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Research Article

On the Analyticity for the Generalized Quadratic Derivative Complex Ginzburg-Landau Equation

Chunyan Huang

School of Statistics and Mathematics, Central University of Finance and Economics, Beijing 100081, China

Correspondence should be addressed to Chunyan Huang; hcy@cufe.edu.cn

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We study the analytic property of the (generalized) quadratic derivative Ginzburg-Landau equation ($1/2 \leqslant \alpha \leqslant 1$) in any spatial dimension $n \geqslant 1$ with rough initial data. For $1/2 < \alpha \leqslant 1$, we prove the analyticity of local solutions to the (generalized) quadratic derivative Ginzburg-Landau equation with large rough initial data in modulation spaces $M^{1-2\alpha}_{p,1}$ ($1 \leqslant p \leqslant \infty$). For $\alpha = 1/2$, we obtain the analytic regularity of global solutions to the fractional quadratic derivative Ginzburg-Landau equation with small initial data in $\dot{B}^0_{\infty,1}(\mathbb{R}^n) \cap M^0_{\infty,1}(\mathbb{R}^n)$. The strategy is to develop uniform and dyadic exponential decay estimates for the generalized Ginzburg-Landau semigroup $e^{-(a+i)t(-\Delta)^\alpha}$ to overcome the derivative in the nonlinear term.

1. Introduction

In this paper, we are interested in the Cauchy problem of the following generalized quadratic derivative complex Ginzburg-Landau equation (GDGL):

$$u_t + (a+i)(-\Delta)^\alpha u - \vec{\gamma} \cdot \nabla \left(u^2\right) = 0, \quad u(0,x) = u_0(x), \tag{1}$$

where u is a complex valued function of $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$, $\mathbb{R}^+ = [0,\infty)$, $n \geqslant 1$. $a > 0$ is the dissipative coefficient, $1/2 \leqslant \alpha \leqslant 1$. $\vec{\gamma}$ is a given complex vector in $\mathbb{R}^n$. $u_0(x)$ is a given complex valued function of $x \in \mathbb{R}^n$. $u_t = \partial u/\partial t$ and $(-\Delta)^\alpha$ denotes the fractional Laplacian defined by $\widehat{(-\Delta)^\alpha u}(t,\xi) = |\xi|^{2\alpha}\widehat{u}(t,\xi)$. It is well known that (1) can be rewritten into an integral equation as follows:

$$u(t) = G_{2\alpha}(t)u_0 + \mathscr{A}_{2\alpha}\left(\vec{\gamma} \cdot \nabla\left(u^2\right)\right), \tag{2}$$

where

$$\begin{gathered} G_{2\alpha}(t) := e^{-(a+i)t(-\Delta)^\alpha} = \mathscr{F}^{-1} e^{-(a+i)t|\xi|^{2\alpha}} \mathscr{F}, \\ \left(\mathscr{A}_{2\alpha} f\right)(t) := \int_0^t G_{2\alpha}(t-\tau) f(\tau,x)\, d\tau. \end{gathered} \tag{3}$$

Complex Ginzburg-Landau type equation is one of the most-studied equations in physics. It describes a lot of phenomena including nonlinear waves and the evolution of amplitudes of unstable modes for any process exhibiting a Hopf bifurcation. GDGL (1) is also called derivative fractional Ginzburg-Landau equation. For details of physical backgrounds of the fractional Ginzburg-Landau equation (1), one can refer to [1–3]. Equation (1) is both dissipative and dispersive. If $\alpha = 1$, (1) is the quadratic derivative Ginzburg-Landau equation (QDGL):

$$u_t - (a+i)\Delta u - \vec{\gamma} \cdot \nabla\left(u^2\right) = 0, \quad u(0,x) = u_0(x). \tag{4}$$

If $\alpha = 1$, $a = 0$, (1) reduces to the well-known quadratic derivative Schrödinger equation (DNLS):

$$u_t - i\Delta u - \vec{\gamma} \cdot \nabla\left(u^2\right) = 0, \quad u(0,x) = u_0(x). \tag{5}$$

For DNLS (5), Christ [4] proved that when space dimension $n = 1$, the flow map $u_0 \to u$ is not continuous in any Sobolev space $H^s(\mathbb{R}^1)$ with any exponent s for any short time in the sense that $\|u_0\|_{H^s} \ll 1$ but $\|u(t)\|_{H^s} \gg 1$ after an arbitrarily short time. For (5), Stefanov [5] established in one space dimension the existence of local solution in H^1 with small total disturbance u_0 in $H^1(\mathbb{R}^1) \cap L^1(\mathbb{R}^1) \cap \{f :$

$\sup_x |\int_{-\infty}^{x} f(y)dy| \leqslant \varepsilon\}$. Han et al. [6] showed that (4) and (5) are locally well-posed in modulation space $M^3_{1,1}(\mathbb{R}^n)$ under the small condition of $\|u_0\|_{L^1}$ norm and they obtained the inviscid limit behavior between the solutions of (4) and (5) with initial data in $M^3_{1,1}(\mathbb{R}^n)$ as the dissipative parameter $a \to 0$. For general α, to the knowledge of the author, there are few results on (1). In this paper, we will study the analyticity of solutions of (1) for $1/2 \leqslant \alpha \leqslant 1$ with rough initial data in certain modulation space. In the case $1/2 < \alpha \leqslant 1$, we prove that the local solution of (1) is real analytic with initial data in $M^{1-2\alpha}_{p,1}(1 \leqslant p \leqslant \infty)$; in the case $\alpha = 1/2$, we show that (1) is globally well-posed with small initial data in $M^0_{\infty,1} \cap \dot{B}^0_{\infty,1}$ and moreover the global solution of (1) is real analytic for any $t > 0$.

We now briefly sketch the idea of the proof. The basic strategy is to choose the working space to be some time dependent type exponential modulation space, say $\widetilde{L}^{\infty}(I; E^s_{p,q})$ with $s > 0$, and consider the map:

$$\mathcal{T} : u(t) \longrightarrow G_{2\alpha}(t)\, u_0 + \mathcal{A}_{2\alpha}(\vec{\gamma} \cdot \nabla)\, u^2; \tag{6}$$

then use the standard contraction mapping method to prove that there exists a unique solution in this space. Due to the nice property of $E^s_{p,q}$, the solution is naturally analytic for any $s > 0$. However, the main obstacle comes from the derivative in the nonlinear term. To resolve this difficulty, our idea is to make full use of the strong dissipative property of GDGL (1) when $a > 0$. Motivated by the work in [7, 8], we prove two exponential decay estimates of the generalized Ginzburg-Landau semigroup $G_{2\alpha}(t) = e^{-(a+i)t(-\Delta)^{\alpha}}$ combined with frequency uniform decomposition operator $\square_k$ and frequency dyadic decomposition operator Δ_j:

$$\begin{gathered} \left\|\square_k G_{2\alpha}(t) f\right\|_p \lesssim e^{-2cat|k|^{2\alpha}} \left\|\square_k f\right\|_p, \\ \left\|\Delta_j G_{2\alpha}(t)\, f\right\|_p \lesssim e^{-cat2^{2\alpha j}} \left\|\Delta_j f\right\|_p, \quad 1 \leqslant p \leqslant \infty, \end{gathered} \tag{7}$$

for all $f \in L^p$. Then we gain 2α derivative in space from (7) in suitable space time norm which is sufficient to balance the one order derivative in the nonlinear term. More precisely, when $1/2 < \alpha \leqslant 1$, we choose the resolution space as $\mathcal{D} = \{u : \widetilde{L}^{2\alpha/(2\alpha-1)}(I; E^{cat}_{p,1}) \leqslant \delta\}$ and establish some linear estimates of $G_{2\alpha}(t)$ and $\mathcal{A}_{2\alpha} f$ in this space like

$$\begin{gathered} \left\|G_{2\alpha}(t)\, u_0\right\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I,\mathbb{A};E^{cat}_{p,1})} \lesssim \sum_{k \in \mathbb{A}} \langle k \rangle^{1-2\alpha} \left\|\square_k u_0\right\|_p, \\ \mathbb{A} \subset \mathbb{Z}^n \setminus \{0\}, \end{gathered} \tag{8}$$

$$\left\|\square_0 G_{2\alpha}(t) u_0\right\|_{L^{2\alpha/(2\alpha-1)}_{t\in I} L^p_x} \lesssim |I|^{(2\alpha-1)/(2\alpha)} \left\|\square_0 u_0\right\|_p.$$

And $\mathcal{A}_{2\alpha} f$ satisfies similar estimates. Since

$$\begin{aligned} &\left\|G_{2\alpha}(t)\, u_0\right\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I,\{k:|k|<J\};E^{cat}_{p,1})} \\ &\quad \leqslant C|I|^{(2\alpha-1)/2\alpha} \langle J \rangle^{2\alpha-1} \left\|u_0\right\|_{M^{1-2\alpha}_{p,1}}, \end{aligned} \tag{9}$$

one can choose $|I|$ sufficiently small to make sure that

$$C|I|^{(2\alpha-1)/(2\alpha)} \langle J \rangle^{2\alpha-1} \left\|u_0\right\|_{M^{1-2\alpha}_{p,1}} \leqslant \frac{\delta}{4}, \tag{10}$$

and finally verify that $\mathcal{T}$ is a contractive map on $\mathcal{D}$ which ensures that (1) has a local solution satisfying $\|u\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I;E^{cat}_{p,1})} \leqslant \delta$. Moreover, we can prove that

$$\sum_k \langle k \rangle^{1-2\alpha} \left\|2^{cat|k|} \square_k u\right\|_{L^{\infty}_{t\in I} L^p_x} \lesssim \left\|u_0\right\|_{M^{1-2\alpha}_{p,1}} + \delta^2, \tag{11}$$

which implies that $(I - \Delta)^{(1-2\alpha)/2} u \in \widetilde{L}^{\infty}(I, E^{cat}_{p,1})$ and hence the local solution is analytic for any $t > 0$. However, when $\alpha = 1/2$, it is impossible to choose I satisfying (10). To make this bound valid, one needs to impose additional small condition on $\|u_0\|_{M^0_{p,1}}$. Then we will only obtain the existence of local solution with small initial data which is not ideal. So we intend to seek for different approach in this critical situation. The preferred working space would be $\{u : \|u\|_{\widetilde{L}^{\infty}(I;E^{cat}_{\infty,1})} \leqslant \delta\}$. But when we bound

