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Research Article

Sharp Geometric Mean Bounds for Neuman Means

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We find the best possible constants $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1/2]$ and $\alpha_3, \alpha_4, \beta_3, \beta_4 \in [1/2, 1]$ such that the double inequalities $G(\alpha_1 a + (1-\alpha_1)b, \alpha_1 b + (1-\alpha_1)a) < N_{AG}(a,b) < G(\beta_1 a + (1-\beta_1)b, \beta_1 b + (1-\beta_1)a)$, $G(\alpha_2 a + (1-\alpha_2)b, \alpha_2 b + (1-\alpha_2)a) < N_{GA}(a,b) < G(\beta_2 a + (1-\beta_2)b, \beta_2 b + (1-\beta_2)a)$, $Q(\alpha_3 a + (1-\alpha_3)b, \alpha_3 b + (1-\alpha_3)a) < N_{QA}(a,b) < Q(\beta_3 a + (1-\beta_3)b, \beta_3 b + (1-\beta_3)a)$, $Q(\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a) < N_{AQ}(a,b) < Q(\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a)$ hold for all $a, b > 0$ with $a \neq b$, where G, A, and Q are, respectively, the geometric, arithmetic, and quadratic means and N_{AG}, N_{GA}, N_{QA}, and N_{AQ} are the Neuman means.

1. Introduction

For $a, b > 0$ with $a \neq b$, the Schwab-Borchardt mean $\mathrm{SB}(a,b)$ [1, 2] of a and b is given by

$$\mathrm{SB}(a,b) = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}, & a < b, \\ \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & a > b, \end{cases} \quad (1)$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = \log(x + \sqrt{x^2-1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively. Recently, the Schwab-Borchardt mean has been the subject of intensive research. In particular, many remarkable inequalities for Schwab-Borchardt mean and its generated means can be found in the literature [1–6].

Very recently, Neuman [7] found a new bivariate mean $N(a,b)$ derived from the Schwab-Borchardt mean as follows:

$$N(a,b) = \frac{1}{2}\left(a + \frac{b^2}{\mathrm{SB}(a,b)}\right). \quad (2)$$

Let $N_{AG}(a,b) = N(A(a,b), G(a,b))$, $N_{GA}(a,b) = N(G(a,b), A(a,b))$, $N_{QA}(a,b) = N(Q(a,b), A(a,b))$, and $N_{AQ}(a,b) = N(A(a,b), Q(a,b))$ be the Neuman means, where $G(a,b) = \sqrt{ab}$, $A(a,b) = (a+b)/2$, and $Q(a,b) = \sqrt{(a^2+b^2)/2}$ are the classical geometric, arithmetic, and quadratic means of a and b, respectively. Then the inequalities

$$\begin{aligned} G(a,b) &< N_{AG}(a,b) < N_{GA}(a,b) < A(a,b) \\ &< N_{QA}(a,b) < N_{AQ}(a,b) < Q(a,b) \end{aligned} \quad (3)$$

for all $a, b > 0$ with $a \neq b$, were established by Neuman [7].

Let $a > b > 0$ and $v = (a-b)/(a+b) \in (0,1)$. Then the following explicit formulas for $N_{AG}(a,b)$, $N_{GA}(a,b)$, $N_{QA}(a,b)$, and $N_{AQ}(a,b)$ are presented in [7]

$$N_{AG}(a,b) = \frac{1}{2}A(a,b)\left[1 + \left(1-v^2\right)\frac{\tanh^{-1}v}{v}\right], \quad (4)$$

$$N_{GA}(a,b) = \frac{1}{2}A(a,b)\left[\sqrt{1-v^2} + \frac{\sin^{-1}v}{v}\right], \quad (5)$$

$$N_{QA}(a,b) = \frac{1}{2}A(a,b)\left[\sqrt{1+v^2} + \frac{\sinh^{-1}v}{v}\right], \quad (6)$$

$$N_{AQ}(a,b) = \frac{1}{2}A(a,b)\left[1 + \left(1+v^2\right)\frac{\tan^{-1}v}{v}\right], \quad (7)$$

where $\tanh^{-1}(x) = \log[(1+x)/(1-x)]/2$, $\sin^{-1}(x)$, $\sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$, and $\tan^{-1}(x)$ are the inverse hyperbolic tangent, inverse sine, inverse hyperbolic sine, and inverse tangent functions, respectively.

