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Research Article

A Korovkin Type Approximation Theorem and Its Applications

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We present a Korovkin type approximation theorem for a sequence of positive linear operators defined on the space of all real valued continuous and periodic functions via A-statistical approximation, for the rate of the third order Ditzian-Totik modulus of smoothness. Finally, we obtain an interleave between Riesz's representation theory and Lebesgue-Stieltjes integral-i, for Riesz's functional supremum formula via statistical limit.

1. Introduction and Main Results

Some will accept the notes and definitions used in this paper. The concept of A-statistical approximation for regular summability matrix (see [1, 2]). Let $A = (a_{nk})$, $n, k = 1, 2, \ldots$, be an infinite summability matrix. For a given sequence $x = (x_k)$, the A-transform of x, denoted by $Ax = (Ax)_n$, is given by $(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k$, provided that the series converges for each n. A is said to be regular if $\lim_{n\to\infty}(Ax)_n = L$, whenever $\lim x = L$. Then $\lim_{n\to\infty} a_{nk} = 0$, for all $k \in N$. In [3], Dzyubenko and Gilewicz have given the notion.

A is nonnegative regular summability matrix. Then x is A-statistically convergent to L, if, for every $\epsilon > 0$, $\lim_{n\to\infty} \sum_{k:|x_k - L| \geq \epsilon} a_{nk} = 0$.

We denote by $\mathsf{C}_{2\pi}(\mathscr{R})$ the space of all 2π-periodic and continuous functions on $\mathscr{R}$. Endowed with the norm $\|\cdot\|_{2\pi}$, this space is a Banach space, where $\|f\|_{2\pi} = \sup\{|f(t)| : f \in \mathsf{C}_{2\pi}(\mathscr{R}), t \in \mathscr{R}\}$. Now, recall that, in [4], the mth order Ditzian-Totik modulus of smoothness in the uniform metric is given by

$$\omega_m^{\phi}(f, \delta, [a,b]) = \sup_{0<h\leq\delta} \left\| \Delta_{h\phi(x)}^{m}(f, x, [a,b]) \right\|_{[a,b]}, \quad (1)$$

where

$$\Delta_h^m(f, x, [a,b]) = \begin{cases} \displaystyle\sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} \\ \quad \times f\left(x - \dfrac{mh}{2} + ih\right), & \text{if } x \pm \dfrac{mh}{2} \in [a,b] \\ 0, & \text{o.w} \end{cases} \quad (2)$$

is the symmetric mth difference. We have to recall the Korovkin type theorem.

Theorem 1 (see [2]). *Let $A = (A^n)_{n\geq 1}$ be a sequence of infinite nonnegative real matrices such that $\sup_{n,k} \sum_{j=1}^{\infty} a_{kj}^n < \infty$ and let $\{L_j\}$ be a sequence of positive linear operators mapping $\mathsf{C}_{2\pi}(\mathscr{R})$ into $\mathsf{C}_{2\pi}(\mathscr{R})$. Then, for all $f \in \mathsf{C}_{2\pi}(\mathscr{R})$, we have $\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^n \|L_j f - f\|_{2\pi} = 0$ uniformly in n, if and only if $\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^n \|L_j f_i - f_i\|_{2\pi} = 0$ $(i = 1, 2, 3)$, uniformly in n, where $f_1(t) = 1$, $f_2(t) = \cos t$, and $f_3(t) = \sin t$, for all $t \in \mathscr{R}$.*

It is worth noting that the statistical analog of Theorem 1 has been studied by Radu [2], as follows.

Theorem 2. *Let $A = (A^n)_{n\in N}$ be a sequence of nonnegative regular summability matrices and let $\{L_j\}$ be a sequence of positive linear operators mapping $C_{2\pi}(\mathscr{R})$ into $C_{2\pi}(\mathscr{R})$. Then, for all $f \in C_{2\pi}(\mathscr{R})$, we have $st_A - \lim_{j\to\infty}\|L_j f - f\|_{2\pi} = 0$, uniformly in n, if and only if $st_A - \lim_{j\to\infty}\|L_j f_i - f_i\|_{2\pi} = 0$ $(i = 1, 2, 3)$, uniformly in n, where $f_1(t) = 1$, $f_2(t) = \cos t$, and $f_3(t) = \sin t$, for all $t \in \mathscr{R}$.*

The following notations are used this paper (see [5, 6]).

