

Research Article

A Class of Generalized Differential Operators with Infinitely Many Discontinuous Points

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We define a class of differential operators T which have infinitely many discontinuous points, and investigate the kernel of T in terms of operator theory. It is shown that the solutions of T have exponential behavior, and the dimension of the kernel is given.

1. Introduction and Motivations

In the study of ergodicity of billiard flows, a very important question concerns the hyperbolicity of these flows [1, 2]. Hyperbolicity is defined in the language of the linearized system; this is the situation where trajectories close to a given trajectory either diverge exponentially in time from it or converge in the same way. The differential operator T we study is associated with the linearization of the flow; hence we ask whether solutions to $Tf = 0$ have the corresponding property. However, this is a paper not about ergodic theory, but about the spectral theory [3, 4] of a class of generalized differential operators which comes up naturally in this context.

The first class of chaotic billiards was introduced by Sinai in [5]; he proved the ergodicity of plenary dispersing billiards. It took more than 30 years; until 2003, Bálint et al. were able to prove ergodicity for multidimensional dispersing billiard in [6]. However it remains an important and difficult question to study hyperbolicity for nondispersing chaotic billiards, as well as high dimensional Bunimovich type billiards. We plan to give a self-contained approach using spectral theory to study the asymptotic behavior of functions in the kernels of a class of ordinary differential operators, motivated by asymptotic behavior of Jacobian field of certain dynamical systems with singularities, especially billiards.

In this paper we construct a new class of generalized differential operators associated with the impulsive equations.

The differential operators we deal with are second order, matrix coefficient Schrödinger operators with infinitely many discontinuous points. These kinds of operators are more general than those occurring in billiard flow, but include these as a special case. In this case, the jump conditions correspond to the reflections. We investigate the exponential behavior of functions in the null-space of T in terms of operator theory. On this basis, we obtain the relation between the minimal and maximal operators associated with the weighted operators and then characterize the dimension of the kernel. The results of this paper extend the result in the papers of Kauffman and Zhang [7] and Zhang and Lian [8] to more general case, which gives some hope that the structure of the differential operators may be used later to analyze some of the problems of greater interest in multidimensional billiards.

This paper is divided into five sections: the first section introduces the research background; the second section describes the main results derived in this paper; they are formulated in Theorems 7 and 8; in the third section, we studied the differential operator T ; then, in the last two sections, we give the proof of the two main theorems.

2. Statement of the Main Results

Throughout this paper, we will let $0 = t_0 < t_1 < \dots < t_j < \dots$ be a partition \mathcal{P} of $[0, \infty)$, $m \geq 1$. Denote $I_i = [t_i, t_{i+1})$, for all $i = 0, 1, 2, \dots$. Denote $u \cdot v$ as the inner product of

vectors $u, v \in \mathbb{C}^m$. Let I be a real interval; denote by $L^2(I)$ the Hilbert space of all measurable functions f from I to \mathbb{C}^m , such that

$$\int_I |f|^2 dt < \infty, \quad (1)$$

with inner product

$$[f, g] = \int_I (f \cdot g) dt. \quad (2)$$

Definition 1. Let l be the differential expression on an interval I defined by

$$lf = -f'' + A(t)f, \quad (3)$$

where $A(t)$ is a positive semidefinite symmetric real $m \times m$ matrix for each $t \in \mathbb{R}^+$ and a continuous function of t .

Definition 2. Let l_p be a set of all sequences of complex numbers $\{\xi_k\}_{k=1}^\infty$, which satisfy the condition $\sum_{k=1}^\infty |\xi_k|^p < \infty$. In this paper, every element of the sequence is a q -dimensional column vector; we denote l_p^q as a set, which satisfy the following condition: $\xi = \{\xi_k\}_{k=1}^\infty \in l_p^q$; then $\xi_k = (\xi_k^1, \xi_k^2, \dots, \xi_k^q)^\top$ ($k = 1, 2, \dots$) and $\{\xi_k^j\}_{k=1}^\infty \in l_p$, for $j = 1, \dots, q$.

Definition 3. Let $F([0, \infty), \mathbb{C}^m)$ denote the set of all measurable functions f from $[0, \infty)$ to \mathbb{C}^m , such that f is differentiable (in distribution sense) almost everywhere. For any $f \in F([0, \infty), \mathbb{C}^m)$, and the partition \mathcal{P} of $[0, \infty)$, define the operator:

$$\Lambda : F([0, \infty), \mathbb{C}^m) \longrightarrow l_0^{4m} \\ f \longmapsto a, \quad (4)$$

where $a = \{a_i\}_{i=1}^\infty$ and $a_i = (f(t_i - 0), f'(t_i - 0), f(t_i), f'(t_i))^T$ is a $4m \times 1$ matrix.

