

Research Article

Generalized Convex Functions on Fractal Sets and Two Related Inequalities

Huixia Mo, Xin Sui, and Dongyan Yu

School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China

Correspondence should be addressed to Huixia Mo; huixmo@bupt.edu.cn

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We introduce the generalized convex function on fractal sets R^α ($0 < \alpha \leq 1$) of real line numbers and study the properties of the generalized convex function. Based on these properties, we establish the generalized Jensen's inequality and generalized Hermite-Hadamard's inequality. Furthermore, some applications are given.

1. Introduction

Let $f : I \subseteq R \rightarrow R$. For any $x_1, x_2 \in I$ and $\lambda \in [0, 1]$, if the following inequality

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \quad (1)$$

holds, then f is called a convex function on I .

The convexity of functions plays a significant role in many fields, for example, in biological system, economy, optimization, and so on [1, 2]. And many important inequalities are established for the class of convex functions. For example, Jensen's inequality and Hermite-Hadamard's inequality are the best known results in the literature, which can be stated as follows.

Jensen's Inequality [3]. Assume that f is a convex function on $[a, b]$. Then, for any $x_i \in [a, b]$ and $\lambda_i \in [0, 1]$ ($i = 1, 2, \dots, n$) with $\sum_{i=1}^n \lambda_i = 1$, we have

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i). \quad (2)$$

Hermite-Hadamard's Inequality [4]. Let f be a convex function on $[a, b]$ with $a < b$. If f is integral on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (3)$$

In recent years, the fractal theory has received significantly remarkable attention from scientists and engineers. In the sense of Mandelbrot, a fractal set is the one whose Hausdorff dimension strictly exceeds the topological dimension [5–9]. Many researchers studied the properties of functions on fractal space and constructed many kinds of fractional calculus by using different approaches (see [10–14]). Particularly, in [13], Yang stated the analysis of local fractional functions on fractal space systematically, which includes local fractional calculus and the monotonicity of function.

Inspired by these investigations, we will introduce the generalized convex function on fractal sets and establish the generalized Jensen's inequality and generalized Hermite-Hadamard's inequality related to generalized convex function. We will focus our attention on the convexity since a function f is concave if and only if $-f$ is convex. So, every result for the convex function can be easily restated in terms of concave functions.

The paper is organized as follows. In Section 2, we state the operations with real line number on fractal sets and give the definitions of the local fractional derivatives and local fractional integral. In Section 3, we introduce the definition of the generalized convex function on fractal sets and study the properties of the generalized convex functions. In Section 4, we establish the generalized Jensen's inequality and generalized Hermite-Hadamard's inequality on fractal sets. In Section 5, some applications are given on fractal sets by means of the generalized Jensen's inequality.

2. Preliminaries

Recall the set R^α of real line numbers and use Gao-Yang-Kang's idea to describe the definitions of the local fractional derivative and local fractional integral.

Recently, the theory of Yang's fractional sets [13] was introduced as follows.

For $0 < \alpha \leq 1$, we have the following α -type set of element sets.

Z^α : the α -type set of the integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, \dots, \pm n^\alpha, \dots\}$.

Q^α : the α -type set of the rational numbers is defined as the set $\{m^\alpha = (p/q)^\alpha : p \in Z, q \neq 0\}$.

J^α : the α -type set of the irrational numbers is defined as the set $\{m^\alpha \neq (p/q)^\alpha : p \in Z, q \neq 0\}$.

R^α : the α -type set of the real line numbers is defined as the set $R^\alpha = Q^\alpha \cup J^\alpha$.

If a^α, b^α , and c^α belong to the set R^α of real line numbers, then one has the following:

- (1) $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belong to the set R^α ;
- (2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a + b)^\alpha = (b + a)^\alpha$;
- (3) $a^\alpha + (b^\alpha + c^\alpha) = (a + b)^\alpha + c^\alpha$;
- (4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;
- (5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;
- (6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;
- (7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

Let us now state some definitions about the local fractional calculus on R^α .

