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Research Article

The Existence of Solutions for Four-Point Coupled Boundary Value Problems of Fractional Differential Equations at Resonance

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A four-point coupled boundary value problem of fractional differential equations is studied. Based on Mawhin's coincidence degree theory, some existence theorems are obtained in the case of resonance.

1. Introduction

In this paper, we are concerned with the following four-point coupled boundary value problem for nonlinear fractional differential equation. Consider

$$\begin{aligned} &D_{0+}^{\alpha}u(t) = f\left(t, u(t), D_{0+}^{\alpha-1}u(t), v(t), D_{0+}^{\beta-1}v(t)\right),\\ &D_{0+}^{\beta}v(t) = g\left(t, u(t), D_{0+}^{\alpha-1}u(t), v(t), D_{0+}^{\beta-1}v(t)\right),\\ &I_{0+}^{2-\alpha}u(t)\big|_{t=0} = 0, \qquad u(1) = av(\xi),\\ &I_{0+}^{2-\beta}v(t)\big|_{t=0} = 0, \qquad v(1) = bu(\eta), \end{aligned} \tag{1}$$

where $1 < \alpha, \beta < 2$, D_{0+}^{α} and I_{0+}^{α} are the standard Riemann-Liouville differentiation and integration, $f, g \in C([0,1] \times \mathbb{R}^4, \mathbb{R})$, $a, b \in \mathbb{R}$, $\xi, \eta \in (0,1)$, and

$$ab\xi^{\beta-1}\eta^{\alpha-1} = 1. \tag{2}$$

The subject of fractional calculus has gained considerable popularity and importance because of its intensive development of the theory of fractional calculus itself and its varied applications in many fields of science and engineering. As a result, the subject of fractional differential equations has attracted much attention; see [1–11] for a good overview.

At the same time, we notice that coupled boundary value problems, which arise in the study of reaction-diffusion equations and Sturm-Liouville problems, have wide applications in various fields of sciences and engineering, for example, the heat equation [12–14] and mathematical biology [15, 16]. In [17], Asif and Khan used the Guo-Krasnosel'skii fixed-point theorem to show the existence of positive solutions to the nonlinear differential system with coupled four-point boundary value conditions

$$\begin{aligned} &-x''(t) = f(t, x(t), y(t)), \quad t \in (0,1),\\ &-y''(t) = g(t, x(t), y(t)), \quad t \in (0,1),\\ &x(0) = y(0) = 0,\\ &x(1) = \alpha y(\xi), \qquad y(1) = \beta x(\eta), \end{aligned} \tag{3}$$

where $\xi, \eta \in (0,1)$, $0 < \alpha\beta\xi\eta < 1$, and $f, g : (0,1) \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are two given continuous functions.

In [18], the authors considered the existence of positive solutions of four-point coupled boundary value problem for systems of the nonlinear semipositone fractional differential equation

$$\begin{aligned} &D_{0+}^{\alpha}u+\lambda f(t,u,v)=0, \quad t\in(0,1),\ \lambda>0,\\ &D_{0+}^{\alpha}v+\lambda g(t,u,v)=0,\\ &u^{(i)}(0)=v^{(i)}(0)=0, \quad 0\le i\le n-2,\\ &u(1)=av(\xi), \qquad v(1)=bu(\eta), \end{aligned}\tag{4}$$

where λ is a parameter, a,b,ξ,η satisfy $\xi,\eta\in(0,1)$, $0<ab\xi\eta<1$, $\alpha\in(n-1,n]$ is a real number and $n\ge 3$, and $D_{0+}^{\alpha}u$ is Riemann-Liouville's fractional derivative.

Recently, Cui and Sun [19] showed the existence of positive solutions of singular superlinear coupled integral boundary value problems for differential systems

$$\begin{aligned} &-x''(t)=f_1(t,x(t),y(t)), \quad t\in(0,1),\\ &-y''(t)=f_2(t,x(t),y(t)), \quad t\in(0,1),\\ &x(0)=y(0)=0, \qquad x(1)=\alpha[y], \qquad y(1)=\beta[x], \end{aligned}\tag{5}$$

where $\alpha[x],\beta[x]$ are bounded linear functionals on $C[0,1]$ given by

$$\alpha[x]=\int_0^1 x(t)\,dA(t), \qquad \beta[x]=\int_0^1 x(t)\,dB(t)\tag{6}$$

with A,B being functions of bounded variation with positive measures.

A key assumption in the above papers is that the case studied is not at resonance; that is, the associated fractional (or ordinary) linear differential operators are invertible. In this paper, instead, we are interested in the resonance case due to the critical condition (2). Boundary value problems at resonance have been studied by several authors including the most recent works [20–31]. In this paper, we establish new results on the existence of a solution for the couple boundary value problems at resonance. Our method is based on the coincidence degree theorem of Mawhin [32, 33].

Now, we briefly recall some notations and an abstract existence result.