Definition 4. Let

$$T : D(T) \subset L_2([0, \infty), \mathbb{C}^m) \longrightarrow L_2([0, \infty), \mathbb{C}^m) \times l_0^{2m} \quad (5)$$

be a linear operator defined as

$$Tf(t) = (g(t), b), \quad (6)$$

$$g(t) = lf(t), \quad t \in I_i, \quad i = 0, 1, 2, \dots, \quad (7)$$

$$b = \tilde{l}f \quad (8)$$

with domain

$$D(T) = \left\{ f \in L_2([0, \infty), \mathbb{C}^m) \mid \right. \\ \left. f' \in AC(I_i), \sum_{i=0}^\infty \|lf\|_{2, I_i}^2 < \infty \right\}, \quad (9)$$

where $\tilde{l} = (B(t_i), -I)\Lambda$, and $AC(I_i)$ denotes the set of absolutely continuous complex valued functions on I_i , and, for each $i \in \mathbb{N}$,

$$B(t_i) = \begin{pmatrix} U_i & 0 \\ O_i & U_i \end{pmatrix}_{2m \times 2m} \quad (10)$$

and the following holds:

- (1) $U_i^* U_i = I$, where I is an identity $m \times m$ matrix;
- (2) $U_i^* O_i$ is a symmetric operator on the \mathbb{R}^m ;
- (3) there exists a universal constant $C_1 > 0$, such that for any unit vector $u \in \mathbb{R}^m$, any $t > 0$, and any $i \in \mathbb{N}$,

$$C_1 \geq (U_i^* O_i u, u) \geq 0, \quad \|A(t)\| < C_1. \quad (11)$$

Remark 5. In Definition 4, (8) represents the jump conditions at points $\{t_i\}$. The range of b is in l_0^{2m} , as one can check that

$$b = \tilde{l}f = (B(t_i), -I)\Lambda f = B(t_i) \begin{pmatrix} f(t_i - 0) \\ f'(t_i - 0) \end{pmatrix} - \begin{pmatrix} f(t_i) \\ f'(t_i) \end{pmatrix}. \quad (12)$$

Definition 6 ($\Omega_{\epsilon, \delta}$ condition). Let $\epsilon > 0, \delta > 0$. $\Omega_{\epsilon, i}$ be a set of all vectors $v \in \mathbb{R}^m$ such that

$$U_i^* O_i v \cdot v \geq \epsilon |v|^2. \quad (13)$$

For any $f : [0, \infty) \rightarrow \mathbb{R}^m$, we say that $f \in \Omega_{\epsilon, \delta}$ if there exists a subsequence $\{i_k\} \subset \mathbb{N}$ with

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k \leq n} (t_{i_k} - t_{i_{k-1}}) \geq \delta, \quad (14)$$

such that $f(t_{i_k} - 0) \in \Omega_{\epsilon, i_k}$ for all $i_k, k \in \mathbb{N}$.

Note that any function $f \in \Omega_{\epsilon, \delta}$ has a certain nice property that

$$U_{i_k}^* O_{i_k} |f|^2(t_{i_k} - 0) \geq \epsilon |f(t_{i_k} - 0)|^2 \quad (15)$$

for a subsequence t_{i_k} which are not too close to each other.

The main results we derived are Theorems 7 and 8.

Theorem 7. Let $f \in \ker(T)$, and there exist $\delta > 0, \epsilon > 0$ such that $f \in \Omega_{\epsilon, \delta}$ for interval I . Then one has the following.

- (1) If $|f|$ is nonincreasing, then there exists $\alpha_1 > 0$, such that, for all $\lambda \in (0, \alpha_1)$ and $t \in I$,

$$|f(t)| \leq e^{-\lambda t} |f(0)|. \quad (16)$$

- (2) If $|f|$ is nondecreasing, then there exists $\alpha_2 > 0$, such that, for all $\lambda \in (0, \alpha_2)$ and $t \in I$,

$$|f(t)| \geq e^{\lambda t} |f(0)|. \quad (17)$$

Next theorem characterizes the dimension of the kernel of T .

Theorem 8. *There exist exactly m linearly independent bounded solutions to the equation $Tf = 0$.*

Example 9. Consider the one-dimensional case; that is, $m = 1$. Let $U_i = 1$, O_i be a real number, and let $A(t) = a(t)$ be a continuous function of t and satisfy the following conditions: $a(t) \geq 0$, $\exists C_1 > 0$, s.t., $\|a(t)\| < C_1$; then the equation $Tf = 0$ becomes

$$\begin{aligned} f''(t) &= a(t)f, \\ f(t_i - 0) - f(t_i) &= 0, \\ f'(t_i - 0) + O_i f(t_i - 0) - f'(t_i) &= 0. \end{aligned} \tag{18}$$

It describes the movement of the billiard particle in a smooth table; when it reaches the border, its position is not changed, but the direction is changed. Theorems 7 and 8 hold for this special equation; that is, the solutions of (18) have exponential behavior, and there exists exactly one linearly independent bounded solution.

3. Characterization of the Generalized Differential Operator T

In this section, we give a characterization on the generalized differential operator T .

