

Research Article

Analytic Solution for the *RL* Electric Circuit Model in Fractional Order

P. V. Shah,^{1,2} A. D. Patel,³ I. A. Salehbhai,⁴ and A. K. Shukla¹

¹ Department of Applied Mathematics and Humanities, S.V. National Institute of Technology, Surat, Gujarat 395007, India

² C.K. Pithawalla College of Engineering and Technology, Surat 395007, India

³ S.R. Rotary Institute of Chemical Technology, Vataria, Bharuch 393002, India

⁴ Department of Mathematics, V. N. South Gujarat University, Surat 395007, India

Correspondence should be addressed to A. K. Shukla; ajayshukla2@rediffmail.com

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This paper provides an analytic solution of *RL* electrical circuit described by a fractional differential equation of the order $0 < \alpha \leq 1$. We use the Laplace transform of the fractional derivative in the Caputo sense. Some special cases for the different source terms have also been discussed.

1. Introduction

Fractional calculus, involving derivatives of integrals of non-integer order, is the natural generalization of the classical calculus, which during recent years became a powerful and widely used tool for better modeling and control of processes in many areas of science and engineering [1–3]. Many physical phenomena have been discussed by fractional calculus approach [4]. In many applications, fractional calculus provides more accurate models of the physical systems than ordinary calculus does. Fundamental physical considerations in favor of the use of models based on derivatives of non-integer order are given in [5]. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [6]. This is the main advantage of fractional calculus in comparison with the classical integer-order models, in which such effects are in fact neglected. Fractional order models have been already used for modeling of electrical circuits (such as domino ladders and tree structures) and elements (coils, memristor, etc.). The review of such models can be found in [7].

Recently, a fractional differential equation has been suggested that combines the simple harmonic oscillations of an LC circuit with the discharging of an RC circuit. The behavior of this new hybrid circuit without sources has been

analyzed [8]. In the work of [9], the simple current source-wire circuit has been studied fractionally using direct and alternating current source. It was shown that the wire acquires an inducting behavior as the current is initiated in it and gradually recovers its resisting behavior. Recently, Guia et al. [10] have analyzed time delay, rise time, and settling time of an RC circuit. In this paper, in the framework of fractional calculus, we are interested in the solution of an *RL* circuit for different source terms.

2. Preliminary

The function $F(s)$ of the complex variable s defined by

$$F(s) = \mathcal{L}\{f(t); s\} = \int_0^{\infty} e^{-st} f(t) dt, \quad \operatorname{Re}(s) > \alpha, \quad (1)$$

is called the Laplace transform of the function $f(t)$ [3]. For the existence of the integral (1), the function $f(t)$ must be piecewise continuous and of exponential order α .

The original $f(t)$ can be restored from the Laplace transform $F(s)$ with the help of the inverse Laplace transform [3]

$$f(t) = \mathcal{L}^{-1}\{F(s); t\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds, \quad (2)$$

$$c = \operatorname{Re}(s) > c_0,$$

where c_0 lies in the right half plane of the absolute convergence of the Laplace integral (1). In this work, we use the Caputo definition of the fractional derivative [3]

$$\frac{d^\alpha f(t)}{dt^\alpha} = {}^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (3)$$

where $\alpha \in \mathbb{R}$ is the order of the fractional derivative and $n-1 < \alpha \leq n \in \mathbb{N} = \{1, 2, 3, \dots\}$, $f^n(\tau) = (d^n/dt^n)f(\tau)$, and $\Gamma(\cdot)$ is the Euler Gamma function. We consider the case $n = 1$; then $0 < \alpha \leq 1$; that is, in the integrand (3), there is only first derivative. The Caputo definition of the fractional derivative is very useful in the time domain studies, because the initial conditions for the fractional order differential equations with the Caputo derivatives can be given in the same manner as for the ordinary differential equations with a known physical interpretation.

