}_{\overline{M}}$ from $\overline{M}$ into F such that $\tilde{p}_{\overline{M}} = p$ on M. Since by Lemma 1 $\overline{M}$ is $\sigma(E'', E')$-compact, $\tilde{p}_{\overline{M}}$ is uniformly $\sigma(E'', E')$-continuous on $\overline{M}$. Since $\tilde{p}_{\overline{M}} = p$ on A, p is weakly uniformly continuous on A. This implies (4). It is clear that (4) implies (1). This completes the proof. □

3. The Tensor Product Representation of Polynomials in Locally Convex Spaces

For any $u \in L_a({}^nE; F)$, we set

$$s(u)(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} u\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right) \tag{30}$$

for every $(x_1, \ldots, x_n) \in E^n$, where S_n is the permutation group of degree n. Then, $s(u)$ is a symmetric n-linear mapping from E^n to F satisfying

$$u(x, \ldots, x) = s(u)(x, \ldots, x) \tag{31}$$

for every $x \in E$. We denote $L_a^s({}^nE; F)$ by the space of all symmetric n-linear mappings from E^n to F. Let Δ_n be the mapping from E into E^n defined by

$$\Delta_n(x) = (x, \ldots, x) \quad \text{for every } x \in E. \tag{32}$$

For any $u \in L_a({}^nE; F)$, we define an n-homogeneous polynomial $\Delta_n^*(u)$ by

$$\Delta_n^*(u) = u \circ \Delta_n. \tag{33}$$

The mapping

$$\Delta_n^* : L_a^s({}^nE; F) \longrightarrow P_a({}^nE; F) \tag{34}$$

is surjective.

Theorem 6 (polarization formula). *Let $p \in P_a({}^nE;F)$ and let $u \in L_a^s({}^nE;F)$. If $\Delta_n^*(u) = p$, then*

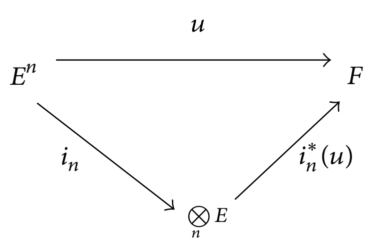
$$u(x_1,\ldots,x_n) = \frac{1}{2^n n!}\sum_{\epsilon_i=\pm1}\epsilon_1\cdots\epsilon_n p\left(\sum_{i=1}^{n}\epsilon_i x_i\right). \quad (35)$$

By the above polarization formula, the mapping $\Delta_n^* : L_a^s({}^nE;F) \to P_a({}^nE;F)$ is a linear isomorphism.

We denote by $\bigotimes_n E$ the n-fold tensor product space of E. Let i_n be the linear mapping from E^n into $\bigotimes_n E$ defined by

$$i_n(x_1,\ldots,x_n) = x_1\otimes\cdots\otimes x_n \quad (36)$$

for every $(x_1,\ldots,x_n) \in E^n$. For any $u \in L_a({}^nE;F)$, there exists a unique $i_n^*(u) \in L_a(\bigotimes_n E;F)$ such that the diagram



(37)

commutes. The mapping

$$i_n^* : L_a({}^nE;F) \longrightarrow L_a\left(\bigotimes_n E;F\right) \quad (38)$$

is a linear isomorphism. Each element of $\bigotimes_n E$ has a representation of the form

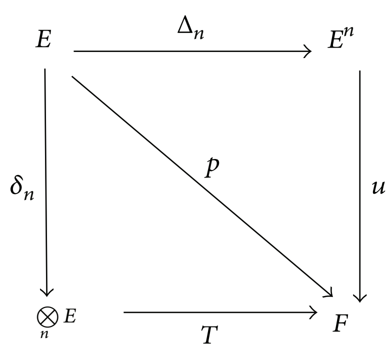
$$\sum_{i=1}^{\ell} x_{i,1}\otimes x_{i,2}\otimes\cdots\otimes x_{i,n}. \quad (39)$$

However, this representation will never be unique. We denote by δ_n the mapping of E into $\bigotimes_n E$ defined by

$$\delta_n(x) = x\otimes\cdots\otimes x \quad (40)$$

for every $x \in E$.