Definition 10. Define an operator $W : D(W) \subset l_2^{2m} \rightarrow l_2^{2m}$, such that

$$\begin{aligned} W(a) &= \hat{a}, \quad \forall a \in l_2^{2m}, \\ \hat{a}_i &= Q_i a_i - X_i a_{i+1}, \end{aligned} \tag{19}$$

where, for $i = 1, 2, \dots, Q_i$ and X_i are $2m \times 2m$ nonsingular matrices with $\|Q_i\|_\infty \leq C$, $\|X_i\|_\infty \leq C$ for some $C > 0$, with

$$D(W) = \{a \in l_2^{2m} \mid W(a) \in l_2^{2m}\}. \tag{20}$$

Lemma 11. *W is surjective.*

Proof. W can be written as

$$W = \begin{pmatrix} Q_1 & -X_1 & 0 & 0 & \cdots \\ 0 & Q_2 & -X_2 & 0 & \cdots \\ 0 & 0 & Q_3 & -X_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{21}$$

Since Q_i and X_i are nonsingular matrices, so W is an invertible matrix. For any $\hat{a} \in l_2^{2m}$, let $a = W^{-1}\hat{a}$. Then we can check that $W(a) = \hat{a}$ and $a \in l_2^{2m}$. \square

In order to study the operator T , we first introduce an operator L which was studied in [7, 9] and review some properties of L .

Definition 12. Let L be a differential operator defined by

$$Lf(t) = lf(t), \quad \forall t \in I_i \cap I, \quad i = 0, 1, 2, \dots, \tag{22}$$

satisfying the following boundary conditions: for any $i \in \mathbb{N}$,

$$(f(t_i), f'(t_i))^T = B(t_i)(f(t_i - 0), f'(t_i - 0))^T, \tag{23}$$

where U_i, O_i are the same as in the definition of T .

Lemma 13 (see [7]). *Let L_M be the maximal operator of L , and let L_0 be the minimal operator. Then the adjoint operator of L_0 satisfies*

$$L_0^* = L_M. \tag{24}$$

Lemma 14 (see [7]). *For all f in the domain of L_0 ,*

$$[Lf, f] \geq \eta \|f\|^2, \tag{25}$$

where η is a universal positive constant.

Lemma 15 (see [9]). *Let B be a densely defined closed operator on a Hilbert space. Denote the range of B by $\mathcal{R}(B)$ and the nullspace of B by $\mathcal{N}(B)$. Then*

$$\mathcal{R}(B^*)^\perp = \mathcal{N}(B). \tag{26}$$

Based on all the above properties for operator L , we obtain the following result.

Lemma 16. *Let $g \in L_2([0, \infty), \mathbb{C}^m)$. Then there exists $f \in D(T)$ such that*

$$Tf(t) = (g, 0). \tag{27}$$

Proof. By Lemma 14, we know that $\mathcal{N}(L_0) = \{0\}$. From Lemmas 13 and 15, it is easy to show that $\mathcal{R}(L_M)^\perp = \mathcal{N}(L_0)$. So L_M is surjective. Then we have, for all $g \in L_2([0, \infty), \mathbb{C}^m)$, $\exists f \in D(L_M)$, such that $L_M f = g$. But by the definition of L , we know that $Tf = (L_M f, 0)$, which gives the result. \square

Lemma 17. *Assuming that there exists $d > 0$, such that*

$$t_{i+1} - t_i \leq d, \quad \forall i \in \mathbb{N}, \tag{28}$$

then the range of T is a subset of $\mathcal{R} := L_2([0, \infty), \mathbb{C}^m) \times l_2^{2m}$.

Proof. Since $f \in L_2([0, \infty), \mathbb{C}^m)$ and $lf \in L_2([0, \infty), \mathbb{C}^m)$, so

$$\begin{aligned} (\|f\|_{2, I_0}, \|f\|_{2, I_1}, \dots) &\in l_1, \\ (\|lf\|_{2, I_0}, \|lf\|_{2, I_1}, \dots) &\in l_1. \end{aligned} \tag{29}$$

There exists $K > 0$, such that, on each I_i ,

$$\|f\|_{\infty, I_i} + \|f'\|_{\infty, I_i} \leq K (\|f\|_{2, I_i} + \|lf\|_{2, I_i}). \tag{30}$$

Therefore $\{f(t_i - 0)\}_{i=1}^\infty, \{f'(t_i - 0)\}_{i=1}^\infty, \{f(t_i)\}_{i=0}^\infty$, and $\{f'(t_i)\}_{i=0}^\infty$ are all in l_2^m . It follows that

$$\{U_i f(t_i - 0) - f(t_i)\}_{i=1}^\infty \in l_2^m, \tag{31}$$

$$\{U_i f'(t_i - 0) + O_i f(t_i - 0) - f'(t_i)\}_{i=1}^\infty \in l_2^m.$$

Thus the range of T is contained in $L_2([0, \infty), \mathbb{C}^m) \times l_2^{2m}$. \square

Next we recall Gronwall's inequality, which will be used in our proof that T is subjective below.