Definition 1 (see [13]). A nondifferentiable function $f : R \rightarrow R^\alpha, x \rightarrow f(x)$ is called local fractional continuous at x_0 , if, for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|f(x) - f(x_0)| < \varepsilon^\alpha \tag{4}$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in R$. If $f(x)$ is local fractional continuous on the interval (a, b) , one denotes $f(x) \in C_\alpha(a, b)$.

Definition 2 (see [13]). The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = \left. \frac{d^\alpha f(x)}{dx^\alpha} \right|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \tag{5}$$

where $\Delta^\alpha (f(x) - f(x_0)) = \Gamma(1 + \alpha)(f(x) - f(x_0))$.

If there exists $f^{((k+1)\alpha)}(x) = \overbrace{D_x^\alpha \cdots D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then one denotes $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, \dots$

Definition 3 (see [13]). The local fractional integral of the function $f(x)$ of order α is defined by

$$\begin{aligned} {}_a I_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(t) (dt)^\alpha \\ &= \frac{1}{\Gamma(1 + \alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha, \end{aligned} \tag{6}$$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_j \mid j = 1, 2, \dots, N - 1\}$, where $[t_j, t_{j+1}]$, $j = 0, \dots, N - 1$, and $t_0 = a < t_1 < \dots < t_i < \dots < t_{N-1} < t_N = b$ is a partition of the interval $[a, b]$.

Here, it follows that ${}_a I_a^{(\alpha)} f(x) = 0$ if $a = b$ and ${}_a I_b^{(\alpha)} f(x) = -{}_b I_a^{(\alpha)} f(x)$ if $a < b$. If, for any $x \in [a, b]$, there exists ${}_a I_x^{(\alpha)} f(x)$, then it is denoted by $f(x) \in I_x^{(\alpha)}[a, b]$.

Lemma 4 (see [13] generalized local fractional Taylor theorem). Suppose that $f^{(k+1)\alpha}(x) \in C_\alpha(I)$, for interval $I \subseteq R$, $k = 0, 1, \dots, n$, $0 < \alpha \leq 1$. And let $x_0 \in [a, b]$. Then, for any $x \in I$, there exists at least one point ξ , which lies between the points x and x_0 , such that

$$\begin{aligned} f(x) &= \sum_{k=0}^n \frac{f^{(k\alpha)}(x_0)}{\Gamma(1 + k\alpha)} (x - x_0)^{k\alpha} \\ &\quad + \frac{f^{((n+1)\alpha)}(\xi)}{\Gamma(1 + (n + 1)\alpha)} (x - x_0)^{(n+1)\alpha}. \end{aligned} \tag{7}$$

Remark 5. When $I \subseteq R$ is an open interval (a, b) , Yang [13] has given the proof for the generalized local fractional Taylor theorem. In fact, using the generalized local fractional Lagrange's theorem and following the proof of the class Taylor theorem, we can show that, for any interval $I \subseteq R$, the formula is also true.

3. Generalized Convex Functions

From an analytical point of view, we have the following definition.

Definition 6. Let $f : I \subseteq R \rightarrow R^\alpha$. For any $x_1, x_2 \in I$ and $\lambda \in [0, 1]$, if the following inequality

$$f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda^\alpha f(x_1) + (1 - \lambda)^\alpha f(x_2) \tag{8}$$

holds, then f is called a generalized convex function on I .

Definition 7. Let $f : I \rightarrow R^\alpha$. For any $x_1 \neq x_2 \in I$ and $\lambda \in [0, 1]$, if the following inequality

$$f(\lambda x_1 + (1 - \lambda) x_2) < \lambda^\alpha f(x_1) + (1 - \lambda)^\alpha f(x_2) \tag{9}$$

holds, then f is called a generalized strictly convex function on $I \subseteq R$.

It follows immediately, from the given definitions, that a generalized strictly convex function is also generalized convex. But, the converse is not true. And if these two

inequalities are reversed, then f is called a generalized concave function or generalized strictly concave function, respectively.

Here are two basic examples of generalized strictly convex functions:

- (1) $f(x) = x^{\alpha p}, x \geq 0, p > 1;$
- (2) $f(x) = E_{\alpha}(x^{\alpha}), x \in R,$ where $E_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} (x^{\alpha k} / \Gamma(1 + k\alpha))$ is the Mittag-Leffler function.