Let n be fixed and sufficiently large. If $y_i \in I_{j(i)}$ and $1 \le i \le k$, then it is convenient to denote

$$\begin{aligned}
y_i' &= x_{j(i)+1}, \\
y_i'' &= x_{j(i)-2}, \\
I_i' &= \left[y_i', y_i''\right] = I_{j(i)+1} \cup I_{j(i)} \cup I_{j(i)-1} \\
&= \left[x_{j(i)+1}, x_{j(i)}\right] \cup \left[x_{j(i)}, x_{j(i)-1}\right] \cup \left[x_{j(i)-1}, x_{j(i)-2}\right], \\
\rho_i &= \left[\frac{y_i + y_i'}{2}, \frac{y_i + y_i''}{2}\right], \quad \text{for } 1 \le i \le k, \\
\frac{5}{3}h_{j(i)} &= \frac{5}{3}\left(x_{j(i)-1} - x_{j(i)}\right) < \left(x_{j(i)-2} - x_{j(i)+1}\right) = \left|I_i'\right| \\
&= 2\left|\rho_i\right| = \left(y_i'' - y_i'\right) = \left(x_{j(i)-2} - x_{j(i)+1}\right) < 7, \\
h_{j(i)} &= 7\left(x_{j(i)-1} - x_{j(i)}\right), \quad 1 \le i \le k,
\end{aligned} \tag{3}$$

and therefore $|I_i'| \sim |\rho_i| \sim h_{j(i)}$, for $x \in I_i'$. Recall that

$$\operatorname{sgn}\left(f(x)\right) = \begin{cases} 1; & \text{if } x \in [a, b], \\ -1; & \text{if } x \notin [a, b], \end{cases} \tag{4}$$

is the sign of f on $[a, b]$.

Now, let us introduce our theorems as follows.

Theorem 3. *Let $A = (A^n)_{n\ge 1}$ be a sequence of infinite nonnegative real matrices such that $\sup_{n,k}\sum_{j=1}^{\infty} a_{kj}^n < \infty$ and let $\{L_j\}$ be a sequence of positive linear operators mapping $C_{2\pi}(\mathscr{R})$ into $C_{2\pi}(\mathscr{R})$. Then, for all $f \in C_{2\pi}(\mathscr{R})$, we have*

$$\sum_{j=1}^{\infty} a_{kj}^n \left\|L_j f - f\right\|_{2\pi} \le c\omega_3^{\phi}\left(f, \frac{\pi}{n}, [-\pi, \pi]\right), \tag{5}$$

uniformly in n, if and only if

$$\sum_{j=1}^{\infty} a_{kj}^n \left\|L_j f_\zeta - f_\zeta\right\|_{2\pi} \le c\omega_3^{\phi}\left(f_\zeta, \frac{\pi}{n}, [-\pi, \pi]\right), \quad \zeta = 1, 2, 3, \tag{6}$$

uniformly in n, where $f_1(t) = 1$, $f_2(t) = (t - y_i')/(y_i'' - y_i)$, and $f_3(t) = (t - y_i'')/(y_i - y_i')$, for all $t \in \mathscr{R}$. And c the constant does not depend on j.