Let Y,Z be real Banach spaces and let $L:\operatorname{dom}L\subset Y\to Z$ be a Fredholm operator of index zero. $P:Y\to Y$ and $Q:Z\to Z$ are continuous projectors such that

$$\begin{aligned} &\operatorname{Im}P=\operatorname{Ker}L, \qquad \operatorname{Ker}Q=\operatorname{Im}L,\\ &Y=\operatorname{Ker}L\oplus\operatorname{Ker}P, \qquad Z=\operatorname{Im}L\oplus\operatorname{Im}Q. \end{aligned}\tag{7}$$

It follows that $L|_{\operatorname{dom}L\cap\operatorname{Ker}P}:\operatorname{dom}L\cap\operatorname{Ker}P\to\operatorname{Im}L$ is invertible. We denote the inverse of the mapping by K_P (generalized inverse operator of L). If Ω is an open bounded subset of Y such that $\operatorname{dom}L\cap\Omega\neq\emptyset$, the mapping $N:Y\to Z$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N:\overline{\Omega}\to Y$ is compact.

Theorem 1 (see [32, 33]). *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied.*

(i) $Lx\neq\lambda Nx$ *for every* $(x,\lambda)\in[(\operatorname{dom}L\setminus\operatorname{Ker}L)\cap\partial\Omega]\times(0,1)$.

(ii) $Nx\notin\operatorname{Im}L$ *for every* $x\in\operatorname{Ker}L\cap\partial\Omega$.

(iii) $\deg(QN|_{\operatorname{Ker}L},\operatorname{Ker}L\cap\Omega,0)\neq 0$, *where* $Q:Z\to Z$ *is a projector as above with* $\operatorname{Im}L=\operatorname{Ker}Q$.

Then the equation $Lx=Nx$ *has at least one solution in* $\operatorname{dom}L\cap\overline{\Omega}$.

For convenience, let us set the following notations:

$$\begin{aligned} \Delta_1&=\max\left\{1+\frac{1}{\Gamma(\beta)},\left(1+\frac{1}{\Gamma(\alpha)}\right)\times\left(1+a\xi^{\beta-1}\Gamma(\alpha)\left[\frac{b}{\Gamma(\alpha)}+\frac{1}{\Gamma(\beta)}\right]\right)\right\},\\ \Delta_2&=\frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\alpha\xi^{\beta-1}\Gamma(\alpha+1)(\xi-1)+\Gamma(\beta+1)(\eta-1)},\\ \Delta_3&=\frac{1}{\Gamma(\beta)}\max\left\{a\xi^{\beta-1}(\Gamma(\alpha)+1),\Gamma(\beta)+1\right\},\\ \Delta_4&=\max\left\{\Delta_1\|a_1\|_1+\Delta_3\|a_2\|_1,\Delta_1\|b_1\|_1+\Delta_3\|a_2\|_1\right\},\\ \Delta_5&=\max\left\{\Delta_1\|c_1\|_1+\Delta_3\|c_2\|_1,\Delta_1\|d_1\|_1+\Delta_3\|d_2\|_1\right\}. \end{aligned}\tag{8}$$

2. Preliminaries and Lemmas

In this section, first we provide recall of some basic definitions and lemmas of the fractional calculus, which will be used in this paper. For more details, we refer to books [1, 2, 4].

Definition 2 (see [1, 4]). The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $u:(0,\infty)\to\mathbb{R}$ is given by

$$I_{0+}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)\,ds,\tag{9}$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 3 (see [1, 4]). The Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $u:(0,\infty)\to\mathbb{R}$ is given by

$$D_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t\frac{u(s)}{(t-s)^{\alpha-n+1}}ds,\tag{10}$$

where $n-1\le\alpha<n$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

We use the classical Banach space $C[0,1]$ with the norm $\|u\|_\infty=\max_{t\in[0,1]}|u(t)|$ and $L^1[0,1]$ with the norm $\|u\|_1=\int_0^1|u(t)|dt$. We also use the space $AC^n[0,1]$ defined by

$$AC^n[0,1]=\left\{u:[0,1]\longrightarrow\mathbb{R}\mid u^{(n-1)}\ \text{are absolutely continuous on }[0,1]\right\}\tag{11}$$

and the Banach space $C^{\mu}[0,1]$ ($\mu > 0$)

$$\begin{aligned} C^{\mu}[0,1] &= \{u(t) \mid u(t) = I_{0+}^{\mu}x(t) + c_1 t^{\mu-1} + c_2 t^{\mu-2} + \cdots \\ &\quad + c_{N-1}t^{\mu-(N-1)}, x \in C[0,1], t \in [0,1], \\ &\quad c_i \in \mathbb{R}, i = 1,2,\ldots,N = [\mu]+1\} \end{aligned} \tag{12}$$

with the norm $\|u\|_{C^{\mu}} = \|D_{0+}^{\mu}u\|_{\infty} + \cdots + \|D_{0+}^{\mu-(N-1)}u\|_{\infty} + \|u\|_{\infty}$.

Lemma 4 (see [1]). *Let $\alpha > 0$, $n = [\alpha]+1$. Assume that $u \in L^1(0,1)$ with a fractional integration of order $n-\alpha$ that belongs to $AC^n[0,1]$. Then the equality*

$$\left(I_{0+}^{\alpha}D_{0+}^{\alpha}u\right)(t) = u(t) - \sum_{i=1}^{n}\frac{\left(\left(I_{0+}^{n-\alpha}u\right)(t)\right)^{(n-i)}|_{t=0}}{\Gamma(\alpha-i+1)}t^{\alpha-i} \tag{13}$$

holds almost everywhere on $[0,1]$.

In the following lemma, we use the unified notation of both for fractional integrals and fractional derivatives assuming that $I_{0+}^{\alpha} = D_{0+}^{-\alpha}$ for $\alpha < 0$.