The formula for the Laplace transform of the Caputo fractional derivative (3) has the form [3]

$$\mathcal{L} \left[\frac{d^\alpha f(t)}{dt^\alpha} \right] = s^\alpha F(s) - \sum_{j=0}^{n-1} s^{\alpha-j-1} f^j(0), \quad (4)$$

where f^k is the ordinary derivative. The inverse Laplace transform requires the introduction of the Mittag-Leffler function [3], which is defined as

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)} \quad (\alpha > 0), \quad (5)$$

where $\Gamma(\cdot)$ is the Gamma function. When $\alpha = 1$, from (5), we have

$$E_1(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t. \quad (6)$$

Therefore, the Mittag-Leffler function includes the exponential function as a special case.

3. Formulation of Fractional Differential Equation Models for Flow of Electricity in Resistance-Inductance Circuit

The differential equation for the RL circuit shown in Figure 1 is given by

$$L \frac{dI}{dt} + RI = E(t), \quad (7)$$

where I is the current and L is the inductance.

The solution of RL circuit is reported by Kreyszig [11] as

$$I(t) = e^{(-R/L)t} \left[\int e^{(R/L)t} \frac{E}{L} dt + c \right]. \quad (8)$$

In this paper, we develop the resistance-inductance circuit model in the form of fractional differential equation as

$$D_t^\alpha I(t) + \frac{R}{L} I(t) = \frac{E(t)}{L} \quad \text{where } D_t^\alpha I(t) = \frac{d^\alpha I}{dt^\alpha}, \quad 0 < \alpha \leq 1. \quad (9)$$

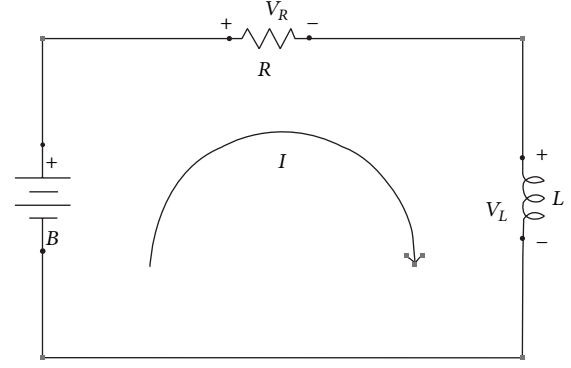


FIGURE 1: RL circuit.

If $\lim_{\alpha \rightarrow 1} (d^\alpha I/dt^\alpha) = dI/dt$, then (9) reduces to its original form $L(dI/dt) + RI = E(t)$.

Since $dI/dt = \lim_{\delta t \rightarrow 0} (\delta I/\delta t)$, it means that $\delta t \rightarrow 0$ is very, very small.

Throughout the paper, we consider $\lim_{\alpha \rightarrow 0} (d^\alpha I/dt^\alpha) = I$ and $\lim_{\alpha \rightarrow 1} (d^\alpha I/dt^\alpha) = dI/dt$. Therefore, we are not interested to consider the dimensionless variables.

Solution of (9) Using the Initial Condition ($I(0) = c, (c > 0)$). Applying the Laplace transform on (9) using the initial condition $I(0) = c, (c > 0)$, we have

$$I(s) = c \left(\frac{s^{\alpha-1}}{s^\alpha + R/L} \right) + \frac{1}{L} \left(\frac{E(s)}{s^\alpha + R/L} \right). \quad (10)$$

Now, applying the inverse Laplace transform and convolution, we have

$$I(t) = c E_\alpha \left(-\frac{R}{L} t^\alpha \right) - \frac{1}{R} \int_0^t E(t-u) E'_\alpha \left(-\frac{R}{L} u^\alpha \right) du. \quad (11)$$

Here, we obtain the solution of different cases of the resistance-inductance circuit model (9) for different source terms.