Lemma 18 (Gronwall's inequality). *Let $f(t) > 0$ be continuous on $[a, b]$ and satisfy*

$$f(t) \leq \alpha + \int_a^t \beta(s) f(s) ds, \quad (32)$$

where $\alpha > 0$ and $\beta > 0$ is continuous on $[a, b]$. Then

$$f(t) \leq \alpha \exp \int_a^t \beta(s) ds, \quad (33)$$

for $t \in [a, b]$.

Proposition 19. $T : L_2([0, \infty), \mathbb{C}^m) \rightarrow \mathcal{H}$ is surjective.

Proof. Let $(g, x, y) \in \mathcal{H}$. By Lemma 16, there exists $f \in D(T)$, such that

$$Tf(t) = (g, 0, 0). \quad (34)$$

We just need to find a $h \in D(T)$, such that $Th = (0, x, y)$. By Definition 4, that is,

$$h''(t) = A(t)h(t) \quad t \in I_i, \quad i = 0, 1, 2, \dots,$$

$$x_i = U_i h(t_i - 0) - h(t_i), \quad i \in \mathbb{N},$$

$$y_i = U_i h'(t_i - 0) + O_i h(t_i - 0) - h'(t_i), \quad i = 0, 1, 2, \dots \quad (35)$$

For each $i \in \mathbb{N}$, let $F_i(t)$ be the fundamental solution matrix of

$$\frac{d}{dt} \begin{pmatrix} h(t) \\ h'(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ A(t) & 0 \end{pmatrix} \begin{pmatrix} h(t) \\ h'(t) \end{pmatrix}, \quad t \in I_i; \quad (36)$$

with $F_i(t_i) = I$. Since we assumed that $|I_i| \leq d$, for all $i \in \mathbb{N}$, so there exists a uniform constant C_d , such that $\|F_i(t)\|_\infty \leq C_d$. By Lemma 11, we choose $Q_i = \begin{pmatrix} U_i & 0 \\ 0 & U_i \end{pmatrix}$, and $X_i^{-1} = F_i(t_{i+1} - 0)$, then we know that there exist $a, b \in \mathbb{C}^m$, such that

$$W(a, b)^T = (x, y)^T. \quad (37)$$

That is,

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = Q_i \begin{pmatrix} a_i \\ b_i \end{pmatrix} - X_i \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix}. \quad (38)$$

Letting $h : [0, \infty) \rightarrow \mathbb{C}^m$ satisfy $h' \in AC(I_i)$ and

$$h(t_i - 0) = a_i, \quad h'(t_i - 0) = b_i, \quad (39)$$

then we have

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = Q_i \begin{pmatrix} h(t_i - 0) \\ h'(t_i - 0) \end{pmatrix} - X_i \begin{pmatrix} h(t_{i+1} - 0) \\ h'(t_{i+1} - 0) \end{pmatrix}. \quad (40)$$

Let

$$h(t_i) = U_i h(t_i - 0) - x_i, \quad (41)$$

$$h'(t_i) = U_i h'(t_i - 0) + O_i h(t_i - 0) - y_i,$$

and then we get

$$x_i = U_i h(t_i - 0) - h(t_i),$$

$$y_i = U_i h'(t_i - 0) + O_i h(t_i - 0) - h'(t_i),$$

$$\begin{pmatrix} h(t_{i+1} - 0) \\ h'(t_{i+1} - 0) \end{pmatrix} = X_i^{-1} \begin{pmatrix} h(t_i) \\ h'(t_i) \end{pmatrix} \quad (42)$$

$$= F_i(t_{i+1} - 0) \begin{pmatrix} h(t_i) \\ h'(t_i) \end{pmatrix}.$$

It follows that $Th = (0, x, y)$.

In the following, it is enough to show that $h \in L_2([0, \infty), \mathbb{C}^m)$. On each I_i , we have

$$\frac{d}{dt} \begin{pmatrix} h(t) \\ h'(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ A(t) & 0 \end{pmatrix} \begin{pmatrix} h(t) \\ h'(t) \end{pmatrix}. \quad (43)$$

Denote

$$B(t) = \begin{pmatrix} 0 & I \\ A(t) & 0 \end{pmatrix}, \quad X(t) = \begin{pmatrix} h(t) \\ h'(t) \end{pmatrix}. \quad (44)$$

Then

$$X'(t) = B(t)X(t). \quad (45)$$

By hypothesis of $A(t)$, we know that $\exists K$, such that $\|B(t)\|_\infty \leq K$ and $|I_i| \leq d$; thus on each I_i

$$\int_{t_i}^t X'(s) ds = \int_{t_i}^t B(s)X(s) ds = X(t) - X(t_i), \quad (46)$$

$$\|X(t)\|_{\infty, I_i} \leq \|X(t_i)\|_\infty + \int_{t_i}^t \|B(s)\|_\infty \|X(s)\|_\infty ds.$$

By Gronwall's inequality,

$$\|X(t)\|_{\infty, I_i} \leq \|X(t_i)\|_\infty \exp \int_{t_i}^t \|B(s)\|_\infty ds \leq \|X(t_i)\|_\infty e^{dK}. \quad (47)$$