$$\begin{aligned} &\left\|\mathcal{A}_1(\vec{\gamma} \cdot \nabla)\, u^2\right\|_{\widetilde{L}^{\infty}(I;\mathbb{Z}^n_*;E^{cat}_{\infty,1})} \\ &\quad \lesssim \left\|(\vec{\gamma} \cdot \nabla)\, u^2\right\|_{\widetilde{L}^1(I;\mathbb{Z}^n_*;E^{cat}_{\infty,1})} \\ &\quad \lesssim \sum_{j=1}^{n} \left\|\partial_{x_j} u\right\|_{\widetilde{L}^1(I;\mathbb{Z}^n_*;E^{cat}_{\infty,1})} \|u\|_{\widetilde{L}^{\infty}(I;\mathbb{Z}^n_*;E^{cat}_{\infty,1})}, \end{aligned} \tag{12}$$

it is easy to see that to control the right-hand side of (12), $\sum_{j=1}^{n} \|\partial_{x_j} u\|_{\widetilde{L}^1(I;\mathbb{Z}^n_*;E^{cat}_{\infty,1})}$ should be involved in the working space. So, the natural working space would be $\mathcal{D} = \{u : \|u\|_Y \leqslant \delta\}$ where $\|u\|_Y = \sum_{i=1}^{n} \|\partial_{x_i} u\|_{\widetilde{L}^1(I;E^{cat}_{\infty,1})} + \|u\|_{\widetilde{L}^{\infty}(I;E^{cat}_{\infty,1})}$. However, the obstacle comes again. The following low frequency projection term could not be bounded in this working space:

$$\left\|\square_0 \partial_{x_i} \mathcal{A}_1(\vec{\gamma} \cdot \nabla)\, u^2\right\|_{L^1_{t\in I} L^{\infty}_x} + \left\|\square_0 \mathcal{A}_1(\vec{\gamma} \cdot \nabla)\, u^2\right\|_{L^{\infty}_{t\in I} L^{\infty}_x}. \tag{13}$$

To overcome this difficulty, our idea is to make use of the property $\|\square_0 f\|_{\infty} \lesssim \|f\|_{\infty} \lesssim \|f\|_{\dot{B}^0_{\infty,1}}$ and bound (13) by

$$\begin{aligned} &\left\|\mathcal{A}_1(\vec{\gamma} \cdot \nabla)\, u^2\right\|_{\widetilde{L}^1(I;\dot{B}^1_{\infty,1})} + \left\|\mathcal{A}_1(\vec{\gamma} \cdot \nabla)\, u^2\right\|_{\widetilde{L}^{\infty}(I;\dot{B}^0_{\infty,1})} \\ &\quad \lesssim \left\|(\vec{\gamma} \cdot \nabla)\, u^2\right\|_{\widetilde{L}^1(I;\dot{B}^0_{\infty,1})} \lesssim \left\|u^2\right\|_{\widetilde{L}^1(I;\dot{B}^1_{\infty,1})} \\ &\quad \lesssim \|u\|_{\widetilde{L}^{\infty}(I;\dot{B}^0_{\infty,1})} \|u\|_{\widetilde{L}^1(I;\dot{B}^1_{\infty,1})}. \end{aligned} \tag{14}$$

Then the time dependent type Besov norm $\|u\|_X = \|u\|_{\widetilde{L}^1(I;\dot{B}^1_{\infty,1}) \cap \widetilde{L}^{\infty}(I;\dot{B}^0_{\infty,1})}$ should be included in the working space. So, finally we choose the resolution space as

$$\mathcal{D} = \left\{u : \|u\|_X + \|u\|_Y \leqslant \delta\right\}. \tag{15}$$

The corresponding condition imposed on the initial data would be stronger, $u_0 \in M^0_{\infty,1} \cap \dot{B}^0_{\infty,1}$, and sufficiently small.

In Section 4, we develop estimates in this resolution space and combine them with the contraction mapping argument; finally we show that there exists a unique global solution to (1) in $\mathscr{D}$ which is naturally analytic for $t > 0$.

Now let us recall some notations and basic facts that will be used in the sequel. $C \geqslant 1$, $c \leqslant 1$ will denote universal positive constants which can be different at different places. $X \lesssim Y$ (for $X, Y > 0$) means that $X \leqslant CY$. For any $x \in \mathbb{R}^n$, we write $|x|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$ and $|x| = |x|_2$. Now we introduce some spaces. We denote by $L^p = L^p(\mathbb{R}^n)$ the Lebesgue space on which the norm is written as $\|\cdot\|_p$. Let X be a Banach space. For any $I \subset \mathbb{R}^+$, we define

$$\|u\|_{L^\gamma(I;X)} = \left(\int_I \|u(t,\cdot)\|_X^\gamma dt\right)^{1/\gamma} \tag{16}$$

for $1 \leqslant \gamma < \infty$ and with usual modification for $\gamma = \infty$. If $X = L^p$, we will write $\|u\|_{L^\gamma_{t\in I}L^p_x} = \|u\|_{L^\gamma(I;L^p)}$ and simply denote $\|u\|_{L^\gamma_t L^p_x}$ if $I = [0, +\infty)$. Now let us recall the notation and definitions in Littlewood-Paley theory [9]. Let $\psi : \mathbb{R}^n \to [0,1]$ be a smooth radial cutoff function satisfying

$$\psi(\xi) = \begin{cases} 1, & |\xi| \leqslant 1, \\ \text{smooth}, & 1 < |\xi| < 2, \\ 0, & |\xi| \geqslant 2. \end{cases} \tag{17}$$

Denote $\varphi(\xi) := \psi(\xi) - \psi(2\xi)$ and we introduce the function sequence $\varphi_k(\xi) = \varphi(2^{-k}\xi)$, $k \in \mathbb{Z}$. Then $\Delta_k := \mathscr{F}^{-1}\varphi_k\mathscr{F}$, $k \in \mathbb{Z}$, are said to be the homogeneous dyadic decomposition operators and satisfying the operator identity: $I = \sum_{k=-\infty}^{+\infty}\Delta_k$. The low frequency projection operators S_k are defined by $S_k := \sum_{j=-\infty}^{k}\Delta_j$. It is easy to see that $S_k u \to u$ as $k \to \infty$ in the sense of distributions. With this decomposition, the norms in homogeneous Besov spaces are defined as follows:

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\displaystyle\sum_{j=-\infty}^{+\infty} 2^{jsq}\left\|\Delta_j f\right\|_p^q\right)^{1/q}, & \text{if } q < \infty, \\ \displaystyle\sup_{-\infty<j<+\infty} 2^{js}\left\|\Delta_j f\right\|_p, & \text{if } q = \infty. \end{cases} \tag{18}$$

And the space time homogenous Besov norms are defined by

$$\|f\|_{\widetilde{L}^\gamma(I;\dot{B}^s_{p,q})} = \left(\sum_{j=-\infty}^{+\infty} 2^{jsq}\left\|\Delta_j f\right\|^q_{L^\gamma_{t\in I}L^p_x}\right)^{1/q} < \infty \tag{19}$$

with the usual modification for $q = \infty$. Such a kind of space was first used in Chemin [10]. It is easy to see by Minkowski inequality that

$$\|f\|_{\widetilde{L}^\gamma(I;\dot{B}^s_{p,q})} \begin{cases} \leqslant \|f\|_{L^\gamma(I;\dot{B}^s_{p,q})}, & \text{if } \gamma \leqslant q, \\ \geqslant \|f\|_{L^\gamma(I;\dot{B}^s_{p,q})}, & \text{if } \gamma \geqslant q. \end{cases} \tag{20}$$

We now recall the definition of Modulation space which was first introduced by Feichtinger [11] in 1983 (see also Gröchenig [12]). Let σ be a smooth cutoff function with supp $\sigma \subset [-3/4, 3/4]^n$, $\sigma_k = \sigma(\cdot - k)$, and

$$\sum_{k\in\mathbb{Z}^n}\sigma_k(\xi) \equiv 1, \quad \forall \xi \in \mathbb{R}^n. \tag{21}$$

Then the frequency uniform decomposition operator $\square_k$ is defined as

$$\square_k = \mathscr{F}^{-1}\sigma_k\mathscr{F}, \quad k \in \mathbb{Z}^n. \tag{22}$$

Using this decomposition operator, for any $0 < p, q \leqslant \infty$, $s \in \mathbb{R}$, we define

$$\|f\|_{M^s_{p,q}} = \left(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{sq}\|\square_k f\|_p^q\right)^{1/q}, \tag{23}$$

with usual modification for $q = \infty$. $M^s_{p,q}$ is said to be a modulation space and it has been successfully applied to study nonlinear evolutions in recent years [8, 13–16]. Let $0 < p, q \leqslant \infty$, $s \geqslant 0$; the exponential modulation space $E^s_{p,q}$ was introduced in [8] with the following norm:

$$\|f\|_{E^s_{p,q}} = \left(\sum_k 2^{qs|k|}\|\square_k f\|_p^q\right)^{1/q}. \tag{24}$$

We remark that when $s > 0$, this space can be viewed as modulation space with analytic regularity and when $s = 0$, it reduces to normal modulation space $M^0_{p,q}$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha! = \alpha_1! \ldots \alpha_n!$ and $\partial^\alpha = \partial^{\alpha_1}_{x_1} \ldots \partial^{\alpha_n}_{x_n}$. Recall that the Gevrey class is defined as follows:

$$\begin{aligned} G_{1,p} = \Big\{ & f \in C^\infty(\mathbb{R}^n) : {}^\exists \rho, M > 0 \\ & \text{s.t. } \left\|\partial^\alpha f(x)\right\|_p \leqslant \frac{M\alpha!}{\rho^{|\alpha|}}, \forall \alpha \in \mathbb{Z}^n_+, x \in \mathbb{R}^n \Big\}. \end{aligned} \tag{25}$$

It is proved that $G_{1,\infty}$ is the Gevrey 1-class and any function in this space is real analytic [17]. One can easily check that $G_{1,p_1} \subset G_{1,p_2}$ for $p_1 \leqslant p_2$. Therefore, any function in $G_{1,p}$ ($0 < p \leqslant \infty$) is real analytic. There is a very nice relationship between Gevrey class and exponential modulation spaces which is shown in Huang and Wang [7].