Note that the linear function $f(x) = a^{\alpha}x^{\alpha} + b^{\alpha}, x \in R$ is generalized convex and also generalized concave.

We will focus our attention on the convexity since a function f is concave if and only if $-f$ is convex. So, every result for the convex function can be easily restated in terms of concave functions.

In the following, we will study the properties of the generalized convex functions.

Theorem 8. *Let $f : I \rightarrow R^{\alpha}$. Then f is a generalized convex function if and only if the inequality*

$$\frac{f(x_1) - f(x_2)}{(x_1 - x_2)^{\alpha}} \leq \frac{f(x_3) - f(x_2)}{(x_3 - x_2)^{\alpha}} \tag{10}$$

holds, for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$.

Proof. In fact, taking $\lambda = (x_3 - x_2)/(x_3 - x_1)$, then $x_2 = \lambda x_1 + (1 - \lambda)x_3$. And by the generalized convexity of f , we get

$$\begin{aligned} f(x_2) &= f(\lambda x_1 + (1 - \lambda)x_3) \leq \lambda^{\alpha} f(x_1) + (1 - \lambda)^{\alpha} f(x_3) \\ &= \left(\frac{x_3 - x_2}{x_3 - x_1}\right)^{\alpha} f(x_1) + \left(\frac{x_2 - x_1}{x_3 - x_1}\right)^{\alpha} f(x_3). \end{aligned} \tag{11}$$

From the above formula, it is easy to see that

$$\frac{f(x_1) - f(x_2)}{(x_1 - x_2)^{\alpha}} \leq \frac{f(x_3) - f(x_2)}{(x_3 - x_2)^{\alpha}}. \tag{12}$$

Reversely, for any two points x_1, x_3 ($x_1 < x_3$) on $I \subseteq R$, we take $x_2 = \lambda x_1 + (1 - \lambda)x_3$ for $\lambda \in (0, 1)$. Then $x_1 < x_2 < x_3$ and $\lambda = (x_3 - x_2)/(x_3 - x_1)$. Using the above inverse process, we have

$$f(\lambda x_1 + (1 - \lambda)x_3) \leq \lambda^{\alpha} f(x_1) + (1 - \lambda)^{\alpha} f(x_3). \tag{13}$$

So, f is a convex function on $I \subseteq R$.

In the same way, it can be shown that f is a generalized convex function on $I \subseteq R$ if and only if

$$\frac{f(x_2) - f(x_1)}{(x_2 - x_1)^{\alpha}} \leq \frac{f(x_3) - f(x_1)}{(x_3 - x_1)^{\alpha}} \leq \frac{f(x_3) - f(x_2)}{(x_3 - x_2)^{\alpha}}, \tag{14}$$

for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$. □

Theorem 9. *Letting $f \in D_{\alpha}(I)$, then the following conditions are equivalent:*

- (1) f is a generalized convex function on I ,
- (2) $f^{(\alpha)}$ is an increasing function on I ,
- (3) for any $x_1, x_2 \in I$,

$$f(x_2) \geq f(x_1) + \frac{f^{(\alpha)}(x_1)}{\Gamma(1 + \alpha)}(x_2 - x_1)^{\alpha}. \tag{15}$$

Proof. (1 \rightarrow 2) Let $x_1, x_2 \in I$ with $x_1 < x_2$. And take $h > 0$ which is small enough such that $x_1 - h, x_2 + h \in I$. Since $x_1 - h < x_1 < x_2 < x_2 + h$, then using Theorem 8 we have

$$\begin{aligned} \Gamma(1 + a) \frac{f(x_1) - f(x_1 - h)}{h^{\alpha}} &\leq \Gamma(1 + a) \frac{f(x_2) - f(x_1)}{(x_2 - x_1)^{\alpha}} \\ &\leq \Gamma(1 + a) \frac{f(x_2 + h) - f(x_2)}{h^{\alpha}}. \end{aligned} \tag{16}$$

Since $f \in D_{\alpha}(I)$, then letting $h \rightarrow 0^+$, it follows that

$$f^{(\alpha)}(x_1) \leq \Gamma(1 + a) \frac{f(x_2) - f(x_1)}{(x_2 - x_1)^{\alpha}} \leq f^{(\alpha)}(x_2). \tag{17}$$

So, $f^{(\alpha)}$ is increasing in I .