Lemma 5 (see [1]). *Assume that $\alpha > 0$; then*

(i) *let $k \in \mathbb{N}$. If $D_{a+}^{\alpha}u(t)$ and $(D_{a+}^{\alpha+k}u)(t)$ exist, then*

$$\left(D^k D_{a+}^{\alpha}\right)u(t) = \left(D_{a+}^{\alpha+k}u\right)(t); \tag{14}$$

(ii) *if $\beta > 0, \alpha+\beta > 1$, then*

$$\left(I_{a+}^{\alpha}I_{a+}^{\beta}\right)u(t) = \left(I_{a+}^{\alpha+\beta}u\right)(t) \tag{15}$$

satisfies at any point on $[a,b]$ for $u \in L_p(a,b)$ and $1 \le p \le +\infty$;

(iii) *let $u \in C[a,b]$. Then $(D_{a+}^{\alpha}I_{a+}^{\alpha})u(t) = u(t)$ holds on $[a,b]$;*

(iv) *note that, for $\lambda > -1$, $\lambda \neq \alpha-1, \alpha-2, \ldots, \alpha-n$, one has*

$$D^{\alpha}t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha}, \qquad D^{\alpha}t^{\alpha-i} = 0, \quad i = 1,2,\ldots,n. \tag{16}$$

Remark 6. If $1 < \alpha < 2$ and u satisfies $D_{0+}^{\alpha}u = f(t) \in L^1(0,1)$ and $I_{0+}^{2-\alpha}u|_{t=0} = 0$, then $u \in C^{\alpha-1}[0,1]$. In fact, with Lemma 4, one has

$$u(t) = I_{0+}^{\alpha}f(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}. \tag{17}$$

Combined with $I_{0+}^{2-\alpha}u|_{t=0} = 0$, there is $c_2 = 0$. So

$$u(t) = I_{0+}^{\alpha}f(t) + c_1 t^{\alpha-1} = I_{0+}^{\alpha-1}\left[I_{0+}^{1}f(t) + c_1\Gamma(\alpha)\right]. \tag{18}$$

Lemma 7 (see [34]). *$F \subset C^{\mu}[0,1]$ is a sequentially compact set if and only if F is uniformly bounded and equicontinuous. Here to be uniformly bounded means that there exists $M > 0$ such that for every $u \in F$*

$$\|u\|_{C^{\mu}} = \left\|D_{0+}^{\mu}u\right\|_{\infty} + \cdots + \left\|D_{0+}^{\mu-(N-1)}u\right\|_{\infty} + \|u\|_{\infty} < M \tag{19}$$

and to be equicontinuous means that for all $\epsilon > 0$, $\exists\delta > 0$ and for all $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$, $u \in F$, and $i \in \{1,\ldots,[\mu]\}$, the following holds:

$$\left|u(t_1) - u(t_2)\right| < \varepsilon, \qquad \left|D_{0+}^{\mu-i}u(t_1) - D_{0+}^{\mu-i}u(t_2)\right| < \varepsilon. \tag{20}$$

We also use the following two Banach spaces $Y = C^{\alpha-1}[0,1] \times C^{\beta-1}[0,1]$ with the norm

$$\|(x,y)\|_Y = \max\left\{\|x\|_{C^{\alpha-1}}, \|y\|_{C^{\beta-1}}\right\} \tag{21}$$

and $Z = L^1[0,1] \times L^1[0,1]$ with the norm

$$\|(x,y)\|_Z = \max\left\{\|x\|_1, \|y\|_1\right\}. \tag{22}$$

Let the linear operator $L : \operatorname{dom} L \subset Y \to Z$ with

$$\begin{aligned} \operatorname{dom} L = \{&(u,v) \in Y : I_{0+}^{2-\alpha}u(t)|_{t=0} = 0,\ u(1) = av(\xi), \\ &I_{0+}^{2-\beta}v(t)|_{t=0} = 0,\ v(1) = bu(\eta)\} \end{aligned} \tag{23}$$

be defined by

$$L(u,v) = (L_1 u, L_2 v), \tag{24}$$

where $L_1 : C^{\alpha-1}[0,1] \to L^1[0,1]$ and $L_2 : C^{\beta-1}[0,1] \to L^1[0,1]$ are defined by

$$L_1 u = D_{0+}^{\alpha}u(t), \qquad L_2 v = D_{0+}^{\beta}v(t). \tag{25}$$

Let the nonlinear operator $N : Y \to Z$ be defined by

$$(N(u,v))(t) = \left(N_1(u,v)(t), N_2(u,v)(t)\right), \tag{26}$$

where $N_1, N_2 : Y \to L^1[0,1]$ are defined by

$$\begin{aligned} N_1(u,v)(t) &= f\left(t, u(t), D_{0+}^{\alpha-1}u(t), v(t), D_{0+}^{\beta-1}v(t)\right), \\ N_2(x,y)(t) &= g\left(t, u(t), D_{0+}^{\alpha-1}u(t), v(t), D_{0+}^{\beta-1}v(t)\right). \end{aligned} \tag{27}$$