Combining all these facts together, we get

$$\|X(t)\|_{2, I_i} \leq \sqrt{dm} \|X(t_i)\|_\infty e^{dK}. \quad (48)$$

Therefore

$$\begin{aligned} \|h\|_{2, [0, \infty)}^2 &\leq \|X(t)\|_{2, [0, \infty)}^2 \leq \sum_{i=0}^{\infty} \|X(t)\|_{2, I_i}^2 \\ &\leq dme^{2dK} \sum_{i=0}^{\infty} \|X(t_i)\|_\infty^2 \\ &\leq dme^{2dK} \sum_{i=0}^{\infty} (|h(t_i)|^2 + |h'(t_i)|^2) < \infty. \end{aligned} \quad (49)$$

Thus $f + h \in D(T)$ satisfies

$$T(f + h)(t) = (g(t), x, y). \quad (50)$$

This completes the proof of Proposition 19. \square

4. Exponential Behavior of Solutions: Proof of Theorem 7

In this section, we give the proof of Theorem 7 by considering the exponential behavior of the null space of T .

For any $f \in \ker(T)$, we denote

$$k_f(t) = \frac{|f|'}{|f|} = \frac{f' \cdot f}{f \cdot f}. \quad (51)$$

Note that $|f| = \sqrt{(f \cdot f)}$, so

$$|f|' = \left(\sqrt{(f \cdot f)} \right)' = \frac{(f \cdot f)'}{2\sqrt{f \cdot f}} = \frac{2f' \cdot f}{2\sqrt{f \cdot f}} = \frac{f' \cdot f}{\sqrt{f \cdot f}}. \quad (52)$$

Lemma 20. *Let $f \in \Omega_{\epsilon, \delta}$ for interval I . Then there exists a constant $r > 0$, and $\lambda = \lambda_{\epsilon, \delta} > 0$, such that for any $t > r$*

$$\frac{1}{t} \int_{[0, t]} |k_f(s)| ds > \lambda. \quad (53)$$

Proof. Assume that there exists $\{i_k\} \subset \mathbb{N}$, such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k \leq n} (t_{i_k} - t_{i_{k-1}}) \geq \delta, \quad (54)$$

since $f \in \ker(T)$ and $f \in \Omega_{\epsilon, \delta}$, so

$$\begin{aligned} & (f' \cdot f)(t_{i_k}) - (f' \cdot f)(t_{i_k} - 0) \\ &= U_{i_k} O_{i_k} |f|^2(t_{i_k} - 0) \geq \epsilon |f|^2(t_{i_k} - 0). \end{aligned} \quad (55)$$

This implies that

$$|k_f(t_{i_k}) - k_f(t_{i_k} - 0)| = \frac{U_{i_k} O_{i_k} |f|^2(t_{i_k} - 0)}{f \cdot f} \geq \epsilon. \quad (56)$$

Thus

$$\frac{1}{t} \int_{[0, t]} |k_f(s)| ds > \frac{1}{t} \sum_{t_{i_k} < t} (k_f(t_{i_k}) - k_f(t_{i_k} - 0)) > \frac{\epsilon}{t} b_t, \quad (57)$$

where b_t is the number of t_{i_k} , such that $t_{i_k} < t$. But we know that

$$\liminf_{t \rightarrow \infty} \frac{b_t}{t} \geq \delta, \quad (58)$$

so there exists $r > 0$, such that, for any $t > r$,

$$\frac{1}{t} \int_{[0, t]} |k_f(s)| ds > \lambda, \quad (59)$$

where $\lambda = \delta\epsilon$. □

Now we are ready to prove Theorem 7.

Proof. By the definition of k_f , we know that

$$k_f(t) = \frac{|f|'}{|f|} = (\ln |f|)'(t). \quad (60)$$

Thus

$$\ln |f| = \int_0^t k_f(s) ds + c, \quad (61)$$

$$|f|(t) = ce^{\int_0^t k_f(s) ds} = |f(0)| e^{\int_0^t k_f(s) ds}.$$

(1) Since $|f|$ is nonincreasing, then $k_f(t) \leq 0$; by Lemma 20, we have

$$\int_0^t k_f(s) ds = - \int_0^t |k_f(s)| ds < -\lambda t, \quad (62)$$

so

$$|f(t)| \leq e^{-\lambda t} |f(0)|. \quad (63)$$

(2) Since $|f|$ is nondecreasing, then $k_f(t) \geq 0$; by Lemma 20, we have

$$\int_0^t k_f(s) ds = \int_0^t |k_f(s)| ds > \lambda t, \quad (64)$$

so

$$|f(t)| \geq e^{\lambda t} |f(0)|. \quad (65)$$

□

5. Proof of Theorem 8

In this section, we give the proof of Theorem 8. First we obtain the relation between the minimal and maximal operators associated with the weighted operators M_I and then characterize the dimension of the kernel of T .