(2 \rightarrow 3) Take $x_1, x_2 \in I$. Without loss of generality, we can assume that $x_1 < x_2$. Since $f^{(\alpha)}$ is increasing in the interval I , then applying the generalized local fractional Taylor theorem, we have

$$\begin{aligned} f(x_2) - f(x_1) &= \frac{f^{(\alpha)}(\xi)}{\Gamma(1 + a)}(x_2 - x_1)^{\alpha} \\ &\geq \frac{f^{(\alpha)}(x_1)}{\Gamma(1 + a)}(x_2 - x_1)^{\alpha}, \end{aligned} \tag{18}$$

where $\xi \in (x_1, x_2)$. That is to say,

$$f(x_2) \geq f(x_1) + \frac{f^{(\alpha)}(x_1)}{\Gamma(1 + a)}(x_2 - x_1)^{\alpha}. \tag{19}$$

(3 \rightarrow 1) For any $x_1, x_2 \in I$, we let $x_3 = \lambda x_1 + (1 - \lambda)x_2$, where $0 < \lambda < 1$. It is easy to see that $x_1 - x_3 = (1 - \lambda)(x_1 - x_2)$ and $x_2 - x_3 = \lambda(x_2 - x_1)$. Then from the third condition, we have

$$\begin{aligned} f(x_1) &\geq f(x_3) + \frac{f^{(\alpha)}(x_3)}{\Gamma(1 + a)}(x_1 - x_3)^{\alpha} \\ &= f(x_3) + (1 - \lambda)^{\alpha} \frac{f^{(\alpha)}(x_3)}{\Gamma(1 + a)}(x_1 - x_2)^{\alpha}, \end{aligned} \tag{20}$$

$$\begin{aligned} f(x_2) &\geq f(x_3) + \frac{f^{(\alpha)}(x_3)}{\Gamma(1 + a)}(x_2 - x_3)^{\alpha} \\ &= f(x_3) + \lambda^{\alpha} \frac{f^{(\alpha)}(x_3)}{\Gamma(1 + a)}(x_2 - x_1)^{\alpha}. \end{aligned}$$

At the above two formulas, multiply λ^α and $(1 - \lambda)^\alpha$, respectively; then we obtain

$$\begin{aligned} &\lambda^\alpha f(x_1) + (1 - \lambda)^\alpha f(x_2) \\ &\geq f(x_3) = f(\lambda x_1 + (1 - \lambda)x_2). \end{aligned} \tag{21}$$

So f is a generalized convex function on I . □

Corollary 10. Let $f \in D_{2\alpha}(a, b)$. Then f is a generalized convex function (or a generalized concave function) if and only if

$$f^{(2\alpha)}(x) \geq 0 \quad (\text{or } f^{(2\alpha)}(x) \leq 0), \tag{22}$$

for any $x \in (a, b)$.

4. Some Inequalities

Theorem 11 (generalized Jensen’s inequality). Assume that f is a generalized convex function on $[a, b]$. Then for any $x_i \in [a, b]$ and $\lambda_i \in [0, 1]$ ($i = 1, 2, \dots, n$) with $\sum_{i=1}^n \lambda_i = 1$, we have

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i^\alpha f(x_i). \tag{23}$$

Proof. When $n = 2$, the inequality is obviously true. Assume that for $n = k$ the inequality is also true. Then, for any $x_1, x_2, \dots, x_k \in [a, b]$ and $\gamma_i > 0$, $i = 1, 2, \dots, k$, with $\sum_{i=1}^k \gamma_i = 1$, we have

$$f\left(\sum_{i=1}^k \gamma_i x_i\right) \leq \sum_{i=1}^k \gamma_i^\alpha f(x_i). \tag{24}$$

If $x_1, x_2, \dots, x_k, x_{(k+1)} \in [a, b]$ and $\lambda_i > 0$ for $i = 1, 2, \dots, k + 1$ with $\sum_{i=1}^{k+1} \lambda_i = 1$, then one sets up $\gamma_i = \lambda_i / (1 - \lambda_{k+1})$, $i = 1, 2, \dots, k$. It is easy to see $\sum_{i=1}^k \gamma_i = 1$.