Then four-point coupled boundary value problems (1) can be written as

$$L(u,v) = N(u,v). \tag{28}$$

Lemma 8. *Let L be the linear operator defined as above. If* (2) *holds, then*

$$\begin{aligned} \operatorname{Ker} L = \{&(u,v) \in \operatorname{dom} L : (u,v) = c\left(a\xi^{\beta-1}t^{\alpha-1}, t^{\beta-1}\right), \\ &c \in \mathbb{R}, t \in [0,1]\}, \end{aligned} \tag{29}$$

$$\begin{aligned} \operatorname{Im} L = \{&(x,y) \in Z : aI_{0+}^{\beta}y(\xi) - I_{0+}^{\alpha}x(1) \\ &+ a\xi^{\beta-1}\left[bI_{0+}^{\alpha}x(\eta) - I_{0+}^{\beta}y(1)\right] = 0\}. \end{aligned} \tag{30}$$

Proof. Let $u(t) = a\xi^{\beta-1}t^{\alpha-1}$and let $v(t) = t^{\beta-1}$. Then by Lemma 5, we have $I_{0+}^{2-\alpha}u(t)|_{t=0} = I_{0+}^{2-\beta}v(t)|_{t=0} = 0$, $u(1) = a\xi^{\beta-1} = av(\xi)$, $v(1) = 1 = bu(\eta)$, and $D_{0+}^{\alpha}u(t) = D_{0+}^{\beta}v(t) = 0$. So

$$\left\{(u,v) \in \operatorname{dom} L : (u,v) = c\left(a\xi^{\beta-1}t^{\alpha-1}, t^{\beta-1}\right), c \in \mathbb{R}\right\} \subset \operatorname{Ker} L. \tag{31}$$

For every $(u,v) \in \operatorname{Ker} L$, if $D_{0+}^{\alpha}u(t) = D_{0+}^{\beta}v(t) = 0$, then

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \qquad v(t) = c_3 t^{\beta-1} + c_4 t^{\beta-2}. \tag{32}$$

Considering that $I_{0+}^{2-\alpha}u(t)|_{t=0} = I_{0+}^{2-\beta}v(t)|_{t=0} = 0$, $u(1) = av(\xi)$ and $v(1) = bu(\eta)$, we can obtain that $c_2 = c_4 = 0$ and $c_1 : c_3 = a\xi^{\beta-1}$. It yields the following:

$$\begin{aligned}\operatorname{Ker} L \subset \Big\{ (u,v) &\in \operatorname{dom} L : (u,v) \\ &= c\left(a\xi^{\beta-1}t^{\alpha-1}, t^{\beta-1}\right), c \in \mathbb{R}\Big\}.\end{aligned} \tag{33}$$

Let $(x,y) \in \operatorname{Im} L$; then there is $(u,v) \in \operatorname{dom} L$ such that $(x,y) = L(u,v)$; that is, $u \in C^{\alpha-1}[0,1]$, $D_{0+}^{\alpha}u(t) = x(t)$ and $v \in C^{\beta-1}[0,1]$, $D_{0+}^{\beta}v(t) = y(t)$. By Lemma 4,

$$\begin{aligned} I_{0+}^{\alpha}x(t) &= u(t) - \frac{\left(\left(D_{0+}^{\alpha-1}u\right)(t)\right)\big|_{t=0}}{\Gamma(\alpha)}t^{\alpha-1} \\ &\quad - \frac{\left(\left(I_{0+}^{2-\alpha}u\right)(t)\right)\big|_{t=0}}{\Gamma(\alpha-1)}t^{\alpha-2}, \\ I_{0+}^{\beta}y(t) &= v(t) - \frac{\left(\left(D_{0+}^{\beta-1}v\right)(t)\right)\big|_{t=0}}{\Gamma(\beta)}t^{\beta-1} \\ &\quad - \frac{\left(\left(I_{0+}^{2-\beta}v\right)(t)\right)\big|_{t=0}}{\Gamma(\beta-1)}t^{\beta-2} \end{aligned} \tag{34}$$

and by the couple boundary conditions, we have

$$\begin{gathered} u(t) = I_{0+}^{\alpha}x(t) + c_1 t^{\alpha-1}, \qquad v(t) = I_{0+}^{\beta}y(t) + c_2 t^{\beta-1}, \\ u(1) = I_{0+}^{\alpha}x(1) + c_1 = av(\xi) = a\left[I_{0+}^{\beta}y(\xi) + c_2\xi^{\beta-1}\right], \\ v(1) = I_{0+}^{\beta}y(1) + c_2 = bu(\eta) = b\left[I_{0+}^{\alpha}y(\eta) + c_1\eta^{\alpha-1}\right]. \end{gathered} \tag{35}$$

It yields the following:

$$aI_{0+}^{\beta}y(\xi) - I_{0+}^{\alpha}x(1) + a\xi^{\beta-1}\left[bI_{0+}^{\alpha}x(\eta) - I_{0+}^{\beta}y(1)\right] = 0. \tag{36}$$