Lemma 1. *Let $0 < p, q \leqslant \infty$. Then*

$$G_{1,p} = \bigcup_{s>0} E^s_{p,q}. \tag{26}$$

Remark 2. From this property we easily see that if we can prove the solution in exponential modulation space $E^s_{p,q}$ with positive regularity, then it is naturally analytic.

Inspired by (19), we define the following space time exponential modulation norm:

$$\|f\|_{\widetilde{L}^\gamma(I;E^s_{p,q})} = \left(\sum_{k\in\mathbb{Z}^n}\left\|2^{s|k|}\square_k f\right\|^q_{L^\gamma_{t\in I}L^p_x}\right)^{1/q} \tag{27}$$

with usual modification if $q = \infty$.

In the end, let us recall the definition of multiplier space M_p [18, 19]. Let $\rho \in \mathcal{S}'$. If there exists a $C > 0$ such that $\|\mathcal{F}^{-1}\rho\mathcal{F}f\|_{L^p} \leqslant C\|f\|_{L^p}$ holds for all $f \in \mathcal{S}$, then ρ is called a Fourier multiplier on L^p. The linear space of all multipliers on L^p is denoted by M_p and the norm on which is defined as $\|\rho\|_{M_p} = \sup\{\|\mathcal{F}^{-1}\rho\mathcal{F}f\|_{L^p} : f \in \mathcal{S}, \|f\|_{L^p} = 1\}$. Concerning the multipliers, there holds the following famous inequality which is also called the multiplier theorem.

Proposition 3 (see [9], Nikol'skij's inequality). *Let $\Omega \subset \mathbb{R}^n$ be a compact set, $0 < r \leqslant \infty$. Denote $\sigma_r = n(1/(r \wedge 1) - 1/2)$ and assume that $s > \sigma_r$. Then there exists a constant $C > 0$ such that*

$$\left\|\mathcal{F}^{-1}\varphi\mathcal{F}f\right\|_r \leqslant C\|\varphi\|_{H^s}\|f\|_r \tag{28}$$

holds for all $f \in L^r_\Omega := \{f \in L^r : \operatorname{supp}\widehat{f} \subset \Omega\}$ and $\varphi \in H^s$. In particular, if $r \geqslant 1$, then (28) holds for all $f \in L^r$.

The remaining part of this paper is organized as follows. In Section 2, we develop two decay estimates (7) associated with the generalized Ginzburg-Landau semigroup. In Section 3, we prove the analytic regularity property of the solutions to (1) when $1/2 < \alpha \leqslant 1$. In Section 4, we deal with the analytic property of (1) in the critical case $\alpha = 1/2$. Finally, a short conclusion is given in Section 5.

2. Decay Estimates for GCGL Semigroup

In this part, we will set up some decay estimates for the generalized Ginzburg-Landau semigroup $G_{2\alpha}(t) = e^{-(a+i)t(-\Delta)^\alpha}$ together with the frequency uniform decomposition operator $\square_k$ and the dyadic operators Δ_j. As explained in the introduction, these estimates are crucial to the proof of the main theorems.

Proposition 4 (uniform decay estimate). *Suppose that $1 \leqslant p \leqslant \infty$, $1/2 \leqslant \alpha \leqslant 1$, $a > 0$. Then there exists $c > 0$ (say $0 < c \leqslant 2^{-10}$) such that*

$$\left\|\square_k G_{2\alpha}(t) f\right\|_p \lesssim e^{-2cat|k|^{2\alpha}}\left\|\square_k f\right\|_p \tag{29}$$

holds for all $f \in L^p$ and $k \in \mathbb{Z}^n$.

Proof. First, we choose a smooth cutoff function $\widetilde{\sigma} : \mathbb{R}^n \to [0,1]$ satisfying $\widetilde{\sigma}(\xi) = 1$ for $|\xi|_\infty \leqslant 3/4$ and $\widetilde{\sigma}(\xi) = 0$ for $|\xi|_\infty > 7/8$. It is easy to see that $\widetilde{\sigma}$ equals 1 on the support of σ and has similar property as σ. Applying this fact, we deduce that

$$\begin{aligned}
&\left\|\square_k G_{2\alpha}(t) f\right\|_p \\
&\quad = \left\|\mathcal{F}^{-1}\sigma(\xi - k)\,\widetilde{\sigma}(\xi - k)\, e^{-t(a+i)|\xi|^{2\alpha}}\mathcal{F}f\right\|_p \\
&\quad = \left\|\mathcal{F}^{-1}\widetilde{\sigma}(\xi - k)\, e^{-t(a+i)|\xi|^{2\alpha}}\mathcal{F}\mathcal{F}^{-1}\sigma(\xi - k)\,\mathcal{F}f\right\|_p \\
&\quad \lesssim \left\|\widetilde{\sigma}(\xi - k)e^{-t(a+i)|\xi|^{2\alpha}}\right\|_{M_p}\left\|\square_k f\right\|_p.
\end{aligned} \tag{30}$$

In view of Nikol'skij's inequality,

$$\begin{aligned}
\left\|\widetilde{\sigma}(\xi - k)e^{-t(a+i)|\xi|^{2\alpha}}\right\|_{M_p} &\lesssim \left\|\widetilde{\sigma}(\xi)\, e^{-t(a+i)|\xi+k|^{2\alpha}}\right\|_{H^L}, \\
& L > \frac{n}{2}.
\end{aligned} \tag{31}$$

Since for $|k| \geqslant 1$, applying Leibniz's Rule, we infer that

$$\begin{aligned}
&\partial^L_{\xi_i}\left(\widetilde{\sigma}(\xi)\, e^{-t(a+i)|\xi+k|^{2\alpha}}\right) \\
&\quad = \sum_{L_1+L_2=L} C_{L\beta\gamma}\partial^{L_1}_{\xi_i}\widetilde{\sigma}(\xi)\,\partial^{L_2}_{\xi_i}e^{-t(a+i)|\xi+k|^{2\alpha}},
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
&\left|\partial^{L_2}_{\xi_i}e^{-(a+i)t|\xi+k|^{2\alpha}}\right| \\
&\quad \lesssim \left|e^{-(a+i)t|\xi+k|^{2\alpha}}\right| \\
&\qquad \times \sum_{\substack{\beta_1+\cdots+\beta_q=L_2\\ 1\leqslant q\leqslant\gamma}}\left|\partial^{\beta_1}_{\xi_i}\left[(a+i)\,t\left|\xi+k\right|^{2\alpha}\right]\right. \\
&\qquad\qquad \left.\cdots\partial^{\beta_q}_{\xi_i}\left[(a+i)\,t\left|\xi+k\right|^{2\alpha}\right]\right| \\
&\quad \lesssim e^{-at|\xi+k|^{2\alpha}/2}.
\end{aligned} \tag{33}$$

Due to the support property $\operatorname{supp}\widetilde{\sigma}(\xi) \subset [-7/8, 7/8]^n$, (30)–(33), we conclude that, for $|k| \geqslant 1$,

$$\left\|\widetilde{\sigma}(\xi - k)e^{-t(a+i)|\xi|^{2\alpha}}\right\|_{M_p} \lesssim e^{-at|k|^{2\alpha}/64}. \tag{34}$$

Note that (34) also holds for $k = 0$. Hence, we complete the proof of (29). ☐

Proposition 5 (dyadic decay estimate). *Suppose that $1 \leqslant p \leqslant \infty$, $1/2 \leqslant \alpha \leqslant 1$, $a > 0$. Then there exists $c > 0$ (say $0 < c \leqslant 2^{-10}$) such that*

$$\left\|\Delta_j G_{2\alpha}(t) f\right\|_p \lesssim e^{-cat2^{2\alpha j}}\left\|\Delta_j f\right\|_p \tag{35}$$

holds for all $f \in L^p$.

Proof. When $a > 0$, Ginzburg-Landau semigroup $e^{-t(a+i)(-\Delta)^{2\alpha}}$ is strong dissipative. Using the exponential decay property of $e^{-at|\xi|^{2\alpha}}$ and Nikol'skij's inequality, we deduce that, for $L > n/2$,

$$\begin{aligned}
\left\|\Delta_j G_{2\alpha}(t) f\right\|_p &\leqslant \left\|\varphi_j(\xi)\, e^{-(a+i)t|\xi|^{2\alpha}}\right\|_{M_p}\|f\|_p \\
&= \left\|\varphi(\xi)\, e^{-(a+i)t2^{2\alpha j}|\xi|^{2\alpha}}\right\|_{M_p}\|f\|_p \\
&\lesssim \left\|\varphi(\xi)\, e^{-(a+i)t2^{2\alpha j}|\xi|^{2\alpha}}\right\|_{H^L}\|f\|_p \\
&\lesssim e^{-cat2^{2\alpha j}}\|f\|_p.
\end{aligned} \tag{36}$$

By the support property of dyadic decomposition operator Δ_j, there holds the following identity:

$$\Delta_j = \Delta_j \left(\Delta_{j-1} + \Delta_j + \Delta_{j+1}\right), \tag{37}$$

so

$$\left\|\Delta_j G_{2\alpha}(t) f\right\|_p \lesssim e^{-cat2^{2\alpha j}} \left\|\Delta_j f\right\|_p. \tag{38}$$

This completes the proof as desired. □

3. Analytic Regularity for GDGL: $1/2<\alpha \leqslant 1$

The main results of this paper are the following theorems.

Theorem 6 (analyticity for QDGL). *Let $\alpha = 1$, $1 \leqslant p \leqslant \infty$, $a > 0$, and $n \geqslant 1$. Assume that $u_0 \in M^{-1}_{p,1}$. Then there exists a $T_{\max} = T_{\max}(u_0) > 0$ such that (1) has a unique solution $u \in \widetilde{L}^2_{loc}(0, T_{\max}; E^{cat}_{p,1})$, $c = 2^{-10}$. Moreover, the solution enjoys the following properties.*

(i) $(I - \Delta)^{-1/2} u \in \widetilde{L}^{\infty}([0, T_{\max}); E^{cat}_{p,1})$.