Thus,

$$\begin{aligned} &f(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k + \lambda_{k+1} x_{k+1}) \\ &= f\left(\left(1 - \lambda_{k+1}\right) \frac{\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k}{1 - \lambda_{k+1}} + \lambda_{k+1} x_{k+1}\right) \\ &\leq (1 - \lambda_{k+1})^\alpha f(\gamma_1 x_1 + \gamma_2 x_2 + \dots + \gamma_k x_k) + \lambda_{k+1}^\alpha f(x_{k+1}) \\ &\leq (1 - \lambda_{k+1})^\alpha [\gamma_1^\alpha f(x_1) + \gamma_2^\alpha f(x_2) + \dots + \gamma_k^\alpha f(x_k)] \\ &\quad + \lambda_{k+1}^\alpha f(x_{k+1}) \\ &= (1 - \lambda_{k+1})^\alpha \left[\left(\frac{\lambda_1}{1 - \lambda_{k+1}}\right)^\alpha f(x_1) + \left(\frac{\lambda_2}{1 - \lambda_{k+1}}\right)^\alpha f(x_2) \right. \\ &\quad \left. + \dots + \left(\frac{\lambda_k}{1 - \lambda_{k+1}}\right)^\alpha f(x_k) \right] + \lambda_{k+1}^\alpha f(x_{k+1}) \\ &= \sum_{i=1}^k \lambda_i^\alpha f(x_i). \end{aligned} \tag{25}$$

So, the mathematical induction gives the proof of Theorem 11. □

Corollary 12. Let $f \in D_{2\alpha}[a, b]$ and $f^{(2\alpha)}(x) \geq 0$ for any $x \in [a, b]$. Then for any $x_i \in [a, b]$ and $\lambda_i \in [0, 1]$ ($i = 1, 2, \dots, n$) with $\sum_{i=1}^n \lambda_i = 1$ we have

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i^\alpha f(x_i). \tag{26}$$

Using the generalized Jensen’s inequality and the convexity of functions, we can also get some integral inequalities.

In [13], Yang established the generalized Cauchy-Schwarz’s inequality by the estimate $a^{\alpha/p} b^{\alpha/q} \leq (a^\alpha/p)(b^\alpha/q)$, where $a^\alpha, b^\alpha > 0$, $p, q \geq 1$, and $(1/p) + (1/q) = 1$.

Now, via the generalized Jensen’s inequality, we will give another proof for the generalized Cauchy-Schwarz’s inequality.

Corollary 13 (generalized Cauchy-Schwarz’s inequality). Let $|a_k| > 0$, $|b_k| > 0$, $k = 1, 2, \dots, n$. Then we have

$$\sum_{k=1}^n |a_k|^\alpha |b_k|^\alpha \leq \left(\sum_{k=1}^n |a_k|^{2\alpha}\right)^{1/2} \left(\sum_{k=1}^n |b_k|^{2\alpha}\right)^{1/2}. \tag{27}$$

Proof. Take $f(x) = x^{2\alpha}$. It is easy to see that $f^{(2\alpha)}(x) \geq 0$ for any $x \in (a, b)$.

Take

$$\lambda_k = \frac{|b_k|^2}{\sum_{k=1}^n |b_k|^2}, \quad x_k = \frac{|a_k|}{|b_k|}. \tag{28}$$

Then $0 \leq \lambda_k \leq 1$ ($k = 1, 2, \dots, n$) with $\sum_{k=1}^n \lambda_k = 1$.