Theorem 4. *Let $A = (A^n)_{n\in N}$ be a sequence of nonnegative regular summability matrices and let $\{L_j\}$ be a sequence of positive linear operators mapping $C_{2\pi}(\mathscr{R})$ into $C_{2\pi}(\mathscr{R})$. Then, if there exists $f \in C_{2\pi}(\mathscr{R})$, we have*

$$st_A - \lim_n \left\|L_j f - f\right\|_{2\pi} \ge c(j)\,\omega_3^{\phi}\left(f, \frac{\pi}{n}, [-\pi, \pi]\right), \tag{7}$$

uniformly in n, if and only if

$$st_A - \lim_n \left\|L_j f_\zeta - f_\zeta\right\|_{2\pi} \ge c(j)\,\omega_3^{\phi}\left(f_\zeta, \frac{\pi}{n}, [-\pi, \pi]\right) \quad \zeta = 1, 2, 3, \tag{8}$$

uniformly in n, where $f_1(t) = 1$, $f_2(t) = (t - y_i')/(y_i'' - y_i)$, and $f_3(t) = (t - y_i'')/(y_i - y_i')$, for all $t \in \mathscr{R}$.

2. Proofs of Theorems 3 and 4

Proof of Theorem 3. Since f_ζ $(\zeta = 1, 2, 3)$ belong to $C_{2\pi}(\mathscr{R})$, implications (5) ⇒ (6) are obvious. Now, assume that (6) holds. Let $f \in C_{2\pi}(\mathscr{R})$, and, I be a closed subinterval of length 2π of $\mathscr{R}$. And let L_j be defined by

$$\begin{gathered}
L_j(x) = \frac{x_n - y_i}{y_i'' - y_i'}\left(\frac{x - y_i'}{y_i'' - yi}L_j\left(y_i''\right) - \frac{x - y_i''}{y_i - y_i'}L_j\left(y_i'\right)\right), \\
y_i \in \left[\frac{y_i + y_i'}{2}, \frac{y_i + y_i''}{2}\right] \quad \text{for } i = 1, 2, 3,
\end{gathered} \tag{9}$$

and also where $L_j(y_i')$ and $L_j(y_i'')$ are chosen so that

$$\begin{gathered}
L_j\left(y_i'\right) = \begin{cases} \dfrac{\mathsf{C}}{\omega_3^{\phi}\left(f, n^{-1}\right)\operatorname{sgn}\left(f\left(y_i'\right)\right)}; & \text{if } \left|f\left(y_i'\right)\right| \le c\omega_3^{\phi}\left(f, n^{-1}\right), \\ f\left(y_i'\right); & \text{o.w,} \end{cases} \\
L_j\left(y_i''\right) = \begin{cases} \dfrac{\mathsf{C}}{\omega_3^{\phi}\left(f, n^{-1}\right)\operatorname{sgn}\left(f\left(y_i''\right)\right)}; & \text{if } \left|f\left(y_i''\right)\right| \le c\omega_3^{\phi}\left(f, n^{-1}\right), \\ f\left(y_i''\right); & \text{o.w.} \end{cases}
\end{gathered} \tag{10}$$

In [5] Kopotun, we have $|f(x) - L(f;x)| \le c\omega_3^{\phi}(f, n^{-1})$ and $x \in I$, where

$$L(f;x) = L(f;x \mid y_i, y_i', y_i'') = \frac{x_n - y_i}{y_i'' - y_i'}\left(\frac{x - y_i'}{y_i'' - y_i} L_j(y_i'') - \frac{x - y_i''}{y_i - y_i'} L_j(y_i')\right) \tag{11}$$

is the Lagrange polynomial of degree ≤ 2, which interpolates f at y_i, y_i', and y_i''. Inequality (11) is an analog of Whitney's inequality for Ditzian-Totik moduli. Using (11) and the above presentations of L_j and $L(f;x)$, we write, for $x \in I$,

$$\begin{aligned} &\left|L_j(f;x) - f(x)\right| \\ &\quad \le \left|L_j(f;x) - L(f;x)\right| + \left|L(f;x) - f(x)\right| \\ &\quad \le \left|\frac{(x_n - y_i)(x - y_i')}{(y_i'' - y_i')(y_i'' - y_i)}\right| \left|L_j(y_i'') - f(y_i'')\right| \\ &\qquad + \left|\frac{(x_n - y_i)(x - y_i'')}{(y_i'' - y_i')(y_i'' - y_i)}\right| \\ &\qquad \times \left|L_j(y_i') - f(y_i')\right| + \left|L(f;x) - f(x)\right|. \end{aligned} \tag{12}$$