In [7], Neuman also proved that the double inequalities

$$\begin{aligned}
&\alpha_1 A(a,b) + (1-\alpha_1) G(a,b) < N_{GA}(a,b) \\
&\qquad < \beta_1 A(a,b) + (1-\beta_1) G(a,b), \\
&\alpha_2 Q(a,b) + (1-\alpha_2) A(a,b) < N_{AQ}(a,b) \\
&\qquad < \beta_2 Q(a,b) + (1-\beta_2) A(a,b), \\
&\alpha_3 A(a,b) + (1-\alpha_3) G(a,b) < N_{AG}(a,b) \\
&\qquad < \beta_3 A(a,b) + (1-\beta_3) G(a,b), \\
&\alpha_4 Q(a,b) + (1-\alpha_4) A(a,b) < N_{QA}(a,b) \\
&\qquad < \beta_4 Q(a,b) + (1-\beta_4) A(a,b)
\end{aligned} \tag{8}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 2/3$, $\beta_1 \geq \pi/4$, $\alpha_2 \leq 2/3$, $\beta_2 \geq (\pi-2)/[4(\sqrt{2}-1)] = 0.689\ldots$, $\alpha_3 \leq 1/3$, $\beta_3 \geq 1/2$, $\alpha_4 \leq 1/3$, and $\beta_4 \geq [\log(1+\sqrt{2}) + \sqrt{2} - 2]/[2(\sqrt{2}-1)] = 0.356\ldots$.

Let $a, b > 0$ with $a \neq b$, $x \in [0, 1/2]$, $y \in [1/2, 1]$,

$$\begin{aligned}
f(\lambda) &= G[\lambda a + (1-\lambda) b, \lambda b + (1-\lambda) a], \\
g(\mu) &= Q[\mu a + (1-\mu) b, \mu b + (1-\mu) a].
\end{aligned} \tag{9}$$

Then, it is not difficult to verify that $f(\lambda)$ and $g(\mu)$ are continuous and strictly increasing on $[0, 1/2]$ and $[1/2, 1]$, respectively. Note that

$$\begin{aligned}
f(0) &= G(a,b) < N_{AG}(a,b) \\
&< N_{GA}(a,b) < A(a,b) = f\left(\frac{1}{2}\right), \\
g\left(\frac{1}{2}\right) &= A(a,b) < N_{QA}(a,b) \\
&< N_{AQ}(a,b) < Q(a,b) = g(1).
\end{aligned} \tag{10}$$

Therefore, it is natural to ask what the best possible constants $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1/2]$ and $\alpha_3, \alpha_4, \beta_3, \beta_4 \in [1/2, 1]$ are such that the double inequalities

$$\begin{aligned}
&G(\alpha_1 a + (1-\alpha_1) b, \alpha_1 b + (1-\alpha_1) a) \\
&\quad < N_{AG}(a,b) < G(\beta_1 a + (1-\beta_1) b, \beta_1 b + (1-\beta_1) a), \\
&G(\alpha_2 a + (1-\alpha_2) b, \alpha_2 b + (1-\alpha_2) a) \\
&\quad < N_{GA}(a,b) < G(\beta_2 a + (1-\beta_2) b, \beta_2 b + (1-\beta_2) a), \\
&Q(\alpha_3 a + (1-\alpha_3) b, \alpha_3 b + (1-\alpha_3) a) \\
&\quad < N_{QA}(a,b) < Q(\beta_3 a + (1-\beta_3) b, \beta_3 b + (1-\beta_3) a), \\
&Q(\alpha_4 a + (1-\alpha_4) b, \alpha_4 b + (1-\alpha_4) a) \\
&\quad < N_{AQ}(a,b) < Q(\beta_4 a + (1-\beta_4) b, \beta_4 b + (1-\beta_4) a)
\end{aligned} \tag{11}$$

hold for all $a, b > 0$ with $a \neq b$. The main purpose of this paper is to answer this question.

2. Lemmas

In order to prove our main results, we need several lemmas, which we present in this section.