On the other hand, suppose that $(x,y) \in Z$ satisfy (36). Let $u(t) = I_{0+}^{\alpha}x(t) + a\xi^{\beta-1}t^{\alpha-1}$ and $v(t) = I_{0+}^{\beta}y(t) + [bI_{0+}^{\alpha}x(\eta) - I_{0+}^{\beta}y(1) + 1]t^{\beta-1}$, and then $D_{0+}^{\alpha}u(t) = x(t)$, $D_{0+}^{\beta}v(t) = y(t)$, and

$$\begin{aligned} I_{0+}^{2-\alpha}u(t)\,|_{t=0} &= I_{0+}^{2-\beta}v(t)\,|_{t=0} = 0, \\ u(1) &= I_{0+}^{\alpha}x(1) + a\xi^{\beta-1} \\ &= aI_{0+}^{\beta}y(\xi) + a\left[bI_{0+}^{\alpha}x(\eta) - I_{0+}^{\beta}y(1) + 1\right]\xi^{\beta-1} \\ &= av(\xi), \\ v(1) &= I_{0+}^{\beta}y(1) + \left[bI_{0+}^{\alpha}x(\eta) - I_{0+}^{\beta}y(1) + 1\right] \\ &= bI_{0+}^{\alpha}x(\eta) + ab\xi^{\beta-1}\eta^{\alpha-1} = bu(\eta). \end{aligned} \tag{37}$$

Therefore, (30) holds. □

Lemma 9. *If* (2) *holds, then L is a Fredholm operator of index zero and* $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1$. *Furthermore, the linear operator $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be defined by*

$$\begin{aligned} &K_P(u,v)(t) \\ &\quad = \left\{I_{0+}^{\alpha}u(t) - a\xi^{\beta-1}t^{\alpha-1}\left[bI_{0+}^{\alpha}u(\eta) - I_{0+}^{\beta}v(1)\right], I_{0+}^{\beta}v(t)\right\}. \end{aligned} \tag{38}$$

Also

$$\left\|K_p(u,v)\right\|_Y \le \Delta_1 \|(u,v)\|_Z. \tag{39}$$

Proof. Define operator $Q : Z \to Z$ as follows:

$$Q(u,v) = \Delta_2 Q_1(u,v)(1,1), \tag{40}$$

where $Q_1 : Z \to \mathbb{R}$ is defined by

$$\begin{aligned} Q_1(u,v) &= aI_{0+}^{\beta}v(\xi) - I_{0+}^{\alpha}u(1) \\ &\qquad + a\xi^{\beta-1}\left[bI_{0+}^{\alpha}u(\eta) - I_{0+}^{\beta}v(1)\right], \\ \Delta_2 &= \frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\alpha\xi^{\beta-1}\Gamma(\alpha+1)(\xi-1) + \Gamma(\beta+1)(\eta-1)} \neq 0. \end{aligned} \tag{41}$$

It is easy to see that $Q^2(u,v) = Q(u,v)$; that is, $Q : Z \to Z$ is a continuous linear projector. Furthermore, $\operatorname{Ker} Q = \operatorname{Im} L$. For $(u,v) \in Z$, set $(u,v) = [(u,v) - Q(u,v)] + Q(u,v)$. Then $(u,v) - Q(u,v) \in \operatorname{Ker} Q$ and $Q(u,v) \in \operatorname{Im} Q$. It follows from $\operatorname{Ker} Q = \operatorname{Im} L$ and $Q^2(u,v) = Q(u,v)$ that $\operatorname{Im} L \cap \operatorname{Im} Q = (0,0)$. So we have

$$Z = \operatorname{Im} L \oplus \operatorname{Im} Q. \tag{42}$$

Now, $\operatorname{Ind} L = \dim \operatorname{Ker} L - \operatorname{codim} \operatorname{Im} L = \dim \operatorname{Ker} L - \dim \operatorname{Im} Q = 0$, and so L is a Fredholm operator of index 0.

Let $P : Y \to Y$ be continuous linear operator defined by

$$P(u,v) = \frac{D_{0+}^{\beta-1} v(0)}{\Gamma(\beta)} \left(a\xi^{\beta-1} t^{\alpha-1}, t^{\beta-1}\right). \tag{43}$$

Obviously, P is a linear projector and

$$\operatorname{Ker} P = \left\{(u,v) \in Y : D_{0+}^{\beta-1} v(0) = 0\right\}. \tag{44}$$

It is easy to know that $Y = \operatorname{Ker} P \oplus \operatorname{Ker} L$.

Define $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ by

$$\begin{aligned} &K_P(u,v)(t) \\ &\quad = \left(I_{0+}^{\alpha} u(t) - a\xi^{\beta-1} t^{\alpha-1} \left[b I_{0+}^{\alpha} u(\eta) - I_{0+}^{\beta} v(1)\right], I_{0+}^{\beta} v(t)\right). \end{aligned} \tag{45}$$

Since

$$\begin{aligned} \left|b I_{0+}^{\alpha} u(\eta) - I_{0+}^{\beta} v(1)\right| &\le \left|\frac{b}{\Gamma(\alpha)} \int_0^{\eta} (\eta - s)^{\alpha-1} u(s)\, ds\right| \\ &\quad + \left|\frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\alpha-1} v(s)\, ds\right| \\ &\le \frac{b}{\Gamma(\alpha)} \|u\|_1 + \frac{1}{\Gamma(\beta)} \|v\|_1, \\ D_{0+}^{\alpha-1} I_{0+}^{\alpha} u(t) &= \int_0^t u(s)\, ds, \\ I_{0+}^{\alpha} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \\ D_{0+}^{\beta-1} I_{0+}^{\beta} v(t) &= \int_0^t v(s)\, ds, \\ I_{0+}^{\beta} v(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s)\, ds, \end{aligned} \tag{46}$$

then

$$\|K_P(u,v)\|_Y \le \Delta_1 \|(u,v)\|_Z. \tag{47}$$

In fact, if $(u,v) \in \operatorname{Im} L$, then

$$\begin{aligned} &LK_P(u,v)(t) \\ &\quad = \left\{D_{0+}^{\alpha} \left[I_{0+}^{\alpha} u(t) - a\xi^{\beta-1} t^{\alpha-1} \left[b I_{0+}^{\alpha} u(\eta) - I_{0+}^{\beta} v(1)\right]\right],\right. \\ &\qquad \left. D_{0+}^{\beta} I_{0+}^{\beta} v(t)\right\} = (u,v). \end{aligned} \tag{48}$$