Proposition 7. *A mapping $p : E \to F$ is an n-homogeneous polynomial if and only if there exists $T \in L_a({}^nE;F)$ such that the diagram*



(41)

commutes.

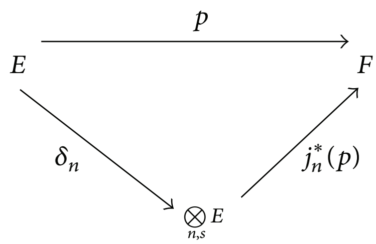
For any $x_1\otimes\cdots\otimes x_n \in \bigotimes_n E$, we set

$$s(x_1\otimes\cdots\otimes x_n) = \frac{1}{n!}\sum_{\sigma\in S_n} x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}. \quad (42)$$

We denote by $\bigotimes_{n,s}E$ the subspace of $\bigotimes_n E$ generated by $s(x_1\otimes\cdots\otimes x_n)$, $x_i \in E$. The space $\bigotimes_{n,s}E$ is called *the n-fold symmetric tensor product space* of E. Elements of $\bigotimes_{n,s}E$ are called *n-symmetric tensors*. Clearly, every tensor of the form $x\otimes\cdots\otimes x$ is a symmetric tensor. Moreover, each element θ in $\bigotimes_{n,s}E$ can be expressed as a finite sum (not necessarily unique) of the form

$$\sum_i x_i\otimes\cdots\otimes x_i. \quad (43)$$

For any $p \in P_a({}^nE;F)$, there exists a unique $j_n^*(p) \in L_a(\bigotimes_{n,s}E;F)$ such that the diagram



(44)

commutes. The mapping

$$j_n^* : P_a({}^nE;F) \longrightarrow L_a\left(\bigotimes_{n,s}E;F\right) \quad (45)$$

is a linear isomorphism. Let $P({}^nE;F)$, $L({}^nE;F)$, and $L^s({}^nE;F)$ be, respectively, the spaces of continuous n-homogeneous polynomials from E into F and the continuous symmetric n-linear mappings from E into F. The restrictions

$$\begin{gathered}\Delta_n^* : L({}^nE;F) \longrightarrow P({}^nE;F),\\ \Delta_n^* : L^s({}^nE;F) \longrightarrow P({}^nE;F),\\ s : L({}^nE;F) \longrightarrow L^s({}^nE;F)\end{gathered} \quad (46)$$

are well-defined. For each $\alpha \in cs(E)$ and $\theta = \sum_i x_i\otimes\cdots\otimes x_i \in \bigotimes_{n,s}E$, we set

$$\pi_{\alpha,n}(\theta) = \inf\left\{\sum_i \alpha(x_i)^n \mid \theta = \sum_i x_i\otimes\cdots\otimes x_i\right\}. \quad (47)$$

$\pi_{\alpha,n}$ is a seminorm on $\bigotimes_{n,s}E$. We define the π-topology or the projective topology on $\bigotimes_{n,s}E$ as the locally convex topology generated by $\{\pi_{\alpha,n}\}_{\alpha\in cs(E)}$. We denote by $\bigotimes_{n,s,\pi}E$ the space endowed with π-topology and denote by $\overline{\bigotimes}_{n,s,\pi}E$ the completion $\bigotimes_{n,s,\pi}E$. Then, the following is valid (cf. Dineen [2]).

Proposition 8. *Let E be a locally convex space, then we have*

$$\begin{gathered}L({}^nE;F) \cong L\left(\bigotimes_{n.\pi}E;F\right),\\ L^s({}^nE;F) \cong L\left(\bigotimes_{n.s,\pi}E;F\right) \cong P({}^nE;F).\end{gathered} \quad (48)$$

Let $\mathscr{E}(E'')$ be the family of all equicontinuous subsets of E'' with respect to the dual pair $\langle E'', E'\rangle$. We denote by $\mathscr{O}_{w^*,\epsilon}$ the family of subsets of E'' defined by

$$\mathscr{O}_{w^*,\epsilon} = \{V \mid V \subset E'', V \cap M \text{ are } \sigma(E'', E')\text{-open in } M \text{ for all } M \in \mathscr{E}(E'')\}. \quad (49)$$

We denote by $\tau_{w^*,\epsilon}$ the topology on E'' such that the family of all $\tau_{w^*,\epsilon}$-open sets coincides with $\mathscr{O}_{w^*,\epsilon}$. By Theorem 5, the following is valid.