Lemma 1 (see [8, Theorem 1.25]). *Let $-\infty < a < b < \infty$, $f, g : [a,b] \to (-\infty, \infty)$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)}, \qquad \frac{f(x) - f(b)}{g(x) - g(b)}. \tag{12}$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2 (see [9, Lemma 1.1]). *Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ and $b_n > 0$, for all $n \geq 0$. If the sequence $\{a_n/b_n\}$ is (strictly) increasing (decreasing), for all $n \geq 0$, then the function $f(x)/g(x)$ is also (strictly) increasing (decreasing) on $(0, r)$.*

Lemma 3. *The function*

$$f_1(x) = \frac{3\sinh^2(x) - 2x\sinh(x) - x^2}{[\cosh(x) - 1]^2} \tag{13}$$

is strictly increasing from $(0, \infty)$ onto $(8/3, 3)$.

Proof. Making use of the power series expansion, we have

$$\begin{aligned}
&3\sinh^2(x) - 2x\sinh(x) - x^2 \\
&\quad = \frac{3}{2}\cosh(2x) - 2x\sinh(x) - x^2 - \frac{3}{2} \\
&\quad = \frac{3}{2}\sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} x^{2n} \\
&\qquad - 2x\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} - x^2 - \frac{3}{2} \\
&\quad = \frac{3}{2}\sum_{n=2}^{\infty} \frac{2^{2n}}{(2n)!} x^{2n} - 2\sum_{n=1}^{\infty} \frac{1}{(2n+1)!} x^{2n+2} \\
&\quad = x^4 \sum_{n=0}^{\infty} \frac{3 \cdot 2^{2n+3} - 4(n+2)}{(2n+4)!} x^{2n}, \\
&[\cosh(x) - 1]^2 = \frac{1}{2}\cosh(2x) - 2\cosh(x) + \frac{3}{2} \\
&\quad = \sum_{n=1}^{\infty} \frac{2^{2n-1} - 2}{(2n)!} x^{2n} = x^4 \sum_{n=0}^{\infty} \frac{2^{2n+3} - 2}{(2n+4)!} x^{2n}.
\end{aligned} \tag{14}$$

Let

$$a_n = \frac{3 \cdot 2^{2n+3} - 4(n+2)}{(2n+4)!}, \qquad b_n = \frac{2^{2n+3} - 2}{(2n+4)!}. \tag{15}$$

Then

$$b_n > 0,$$
$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = \frac{(6n+1)\,2^{2n+2} + 2}{(2^{n+4}-1)(2^{2n+2}-1)} > 0 \tag{16}$$

for all $n \geq 0$. Note that

$$f_1(0^+) = \frac{a_0}{b_0} = \frac{8}{3}, \qquad \lim_{x\to\infty} f_1(x) = \lim_{n\to\infty} \frac{a_n}{b_n} = 3. \tag{17}$$

Therefore, Lemma 3 follows easily from Lemma 2 and (14)–(17). □

Lemma 4. *The function*

$$f_2(x) = \frac{1-(1/4)(\cos(x)+x/\sin(x))^2}{\sin^2(x)} \tag{18}$$

is strictly increasing from $(0, \pi/2)$ *onto* $(1/3, 1-\pi^2/16)$.

Proof. It is not difficult to verify that

$$f_2(0^+) = \frac{1}{3}, \qquad f_2\left(\frac{\pi}{2}\right) = 1 - \frac{\pi^2}{16}. \tag{19}$$

Let $g(x) = 1 - [\cos(x) + x/\sin(x)]^2/4$ and $h(x) = \sin^2(x)$. Then,

$$f_2(x) = \frac{g(x)}{h(x)}, \qquad g(0^+) = h(0) = 0,$$
$$\frac{g'(x)}{h'(x)} = \frac{x^2 - \sin^2(x)\cos^2(x)}{4\sin^4(x)} = \frac{(2x)^2 - \sin^2(2x)}{4[1-\cos(2x)]^2}. \tag{20}$$

From Lemma 1 and (19)-(20), we know that we just need to prove that the function

$$F(y) = \frac{y^2 - \sin^2(y)}{[1-\cos(y)]^2} \tag{21}$$

is strictly increasing on $(0, \pi)$.