By Lemma 4, for $(u,v) \in \operatorname{dom} L \cap \operatorname{Ker} P$,

$$\begin{aligned} &K_P L(u,v)(t) \\ &\quad = \left(I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) - a\xi^{\alpha-1} t^{\alpha-1} \left[b I_{0+}^{\alpha} D_{0+}^{\alpha} u(\eta) - I_{0+}^{\beta} D_{0+}^{\beta} v(1)\right],\right. \\ &\qquad \left. I_{0+}^{\beta} D_{0+}^{\beta} v(t)\right) \\ &\quad = \left(u(t) - \frac{D_{0+}^{\alpha-1} u(0)}{\Gamma(\alpha)} t^{\alpha-1} - \frac{I_{0+}^{2-\alpha} u(0)}{\Gamma(\alpha-1)} t^{\alpha-2} - a\xi^{\beta-1} t^{\alpha-1}\right. \\ &\qquad \times \left[b u(\eta) - b \frac{D_{0+}^{\alpha-1} u(0)}{\Gamma(\alpha)} \eta^{\alpha-1} - \frac{I_{0+}^{2-\alpha} u(0)}{\Gamma(\alpha-1)} \eta^{\alpha-2}\right. \\ &\qquad \left.\left. - v(1) + \frac{D_{0+}^{\beta-1} v(0)}{\Gamma(\beta)}\right], v(t) - \frac{D_{0+}^{\beta-1} v(0)}{\Gamma(\beta)} t^{\beta-1}\right) \\ &\quad = \left(u(t) - \frac{D_{0+}^{\alpha-1} u(0)}{\Gamma(\alpha)} t^{\alpha-1}\right. \\ &\qquad \left. + a\xi^{\beta-1} t^{\alpha-1} b \frac{D_{0+}^{\alpha-1} u(0)}{\Gamma(\alpha)} \eta^{\alpha-1}, v(t)\right) \\ &\quad = (u(t), v(t)) \end{aligned} \tag{49}$$

($D_{0+}^{\beta-1} v(0) = 0$ since $(u,v) \in \operatorname{Ker} P$ and $I_{0+}^{2-\alpha} u(0) = 0$ since $(u,v) \in \operatorname{dom} L$). Hence,

$$K_p = \left(L|_{\operatorname{dom} L \cap \operatorname{Ker} P}\right)^{-1}. \tag{50}$$

The proof is complete. □

3. Main Results

In this section, we will use Theorem 1 to prove the existence of solutions to BVP (1). To obtain our main theorem, we use the following assumptions.

(H1) There exist functions $a_i, b_i, c_i, d_i, e_i \in L^1[0,1]$ $(i = 1, 2)$ such that for all $(x,y,z,w) \in \mathbb{R}^4$, $t \in [0,1]$,

$$\begin{aligned} |f(t,x,y,z,w)| &\le e_1(t) + a_1(t)|x| + b_1(t)|y| \\ &\quad + c_1(t)|z| + d_1(t)|w|, \\ |g(t,x,y,z,w)| &\le e_2(t) + a_2(t)|x| + b_2(t)|y| \\ &\quad + c_2(t)|z| + d_2(t)|w|. \end{aligned} \tag{51}$$

(H2) There exists a constant $A > 0$ such that, for $(u,v) \in \operatorname{dom} L$, if $|D_{0+}^{\beta-1} v(t)| > A$ for all $t \in [0,1]$, then $Q_1(N_1(u,v)) \neq 0$ or $Q_1(N_2(u,v)) \neq 0$.

(H3) There exists a constant $B > 0$ such that either, for each $c \in \mathbb{R} : |c| > B$,

$$cN_1\left(ca\xi^{\beta-1} t^{\alpha-1}, ct^{\beta-1}\right) > 0, \quad cN_2\left(ca\xi^{\beta-1} t^{\alpha-1}, ct^{\beta-1}\right) > 0 \tag{52}$$

or, for each $a \in \mathbb{R} : |a| > B$,

$$\begin{aligned} &cN_1\left(ca\xi^{\beta-1} t^{\alpha-1}, ct^{\beta-1}\right) < 0, \\ &cN_2\left(ca\xi^{\beta-1} t^{\alpha-1}, ct^{\beta-1}\right) < 0. \end{aligned} \tag{53}$$

Theorem 10. *Suppose* (2) *and (H1)–(H3) hold. Then* (1) *has at least one solution in Y, provided that*

$$\max\{(\Delta_1+\Delta_3)\max\{\|a_2\|_1,\|b_2\|_1,\|c_2\|_1,\|d_2\|_1\},\Delta_4,\Delta_5\} < 1. \tag{54}$$

Proof. Set

$$\Omega_1 = \{(u,v)\in \operatorname{dom} L\setminus \operatorname{Ker} L : L(u,v)=\lambda N(u,v) \text{ for some } \lambda\in[0,1]\}. \tag{55}$$