(ii) *If $T_{\max} < \infty$, we have $\|u\|_{\widetilde{L}^2(0,T_{\max};E^{cat}_{p,1})} = \infty$.*

Theorem 7 (analyticity for GDGL (I)). *Let $1/2 < \alpha < 1$, $1 \leqslant p \leqslant \infty$, $a > 0$, $n \geqslant 1$. Suppose that $u_0 \in M^{1-2\alpha}_{p,1}$. There exists a $T_{\max} = T_{\max}(u_0) > 0$ such that (1) has a unique solution $u \in \widetilde{L}^{2\alpha/(2\alpha-1)}_{loc}(0, T_{\max}; E^{cat}_{p,1})$, $c = 2^{-10}$. Moreover, the solution satisfies the following properties.*

(i) $(I - \Delta)^{(1-2\alpha)/2} u \in \widetilde{L}^{\infty}([0, T_{\max}); E^{cat}_{p,1})$.

(ii) *If $T_{\max} < \infty$, then $\|u\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(0,T_{\max};E^{cat}_{p,1})} = \infty$.*

Theorems 6 and 7 tell us that when $1/2 < \alpha \leqslant 1$, (1) is locally well-posed with any initial data in $M^{1-2\alpha}_{p,1}$ and moreover the local solution is analytic. However, the method used for Theorems 6 and 7 does not work for the critical case: $\alpha = 1/2$. We need to impose stronger conditions on the initial data, that is, $u_0 \in \dot{B}^0_{\infty,1} \cap M^0_{\infty,1}$ and be sufficiently small. We remark that there is no inclusion between $\dot{B}^0_{\infty,1}$ and $M^0_{\infty,1}$ since $S_1 \dot{B}^{-1}_{\infty,1} \subset S_1 M^{-1}_{\infty,1}$ while $(I - S_1) M^{-1}_{\infty,1} \subset (I - S_1) \dot{B}^{-1}_{\infty,1}$.

Theorem 8 (analyticity for GDGL (II)). *Let $\alpha = 1/2$, $a > 0$, and $n \geqslant 1$. Assume that $u_0 \in \dot{B}^0_{\infty,1} \cap M^0_{\infty,1}$ is sufficiently small; then there exists a unique global solution u to (1) satisfying $u \in \widetilde{L}^{\infty}(\mathbb{R}^+; E^{s(t)}_{\infty,1})$ and $\partial_{x_i} u \in \widetilde{L}^1(\mathbb{R}^+; E^{s(t)}_{\infty,1})$ with $s(t) = 2^{-5}(a \wedge at)$.*

Theorem 8 states that (1) is globally well-posed with small initial data in $u_0 \in \dot{B}^0_{\infty,1} \cap M^0_{\infty,1}$ and the solution $u(t)$ is actually real analytic for any $t > 0$.

In this section, we unify the proof of Theorems 6 and 7 in one part. The proof of Theorem 8 is left to Section 4. For convenience, we denote, for any $\mathbb{A} \subset \mathbb{Z}^n$,

$$\|f\|_{\widetilde{L}^{\bar{q}}(I,\mathbb{A};E^s_{p,q})} = \left(\sum_{k\in\mathbb{A}} \left\|2^{s|k|} \Box_k f\right\|^q_{L^{\bar{q}}_{t\in I} L^p_x}\right)^{1/q}. \tag{39}$$

We first build up some linear estimates for $G_{2\alpha}(t)$ and $\mathscr{A}_{2\alpha} f(t, x) = \int_0^t G_{2\alpha}(t - \tau) f(\tau, x) d\tau$.

Proposition 9. *Let $1 \leqslant p \leqslant \infty$, $1/2 < \alpha \leqslant 1$, and $a > 0$. There exists a constant $c > 0$ ($0 < c \leqslant 2^{-10}$) such that, for $0 < t_0 < \infty$ and $I = [0, t_0]$,*

$$\left\|G_{2\alpha}(t) u_0\right\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I,\mathbb{A};E^{cat}_{p,1})} \lesssim \sum_{k\in\mathbb{A}} \langle k\rangle^{1-2\alpha} \left\|\Box_k u_0\right\|_p, \quad \mathbb{A} \subset \mathbb{Z}^n \setminus \{0\}, \tag{40}$$

$$\left\|\Box_0 G_{2\alpha}(t) u_0\right\|_{L^{2\alpha/(2\alpha-1)}_{t\in I} L^p_x} \lesssim |I|^{(2\alpha-1)/(2\alpha)} \left\|\Box_0 u_0\right\|_p, \tag{41}$$

holds for all $u_0 \in M^{1-2\alpha}_{p,1}$.

Proof. For any $\mathbb{A} \subset \mathbb{Z}^n \setminus \{0\}$, multiplying (29) by $2^{cat|k|}$ and taking $L^{2\alpha/(2\alpha-1)}_t$ norm, we get

$$\begin{aligned}
&\left\|2^{cat|k|} \Box_k G_{2\alpha}(t) u_0\right\|_{L^{2\alpha/(2\alpha-1)}_{t\in I} L^p_x} \\
&\quad\lesssim \left\|2^{-cat|k|^{2\alpha}} 2^{cat(|k|-|k|^{2\alpha})}\right\|_{L^{2\alpha/(2\alpha-1)}_{t\in I}} \left\|\Box_k u_0\right\|_{L^p_x} \\
&\quad\lesssim \left\|2^{-cat|k|^{2\alpha}}\right\|_{L^{2\alpha/(2\alpha-1)}_{t\in I}} \left\|\Box_k u_0\right\|_{L^p_x} \\
&\quad\lesssim \langle k\rangle^{1-2\alpha} \left\|\Box_k u_0\right\|_{L^p_x}.
\end{aligned} \tag{42}$$

Taking the sequence l^1 norm on set $\mathbb{A}$ to inequality (42), we obtain the estimate in (40). For $k = 0$, from (29), we see that

$$\left\|\Box_0 G_{2\alpha} u_0\right\|_{L^p_x} \lesssim \left\|\Box_0 u_0\right\|_{L^p_x}, \tag{43}$$

taking $L^{2\alpha/(2\alpha-1)}_t(I)$ norm on (43) which implies (41). □

Now let $0 < t_0 \leqslant \infty$. We consider the estimate of $\mathscr{A}_{2\alpha} f$. Again using uniform decay estimate (29), we have

$$\left\|\Box_k \mathscr{A}_{2\alpha} f\right\|_{L^p_x} \lesssim \int_0^t 2^{-2ca(t-\tau)|k|^{2\alpha}} \left\|\Box_k f(\tau)\right\|_{L^p_x} d\tau. \tag{44}$$

It follows from (44) that for $|k| \neq 0$

$$\begin{aligned}
&2^{cat|k|} \left\|\Box_k \mathscr{A}_{2\alpha} f\right\|_{L^p_x} \\
&\quad\lesssim \int_0^t 2^{-2ca(t-\tau)|k|^{2\alpha}} 2^{cat|k|} \left\|\Box_k f(\tau)\right\|_{L^p_x} d\tau \\
&\quad\lesssim \int_0^t 2^{-ca(t-\tau)|k|^{2\alpha}} 2^{-ca(t-\tau)|k|} 2^{cat|k|} \left\|\Box_k f(\tau)\right\|_{L^p_x} d\tau \\
&\quad= \int_0^t 2^{-ca(t-\tau)|k|^{2\alpha}} 2^{ca\tau|k|} \left\|\Box_k f(\tau)\right\|_{L^p_x} d\tau.
\end{aligned} \tag{45}$$

Taking $L^{2\alpha/(2\alpha-1)}_t$ norm on (45) and applying Young's inequality, we get

$$\begin{aligned}
&\left\|2^{cat|k|} \Box_k \mathscr{A}_{2\alpha} f\right\|_{L^{2\alpha/(2\alpha-1)}_{t\in[0,t_0]} L^p_x} \\
&\quad\lesssim \left\|2^{-cat|k|^{2\alpha}}\right\|_{L^{2\alpha}_t} \left\|2^{cat|k|} \Box_k f\right\|_{L^{\alpha/(2\alpha-1)}_t L^p_x} \\
&\quad\lesssim \langle k\rangle^{-1} \left\|2^{cat|k|} \Box_k f\right\|_{L^{\alpha/(2\alpha-1)}_t L^p_x}.
\end{aligned} \tag{46}$$

In view of $\|\partial_{x_i}\square_k f\|_{L_x^p} \lesssim \langle k_i\rangle \|\square_k f\|_{L_x^p}$, we deduce that

$$\left\|2^{cat|k|}\square_k \mathscr{A}_{2\alpha}(\vec{\gamma}\cdot\nabla) f\right\|_{L^{2\alpha/(2\alpha-1)}_{t\in[0,t_0]}L_x^p} \lesssim \left\|2^{cat|k|}\square_k f\right\|_{L^{\alpha/(2\alpha-1)}_{t\in[0,t_0]}L_x^p}. \quad (47)$$

So, taking the sequence l^q norm on both sides of (47), we have the following.

Proposition 10. *Let $1 \leqslant p \leqslant \infty$, $a > 0$. There exists a constant $c > 0$ $(0 < c \leqslant 2^{-10})$ such that, for $0 < t_0 < \infty$ and $I = [0, t_0]$,*

$$\left\|\partial_{x_i}\mathscr{A}_{2\alpha} f\right\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I,\mathbb{A};E^{cat}_{p,1})} \lesssim \|f\|_{\widetilde{L}^{\alpha/(2\alpha-1)}(I,\mathbb{A};E^{cat}_{p,1})}, \quad \mathbb{A}\subset\mathbb{Z}^n\setminus\{0\}, \quad (48)$$

$$\left\|\square_0\mathscr{A}_{2\alpha} f\right\|_{L^{2\alpha/(2\alpha-1)}_{t\in I}L_x^p} \lesssim |I|^{1/(2\alpha)}\left\|\square_0 f\right\|_{L^{\alpha/(2\alpha-1)}_{t\in I}L_x^p}, \quad (49)$$

holds for all $f \in \widetilde{L}^{\alpha/(2\alpha-1)}(I; E^{cat}_{p,1})$.

For the proof of (49), one only needs to take $k = 0$ in (44) and then $L_t^{2\alpha/(2\alpha-1)}$ norm together with Young's inequality will imply the result.

Now we recall a nonlinear mapping estimate in $\widetilde{L}^{\widetilde{q}}(I; E^{s(t)}_{p,q})$ which was shown in [7].