Taking supremum over x and $= 1/(\omega_3^{\phi}(f, n^{-1})\,\mathrm{sgn}(f))$, we obtain

$$\begin{aligned} &\left\|L_j(f;x) - f(x)\right\|_{2\pi} \\ &\quad \le \left\|L_j(f;x) - L(f;x)\right\|_{2\pi} + \left\|L(f;x) - f(x)\right\|_{2\pi} \\ &\quad \le \left|\frac{(x_n - y_i)(x - y_i')}{(y_i'' - y_i')(y_i'' - y_i)}\right| \left\|L_j(y_i'') - f(y_i'')\right\|_{2\pi} \\ &\qquad + \left|\frac{(x_n - y_i)(x - y_i'')}{(y_i'' - y_i')(y_i'' - y_i)}\right| \left\|L_j(y_i') - f(y_i')\right\|_{2\pi} \\ &\qquad + \left\|L(f;x) - f(x)\right\|_{2\pi} \\ &\quad \le c(K, y_i'') + c(K, y_i') + c\omega_3^{\phi}(f, n^{-1}, [-1,1]). \end{aligned} \tag{13}$$

Suppose $B > 0$, let us write sets as follows:

$$\begin{aligned} \vartheta = \Bigg\{ j : &\left\|L_j(1;x) - 1\right\|_{2\pi} \\ &+ \left\|L_j\left(\frac{t - y_i'}{y_i'' - y_i}; x\right) - \frac{x - y_i'}{y_i'' - y_i}\right\|_{2\pi} \\ &+ \left\|L_j\left(\frac{t - y_i''}{y_i - y_i'}; x\right) - \frac{x - y_i''}{y_i - y_i'}\right\|_{2\pi} \ge KB \Bigg\}, \end{aligned}$$

$$\begin{aligned} \vartheta_1 &= \left\{ j : \left\|L_j(1;x) - 1\right\|_{2\pi} \ge KB \right\}, \\ \vartheta_2 &= \left\{ j : \left\|L_j\left(\frac{t - y_i'}{y_i'' - y_i}; x\right) - \frac{x - y_i'}{y_i'' - y_i}\right\|_{2\pi} \ge KB \right\}, \\ \vartheta_3 &= \left\{ j : \left\|L_j\left(\frac{t - y_i''}{y_i - y_i'}; x\right) - \frac{x - y_i''}{y_i - y_i'}\right\|_{2\pi} \ge KB \right\}. \end{aligned} \tag{14}$$

Consequently, we get $\vartheta \subset \vartheta_1 \cup \vartheta_2 \cup \vartheta_3$ and $\sum_{j\in\vartheta} a_{kj}^n \ge \sum_{j\in\vartheta_1} a_{kj}^n \ge \sum_{j\in\vartheta_2} a_{kj}^n \ge \sum_{j\in\vartheta_3} a_{kj}^n$ implies

$$\sum_{j=1}^{\infty} a_{kj}^n \left\|L_j f - f\right\|_{2\pi} \le c\omega_3^{\phi}\left(f, \frac{\pi}{n}, [-\pi, \pi]\right). \tag{15}$$

□

Proof of Theorem 4. Since f_ζ $(\zeta = 1,2,3)$ belong to $C_{2\pi}(\mathscr{R})$, implications (8) ⇒ (7) are obvious. Assume that the condition (7) is satisfied. Let $f \in C_{2\pi}(\mathscr{R})$ and I be a closed subinterval of length 2π of $\mathscr{R}$; we have

$$st_A - \lim_n \left\|L_j f - f\right\|_{2\pi} \ge c(j)\,\omega_3^{\phi}\left(f, \frac{\pi}{n}, [-\pi, \pi]\right). \tag{16}$$