Take $(u,v)\in\Omega_1$. Since $Lu=\lambda Nu$, so $\lambda\neq 0$ and $N(u,v)\in \operatorname{Im} L=\operatorname{Ker} Q$; hence,

$$QN(u,v)=Q_1(N_1(u,v),N_2(u,v))(1,1)=0. \tag{56}$$

Thus, from (H2), there exist $t_0\in[0,1]$ such that

$$\left|D_{0+}^{\beta-1}v(t_0)\right|\le A. \tag{57}$$

Noticing that

$$D_{0+}^{\beta-1}v(t)=D_{0+}^{\beta-1}v(t_0)+\int_{t_0}^{t}D_{0+}^{\beta}v(s)\,ds \tag{58}$$

so

$$\begin{aligned}\left|D_{0+}^{\beta-1}v(0)\right| &\le \left\|D_{0+}^{\beta-1}v(t)\right\|_\infty \le \left|D_{0+}^{\beta-2}v(t_0)\right|+\left\|D_{0+}^{\beta}v(t)\right\|_1\\ &\le A+\|L_2v\|_1\le A+\|N_2(u,v)\|_1.\end{aligned} \tag{59}$$

Thus

$$\begin{aligned}\|P(u,v)\|_Y &= \left\|\frac{D_{0+}^{\beta-1}v(0)}{\Gamma(\beta)}\left(a\xi^{\beta-1}t^{\alpha-1},t^{\beta-1}\right)\right\|_Y\\ &=\Delta_3\left|D_{0+}^{\beta-1}v(0)\right|\le\Delta_3\left(A+\|N_2(u,v)\|_1\right).\end{aligned} \tag{60}$$

For all $(u,v)\in\Omega_1$, $(I-P)(u,v)\in\operatorname{dom} L\cap\operatorname{Ker} P$. Considering Lemma 9, we get $LP(u,v)=(0,0)$. Together with (39), we have

$$\begin{aligned}\|(I-P)(u,v)\|_Y &= \|K_PL(I-P)(u,v)\|_Y\\ &\le\Delta_1\|L(I-P)(u,v)\|_Z\\ &\le\Delta_1\|N(u,v)\|_Z.\end{aligned} \tag{61}$$

From (60) and (61), we have

$$\begin{aligned}&\|(u,v)\|_Y\\ &\le\|P(u,v)\|_Y+\|(I-P)(u,v)\|_Y\\ &\le\Delta_3\left(A+\|N_2(u,v)\|_1\right)+\Delta_1\|N(u,v)\|_Z\\ &=A\Delta_3+\max\{(\Delta_1+\Delta_3)\|N_2(u,v)\|_1,\Delta_1\|N_1(u,v)\|_1\\ &\qquad+\Delta_3\|N_2(u,v)\|_1\}.\end{aligned} \tag{62}$$

From (62), we discuss various cases.

Case 1 ($\|(u,v)\|_Y\le A\Delta_3+(\Delta_1+\Delta_3)\|N_2(u,v)\|_1$). From (H1), we have

$$\begin{aligned}&\|(u,v)\|_Y\\ &\le A\Delta_3+(\Delta_1+\Delta_3)\\ &\quad\times\left(\|a_2\|_1\|u\|_\infty+\|b_2\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty+\|c_2\|_1\|v\|_\infty\right.\\ &\qquad\left.+\|d_2\|_1\left\|D_{0+}^{\beta-1}v\right\|_\infty+\|e_2\|_1\right)\\ &\le A\Delta_3+(\Delta_1+\Delta_3)\|e_2\|_1\\ &\quad+(\Delta_1+\Delta_3)\max\{\|a_2\|_1,\|b_2\|_1\}\|u\|_{C^{\alpha-1}}\\ &\quad+(\Delta_1+\Delta_3)\max\{\|c_2\|_1,\|d_2\|_1\}\|v\|_{C^{\beta-1}}\\ &\le A\Delta_3+(\Delta_1+\Delta_3)\|e_2\|_1\\ &\quad+(\Delta_1+\Delta_3)\max\{\|a_2\|_1,\|b_2\|_1,\|c_2\|_1,\|d_2\|_1\}\|(u,v)\|_Y,\end{aligned} \tag{63}$$

which yield

$$\|(u,v)\|_Y\le\frac{A\Delta_2+(\Delta_1+\Delta_3)\|e_2\|_1}{1-(\Delta_1+\Delta_3)\max\{\|a_2\|_1,\|b_2\|_1,\|c_2\|_1,\|d_2\|_1\}}. \tag{64}$$

Thus, Ω_1 is bounded.