Proposition 11. *Let $1 \leqslant p, p_1, p_2, q, q_1, q_2, \widetilde{q}, \widetilde{q}_1, \widetilde{q}_2 \leqslant \infty$. Consider $1/p = 1/p_1 + 1/p_2$, $1/\widetilde{q} = 1/\widetilde{q}_1 + 1/\widetilde{q}_2$ and $1/q = 1/q_1 + 1/q_2 - 1$, $s(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, and $I \subset \mathbb{R}_+$ with $\sup_{t\in I} s(t) < \infty$. Then one has*

$$\|fg\|_{\widetilde{L}^{\widetilde{q}}(I;E^{s(t)}_{p,q})} \lesssim \sup_{t\in I} 2^{4s(t)}\|f\|_{\widetilde{L}^{\widetilde{q}_1}(I;E^{s(t)}_{p_1,q_1})}\|g\|_{\widetilde{L}^{\widetilde{q}_2}(I;E^{s(t)}_{p_2,q_2})}. \quad (50)$$

Now let us consider the following map:

$$\mathscr{T} : u(t) \longrightarrow G_{2\alpha}(t)u_0 + \mathscr{A}_{2\alpha}\left((\vec{\gamma}\cdot\nabla)u^2\right). \quad (51)$$

Let $c = 2^{-10}$, $I = [0, t_0]$. For any $\delta > 0$, one can choose $J := J(u_0) > 0$ satisfying $\sum_{|k|\geqslant J}\langle k\rangle^{1-2\alpha}\|\square_k u_0\|_p \leqslant \delta/4C$. By Proposition 9,

$$\left\|G_{2\alpha}(t)u_0\right\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I,\{k:|k|\geqslant J\};E^{cat}_{p,1})} \leqslant C\sum_{|k|\geqslant J}\langle k\rangle^{1-2\alpha}\left\|\square_k u_0\right\|_p \leqslant \frac{\delta}{4}. \quad (52)$$

On the other hand, one can choose I satisfying

$$|I|^{(2\alpha-1)/(2\alpha)}\langle J\rangle^{2\alpha-1} \leqslant \frac{\delta}{4C\|u_0\|_{M^{1-2\alpha}_{p,1}}}, \quad (53)$$

so,

$$\begin{aligned}\|G_{2\alpha}(t)u_0\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I,\{k:|k|<J\};E^{cat}_{p,1})} &\leqslant C\sum_{|k|<J}\left\|\left\|2^{-2cat|k|^{2\alpha}}\square_k u_0\right\|_p\right\|_{L^{2\alpha/(2\alpha-1)}_{t\in I}} \\ &\leqslant C|I|^{(2\alpha-1)/(2\alpha)}\langle J\rangle^{2\alpha-1}\sum_{|k|<J}\langle k\rangle^{1-2\alpha}\left\|\square_k u_0\right\|_p \leqslant \frac{\delta}{4}.\end{aligned} \quad (54)$$

Hence, in view of (52) and (54),

$$\left\|G_{2\alpha}(t)u_0\right\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I;E^{cat}_{p,1})} \leqslant \frac{\delta}{2}. \quad (55)$$

It follows from Propositions 10 and 11 that

$$\begin{aligned}&\left\|\mathscr{A}_{2\alpha}(\vec{\gamma}\cdot\nabla)\left(u^2\right)\right\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I;E^{cat}_{p,1})} \\ &\quad\lesssim \left(1+|I|^{1/(2\alpha)}\right)\left\|u^2\right\|_{\widetilde{L}^{\alpha/(2\alpha-1)}(I;E^{cat}_{p,1})} \\ &\quad\lesssim \left(1+|I|^{1/(2\alpha)}\right)\|u\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I;E^{cat}_{p,1})}\|u\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I;E^{cat}_{\infty,1})} \\ &\quad\lesssim \left(1+|I|^{1/(2\alpha)}\right)\|u\|^2_{\widetilde{L}^{2\alpha/(2\alpha-1)}\left(I;E^{cat}_{p,1}\right)}.\end{aligned} \quad (56)$$

We can assume that $|I| \leqslant 1$. Hence, collecting (55)-(56), we have

$$\|\mathscr{T}u\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I;E^{cat}_{p,1})} \leqslant \frac{\delta}{2} + C\|u\|^2_{\widetilde{L}^{2\alpha/(2\alpha-1)}\left(I;E^{cat}_{p,1}\right)}. \quad (57)$$

Now we can fix δ verifying $C\delta \leqslant 1/4$. Put

$$\mathscr{D} = \left\{u \in \widetilde{L}^{2\alpha/(2\alpha-1)}\left(I;E^{cat}_{p,1}\right) : \|u\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I;E^{cat}_{p,1})} \leqslant \delta\right\}. \quad (58)$$

We have $\mathscr{T}u \in \mathscr{D}$ if $u \in \mathscr{D}$ and

$$\|\mathscr{T}u - \mathscr{T}v\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I;E^{cat}_{p,1})} \leqslant \frac{1}{2}\|u-v\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I;E^{cat}_{p,1})}, \quad u, v \in \mathscr{D}. \quad (59)$$

By the standard contraction mapping argument, there exists a unique solution to (1) in $\mathscr{D}$ satisfying $\|u\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I;E^{cat}_{p,1})} \leqslant \delta$.

Next, we prove that

$$\sum_k\langle k\rangle^{1-2\alpha}\left\|2^{cat|k|}\square_k u\right\|_{L^\infty_{t\in[0,t_0]}L_x^p} \lesssim \left\|u_0\right\|_{M^{1-2\alpha}_{p,1}} + \delta^2. \quad (60)$$

In order to show (60), we need the following.

Proposition 12. *Let $1 \leqslant p \leqslant \infty$, $a > 0$. There exists a constant $c > 0$ $(0 < c \leqslant 2^{-10})$ such that, for $0 < t_0 < \infty$ and $I = [0, t_0]$,*

$$\begin{aligned}\sum_k\langle k\rangle^{1-2\alpha}\left\|2^{cat|k|}\square_k G_{2\alpha}(t)u_0\right\|_{L^\infty_{t\in[0,t_0]}L_x^p} &\lesssim \left\|u_0\right\|_{M^{1-2\alpha}_{p,1}}, \\ \sum_k\langle k\rangle^{1-2\alpha}\left\|2^{cat|k|}\square_k\partial_{x_i}\mathscr{A}_{2\alpha}f\right\|_{L^\infty_{t\in[0,t_0]}L_x^p} &\lesssim \left\|f\right\|_{\widetilde{L}^{\alpha/(2\alpha-1)}(I;E^{cat}_{p,1})}\end{aligned} \quad (61)$$

hold for all $u_0 \in M^{1-2\alpha}_{p,1}$ and $f \in \widetilde{L}^{\alpha/(2\alpha-1)}(I; E^{cat}_{p,1})$.

Proof. In view of (29), we immediately have (61). □

Let us apply Proposition 12:

$$\begin{aligned}&\sum_k\langle k\rangle^{1-2\alpha}\left\|2^{ct|k|}\square_k u\right\|_{L^\infty_{t\in[0,t_0]}L_x^p} \\ &\quad\lesssim \left\|u_0\right\|_{M^{1-2\alpha}_{p,1}} + \left\|u^2\right\|_{\widetilde{L}^{\alpha/(2\alpha-1)}(I;E^{cat}_{p,1})} \\ &\quad\lesssim \left\|u_0\right\|_{M^{1-2\alpha}_{p,1}} + \|u\|^2_{\widetilde{L}^{2\alpha/(2\alpha-1)}\left(I;E^{cat}_{p,1}\right)},\end{aligned} \quad (62)$$

which implies (60).

We now extend the solution from $I = [0, t_0]$ to $I_1 = [t_0, t_1]$ for some $t_1 > t_0$. Consider the mapping

$$\begin{aligned}\mathcal{T}_1 : u(t) \longrightarrow & G_{2\alpha}(t - t_0) u(t_0) \\ & + \int_{t_0}^{t} G_{2\alpha}(t-\tau)(\vec{\gamma}\cdot\nabla)(u^2)(\tau)\, d\tau,\end{aligned} \tag{63}$$

and the resolution space

$$\mathcal{D}_1 = \left\{ u \in \widetilde{L}^{2\alpha/(2\alpha-1)}\left(I_1; E^{cat}_{p,1}\right) : \|u\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I_1;E^{cat}_{p,1})} \leqslant \delta_1 \right\}, \tag{64}$$

where δ_1 will be chosen below. Taking $t = t_0$ in (60), one has that

$$\sum_k \langle k\rangle^{1-2\alpha} \left\| 2^{cat_0|k|} \square_k u \right\|_p \lesssim \|u_0\|_{M^{1-2\alpha}_{p,1}} + \delta^2. \tag{65}$$

For any $\delta_1 > 0$, in view of (65), we can choose a sufficiently large J such that

$$C \sum_{|k|>J} \langle k\rangle^{1-2\alpha} \left\| 2^{cat_0|k|} \square_k u \right\|_p \leqslant \frac{\delta_1}{4}. \tag{66}$$

Hence, in view of Proposition 9,

$$\begin{aligned}&\left\| G_{2\alpha}(t-t_0) u(t_0) \right\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I_1,\{k:|k|>J\};E^{cat}_{p,1})} \\ &\quad \leqslant C \sum_{|k|>J} \langle k\rangle^{1-2\alpha} \left\| 2^{cat_0|k|} \square_k u \right\|_p \leqslant \frac{\delta_1}{4},\end{aligned} \tag{67}$$

and one can choose $t_1 > t_0$ verifying $C|I_1|^{(2\alpha-1)/(2\alpha)} \langle J\rangle^{2\alpha-1} (\|u_0\|_{M^{1-2\alpha}_{p,1}} + \delta^2) \leqslant \delta_1/4$, so

$$\begin{aligned}&\left\| G_{2\alpha}(t-t_0) u(t_0) \right\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I_1,\{k:|k|\leqslant J\};E^{cat}_{p,1})} \\ &\quad \leqslant C \left|I_1\right|^{(2\alpha-1)/(2\alpha)} \sum_{|k|\leqslant J} \left\| 2^{cat_0|k|} \square_k u \right\|_p \\ &\quad \leqslant C \left|I_1\right|^{(2\alpha-1)/(2\alpha)} \langle J\rangle^{2\alpha-1} \sum_{|k|\leqslant J} \langle k\rangle^{1-2\alpha} \left\| 2^{cat_0|k|} \square_k u \right\|_p \\ &\quad \leqslant C \left|I_1\right|^{(2\alpha-1)/(2\alpha)} \langle J\rangle^{2\alpha-1} \left(\|u_0\|_{M^{1-2\alpha}_{p,1}} + \delta^2 \right) \leqslant \frac{\delta_1}{4}.\end{aligned} \tag{68}$$