Now, given $K(j) > 0$, choose $B > 0$, where $B = \sup\{|f(x)| : x \in I\}$ implied $K < B$, and define the following set:

$$\begin{aligned} \vartheta = \Bigg\{ j : &\left\|L_j(1;x) - 1\right\|_{2\pi} \\ &+ \left\|L_j\left(\frac{t - y_i'}{y_i'' - y_i}; x\right) - \frac{x - y_i'}{y_i'' - y_i}\right\|_{2\pi} \\ &+ \left\|L_j\left(\frac{t - y_i''}{y_i - y_i'}; x\right) - \frac{x - y_i''}{y_i - y_i'}\right\|_{2\pi} \ge KB \Bigg\}. \end{aligned} \tag{17}$$

Thus,

$$\begin{aligned} &st_A - \lim_n \left\|L_j p_3(t) - p_3(x)\right\|_{2\pi} \\ &\quad = st_A - \lim_n \left\|L_j\left(\frac{t - y_i''}{y_i - y_i'}; x\right) - \frac{x - y_i''}{y_i - y_i'}\right\|_{2\pi} \\ &\quad \ge (K - K_\circ)\grave{B}, \end{aligned} \tag{18}$$

where $p_3(x) = (x - y_i'')/(y_i - y_i') \in C_{2\pi}(\mathscr{R})$ polynomial and $x \in \mathscr{R}$. Since x is A-statistically convergent, we can easily show that $\vartheta \supset \vartheta_1 \supset \vartheta_2 \supset \vartheta_3$ implies $\sum_{j\in\vartheta} a_{kj}^n \ge \sum_{j\in\vartheta_1} a_{kj}^n \ge \sum_{j\in\vartheta_2} a_{kj}^n \ge \sum_{j\in\vartheta_3} a_{kj}^n$.

Now, let $\grave{B} = \omega_3^{\phi}(f_\zeta, \pi/n, [-\pi, \pi])$, and using (7) implies

$$st_A - \lim_n \left\|L_j p_3(t) - p_3(x)\right\|_{2\pi} \ge c(j)\,\omega_3^{\phi}\left(p_3, \frac{\pi}{n}, [-\pi, \pi]\right). \tag{19}$$

This is a complete proof. □

3. Application to Functional Approximation

In this section we give some applications which satisfy our theorems, but it's not the classical Korovkin theorem. It has been treated with the Weierstrass second approximation theorem via A-statistical convergence (see [6–8]). If $f \in C_{2\pi}(\mathscr{R})$, then there is a sequence of polynomials and A-statistically uniformly convergent to f on $[-\pi, \pi]$ (not uniformly convergent). Observe that Fejer operators may be written in the form of

$$F_n(f;x) = \frac{a_\circ}{2} + \sum_{k=1}^{n} \frac{n-k}{n}\left(a_k \frac{kx - y_i'}{y_i'' - y_i} + b_k \frac{kx - y_i''}{y_i - y_i'}\right). \quad (20)$$

We now consider the linear operator T_n defined by

$$T_n(f;x) = \frac{a_\circ}{2} + \sum_{k=1}^{n} \xi_k^{(n)}\left(a_k \frac{kx - y_i'}{y_i'' - y_i} + b_k \frac{kx - y_i''}{y_i - y_i'}\right), \quad (21)$$

where $\{\xi_k^{(n)}\}$ $(n = 1, 2, \ldots; k = 1, 2, \ldots, n)$ is a matrix of real numbers and also a_k and b_k are Fourier coefficients. Now, let $A = (a_{nk})$ be a nonnegative regular summability matrice. Assume that the following statements are satisfied:

(i) $st_A - \lim_n \xi_1^{(n)} = 1$;

(ii) $(1/2) + \sum_{k=1}^{n} \xi_k^{(n)}(t - y_i')/(y_i'' - y_i) \geq c(n)\omega_3^{\phi}(f, \pi/n, [-\pi, \pi])$. We get

$$\begin{aligned} & st_A - \lim_n \left\| T_n(f;x) - f(x) \right\|_{2\pi} \\ & \geq c(n)\,\omega_3^{\phi}\left(f, \frac{\pi}{n}, [-\pi, \pi]\right), \quad \forall f \in C_{2\pi}(\mathscr{R}), \end{aligned} \quad (22)$$

where $\{T_n\}$ is the sequence of linear operators given by (21).