Thus, by Jensen’s inequality $f(\sum_{k=1}^n \lambda_k x_k) \leq \sum_{k=1}^n \lambda_k^\alpha f(x_k)$, we have

$$\left[\sum_{k=1}^n \frac{|b_k|^2 |a_k|}{\sum_{k=1}^n |b_k|^2 |b_k|}\right]^{2\alpha} \leq \sum_{k=1}^n \left[\frac{|b_k|^2}{\sum_{k=1}^n |b_k|^2}\right]^\alpha \left[\frac{|a_k|}{|b_k|}\right]^{2\alpha}. \tag{29}$$

The above formula can be reduced to

$$\left[\sum_{k=1}^n \frac{|b_k| |a_k|}{\sum_{k=1}^n |b_k|^2}\right]^{2\alpha} \leq \sum_{k=1}^n \left[\frac{|a_k|^{2\alpha}}{(\sum_{k=1}^n |b_k|^2)^\alpha}\right], \tag{30}$$

which implies that

$$\left[\sum_{k=1}^n |b_k| |a_k|\right]^{2\alpha} \leq \sum_{k=1}^n |a_k|^{2\alpha} \sum_{k=1}^n |b_k|^{2\alpha}. \tag{31}$$

Thus we have

$$\sum_{k=1}^n |a_k|^\alpha |b_k|^\alpha \leq \left(\sum_{k=1}^n |a_k|^{2\alpha}\right)^{1/2} \left(\sum_{k=1}^n |b_k|^{2\alpha}\right)^{1/2}. \tag{32}$$

□

Theorem 14 (generalized Hermite-Hadamard's inequality). Let $f(x) \in I_x^{(\alpha)}[a, b]$ be a generalized convex function on $[a, b]$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^{(\alpha)} f(x) \leq \frac{f(a)+f(b)}{2^\alpha}. \quad (33)$$

Proof. Let $x = a + b - y$. Then

$$\int_a^{(a+b)/2} f(x) (dx)^\alpha = \int_{(a+b)/2}^b f(a+b-y) (dy)^\alpha. \quad (34)$$

Furthermore, when $x \in [(a+b)/2, b]$, $a+b-x \in [a, (a+b)/2]$. And by the convexity of f , we have

$$f(a+b-x) + f(x) \geq 2^\alpha f\left(\frac{a+b}{2}\right). \quad (35)$$

Thus

$$\begin{aligned} & \int_a^b f(x) (dx)^\alpha \\ &= \int_a^{(a+b)/2} f(x) (dx)^\alpha + \int_{(a+b)/2}^b f(x) (dx)^\alpha \\ &= \int_{(a+b)/2}^b [f(a+b-x) + f(x)] (dx)^\alpha \\ &\geq \int_{(a+b)/2}^b 2^\alpha f\left(\frac{a+b}{2}\right) (dx)^\alpha \\ &= (b-a)^\alpha f\left(\frac{a+b}{2}\right). \end{aligned} \quad (36)$$

For another part, we first note that if f is a generalized convex function, then, for $t \in [0, 1]$, it yields

$$\begin{aligned} f(ta + (1-t)b) &\leq t^\alpha f(a) + (1-t)^\alpha f(b), \\ f((1-t)a + tb) &\leq (1-t)^\alpha f(a) + t^\alpha f(b). \end{aligned} \quad (37)$$

By adding these inequalities we have

$$\begin{aligned} & f(ta + (1-t)b) + f((1-t)a + tb) \\ &\leq t^\alpha f(a) + (1-t)^\alpha f(b) + (1-t)^\alpha f(a) + t^\alpha f(b) \\ &= f(a) + f(b). \end{aligned} \quad (38)$$

Then, integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^1 [f(ta + (1-t)b) + f((1-t)a + tb)] (dt)^\alpha \\ &\leq \frac{1}{\Gamma(1+\alpha)} \int_0^1 (f(a) + f(b)) (dt)^\alpha. \end{aligned} \quad (39)$$

It is easy to see that

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^1 [f(ta + (1-t)b) + f((1-t)a + tb)] (dt)^\alpha \\ &= \frac{2^\alpha}{(b-a)^\alpha} {}_a I_b^{(\alpha)} f(x), \\ & \frac{1}{\Gamma(1+\alpha)} \int_0^1 (f(a) + f(b)) (dt)^\alpha = \frac{f(a) + f(b)}{\Gamma(1+\alpha)}. \end{aligned} \quad (40)$$