Definition 21. Let L be defined as Definition 12, $\langle t \rangle = \sqrt{t^2 + 1}$. Define

$$M_I = \langle t \rangle L \langle t \rangle, \quad \forall t \in I_i \cap I, \quad i = 0, 1, 2, \dots \quad (66)$$

Definition 22. Let the maximal operator $M_{M, I}$ associated with M_I be the restriction of M_I on D_M , with

$$\begin{aligned} D_M &= \{f \in L^2(I) \cap D(L) \mid M_I(f) \in L^2(I), \\ & f' \in AC(I_i \cap I), \forall i = 0, 1, 2, \dots\}. \end{aligned} \quad (67)$$

Let \widehat{D}_I be the set of all $f \in D_M$, such that $f \in C^\infty(I_i \cap I)$, for all $i \in \mathbb{N}$ and f vanishes in a neighborhood of both “end points” of I and $\text{supp}(f)$ is compact. Denote \widehat{M}_I to be M_I restricting on \widehat{D}_I ; then we define the minimal operator $M_{0, I}$ to be the smallest closed operator in $L^2(I)$ which extends \widehat{M}_I and denote by D_0 the domain of $M_{0, I}$.

Lemma 23. *If M_I is defined as in Definition 21 on I , then*

$$M_{0,I} \subseteq M_{M,I}, \tag{68}$$

and, for any f in the domain of $M_{0,I}$, we have $f(0) = 0, f'(0) = 0$. If I is compact, say $I = [0, \mathcal{T}] \subset [0, \infty)$, then for any $f \in D_0$, also $f(\mathcal{T}) = f'(\mathcal{T}) = 0$.

Proof. Let $\{f_n\}$ be a Cauchy sequence of functions from \widehat{D}_I converging to f with $M_{0,I}f_n$ converging to $M_{0,I}f$ in $L^2(I)$. Then on any compact interval $[\alpha, \beta] \subseteq I, t \geq 0, f_n$ and f'_n both converge uniformly; therefore f''_n also converges in each $L^2(I_i)$, so f''_n converges in $L^2(I)$. It follows that f' is absolutely continuous on each I_i , and $f'(0) = 0$, if $I = [0, \mathcal{T}]$; then also $f'(\mathcal{T}) = 0$. Since $f_n^{(i)}$ converges to $f^{(i)}$ for $i = 0, 1, 2$, we see that $M_{0,I}f = M_I f, M_I f$ is in L^2 , and so $f \in D_{M,I}$. \square

Definition 24 (see [10]). A densely defined operator S on a Hilbert space is said to be symmetric if $[Sf, g] = [f, Sg]$ for all f, g in the domain of S .

Lemma 25. *$M_{0,I}$ is a symmetric operator, for $I = [0, \mathcal{T}]$ or $I = [0, \infty)$.*

Proof. For any f, g in the domain of $M_{0,I}$, by Lemma 23, we know that $f(0) = f'(0) = 0$ and $g(0) = g'(0) = 0$. Without loss of generality, we suppose that f and g are real vector valued functions:

$$\begin{aligned} & [M_I f, g] - [f, M_I g] \\ &= -[\langle t \rangle (\langle t \rangle f'')', g] + [\langle t \rangle A \langle t \rangle f, g] \\ & \quad + [f, \langle t \rangle (\langle t \rangle g'')'] - [f, \langle t \rangle A \langle t \rangle g] \\ &= -\int_I (\langle t \rangle (\langle t \rangle f'')' \cdot g - f \cdot \langle t \rangle (\langle t \rangle g'')') (t) dt \\ & \quad + [\langle t \rangle^2 A f, g] - [f, \langle t \rangle^2 A g] \\ &= -[\langle t \rangle f' \cdot \langle t \rangle g - \langle t \rangle f \cdot \langle t \rangle g'] \Big|_I \\ &= -[\langle t \rangle^2 f' \cdot g - \langle t \rangle^2 f \cdot g'] \Big|_I. \end{aligned} \tag{69}$$

If I is a compact interval $[0, \mathcal{T}]$, then by the fact that

$$f(\mathcal{T}) = f'(\mathcal{T}) = g(\mathcal{T}) = g'(\mathcal{T}) = 0, \tag{70}$$

we get $[M_I f, g] = [f, M_I g]$.

If $I = [0, \infty)$, by hypotheses on the boundary conditions and $U_i^* U_i = I$, we have

$$\begin{aligned} & [f' \cdot g - f \cdot g'] (t_i) \\ &= [O_i f(t_i - 0) + U_i f'(t_i - 0)] \\ & \quad \cdot U_i g(t_i - 0) - U_i f(t_i - 0) \\ & \quad \cdot [O_i g(t_i - 0) + U_i g'(t_i - 0)] \\ &= [f' \cdot g - f \cdot g'] (t_i - 0). \end{aligned} \tag{71}$$

Thus we get

$$\begin{aligned} & [M_I f, g] - [f, M_I g] \\ &= -\sum_{i=1}^{\infty} \langle t \rangle^2 \{ [f' \cdot g - f \cdot g'] (t_i) - [f' \cdot g - f \cdot g'] (t_i - 0) \} \\ &= 0. \end{aligned} \tag{72}$$