Let $F_1(y) = y^2 - \sin^2(y)$, $F_2(y) = [1-\cos(y)]^2$, $F_3(y) = y - \sin(y)\cos(y)$, and $F_4(y) = \sin(y)(1-\cos(y))$. Then

$$F(y) = \frac{F_1(y)}{F_2(y)}, \qquad F_1(0) = F_2(0) = F_3(0) = F_4(0) = 0, \tag{22}$$

$$\frac{F_1'(y)}{F_2'(y)} = \frac{F_3(y)}{F_4(y)}. \tag{23}$$

Note that

$$\frac{F_3'(y)}{F_4'(y)} = \frac{2(1+\cos(y))}{1+2\cos(y)} = 1 + \frac{1}{1+2\cos(y)} \tag{24}$$

is strictly increasing on $(0, \pi)$. Therefore, the monotonicity of $F(y)$ follows easily from (22) and (23) together with the monotonicity of $F_3'(y)/F_4'(y)$. □

Lemma 5. *The function*

$$f_3(x) = \frac{1-(1/4)(\cosh(x)+x/\sinh(x))^2}{\sinh^2(x)} \tag{25}$$

is strictly increasing from $(0, \log(1+\sqrt{2}))$ *onto* $(-1/3, -(2\sqrt{2}\log(1+\sqrt{2}) + \log^2(1+\sqrt{2}) - 2)/4)$.

Proof. Simple computations lead to

$$f_3(0^+) = -\frac{1}{3},$$
$$f_3\left(\log\left(1+\sqrt{2}\right)\right) = -\frac{2\sqrt{2}\log\left(1+\sqrt{2}\right) + \log^2\left(1+\sqrt{2}\right) - 2}{4}. \tag{26}$$

Let $g_1(x) = 1 - [\cosh(x) + x/\sinh(x)]^2/4$ and $g_2(x) = \sinh^2(x)$. Then

$$f_3(x) = \frac{g_1(x)}{g_2(x)}, \qquad g_1(0^+) = g_2(0) = 0, \tag{27}$$

$$\frac{g_1'(x)}{g_2'(x)} = \frac{4x^2 - \sinh^2(2x)}{4[\cosh(2x)-1]^2} = -\frac{\sum_{n=0}^{\infty}\left(4^{2n+4}/(2n+4)!\right)x^{2n}}{\sum_{n=0}^{\infty}\left((4^{2n+5}-4^{n+4})/(2n+4)!\right)x^{2n}}. \tag{28}$$

Let

$$a_n = \frac{4^{2n+4}}{(2n+4)!}, \qquad b_n = \frac{4^{2n+5} - 4^{n+4}}{(2n+4)!}. \tag{29}$$

Then

$$b_n > 0, \tag{30}$$

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = -\frac{3\cdot 4^n}{(4^{n+1}-1)(4^{n+2}-1)} < 0 \tag{31}$$

for all $n \geq 0$.

It, from Lemma 2 and (28)–(31), that $g_1'(x)/g_2'(x)$ is strictly increasing on $(0, \infty)$. Therefore, Lemma 5 follows easily from (26) and (27) together with Lemma 1 and the monotonicity of $g_1'(x)/g_2'(x)$. □

Lemma 6. *The function*

$$f_4(x) = \frac{3\sin^2(x) - 2x\sin(x) - x^2}{(1-\cos^2(x))} \tag{32}$$

is strictly increasing from $(0, \pi/2)$ *on* $(-8/3, -(\pi^2+4\pi-12)/4)$.

Proof. Differentiating $f_4(x)$ gives

$$f_4'(x) = \frac{2\sin(x)}{(1-\cos(x))^3}\left[x^2 + x\sin(x) + 4\cos(x) - 4\right]. \tag{33}$$

Let

$$H(x) = x^2 + x\sin(x) + 4\cos(x) - 4. \tag{34}$$

Then, simple computations lead to

$$H(0) = 0,$$
$$H'(x) = 2x + x\cos(x) - 3\sin(x), \tag{35}$$

$$H'(0) = 0,$$
$$H''(x) = 2 - x\sin(x) - 2\cos(x), \tag{36}$$

$$H''(0) = 0,$$
$$H'''(x) = \sin(x) - x\cos(x), \tag{37}$$

$$H'''(0) = 0, \tag{38}$$

$$H^{(4)}(x) = x\sin(x) > 0 \tag{39}$$

for $x \in (0, \pi/2)$.