Case 2 ($\|(u,v)\|_Y\le A\Delta_3+\Delta_1\|N_1(u,v)\|_1+\Delta_3\|N_2(u,v)\|_1$). From (H1), we have

$$\begin{aligned}&\|(u,v)\|_Y\\ &\le A\Delta_3+\Delta_1\left(\|a_1\|_1\|u\|_\infty+\|b_1\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty+\|c_1\|_1\|v\|_\infty\right.\\ &\qquad\left.+\|d_1\|_1\left\|D_{0+}^{\beta-1}v\right\|_\infty+\|e_1\|_1\right)\\ &\quad+\Delta_3\left(\|a_2\|_1\|u\|_\infty+\|b_2\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty+\|c_2\|_1\|v\|_\infty\right.\\ &\qquad\left.+\|d_2\|_1\left\|D_{0+}^{\beta-1}v\right\|_\infty+\|e_2\|_1\right)\\ &\le A\Delta_3+\Delta_1\|e_1\|_1+\Delta_3\|e_2\|_1+\Delta_4\|u\|_{C^{\alpha-1}}+\Delta_5\|v\|_{C^{\beta-1}}\\ &\le A\Delta_3+\Delta_1\|e_1\|_1+\Delta_3\|e_2\|_1+\max\{\Delta_4,\Delta_5\}\|(u,v)\|_Y\end{aligned} \tag{65}$$

which yield

$$\|(u,v)\|_Y\le\frac{A\Delta_3+\Delta_1\|e_1\|_1+\Delta_3\|e_2\|_1}{1-\max\{\Delta_4,\Delta_5\}}. \tag{66}$$

Thus, Ω_1 is bounded. Let

$$\Omega_2=\{(u,v)\in\operatorname{Ker} L : N(u,v)\in\operatorname{Im} L\}. \tag{67}$$

For $(u,v)\in\Omega_2$ and $(u,v)\in\operatorname{Ker} L$, so $(u,v)=c(a\xi^{\beta-1}t^{\alpha-1}, t^{\beta-1})$, $t\in[0,1]$, $c\in\mathbb{R}$. Noticing that $\operatorname{Im} L=\operatorname{Ker} Q$, then we get $QN(u,v)=0$, and thus $Q_1N_1(ca\xi^{\beta-1}t^{\alpha-1}, ct^{\beta-1})=0$ and $Q_1N_2(ca\xi^{\beta-1}t^{\alpha-1}, ct^{\beta-1})=0$. From (H2), we get $|c| \le A/\Gamma(\beta)$, and thus Ω_2 is bounded.

We define the isomorphism $J : \operatorname{Ker} L \to \operatorname{Im} Q$ by

$$J\left(ca\xi^{\beta-1}t^{\alpha-1}, ct^{\beta-1}\right) = (c,c). \tag{68}$$

If the first part of (H3) is satisfied, then let

$$\begin{aligned}\Omega_3 = \{&(u,v)\in \ker L : \lambda J(u,v) \\ &+ (1-\lambda) QN(u,v) = 0, \lambda \in [0,1]\}.\end{aligned} \tag{69}$$

For $(u,v) = (ca\xi^{\beta-1}t^{\alpha-1}, ct^{\beta-1}) \in \Omega_3$,

$$\lambda(c,c) = -(1-\lambda)\Delta_2 Q_1 N\left(ca\xi^{\beta-1}t^{\alpha-1}, ct^{\beta-1}\right)(1,1). \tag{70}$$

If $\lambda=1$, then $c=0$. Otherwise, if $|c|>B$, in view of (H3) and $\Delta_2<0$, one has

$$(1-\lambda)\Delta_3 Q_1 N\left(ca\xi^{\beta-1}t^{\alpha-1}, ct^{\beta-1}\right) < 0 \tag{71}$$

which contradict $\lambda c^2 \ge 0$. Thus $\Omega_3 \subset \{(u,v)\in \operatorname{Ker} L \mid (u,v) = (ca\xi^{\beta-1}t^{\alpha-1}, ct^{\beta-1}), |c|\le B\}$ is bounded.

If the second part of (H3) holds, then define the set

$$\begin{aligned}\Omega_3 = \{&(u,v)\in \operatorname{Ker} L : -\lambda J(u,v) \\ &+ (1-\lambda) QN(u,v) = 0, \lambda \in [0,1]\},\end{aligned} \tag{72}$$

and here J is as above. Similar to the above argument, we can show that Ω_3 is bounded too.

In the following, we will prove that all conditions of Theorem 1 are satisfied. Let Ω be a bounded open subset of Y such that $\cup_{i=1}^{3}\overline{\Omega_i} \subset \Omega$. By standard arguments, we can prove that $K_P(I-Q)N : \Omega \to Y$ is compact, and thus N is L-compact on $\overline{\Omega}$. Then by the above argument we have

(i) $Lu \neq \lambda Nu$, for every $(u,\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L)\cap \partial\Omega] \times (0,1)$,

(ii) $Nu \notin \operatorname{Im} L$ for $u \in \operatorname{Ker} L \cap \partial\Omega$.

Finally, we will prove that (iii) of Theorem 1 is satisfied. Let $H(u,\lambda) = \pm\lambda Ju + (1-\lambda)QNu$. According to the above argument, we know that

$$H(u,\lambda) \neq 0 \text{ for } u \in \operatorname{Ker} L \cap \partial\Omega. \tag{73}$$

Thus, by the homotopy property of degree,

$$\begin{aligned}\deg&\left(QN|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0\right) \\ &= \deg\left(H(\cdot,0), \operatorname{Ker} L\cap\Omega, 0\right) \\ &= \deg\left(H(\cdot,1), \operatorname{Ker} L\cap\Omega, 0\right) \\ &= \deg\left(\pm J, \operatorname{Ker} L\cap\Omega, 0\right) \neq 0.\end{aligned} \tag{74}$$

Then by Theorem 1, $L(u,v)=N(u,v)$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$ so that BVP (1) has a solution in $C^{\alpha-1}[0,1]\times C^{\beta-1}[0,1]$. The proof is complete. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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