Case 1 (when no electromotive force is applied (no source term), i.e., $E(t) = 0$). In this case, (9) becomes

$$D_t^\alpha I(t) + \frac{R}{L} I(t) = 0 \quad (12a)$$

and the initial condition is

$$I(0) = c, \quad (c > 0) \quad \left(\text{Since, } \lim_{\alpha \rightarrow 0} \frac{d^\alpha I}{dt^\alpha} = I \right). \quad (12b)$$

Solution. Rewrite (12a) as

$$D_t^\alpha I(t) = -\frac{R}{L} I(t) = 0. \quad (13)$$

Taking the Laplace transform on both sides and using (12b), we get

$$I(s) = c \frac{s^{\alpha-1}}{(s^\alpha + R/L)}. \quad (14)$$

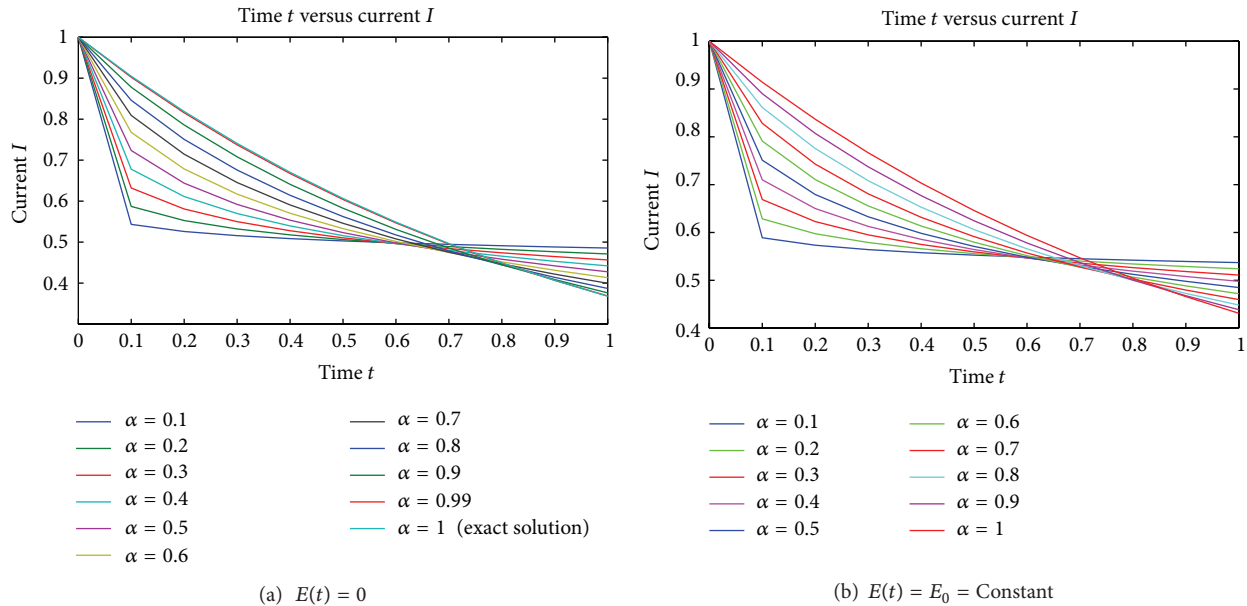


FIGURE 2: Time t v/s current I .

Taking the inverse Laplace transform on both sides, we obtain

$$I(t) = cE_\alpha\left(-\frac{R}{L}t^\alpha\right). \quad (15)$$

Case 2 (when constant electromotive force is applied, i.e., $E(t) = E_0$). In this case, (9) becomes

$$D_t^\alpha I(t) + \frac{R}{L}I(t) = \frac{E_0}{L} \quad (16a)$$

and the initial condition is

$$I(0) = c, \quad (c > 0) \quad \left(\text{Since, } \lim_{\alpha \rightarrow 0} \frac{d^\alpha I}{dt^\alpha} = I\right). \quad (16b)$$

Rewrite (16a) as

$$D_t^\alpha I(t) = \frac{E_0}{L} - \frac{R}{L}I(t). \quad (17)$$

Taking the Laplace transform on both sides and using (16b), we get

$$I(s) = \frac{E_0}{L} \frac{1}{s(s^\alpha + R/L)} + c \frac{s^{\alpha-1}}{(s^\alpha + R/L)}. \quad (18)$$