Proposition 9. *A n-homogeneous polynomial p on E is of weak type if and only if there exists a $\tau_{w^*,\epsilon}$-continuous n-homogeneous polynomial $\tilde{p}$ on E'' such that $\tilde{p} = p$ on E.*

However, in general, the topology $\tau_{w^*,\beta}$ is not a locally convex topology (cf. Kōmura [7]).

Definition 10. A locally convex space E is called a DF-space if it contains a countable fundamental system of bounded sets and if the intersection of any sequence of absolutely convex neighborhoods of 0 which absorbs all bounded sets is itself a neighborhood of 0.

Grothendieck [8, 9] proved that the strong dual space of a metrizable locally convex space is a DF-space and the strong dual space of a DF-space is a Fréchet space.

All Banach spaces are DF-spaces. The following result is known.

Proposition 11 (Banach-Dieudonné theorem). *Let E be a metrizable locally convex space. Then, the topology $\tau_{w^*,\epsilon}$ on E' is the topology of uniform convergence on all compact sets in E.*

Proof. The proof is on Köthe [10, § 21–10]. □

If E is a DF space, then E'_β is a Fréchet space. Therefore, by Proposition 11 the following is valid.

Proposition 12. *The topology $\tau_{w^*,\epsilon}$ on E'' is the topology of uniform convergence on all compact sets in E'.*

We denote by α_K the seminorm on E'' defined by

$$\alpha_K(x'') = \sup\{|\langle x'', x'\rangle| \mid x' \in K\} \quad (50)$$

for every compact subsets K of E'_β. We denote by τ_0 the locally convex topology on E'' defined by the set of seminorms:

$$\{\alpha_K \mid K \text{ is every compact set of } E'\}. \quad (51)$$

We denote by E_{τ_0} the locally convex space E defined the set of seminorms:

$$\{\alpha_K|_E \mid K \text{ is every compact set of } E'\}. \quad (52)$$

An n-homogeneous polynomial p on E'' is $\sigma(E'', E')$-continuous on every equicontinuous subsets of E'' if and only if p is $\tau_{w^*,\epsilon}$-continuous on E''. Thus, by Propositions 9 and 11, we have the following theorem.

Theorem 13. *If an n-homogeneous polynomial p on E is of weak type if and only if p is τ_o-continuous in E and continuous on E with the initial topology of E.*

Then, we can obtain the following topological tensor product representation of polynomials of weak type.

Theorem 14. *Let E be a DF space, then we have*

$$L\left(\bigotimes_{n,s\pi} E_{\tau_0}; F\right) \cap L\left(\bigotimes_{n,s\pi} E;\ F\right) \cong P_w(^nE; F) \quad (53)$$

for every complete locally convex space F.

A topological space X is called a k-space if its topology is localized on its compact set;, that is, $U \subset X$ is open if and only if $U \cap K$ is open in K, with the induced topology, for each compact subset K of X. A mapping from a k-space into a topological space is continuous if and only if its restriction to each compact set is continuous.

Let u be a mapping from a locally convex space E into a locally convex space F. If E is a k-space and if u is $\sigma(E, E')$-continuous on every bounded subset of E, then u is continuous on E with the initial topology of E. Thus, we obtain the following theorem.

Theorem 15. *If E is a DF space and a k-space, then we have*

$$L\left(\bigotimes_{n,s\pi} E_{\tau_0}; F\right) \cong P_w(^nE; F) \quad (54)$$

for every complete locally convex space F.

If E is a Banach space, then E is a DF-space and a k-space. Thus, we have the following corollary.

Corollary 16. *If E is a Banach space, then we have*

$$L\left(\bigotimes_{n,s\pi} E_{\tau_0}; F\right) \cong P_w(^nE; F) \quad (55)$$

for every complete locally convex space F.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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