In [9], Sakaoğlu and Ünver proved the following theorem by using $L_P[a, b; c, d]$ and denoted the space of all functions f defined on $[a, b] \times [c, d]$, for which $\int_c^d \int_a^b |f(x, y)|^P dx\,dy < \infty$, $1 \leq P < \infty$. In this case, the L_P norm of a function f in $L_P[a, b; c, d]$, denoted by $\|f\|_P$, is given by $\|f\|_P = (\int_c^d \int_a^b |f(x, y)|^P dx\,dy)^{1/P}$.

Theorem 5 (see [9]). *Let $A = (a_{jn})$ be a nonnegative regular summability matrix and let $\{T_n\}$ be an A-statistically uniformly bounded sequence of positive linear operators from $L_P[a, b; c, d]$ into $L_P[a, b; c, d]$ and $1 \leq P < \infty$. Then, for any function $f \in L_P[a, b; c, d]$, $st_A - \lim_n \|T_n(f; x, y) - f(x, y)\|_P = 0$ if and only if $st_A - \lim_n \|T_n(f_i; x, y) - f_i(x, y)\|_P = 0$, $i = 1, 2, 3, 4$ where $f_1(t, v) = 1$, $f_2(t, v) = t$, $f_3(t, v) = v$, and $f_4(t, v) = t^2 + v^2$.*

The theory of the Lebesgue integral can be developed in several distinct ways (see [10, 11]). Only one of these methods will be discussed here.

Now, let us introduce our definition as follows.

Definition 6 (Lebesgue-Stieltjes integral-i). Let S be measurable set, $f : S \to R$ be a bounded function, and $g_i : S \to R$ be nondecreasing function for $i \in I$. For $\mathscr{P}$ Lebesgue partition of S, put $\underline{\mathrm{LS}}(f, \mathscr{P}, \underline{g}) = \sum_{j=1}^{n} \prod_{i \in I} m_j g_i(\mu(S_j))$ and $\overline{\mathrm{LS}}(f, \mathscr{P}, \underline{g}) = \sum_{j=1}^{n} \prod_{i \in I} M_j g_i(\mu(S_j))$ such that μ measurable function of S; $m_j = \inf\{f(x) : x \in S_j\}$, $M_j = \sup\{f(x) : x \in S_j\}$, and $\underline{g} = g_1, g_2, \ldots$. Also, $g_i(x_j) - g_i(x_{j-1}) > 0$, $\underline{\mathrm{LS}}(f, \mathscr{P}, \underline{g}) \leq \overline{\mathrm{LS}}(f, \mathscr{P}, \underline{g})$, $\prod_{i \in I} \underline{\int}_i f\,\mathrm{d}\underline{g} = \sup\{\underline{\mathrm{LS}}(f, \underline{g})\}$, and $\prod_{i \in I} \overline{\int}_i f\,\mathrm{d}\underline{g} = \inf\{\overline{\mathrm{LS}}(f, \underline{g})\}$, where $\underline{\mathrm{LS}}(f, \underline{g}) = \{\underline{\mathrm{LS}}(f, \mathscr{P}, \underline{g}) : \mathscr{P}$ part of set $S\}$ and $\overline{\mathrm{LS}}(f, \underline{g}) = \{\overline{\mathrm{LS}}(f, \mathscr{P}, \underline{g}) : \mathscr{P}$ part of set $S\}$. If $\prod_{i \in I} \underline{\int}_i f\,\mathrm{d}\underline{g} = \prod_{i \in I} \overline{\int}_i f\,\mathrm{d}\underline{g}$, where $\mathrm{d}\underline{g} = \mathrm{d}g_1 \times \mathrm{d}g_2 \times \ldots \times \mathrm{d}g_n \ldots$. Then f is integral $\int_i$ according to g_i for $i \in I$.

Now, we can provide our theorem as follows as a case which is an illustrative application of approximation theory in functional analysis using functional supremum to limit convergence that acts as support and reinforcement of the concept of Riesz's representation.