Taking the inverse Laplace transform on both sides, we have

$$I(t) = \frac{E_0}{L} \mathcal{L}^{-1} \left\{ \frac{1}{s(s^\alpha + R/L)} \right\} + c \mathcal{L}^{-1} \left\{ \frac{s^{\alpha-1}}{(s^\alpha + R/L)} \right\} \quad (19)$$

on applying the convolution theorem and using $(\lambda/\Gamma(\alpha)) \int_0^x (x-t)^{\alpha-1} E_\alpha(\lambda t^\alpha) dt = E_\alpha(\lambda x^\alpha) - 1, \alpha > 0$ [12, page 109], we get

$$\mathcal{L}^{-1} \left\{ \frac{1}{s(s^\alpha + R/L)} \right\} = -\frac{L}{R} \left[E_\alpha\left(-\frac{R}{L}t^\alpha\right) - 1 \right] \quad (20)$$

and from (15), we have

$$\mathcal{L}^{-1} \left\{ \frac{s^{\alpha-1}}{(s^\alpha + R/L)} \right\} = E_\alpha\left(-\frac{R}{L}t^\alpha\right). \quad (21)$$

Using (19), (20), and (21), we get

$$I(t) = \left(c - \frac{E_0}{R}\right) E_\alpha\left(-\frac{R}{L}t^\alpha\right) + \frac{E_0}{R}, \quad \text{where } \left(c - \frac{E_0}{R}\right) \geq 0. \quad (22)$$

In Figure 2, we observe the interesting behavior of current by using fractional calculus approach for different values of α . When $\alpha = 0.1$, the current decreases very sharply and it moves towards stability as time increases. On increasing the value of α , the current increases for the specific time and afterwards attains its stability. Finally, when $\alpha = 1$, then current shows its natural behavior. This exhibits the behavior of current for different values of α with respect to time (t) before it attends the natural behavior.

Case 3 (when electromotive force is in terms of unit step function, i.e., $E(t) = u(t)$). In this case, (9) becomes

$$D_t^\alpha I(t) + \frac{R}{L}I(t) = \frac{u(t)}{L}, \quad (23a)$$

where $u(t)$ is a unit step function.

The initial condition is

$$I(0) = c, \quad (c > 0) \quad \left(\text{Since, } \lim_{\alpha \rightarrow 0} \frac{d^\alpha I}{dt^\alpha} = I\right). \quad (23b)$$

Solution. Rewrite (23a) as

$$D_t^\alpha I(t) = \frac{1}{L}u(t) - \frac{R}{L}I(t). \quad (24)$$

Taking the Laplace transform on both sides and using (23b), we get

$$I(s) = \frac{1}{L} \frac{1}{s(s^\alpha + R/L)} + c \frac{s^{\alpha-1}}{(s^\alpha + R/L)}. \quad (25)$$

Taking the inverse Laplace transform, we obtain

$$I(t) = \frac{1}{L} \mathcal{L}^{-1} \left\{ \frac{1}{s(s^\alpha + R/L)} \right\} + c \mathcal{L}^{-1} \left\{ \frac{s^{\alpha-1}}{(s^\alpha + R/L)} \right\}. \quad (26)$$

on applying the convolution theorem and using the result $(\lambda/\Gamma(\alpha)) \int_0^x (x-t)^{\alpha-1} E_\alpha(\lambda t^\alpha) dt = E_\alpha(\lambda x^\alpha) - 1$, $\alpha > 0$ [12, page 109], we get

$$\mathcal{L}^{-1} \left\{ \frac{1}{s(s^\alpha + R/L)} \right\} = -\frac{L}{R} \left[E_\alpha \left(-\frac{R}{L} t^\alpha \right) - 1 \right] \quad (27)$$

and using (15), (26), and (27), we obtain

$$I(t) = \left(c - \frac{1}{R} \right) E_\alpha \left(-\frac{R}{L} t^\alpha \right) + \frac{1}{R}, \quad \text{where } \left(c - \frac{1}{R} \right) \geq 0. \quad (28)$$