In view of (67) and (68),

$$\left\| G_{2\alpha}(t-t_0) u(t_0) \right\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I_1;E^{cat}_{p,1})} \leqslant \frac{\delta_1}{2}. \tag{69}$$

It follows from Propositions 10 and 11

$$\begin{aligned}&\left\| \int_1^t G_{2\alpha}(t-\tau)(\vec{\gamma}\cdot\nabla) u^2(\tau)\, d\tau \right\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I_1;E^{cat}_{p,1})} \\ &\quad \leqslant C \left(1 + \left|I_1\right|^{1/(2\alpha)}\right) 2^{4cat_1} \left\| u^2 \right\|_{\widetilde{L}^{\alpha/(2\alpha-1)}(I_1;E^{cat}_{p,1})} \\ &\quad \lesssim C \left(1 + \left|I_1\right|^{1/(2\alpha)}\right) 2^{4cat_1} \|u\|^2_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I_1;E^{cat}_{p,1})}.\end{aligned} \tag{70}$$

Hence, collecting (69) and (70), we have

$$\begin{aligned}&\left\| \mathcal{T}_1 u \right\|_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I_1;E^{cat}_{p,1})} \\ &\quad \leqslant \frac{\delta_1}{2} + C \left(1 + \left|I_1\right|^{1/(2\alpha)}\right) 2^{4cat_1} \|u\|^2_{\widetilde{L}^{2\alpha/(2\alpha-1)}(I_1;E^{cat}_{p,1})}.\end{aligned} \tag{71}$$

Now we can choose $\delta_1 > 0$ satisfying

$$C \left(1 + \left|I_1\right|^{1/(2\alpha)}\right) 2^{4cat_1} \delta_1 \leqslant \frac{1}{4}. \tag{72}$$

It follows that $\mathcal{T}_1 u \in \mathcal{D}_1$ for any $u \in \mathcal{D}_1$. So, we have extended the solution from $[0, t_0]$ to $[t_0, t_1]$. Noticing that

$$\begin{aligned}&\sum_k \langle k\rangle^{1-2\alpha} \left\| 2^{cat|k|} \square_k G_{2\alpha}(t) u_0 \right\|_{L^\infty_{t\in[t_0,t_1]} L^p_x} \\ &\quad \lesssim \sum_k \langle k\rangle^{1-2\alpha} \left\| 2^{cat_0|k|} \square_k u_0 \right\|_p,\end{aligned} \tag{73}$$

we easily see that the solution can be extended to $[t_1, t_2]$, $\ldots, [t_m, t_{m+1}], \ldots$ and finally find a $T_{\max}$ satisfying the conclusions in Theorems 6 and 7.

Remark 13. As explained in the introduction, the method used here does not work for the critical case $\alpha = 1/2$ and we need to find other ways to deal with this situation in the next section.

4. Analytic Regularity for GDGL: α=1/2

Recall that, in Section 2, we verify the following frequency uniform exponential decay estimate for the generalized Ginzburg-Landau semigroup $G_1(t) = e^{-(a+i)t(-\Delta)^{1/2}}$:

$$\left\| \square_k G_1(t) f \right\|_p \lesssim 2^{-2cat|k|} \left\| \square_k f \right\|_p \tag{74}$$

and the frequency dyadic decay estimate:

$$\left\| \Delta_j G_1(t) f \right\|_p \lesssim e^{-cat2^j} \left\| \Delta_j f \right\|_p. \tag{75}$$

Based on (74) and (75), we will establish some estimates for $G_1(t)$ and $\mathcal{A}_1 f$ in suitable space time exponential modulation and Besov norms, respectively. First, let us consider the linear estimates for $G_1(t)$ in space time exponential modulation norm.

Proposition 14. *Assume that* $1 \leqslant p, q \leqslant \infty$, $a > 0$, $I = [0, 1]$; *then there hold*

$$\begin{aligned}\left\| \partial_{x_i} G_1(t) u_0 \right\|_{\widetilde{L}^1(I;E^{cat}_{p,q})} &\lesssim \|u_0\|_{M^0_{p,q}}, \\ \left\| G_1(t) u_0 \right\|_{\widetilde{L}^\infty(I;E^{cat}_{p,q})} &\lesssim \|u_0\|_{M^0_{p,q}}.\end{aligned} \tag{76}$$

Proof. For $|k| \geqslant 1$, from (74)

$$2^{cat|k|} \left\| \square_k \partial_{x_i} G_1(t) u_0 \right\|_p \lesssim 2^{-cat|k|} \langle k_i\rangle \left\| \square_k u_0 \right\|_p. \tag{77}$$

Taking $L^1_{t\in I}$ norm to (77),

$$\begin{aligned}\left\|2^{cat|k|}\square_k\partial_{x_i}G_1(t)u_0\right\|_{L^1_{t\in I}L^p_x} &\lesssim \langle k\rangle^{-1}\langle k_i\rangle\left\|\square_k u_0\right\|_p\\ &\lesssim \left\|\square_k u_0\right\|_p.\end{aligned}\tag{78}$$

For $k=0$, (78) also holds. Hence taking a sequence l^q norm on both sides of (78) implies the first estimate. For the second estimate, one only needs to take $L^\infty_{t\in I}$ norm to (74) and then a sequence l^q norm. □

More generally, using the method in Proposition 14, one can verify that, for the infinite time interval $I=[1,+\infty)$, there holds the following.

Proposition 15. *Let $1\leqslant p,q\leqslant\infty$, $a>0$, $I=[1,+\infty)$; then for any $\mathbb{A}\subset\mathbb{Z}^n\setminus\{0\}$,*

$$\begin{aligned}\left\|\partial_{x_i}G_1(t-1)u(1)\right\|_{\widetilde{L}^1(I;\mathbb{A};E^{cat}_{p,q})} &\lesssim \|u(1)\|_{E^{ca}_{p,q}},\\ \left\|G_1(t-1)u(1)\right\|_{\widetilde{L}^\infty(I;\mathbb{A};E^{cat}_{p,q})} &\lesssim \|u(1)\|_{E^{ca}_{p,q}}.\end{aligned}\tag{79}$$

Next, we proceed with the estimates of $\mathscr{A}_1 f$ in space time exponential modulation norm.

Proposition 16. *Let $1\leqslant p,q\leqslant\infty$, $1\leqslant\widetilde{q}_1\leqslant\widetilde{q}\leqslant\infty$, $a>0$. Then there exists a constant $c>0$ such that, for $0<t_0\leqslant\infty$, $I=[0,t_0)$, and $\mathbb{A}\subset\mathbb{Z}^n\setminus\{0\}$,*

$$\left\|\mathscr{A}_1 f\right\|_{\widetilde{L}^{\widetilde{q}}(I;\mathbb{A};E^{cat}_{p,q})} \lesssim \sum_{k\in\mathbb{A}}\langle k\rangle^{1/\widetilde{q}_1-1/\widetilde{q}-1}\left\|2^{cat|k|}\square_k f\right\|_{L^{\widetilde{q}_1}_{t\in I}L^p_x}\tag{80}$$

holds for all $f\in\widetilde{L}^{\widetilde{q}}(I;\mathbb{A};E^{cat}_{p,q})$.

Proof. By the decay estimate (74), we easily see that

$$\left\|\square_k\mathscr{A}_1 f\right\|_{L^p_x}\lesssim\int_0^t 2^{-2ca(t-\tau)|k|}\left\|\square_k f(\tau)\right\|_{L^p_x}d\tau.\tag{81}$$

Multiplying (81) by $2^{cat|k|}$,

$$2^{cat|k|}\left\|\square_k\mathscr{A}_1 f\right\|_{L^p_x}\lesssim\int_0^t 2^{-ca(t-\tau)|k|}2^{ca\tau|k|}\left\|\square_k f(\tau)\right\|_{L^p_x}d\tau.\tag{82}$$

When $|k|\geqslant 1$, applying Young's inequality to (82) with $\widetilde{q}\geqslant\widetilde{q}_1$,

$$\left\|2^{cat|k|}\square_k\mathscr{A}_1 f\right\|_{L^{\widetilde{q}}_{t\in I}L^p_x}\lesssim\langle k\rangle^{1/\widetilde{q}_1-1/\widetilde{q}-1}\left\|2^{ca\tau|k|}\square_k f(\tau)\right\|_{L^{\widetilde{q}}_{t\in I}L^p_x}.\tag{83}$$

So, taking the sequence l^q norm on both sides of (83) implied the conclusion. □

In view of $\|\partial_{x_i}\square_k f\|_{L^p_x}\lesssim\langle k_i\rangle\|\square_k f\|_{L^p_x}$ and taking special index in Proposition 16, we obtain the following.

Corollary 17. *Under the same conditions as above, one has*

$$\begin{aligned}\left\|\partial_{x_i}\mathscr{A}_1 f\right\|_{\widetilde{L}^1(I;\mathbb{A};E^{cat}_{\infty,1})} &\lesssim \|f\|_{\widetilde{L}^1(I;\mathbb{A};E^{cat}_{\infty,1})},\\ \left\|\mathscr{A}_1 f\right\|_{\widetilde{L}^\infty(I;\mathbb{A};E^{cat}_{\infty,1})} &\lesssim \|f\|_{\widetilde{L}^1(I;\mathbb{A};E^{cat}_{\infty,1})}.\end{aligned}\tag{84}$$

We now consider the linear estimates of $G_1(t)$ and $\mathscr{A}_1 f$ in space time homogeneous Besov norms based on dyadic decay estimate (75).