Theorem 7. *If a sequence $G_n(f)$ is positive linear functional and bounded on $C(S)$, f is bounded measurable function to S. Then, there exists nondecreasing function to S such that $st_A - \lim_{\mu(S) \to 0}(\sup_n G_n(f) - f) = 0$.*

Proof. Assume that functional supremum G_n is as follows:

$$\sup_n G_n(f) = \sup_n \prod_{i \in I} \int_i f\,\mathrm{d}\underline{\varphi_{t,n}}, \quad (23)$$

where $\varphi_{t,n}(x) = (1 - n(x - t))/(y_i'' - y_i')$ converges to $r \in R$; that is, let $\mathscr{P} = \{S_\ell\}_{\ell=0}^{m}$ be Lebesgue partition such that

$$\begin{gathered} \sup\{\mu(S_\ell) : \ell = 0, \ldots, m\} < \frac{1}{2}\,\delta(\varepsilon), \\ \frac{1}{2} < \inf\{\mu(S_\ell) : \ell = 0, \ldots, m\}, \\ \varphi_{it,n}(\mu(S_\ell)) \leq G(\pi_{t,n}(\mu(S_\ell))) \leq \varphi_{it,n}(\mu(S_\ell)) + \frac{\varepsilon}{m}\|f\|, \end{gathered} \quad (24)$$

where $\pi_{t,n}(x) = \varphi_{t,n}(x)(y_i'' - y_i')$.

Since G positive linear functional and bounded on $C(S)$, then

$$\begin{aligned} & \left| G(f) - G\left(f(t_1)\ \pi_{t_1,n} + \sum_{\mathscr{L}=2}^{m} f(t_{\mathscr{L}})\,\pi_{t_{\mathscr{L}},n}(\mu(S_\ell)) \right) \right| \\ & < \frac{\varepsilon}{m}\|f\|, \end{aligned} \quad (25)$$

also, respect between sum $\mathrm{LS}(f,\mathscr{P},\underline{\varphi_{t,n}})$ and Lebesgue-Stieltjes integral-i are 2ε, we have

$$\begin{aligned}
&\left|\prod_{i\in I}\int_i f\,\mathrm{d}\underline{\varphi_{t,n}} - G(f)\right| \\
&\quad \le \left|\prod_{i\in I}\int_i f\,\mathrm{d}\underline{\varphi_{t,n}}\right. \\
&\qquad \left. - G\left(f(t_1)\,\pi_{t_1,n} + \sum_{\mathscr{L}=2}^{m} f(t_{\mathscr{L}})\,\pi_{t_{\mathscr{L}},n}(\mu(S_\ell))\right)\right| \\
&\qquad + \left|G(f) - G\left(f(t_1)\,\pi_{t_1,n} + \sum_{\mathscr{L}=2}^{m} f(t_{\mathscr{L}})\,\pi_{t_{\mathscr{L}},n}(\mu(S_\ell))\right)\right| \\
&\quad < \frac{\grave{\varepsilon}}{m}
\end{aligned} \tag{26}$$

as $m \to \infty$; hence G satisfies Lebesgue-Stieltjes integral-i of f.

Now, since $G_n(f)$ is functional supremum and satisfies Lebesgue-Stieltjes integral-i, and us Definition 6, we have