That is, $[M_I f, g] = [f, M_I g]$. So by Definition 24, $M_{0,I}$ is symmetric. \square

Lemma 26. *For $h \in L^2(I)$, the equation $M_I f = h$ has a solution f in the domain of $M_{0,I}$ if and only if h is orthogonal to all solutions of $M_{M,I} g = 0$.*

Proof. Assume that, for $h \in L^2(I)$, the equation $M_I f = h$ has a solution $f \in D_0$. For any $g \in D_M$, such that $M_I g = 0$, we have

$$[h, g] = [M_{0,I} f, g] = [f, M_I g] = 0. \tag{73}$$

Conversely, if h is orthogonal to all solutions of $M_{M,I} g = 0$, choose f such that $M_I f = h$ and $f(\alpha) = f'(\alpha) = 0$. We need to show that $f(\beta) = f'(\beta) = 0$. Using the same calculation as in the proof of Lemma 25, we get

$$[M_I f, g] - [f, M_I g] = f(\beta) \cdot g'(\beta) - f'(\beta) \cdot g(\beta). \tag{74}$$

On the other hand, Consider

$$[M_I f, g] - [f, M_{M,I} g] = [M_I f, g] = [h, g] = 0. \tag{75}$$

Thus we get that $f(\beta) \cdot g'(\beta) - f'(\beta) \cdot g(\beta) = 0$ for any g satisfying $M_I g = 0$. Choose $g \in D_M$, s.t. $g(\beta) \neq 0$, but $g'(\beta) = 0$; then we get $f'(\beta) = 0$; similarly, we can get $f(\beta) = 0$. \square

Corollary 27. *$I = [\alpha, \beta] \subset [0, \infty)$; letting τ denote the restriction of M_I to domain*

$$\{f \in D_M(I) : f(\alpha) = f'(\alpha) = 0, f(\beta) = f'(\beta) = 0\}, \tag{76}$$

then $\tau = M_{0,I}$.

Based on all the above results, we can derive the relation between the operators $M_{0,I}$ and $M_{M,I}$.

Proposition 28. *If M_I is defined as in Definition 21 and $I = [\alpha, \beta] \subset [0, \infty)$, then*

$$M_{0,I}^* = M_{M,I}. \tag{77}$$

Proof. Since $M_{0,I}$ is the smallest closed extension of \widehat{M}_I , it follows that

$$M_{0,I}^* = \widehat{M}_I^*. \tag{78}$$

As for any $f \in \widehat{D}_I$ (then $f \in D_0$) and $g \in D_{M,I}$, $[\widehat{M}_I f, g] = [f, M_{M,I} g]$, then

$$M_{M,I} \subseteq \widehat{M}_I^* = M_{0,I}^*. \quad (79)$$

Let f be in the domain of $M_{0,I}^*$. Since $M_{M,I}$ is surjective for compact I , so there is a function g in $D_{M,I}$ such that $M_{M,I} g = M_{0,I}^* f$; thus

$$M_{0,I}^* (f - g) = 0. \quad (80)$$

So if u is in the range of $M_{0,I}$ with $M_{0,I} v = u$, we have

$$[f - g, u] = [f - g, M_{0,I} v] = [M_{0,I}^* (f - g), v] = 0. \quad (81)$$

Therefore $f - g$ is orthogonal to the range of $M_{0,I}$, but, by Lemma 26, $f - g$ is in the null space of $M_{M,I}$. Hence $f \in D_{M,I}$; that is, $M_{0,I}^* \subset M_{M,I}$. \square

Proposition 29. *If M_I is defined as in Definition 21 and $I \subset [0, \infty)$, then*

$$M_{0,I}^* = M_{M,I}. \quad (82)$$

Proof. By Proposition 28, we only need to show $M_{0,I}^* \subset M_{M,I}$. If g is in the domain of $M_{0,I}^*$, then, on any compact subinterval $[\alpha, \beta]$ of I , we have

$$[M_I f, g] = [f, M_{0,I}^* g], \quad \forall f \in \widehat{D}_I. \quad (83)$$

Thus the restriction of g to $[\alpha, \beta]$ is in the domain of $M_{0,I}^*$. But, by Proposition 28, the restriction of $M_{0,I}^* g$ to $[\alpha, \beta]$ must agree with $M_{M,I} g$. Since $[\alpha, \beta]$ is arbitrary, the proposition is proved. \square

We can show that $M_{0,I}$ is semibounded in the next lemma.

Lemma 30. *If M_I is defined as in Definition 21, for all $f \in D_0$,*

$$[M_I f, f] \geq \frac{1}{4} \|f\|^2. \quad (84)$$

Proof. By integration by parts we obtain, for $f \in C_0^\infty(\mathbb{R}^+)$,

$$\begin{aligned} [M_{0,I} f, f] &= [\langle t \rangle L(\langle t \rangle f), f] \\ &= -[\langle t \rangle (\langle t \rangle f)'', f] + [\langle t \rangle A \langle t \rangle f, f] \\ &= [(\langle t \rangle f)', (\langle t \rangle f)'] + [\langle t \rangle^2 A f, f] \\ &\geq [(\langle t \rangle f)', (\langle t \rangle f)'] \\ &\geq \frac{1}{4} \|f\|^2, \end{aligned} \quad (85)$$

where the last inequality follows from Hardy's inequality [11]. \square

In order to prove Theorem 8, we need to introduce the Friedrichs extension of $M_{0,I}$.