Proposition 18. *Assume that $1\leqslant p,q,\gamma\leqslant\infty$, $a>0$; then*

$$\left\|G_1(t)f\right\|_{\widetilde{L}^\gamma(\mathbb{R}^+;\dot{B}^s_{p,q})}\lesssim\|f\|_{\dot{B}^{s-1/\gamma}_{p,q}}.\tag{85}$$

Proof. Let us take L^γ_t norm on both sides of dyadic decay estimate (75):

$$\left\|\Delta_j G_1(t)f\right\|_{L^\gamma_t L^p_x}\lesssim\left\|e^{-cat2^j}\right\|_{L^\gamma_t}\left\|\Delta_j f\right\|_{L^p_x}\lesssim 2^{-j/\gamma}\left\|\Delta_j f\right\|_{L^p_x}.\tag{86}$$

Then multiplying the above inequality by 2^{js} and taking a sequence l^q norm, we obtain the result as desired. □

Proposition 19. *Let $1\leqslant p,q\leqslant\infty$, $1\leqslant\gamma_1\leqslant\gamma\leqslant\infty$, $a>0$. Then*

$$\left\|\mathscr{A}_1 f\right\|_{\widetilde{L}^\gamma(\mathbb{R}^+;\dot{B}^s_{p,q})}\lesssim\|f\|_{\widetilde{L}^{\gamma_1}(\mathbb{R}^+;\dot{B}^{s-(1+1/\gamma-1/\gamma_1)}_{p,q})}.\tag{87}$$

In particular,

$$\begin{aligned}\left\|\mathscr{A}_1 f\right\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^1_{\infty,1})} &\lesssim \|f\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^0_{\infty,1})},\\ \left\|\mathscr{A}_1 f\right\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^0_{\infty,1})} &\lesssim \|f\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^0_{\infty,1})}.\end{aligned}\tag{88}$$

Proof. Using dyadic decay estimate (75), we see that

$$\begin{aligned}\left\|\Delta_j\mathscr{A}_1 f\right\|_{L^p_x} &\leqslant \int_0^t\left\|\Delta_j G_1(t-\tau)f(\tau,x)\right\|_{L^p_x}d\tau\\ &\leqslant \int_0^t e^{-ca(t-\tau)2^j}\left\|\Delta_j f(\tau)\right\|_{L^p_x}d\tau.\end{aligned}\tag{89}$$

Taking L^γ_t norm on both sides of the above estimate and with Young's inequality, we get

$$\left\|\Delta_j\mathscr{A}_1 f\right\|_{L^\gamma_t L^p_x}\leqslant 2^{-j(1+1/\gamma-1/\gamma_1)}\left\|\Delta_j f\right\|_{L^{\gamma_1}_t L^p_x}.\tag{90}$$

Arguing as before, we multiply (90) by 2^{js} and take a sequence l^q norm and then finally get the conclusion as desired. □

With the preparation above, we are now ready to start the proof of Theorem 8. In the following let $I=[0,1]$. We choose the following resolution space:

$$\mathscr{D}=\left\{u:\|u\|_X+\|u\|_Y\leqslant\delta\right\}\tag{91}$$

with metric $d(u,v)=\|u-v\|_{X\cap Y}$, where $\|u\|_X=\|u\|_{\widetilde{L}^1(I;\dot{B}^1_{\infty,1})\cap\widetilde{L}^\infty(I;\dot{B}^0_{\infty,1})}$ and $\|u\|_Y=\sum_{i=1}^n\|\partial_{x_i}u\|_{\widetilde{L}^1(I;E^{cat}_{\infty,1})}+\|u\|_{\widetilde{L}^\infty(I;E^{cat}_{\infty,1})}$. Consider the map

$$\mathscr{T}:u(t)\longrightarrow G_1(t)u_0+\mathscr{A}_1\left[(\vec{\gamma}\cdot\nabla)u^2\right].\tag{92}$$

We will first carry out the computation of $\|G_1(t)u_0\|_Y$. Set $c=2^{-5}$. By (74), for $|k|\geqslant 1$,

$$2^{cat|k|}\left\|\square_k\partial_{x_i}G_1(t)u_0\right\|_\infty\lesssim 2^{-cat|k|}\langle k_i\rangle\left\|\square_k u_0\right\|_\infty.\tag{93}$$

Denote $\mathbb{Z}^n_\star = \mathbb{Z}^n \setminus \{0\}$. Taking $L^1_{t\in I}$ norm and then sequence l^1 norm on $\mathbb{Z}^n_\star$ to (93), we get

$$\left\|\partial_{x_i} G_1(t)\, u_0\right\|_{\widetilde{L}^1(I;\mathbb{Z}^n_\star;E^{cat}_{\infty,1})} \lesssim \langle k\rangle^{-1}\langle k_i\rangle \left\|u_0\right\|_{M^0_{\infty,1}} \lesssim \left\|u_0\right\|_{M^0_{\infty,1}}. \tag{94}$$

For $k = 0$, there holds

$$\left\|\Box_0 \partial_{x_i} G_1(t)\, u_0\right\|_\infty \lesssim \left\|\Box_0 u_0\right\|_\infty. \tag{95}$$

From (94) and (95), it is easy to see that

$$\left\|\partial_{x_i} G_1(t)\, u_0\right\|_{\widetilde{L}^1(I;E^{cat}_{\infty,1})} \lesssim \left\|u_0\right\|_{M^0_{\infty,1}}. \tag{96}$$

Similarly,

$$\left\|G_1(t)\, u_0\right\|_{\widetilde{L}^\infty(I;E^{cat}_{\infty,1})} \lesssim \left\|u_0\right\|_{M^0_{\infty,1}}. \tag{97}$$

Hence

$$\begin{aligned}\left\|G_1(t)u_0\right\|_Y &= \sum_{i=1}^n \left\|\partial_{x_i} G_1(t)\, u_0\right\|_{\widetilde{L}^1(I;E^{cat}_{\infty,1})} \\ &\quad + \left\|G_1(t)\, u_0\right\|_{\widetilde{L}^\infty(I;E^{cat}_{\infty,1})} \lesssim \left\|u_0\right\|_{M^0_{\infty,1}}.\end{aligned} \tag{98}$$

By Corollary 17 and Proposition 11,

$$\begin{aligned}&\sum_{i=1}^n \left\|\partial_{x_i} \mathscr{A}_1(\vec{\gamma}\cdot\nabla)\, u^2\right\|_{\widetilde{L}^1(I;\mathbb{Z}^n_\star;E^{cat}_{\infty,1})} \\ &\qquad + \left\|\mathscr{A}_1(\vec{\gamma}\cdot\nabla)\, u^2\right\|_{\widetilde{L}^\infty(I;\mathbb{Z}^n_\star;E^{cat}_{\infty,1})} \\ &\quad \lesssim \left\|(\vec{\gamma}\cdot\nabla)u^2\right\|_{\widetilde{L}^1(I;\mathbb{Z}^n_\star;E^{cat}_{\infty,1})} \\ &\quad \lesssim \sum_{j=1}^n \left\|\partial_{x_j} u\right\|_{\widetilde{L}^1(I;\mathbb{Z}^n_\star;E^{cat}_{\infty,1})} \|u\|_{\widetilde{L}^\infty(I;\mathbb{Z}^n_\star;E^{cat}_{\infty,1})} \lesssim \|u\|_Y^2.\end{aligned} \tag{99}$$

In view of $\|\Box_0 f\|_\infty \lesssim \|f\|_\infty \lesssim \|f\|_{\dot{B}^0_{\infty,1}}$, we observe that

$$\begin{aligned}&\left\|\Box_0 \partial_{x_i} \mathscr{A}_1(\vec{\gamma}\cdot\nabla)\, u^2\right\|_{L^1_{t\in I} L^\infty_x} + \left\|\Box_0 \mathscr{A}_1(\vec{\gamma}\cdot\nabla)\, u^2\right\|_{L^\infty_{t\in I} L^\infty_x} \\ &\quad \lesssim \left\|\mathscr{A}_1(\vec{\gamma}\cdot\nabla)\, u^2\right\|_{\widetilde{L}^1(I;\dot{B}^1_{\infty,1})} + \left\|\mathscr{A}_1(\vec{\gamma}\cdot\nabla)\, u^2\right\|_{\widetilde{L}^\infty(I;\dot{B}^0_{\infty,1})} \\ &\quad \lesssim \left\|(\vec{\gamma}\cdot\nabla)\, u^2\right\|_{\widetilde{L}^1(I;\dot{B}^0_{\infty,1})} \lesssim \sum_{i=1}^n \left\|\partial_{x_i} u^2\right\|_{\widetilde{L}^1(I;\dot{B}^0_{\infty,1})} \\ &\quad \lesssim \left\|u^2\right\|_{\widetilde{L}^1(I;\dot{B}^1_{\infty,1})}.\end{aligned} \tag{100}$$

To bound the right-hand side of (100), let us recall the para-product decomposition [10, 20]: for any two tempered distributions f, g, fg can be decomposed as the summation of two parts:

$$fg = \sum_{i,j} \Delta_i f \Delta_j g = \sum_j S_{j-1} f \Delta_j g + \sum_i S_i g \Delta_i f. \tag{101}$$

From the support property in frequency space, there holds

$$\Delta_i\left(S_j f \Delta_j g\right) = 0, \quad \text{if } i > j + 3. \tag{102}$$

Thus

$$\Delta_i(fg) = \sum_{i\leqslant j+3} \Delta_i\left(S_{j-1} f \Delta_j g + S_j g \Delta_j f\right). \tag{103}$$

With this tool, we now estimate $\|u^2\|_{\widetilde{L}^1(I;\dot{B}^1_{\infty,1})}$.

$$\begin{aligned}\left\|\Delta_i\left(u^2\right)\right\|_{L^\infty_x} &= \left\|\sum_{i\leqslant j+3} \Delta_i\left(S_{j-1} u \Delta_j u + S_j u \Delta_j u\right)\right\|_{L^\infty_x} \\ &\lesssim \sum_{j\geqslant i-3} \left\|S_j u\right\|_{L^\infty_x} \left\|\Delta_j u\right\|_{L^\infty_x} \\ &\lesssim \sum_{j\geqslant i-3} \sum_{k\leqslant j} \left\|\Delta_k u\right\|_{L^\infty_x} \left\|\Delta_j u\right\|_{L^\infty_x}.\end{aligned} \tag{104}$$

By taking L^1_t norm, we deduce that

$$\begin{aligned}\left\|\Delta_i\left(u^2\right)\right\|_{L^1_t L^\infty_x} &\lesssim \sum_{j\geqslant i-3} \sum_{k\leqslant j} \left\|\Delta_k u\right\|_{L^\infty_t L^\infty_x} \left\|\Delta_j u\right\|_{L^1_t L^\infty_x} \\ &\lesssim \|u\|_{\widetilde{L}^\infty(I;\dot{B}^0_{\infty,1})} \sum_{j\geqslant i-3} \left\|\Delta_j u\right\|_{L^1_t L^\infty_x}.\end{aligned} \tag{105}$$