$$\begin{aligned}
&\sup_n G_n(f) - f \\
&\quad = \sup_n \prod_{i\in I}\int_i f\,\mathrm{d}\underline{\varphi_{t,n}} - f \\
&\quad = \sup_n\left(\sup\sum_{j=1}^{n}\prod_{i\in I} m_j\varphi_{it,n}(\mu(S_j))\right) - f \\
&\quad = \sup_n\left(\sup\sum_{j=1}^{n}\prod_{i\in I}\inf_j f(x)\,\varphi_{it,n}(\mu(S_j))\right) - f \\
&\quad = \sup_n\left(\inf\sum_{j=1}^{n}\prod_{i\in I}\sup_j f(x)\,\varphi_{it,n}(\mu(S_j))\right) - f \\
&\quad = \sup_n\left(\inf\sum_{j=1}^{n}\sup_j f(x)\prod_{i\in I}\varphi_{it,n}(\mu(S_j))\right) - f \\
&\quad = \sup_n\left(\inf\sum_{j=1}^{n}\sup_j f(x)\right. \\
&\qquad \times\left(\left[\left(\frac{1-n(\mu(S_1)-t)}{y_i''-y_i'}\right)\right.\right. \\
&\qquad \left.\left.\left.\cdot\left(\frac{1-n(\mu(S_2)-t)}{y_i''-y_i'}\right)\cdots\right]\right)\right) - f \\
&\quad = \sup_n\left(\inf\left(\sup_1 f(x)\right.\right. \\
&\qquad \times\left[\left(\frac{1-n(\mu(S_1)-t)}{y_i''-y_i'}\right)\right. \\
&\qquad \left.\cdot\left(\frac{1-n(\mu(S_2)-t)}{y_i''-y_i'}\right)\cdots\right] \\
&\qquad + \sup_2 f(x)\left[\left(\frac{1-n(\mu(S_1)-t)}{y_i''-y_i'}\right)\right. \\
&\qquad \left.\cdot\left(\frac{1-n(\mu(S_2)-t)}{y_i''-y_i'}\right)\cdots\right] \\
&\qquad + \cdots + \sup_n f(x) \\
&\qquad \times\left[\left(\frac{1-n(\mu(S_1)-t)}{y_i''-y_i'}\right)\right. \\
&\qquad \left.\left.\left.\cdot\left(\frac{1-n(\mu(S_2)-t)}{y_i''-y_i'}\right)\cdots\right]\right)\right) - f.
\end{aligned} \tag{27}$$

Note that effect sum on measurable function $\mu(S_j)$ by using Lebesgue partition

$$\lim_{\mu(S)\to 0}\prod_{i\in I}\int_i f\,\mathrm{d}\underline{\varphi_{t,n}} = f; \tag{28}$$

let $n \in N$, choose $\mathscr{K}_n > 0$, and define the following sets:

$$\begin{aligned}
\mathbb{L} &= \left\{\mu(S) : \left(\sup_n G_n(f) - f\right) \ge \mathscr{K}_n\right\}, \\
\mathbb{L}_1 &= \left\{\mu(S) : \left(\sup_n G_n(f_1) - f_1\right) \ge \frac{\mathscr{K}_n}{3}\right\}, \\
\mathbb{L}_2 &= \left\{\mu(S) : \left(\sup_n G_n(f_2) - f_2\right) \ge \frac{\mathscr{K}_n}{3}\right\}, \\
\mathbb{L}_3 &= \left\{\mu(S) : \left(\sup_n G_n(f_3) - f_3\right) \ge \frac{\mathscr{K}_n}{3}\right\};
\end{aligned} \tag{29}$$

Then $\mathbb{L} \subset \mathbb{L}_1 \cup \mathbb{L}_2 \cup \mathbb{L}_3$, which gives

$$\sum_{\mu(S)\subset\mathbb{L}} a_{kj}^n \le \sum_{\mu(S)\subset\mathbb{L}_1} a_{kj}^n \cup \sum_{\mu(S)\subset\mathbb{L}_2} a_{kj}^n \cup \sum_{\mu(S)\subset\mathbb{L}_3} a_{kj}^n; \tag{30}$$

we obtain that $st_A - \lim_{\mu(S)\to 0}\sum_{\mu(S)\subset\mathbb{L}} a_{kj}^n = 0$ implies $st_A - \lim_{\mu(S)\to 0}(\sup_n G_n(f) - f) = 0$. □

Now, in this paper we have proved Riesz's representation theory with Lebesgue-Stieltjes integral-i, by using Korovkin type approximation which is one of the threads in the development of Riesz's theorem to support the definition of Lebesgue integral, Rudin [10]. This integration toxicity ratio for the world on behalf of the French Lebesgue, who came in his thesis for a doctorate in 1902.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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