Definition 31. With M_I as in Definition 21, let $Q_I(f, g) = [M_I f, g]$, for all $f, g \in \widehat{D}_I$.

It follows from operator theory [9] that the semibounded symmetric operator $M_{0,I}$ has equal deficiency indices, and therefore, by Von Neumann's theorem, such an operator always has self-adjoint extensions. There is a distinguished extension H , called the Friedrichs extension [11], which is obtained from the quadratic form associated with $M_{0,I}$.

Proposition 32. *Q_I is a closable quadratic form and its closure \widehat{Q}_I is the quadratic form of a unique self-adjoint operator H defined by*

$$\begin{aligned} D(H_I) &= \{f \in L^2(I) : |Q_I(f, g)| \\ &\leq C_f [g, g], \forall g \in \widehat{D}_I, \text{ where } C_f \text{ is a constant}\}, \\ H_I f &= M_{M,I} f, \quad \forall f \in D(H_I). \end{aligned} \quad (86)$$

Furthermore,

$$[H_I f, f] \geq \frac{1}{4} [f, f], \quad \forall f \in D(H_I). \quad (87)$$

Proof. This is a form of the definition of the Friedrichs extension: for the semibounded symmetric operator $M_{0,I}$ and the closable quadratic form Q_I , the restriction of $M_{0,I}^*$ to the domain of the closure of Q_I is in fact the Friedrichs extension. This form is clearly closable. The last inequality follows from the construction of the Friedrichs extension H_I , together with Lemma 30. \square

Next proposition tells us the relation between $\ker(T)$ and $\ker(M_I)$.

Proposition 33. *$f \in \ker(T)$ is nonincreasing if and only if $g = f/\langle t \rangle$ is a nonincreasing solution to $M_I g = 0$.*

Proof. Let $f \in \ker(T)$; if $|f|$ is nonincreasing, then $f' \cdot f < 0$. So by east calculation, we have

$$(g \cdot g)'(t) = 2g'(t) \cdot g(t) = 2 \frac{f' \cdot f (1+t^2) - f^2 t}{(1+t^2)^2} < 0, \quad (88)$$

which implies the result. \square

Now we are ready to prove Theorem 8.

Proof. It is sufficient to consider real solutions. Let f be any real solution of $Tf = 0$; for any $i \in \mathbb{N}$, let $t \in I_i$, $(f \cdot f)'(t) = 2f'(t) \cdot f(t)$. Using the fact that $A(t)$ is positive semidefinite for each $t \geq 0$, we have

$$(f \cdot f)''(t) = 2|f'|^2(t) + 2(A(t) f(t) \cdot f(t)) \geq 0. \quad (89)$$

On the other hand, for any $i \in \mathbb{N}$,

$$(f \cdot f)'(t_i) - (f \cdot f)'(t_i - 0) = (U_i^* O_i f, f)(t_i - 0) \geq 0. \quad (90)$$

This implies that $(f \cdot f)'$ is nondecreasing on $[0, \infty]$. Choosing f , such that $f(0) = 0, f'(0) \neq 0$, then $(f \cdot f)'(t) > (f \cdot f)'(0) = 0$, from (52); $|f|$ is nondecreasing. Then, by Theorem 7, we know that those solutions have exponentially increasing amplitude; that is, they are unbounded solutions. Clearly $\dim(\ker(T)) \leq 2m$. The set $\{f \in \ker(T) \mid f(0) = 0, f'(0) \neq 0\}$ is m dimensional, which means that there are at most m linearly independent bounded solutions for $Tf = 0$.

Now we prove $\dim(\ker(M_{M,I})) \geq m$, using the properties of Friedrichs extension of $M_{0,I}$. Let H_I be the Friedrichs extension of $M_{0,I}$; then, by Proposition 32, H_I is self-adjoint and positive definite. Also for any f in $D(H_I)$, $f(0) = 0$. We choose m compactly supported piecewise smooth functions f_1, f_2, \dots, f_m on I , such that the space spanned by $\{f_k(0) \mid k = 1, \dots, m\}$ has dimension m . Then $\{f_k \mid k = 1, \dots, m\}$ are linearly independent mod $D(H_I)$. It is clear that $f_k \in D_M$ for $k = 1, \dots, m$; this forces the Fredholm index to increase by m . Since H_I is already surjective, this produces an m dimensional nonincreasing solution space to $M_I f = 0$, so $\dim(\ker(M_{M,I})) \geq m$; by Proposition 33, we know that there are at least m linearly independent bounded solutions for $Tf = 0$. This completes the proof. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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