Multiplying (105) by 2^i and taking a sequence l^1 norm, we get

$$\begin{aligned}\left\|u^2\right\|_{\widetilde{L}^1(I;\dot{B}^1_{\infty,1})} &\lesssim \|u\|_{\widetilde{L}^\infty(I;\dot{B}^0_{\infty,1})} \left\|\sum_{j\geqslant i-3} 2^{-(j-i)} 2^j \left\|\Delta_j u\right\|_{L^1_t L^\infty_x}\right\|_{l^1} \\ &\lesssim \|u\|_{\widetilde{L}^\infty(I;\dot{B}^0_{\infty,1})} \|u\|_{\widetilde{L}^1(I;\dot{B}^1_{\infty,1})},\end{aligned} \tag{106}$$

where in the last inequality, we use Young's inequality. From (100) and (106), we obtain

$$\begin{aligned}&\left\|\Box_0 \partial_{x_i} \mathscr{A}_1(\vec{\gamma}\cdot\nabla)\, u^2\right\|_{L^1_{t\in I} L^\infty_x} + \left\|\Box_0 \mathscr{A}_1(\vec{\gamma}\cdot\nabla)\, u^2\right\|_{L^\infty_{t\in I} L^\infty_x} \\ &\quad \lesssim \|u\|_{\widetilde{L}^\infty(I;\dot{B}^0_{\infty,1})} \|u\|_{\widetilde{L}^1(I;\dot{B}^1_{\infty,1})} \lesssim \|u\|_X^2.\end{aligned} \tag{107}$$

Collecting (98), (99), and (107), we see that

$$\|\mathscr{T} u\|_Y \lesssim \left\|u_0\right\|_{M^0_{\infty,1}} + \|u\|_Y^2 + \|u\|_X^2. \tag{108}$$

On the other hand, by Propositions 18 and 19, (100), and (106), similar reasoning yields that

$$\begin{aligned}\|\mathscr{T} u\|_X &\lesssim \left\|u_0\right\|_{\dot{B}^0_{\infty,1}} + \left\|\mathscr{A}_1(\vec{\gamma}\cdot\nabla)\, u^2\right\|_{\widetilde{L}^1(I;\dot{B}^1_{\infty,1})\cap\widetilde{L}^\infty(I;\dot{B}^0_{\infty,1})} \\ &\lesssim \left\|u_0\right\|_{\dot{B}^0_{\infty,1}} + \|u\|_X^2.\end{aligned} \tag{109}$$

Thus from (108) and (109), we have

$$\|\mathscr{T} u\|_X + \|\mathscr{T} u\|_Y \leqslant C\left\|u_0\right\|_{M^0_{\infty,1}\cap\dot{B}^0_{\infty,1}} + C\|u\|_X^2 + C\|u\|_Y^2. \tag{110}$$

Suppose that

$$C\delta \leqslant \frac{1}{4}, \quad C\|u_0\|_{M^0_{\infty,1}\cap \dot{B}^0_{\infty,1}} \leqslant \frac{\delta}{2}. \tag{111}$$

By the standard contraction mapping principle, we see that there exists a unique solution to $\mathcal{T}u = u$ in space $X \cap Y$. In particular, there hold

$$\|u\|_{\widetilde{L}^\infty(0,1;E^{cat}_{\infty,1})} \leqslant \delta; \qquad \|u\|_{\widetilde{L}^\infty(0,1;\dot{B}^0_{\infty,1})} \leqslant \delta. \tag{112}$$

So for $u(1) = u(1,x)$,

$$\|u(1)\|_{E^{ca}_{\infty,1}} \leqslant \delta, \quad \|u(1)\|_{\dot{B}^0_{\infty,1}} \leqslant \delta, \quad c = 2^{-5}. \tag{113}$$

We now extend the regularity of solutions. Consider the integral equation

$$u(t) = G_1(t-1)u(1) + \int_1^t G_1(t-\tau)(\vec{\gamma}\cdot\nabla)(u^2)(\tau)\,d\tau. \tag{114}$$

Let $I_1 = [1,+\infty)$. Denote

$$\begin{aligned} \|u\|_{X_1} &= \|u\|_{\widetilde{L}^1(I_1;\dot{B}^1_{\infty,1})\cap\widetilde{L}^\infty(I_1;\dot{B}^0_{\infty,1})}, \\ \|u\|_{Y_1} &= \sum_{i=1}^n \|\partial_{x_i}u\|_{\widetilde{L}^1(I_1;E^{ca}_{\infty,1})} + \|u\|_{\widetilde{L}^\infty(I_1;E^{ca}_{\infty,1})}. \end{aligned} \tag{115}$$

Choose the resolution space to be

$$\mathcal{D}_1 = \left\{u : \|u\|_{X_1} + \|u\|_{Y_1} \leqslant \delta_1\right\} \tag{116}$$

with metric $d_1(u,v) = \|u-v\|_{X_1\cap Y_1}$. δ_1 is to be determined later. Similar to (109), we have

$$\|\mathcal{T}_1 u\|_{X_1} \leqslant C\|u(1)\|_{\dot{B}^0_{\infty,1}} + C\|u\|^2_{X_1}. \tag{117}$$

To estimate $\|\mathcal{T}_1 u\|_{Y_1}$, by Proposition 15

$$\begin{aligned} &\left\|\partial_{x_i}G_1(t-1)u(1)\right\|_{\widetilde{L}^1(I_1,\mathbb{Z}^n_\star;E^{ca}_{\infty,1})} \\ &\quad + \left\|G_1(t-1)u(1)\right\|_{\widetilde{L}^\infty(I_1,\mathbb{Z}^n_\star;E^{ca}_{\infty,1})} \lesssim \|u(1)\|_{E^{ca}_{\infty,1}}. \end{aligned} \tag{118}$$

For the remaining part concerning $\square_0$, we use $\|\square_0 f\| \leqslant \|f\|_{\dot{B}^0_{\infty,1}}$ and Proposition 18 to bound

$$\begin{aligned} &\left\|\square_0\partial_{x_i}G_1(t-1)u(1)\right\|_{L^1_{t\in I_1}L^\infty_x} + \left\|\square_0 G_1(t-1)u(1)\right\|_{L^\infty_{t\in I_1}L^\infty_x} \\ &\quad \lesssim \left\|G_1(t-1)u(1)\right\|_{\widetilde{L}^1(I;\dot{B}^1_{\infty,1})\cap\widetilde{L}^\infty(I;\dot{B}^0_{\infty,1})} \lesssim \|u(1)\|_{\dot{B}^0_{\infty,1}}. \end{aligned} \tag{119}$$

Arguing as before, with Corollary 17 and Proposition 11, we get

$$\begin{aligned} &\left\|\partial_{x_i}\int_1^t G_1(t-\tau)(\vec{\gamma}\cdot\nabla)u^2(\tau)\right\|_{\widetilde{L}^1(I_1,\mathbb{Z}^n_\star;E^{ca}_{\infty,1})} \\ &\quad + \left\|\int_1^t G_1(t-\tau)(\vec{\gamma}\cdot\nabla)u^2(\tau)\right\|_{\widetilde{L}^\infty(I_1,\mathbb{Z}^n_\star;E^{ca}_{\infty,1})} \\ &\lesssim \left\|(\vec{\gamma}\cdot\nabla)u^2\right\|_{\widetilde{L}^1(I_1,\mathbb{Z}^n_\star;E^{ca}_{\infty,1})} \\ &\lesssim \left\|\partial_{x_i}u\right\|_{\widetilde{L}^1(I_1;E^{ca}_{\infty,1})}\|u\|_{\widetilde{L}^\infty(I_1;E^{ca}_{\infty,1})} \lesssim \|u\|^2_{Y_1}. \end{aligned} \tag{120}$$

Similar as in (107), we have

$$\begin{aligned} &\left\|\square_0\partial_{x_i}\int_1^t G_1(t-\tau)(\vec{\gamma}\cdot\nabla)u^2(\tau)\right\|_{L^1_{t\in I_1}L^\infty_x} \\ &\quad + \left\|\square_0\int_1^t G_1(t-\tau)(\vec{\gamma}\cdot\nabla)u^2(\tau)\right\|_{L^\infty_{t\in I_1}L^\infty_x} \\ &\lesssim \|u\|_{\widetilde{L}^\infty(I_1;\dot{B}^0_{\infty,1})}\|u\|_{\widetilde{L}^1(I_1;\dot{B}^1_{\infty,1})} \lesssim \|u\|^2_{X_1}. \end{aligned} \tag{121}$$

Then

$$\begin{aligned} \|\mathcal{T}_1 u\|_{Y_1} &\leqslant C\|u(1)\|_{\dot{B}^0_{\infty,1}} + C\|u(1)\|_{E^{ca}_{\infty,1}} + C\|u\|^2_{X_1} + C\|u\|^2_{Y_1} \\ &\leqslant C\delta + C\|u\|^2_{X_1\cap Y_1}. \end{aligned} \tag{122}$$

Choose δ_1 satisfying $C\delta \leqslant \delta_1/2$ and $C\delta_1 \leqslant 1/2$; then by the contraction mapping argument, we see that $\mathcal{T}_1 u = u$ has a unique solution in $\mathcal{D}_1$ which completes the proof of Theorem 8.

Remark 20. The same reasoning can show that if $u_0 \in \dot{B}^0_{p,1} \cap M^0_{p,1}(1 \leqslant p \leqslant \infty)$ is sufficiently small, then there exists a unique global solution u to (1) satisfying $u \in \widetilde{L}^\infty(\mathbb{R}^+;E^{s(t)}_{p,1})$ and $\partial_{x_i}u \in \widetilde{L}^1(\mathbb{R}^+;E^{s(t)}_{p,1})$ with $s(t) = 2^{-5}(a \wedge at)$. We omit the details here.

5. Conclusion

In this paper, we investigate and prove some well-posedness results and analytic regularity property of the solutions to the (fractional) quadratic derivative Ginzburg-Landau equation with rough initial data in certain critical modulation spaces and Besov type spaces. The basic strategy is via contraction mapping principle. The main obstacle that comes from the one order derivative in the nonlinearity is finally resolved by developing suitable exponential decay estimates for the (fractional) Ginzburg-Landau semigroup. The suggested approach can be used for more general fractional nonlinear partial differential equations.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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