So,

$$\frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^{(\alpha)} f(x) \leq \frac{f(a) + f(b)}{2^\alpha}. \quad (41)$$

Combining the inequalities (36) and (41), we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^{(\alpha)} f(x) \leq \frac{f(a) + f(b)}{2^\alpha}. \quad (42)$$

Note that it will be reduced to the class Hermite-Hadamard inequality if $\alpha = 1$. \square

5. Applications of Generalized Jensen's Inequality

Using the generalized Jensen's inequality, we can get some inequalities.

Example 15. Let $a > 0, b > 0$ and $a^{3\alpha} + b^{3\alpha} \leq 2^\alpha$. Then $a + b \leq 2$.

Proof. Let $f(x) = x^{3\alpha}, x \in (0, +\infty)$. It is easy to see that f is a generalized convex function.

So,

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a) + f(b)}{2^\alpha}. \quad (43)$$

That is,

$$\frac{(a+b)^{3\alpha}}{8^\alpha} \leq \frac{a^{3\alpha} + b^{3\alpha}}{2^\alpha} \leq 1^\alpha. \quad (44)$$

Thus, we conclude that $a + b \leq 2$. \square

Example 16. Let $x, y \in R$. Then

$$E_\alpha\left(\left(\frac{x+y}{2}\right)^\alpha\right) \leq \frac{1}{2^\alpha} (E_\alpha(x^\alpha) + E_\alpha(y^\alpha)), \quad (45)$$

where $E_\alpha(x^\alpha) = \sum_{k=0}^\infty (x^{\alpha k} / \Gamma(1 + k\alpha))$ is the Mittag-Leffler function.

Proof. Take $f(x) = E_\alpha(x^\alpha)$. It is easy to see $(E_\alpha(x^\alpha))^{(2\alpha)} = E_\alpha(x^\alpha) > 0$. So, the generalized Jensen's inequality gives

$$E_\alpha\left(\left(\frac{x+y}{2}\right)^\alpha\right) \leq \frac{1}{2^\alpha} (E_\alpha(x^\alpha) + E_\alpha(y^\alpha)). \quad (46)$$

\square

Example 17 (power mean inequality). Let $a_1, a_2, \dots, a_n > 0$ and $0 < s < t$ or $s < t < 0$. Denote

$$S_r = \left(\frac{a_1^{\alpha r} + a_2^{\alpha r} + \dots + a_n^{\alpha r}}{n^\alpha} \right)^{1/r}, \quad r \in \mathbb{R}. \quad (47)$$

Then $S_s \leq S_t$. And $S_s = S_t$ if and only if $a_1 = a_2 = \dots = a_n$.

Proof. Consider the following.

Case I ($0 < s < t$). Take $f(x) = x^{(t/s)\alpha}$, $x > 0$. Then

$$f^{(2\alpha)}(x) = \frac{\Gamma(1 + t\alpha/s)}{\Gamma(1 + (t/s - 1)\alpha)} x^{(t/s-2)\alpha} > 0. \quad (48)$$

By the generalized Jensen's inequality, we have

$$f\left(\frac{a_1^s + a_2^s + \dots + a_n^s}{n}\right) \leq \frac{f(a_1^s) + f(a_2^s) + \dots + f(a_n^s)}{n^\alpha}. \quad (49)$$

That is,

$$\begin{aligned} & \left(\frac{a_1^s + a_2^s + \dots + a_n^s}{n}\right)^{(t/s)\alpha} \\ & \leq \frac{(a_1^s)^{(t/s)\alpha} + (a_2^s)^{(t/s)\alpha} + \dots + (a_n^s)^{(t/s)\alpha}}{n^\alpha}. \end{aligned} \quad (50)$$

From the above formula, it is easy to see

$$\left(\frac{a_1^{\alpha s} + a_2^{\alpha s} + \dots + a_n^{\alpha s}}{n^\alpha}\right)^{1/s} \leq \left(\frac{a_1^{\alpha t} + a_2^{\alpha t} + \dots + a_n^{\alpha t}}{n^\alpha}\right)^{1/t}. \quad (51)$$

So, we have $S_s \leq S_t$.