Case 4 (when periodic electromotive force is applied, i.e., $E(t) = E_0 \sin \omega t$). In this case, (9) becomes

$$D_t^\alpha I(t) + \frac{R}{L} I(t) = \frac{E_0}{L} \sin \omega t. \quad (29a)$$

The initial condition is

$$I(0) = c, \quad (c > 0) \quad \left(\text{Since, } \lim_{\alpha \rightarrow 0} \frac{d^\alpha I}{dt^\alpha} = I \right). \quad (29b)$$

Solution. Rewrite (29a) as

$$D_t^\alpha I(t) = \frac{E_0}{L} \sin \omega t - \frac{R}{L} I(t). \quad (30)$$

Taking the Laplace transform on both sides and using (29b), we get

$$I(s) = \frac{E_0}{L} \frac{\omega}{(s^2 + \omega^2)(s^\alpha + R/L)} + c \frac{s^{\alpha-1}}{(s^\alpha + R/L)}. \quad (31)$$

Taking the inverse Laplace transform, we obtain

$$I(t) = \frac{E_0}{L} \mathcal{L}^{-1} \left\{ \frac{\omega}{(s^2 + \omega^2)(s^\alpha + R/L)} \right\} + c \mathcal{L}^{-1} \left\{ \frac{s^{\alpha-1}}{(s^\alpha + R/L)} \right\} \quad (32)$$

on applying the convolution theorem, this yields

$$\mathcal{L}^{-1} \left\{ \frac{\omega}{(s^2 + \omega^2)(s^\alpha + R/L)} \right\} = -\frac{L}{R} \int_0^t \sin \omega(t-u) E'_\alpha \left(-\frac{R}{L} u^\alpha \right) du. \quad (33)$$

Using (15), (32), and (33), we obtain

$$I(t) = c E_\alpha \left(-\frac{R}{L} t^\alpha \right) - \frac{E_0}{R} \int_0^t \sin \omega(t-u) E'_\alpha \left(-\frac{R}{L} u^\alpha \right) du. \quad (34)$$

Case 5 (when periodic electromotive force is applied, i.e., $E(t) = E_0 \cos \omega t$). In this case, (9) becomes

$$D_t^\alpha I(t) + \frac{R}{L} I(t) = \frac{E_0}{L} \cos \omega t. \quad (35)$$

The initial condition is

$$I(0) = c, \quad (c > 0) \quad \left(\text{Since, } \lim_{\alpha \rightarrow 0} \frac{d^\alpha I}{dt^\alpha} = I \right). \quad (36)$$

Solution. Rewrite (35) as

$$D_t^\alpha I(t) = \frac{E_0}{L} \cos \omega t - \frac{R}{L} I(t). \quad (37)$$

Taking the Laplace transform on both sides and using (36), we get

$$I(s) = \frac{E_0}{L} \frac{s}{(s^2 + \omega^2)(s^\alpha + R/L)} + \frac{s^{\alpha-1}}{(s^\alpha + R/L)}. \quad (38)$$

Taking the inverse Laplace transform, we obtain

$$I(t) = \frac{E_0}{L} \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + \omega^2)(s^\alpha + R/L)} \right\} + c \mathcal{L}^{-1} \left\{ \frac{s^{\alpha-1}}{(s^\alpha + R/L)} \right\} \quad (39)$$

on applying the convolution theorem, this gives

$$\mathcal{L}^{-1} \left\{ \frac{\omega}{(s^2 + \omega^2)(s^\alpha + R/L)} \right\} = -\frac{L}{R} \int_0^t \cos \omega(t-u) E'_\alpha \left(-\frac{R}{L} u^\alpha \right) du. \quad (40)$$

Further, using (15), (39), and (40) we get

$$I(t) = c E_\alpha \left(-\frac{R}{L} t^\alpha \right) - \frac{E_0}{R} \int_0^t \cos \omega(t-u) E'_\alpha \left(-\frac{R}{L} u^\alpha \right) du. \quad (41)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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