Case II ($s < t < 0$). Letting $b_i = 1/a_i$ and applying the case for $0 < -t < -s$, we can get the conclusion. \square

Example 18. If $a, b, c > 0$ and $a + b + c = 1$, then find the minimum of

$$\left(a + \frac{1}{a}\right)^{10\alpha} + \left(b + \frac{1}{b}\right)^{10\alpha} + \left(c + \frac{1}{c}\right)^{10\alpha}. \quad (52)$$

Solution. Note that $0 < a, b, c < 1$. Let $f(x) = (x + 1/x)^{10\alpha}$, $x \in (0, 1)$. Then, via the formula

$$\frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k - 1)\alpha)} x^{(k-1)\alpha}, \quad (53)$$

we have

$$\begin{aligned} f^{(2\alpha)}(x) &= \frac{\Gamma(1 + 10\alpha)}{\Gamma(1 + 8\alpha)} \left(x + \frac{1}{x}\right)^{8\alpha} \left(1 - \frac{1}{x^2}\right)^{2\alpha} \\ &+ \frac{\Gamma(1 + \alpha)\Gamma(1 + 10\alpha)}{\Gamma(1 + 9\alpha)} \left(x + \frac{1}{x}\right)^{9\alpha} \left(\frac{2}{x^3}\right)^\alpha > 0. \end{aligned} \quad (54)$$

By the generalized Jensen's inequality,

$$\begin{aligned} \left(\frac{10}{3}\right)^{10\alpha} &= f\left(\frac{a + b + c}{3}\right) \\ &\leq \frac{1}{3^\alpha} [f(a) + f(b) + f(c)] \\ &= \frac{1}{3^\alpha} \left[\left(a + \frac{1}{a}\right)^{10\alpha} + \left(b + \frac{1}{b}\right)^{10\alpha} + \left(c + \frac{1}{c}\right)^{10\alpha}\right]. \end{aligned} \quad (55)$$

So, the minimum is $10^{10\alpha}/3^{9\alpha}$, when $a = b = c = 1/3$.

Example 19. If $a, b, c, d > 0$ and $c^{2\alpha} + d^{2\alpha} = (a^{2\alpha} + b^{2\alpha})^3$, then

$$\frac{a^{3\alpha}}{c^\alpha} + \frac{b^{3\alpha}}{d^\alpha} \geq 1^\alpha. \quad (56)$$

Proof. Let $x_1 = (a^3/c)^{1/2}$, $x_2 = (b^3/d)^{1/2}$, $y_1 = (ac)^{1/2}$, $y_2 = (bd)^{1/2}$. By the generalized Cauchy-Schwartz inequality, we have

$$\begin{aligned} & \left(\frac{a^{3\alpha}}{c^\alpha} + \frac{b^{3\alpha}}{d^\alpha}\right) (a^\alpha c^\alpha + b^\alpha d^\alpha) \\ &= (x_1^{2\alpha} + x_2^{2\alpha})(y_1^{2\alpha} + y_2^{2\alpha}) \\ &\geq (x_1^\alpha y_1^\alpha + x_2^\alpha y_2^\alpha)^2 \\ &= (a^{2\alpha} + b^{2\alpha})^2 \\ &= (a^{2\alpha} + b^{2\alpha})^{1/2} (c^{2\alpha} + d^{2\alpha})^{1/2} \\ &\geq a^\alpha c^\alpha + b^\alpha d^\alpha. \end{aligned} \quad (57)$$

Canceling $a^\alpha c^\alpha + b^\alpha d^\alpha$ on both sides, we get the desired result. \square

6. Conclusion

In the paper, we introduce the definition of generalized convex function on fractal sets. Based on the definition, we study the properties of the generalized convex functions and establish two important inequalities: the generalized Jensen's inequality and generalized Hermite-Hadamard's inequality. At last, we also give some applications for these inequalities on fractal sets.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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