Therefore, Lemma 6 follows easily from (33)–(39) together with the fact that $f_4(0^+) = -8/3$ and $f_4(\pi/2) = -(\pi^2 + 4\pi - 12)/4$. □

3. Main Results

Theorem 7. *Let $\alpha_1, \beta_1 \in [0, 1/2]$. Then the double inequality*

$$G(\alpha_1 a + (1-\alpha_1)b, \alpha_1 b + (1-\alpha_1)a) < N_{AG}(a,b) < G(\beta_1 a + (1-\beta_1)b, \beta_1 b + (1-\beta_1)a) \tag{40}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \le 1/2 - \sqrt{3}/4 = 0.06698\ldots$ and $\beta_1 \ge 1/2 - \sqrt{6}/6 = 0.09175\ldots$.

Proof. Since both the geometric mean $G(a,b)$ and arithmetic mean $A(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b) \in (0,1)$ and $\lambda \in [0,1/2]$. Then, from (4), one has

$$\begin{aligned}
&G(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) - N_{AG}(a,b) \\
&= A(a,b)\sqrt{1 - v^2(1-2\lambda)^2} \\
&\quad - \frac{1}{2}A(a,b)\left[1 + \frac{1-v^2}{v}\tanh^{-1}(v)\right] \\
&= \left(A(a,b)\left[\frac{1}{v^2} - \frac{1}{4}\left(\frac{1}{v} + \left(\frac{1}{v^2} - 1\right)\right.\right.\right. \\
&\qquad \left.\left.\left.\times \tanh^{-1}(v)\right)^2 - (1-2\lambda)^2\right]v\right) \\
&\quad \times \left(\sqrt{\frac{1}{v^2} - (1-2\lambda)^2}\right. \\
&\qquad \left. + \frac{1}{2}\left[\frac{1}{v} + \left(\frac{1}{v^2} - 1\right)\tanh^{-1}(v)\right]\right)^{-1}.
\end{aligned} \tag{41}$$

Let $t = \tanh^{-1}(v)$. Then $t \in (0, \infty)$, and

$$\begin{aligned}
&\frac{1}{v^2} - \frac{1}{4}\left(\frac{1}{v} + \left(\frac{1}{v^2} - 1\right)\tanh^{-1}(v)\right)^2 - (1-2\lambda)^2 \\
&= \frac{3\sinh^2(t)\cosh^2(t) - 2t\sinh(t)\cosh(t) - t^2}{4\sinh^4(t)} \\
&\quad - (1-2\lambda)^2 \\
&= \frac{3\sinh^2(2t) - 4t\sinh(2t) - 4t^2}{4(\cosh(2t) - 1)^2} - (1-2\lambda)^2 \\
&= \frac{1}{4}f_1(2t) - (1-2\lambda)^2,
\end{aligned} \tag{42}$$

where $f_1(t)$ is defined as in Lemma 3.

Therefore, Theorem 7 follows easily from (41) and (42) together with Lemma 3. □

Theorem 8. *Let $\alpha_2, \beta_2 \in [0, 1/2]$. Then the double inequality*

$$G(\alpha_2 a + (1-\alpha_2)b, \alpha_2 b + (1-\alpha_2)a) < N_{GA}(a,b) < G(\beta_2 a + (1-\beta_2)b, \beta_2 b + (1-\beta_2)a) \tag{43}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \le 1/2 - \sqrt{16 - \pi^2}/8 = 0.1905\ldots$ and $\beta_2 \ge 1/2 - \sqrt{3}/6 = 0.2113\ldots$.

Proof. We follow the same idea in the proof of Theorem 7. Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b) \in (0,1)$ and $\mu \in [0,1/2]$. Then, from (5), we get

$$\begin{aligned}
&G(\mu a + (1-\mu)b, \mu b + (1-\mu)a) - N_{GA}(a,b) \\
&= A(a,b)\sqrt{1 - v^2(1-2\mu)^2} \\
&\quad - \frac{1}{2}A(a,b)\left[\sqrt{1-v^2} + \frac{\sin^{-1}(v)}{v}\right] \\
&= \left(A(a,b)\left[\frac{1}{v^2} - \frac{1}{4}\left(\sqrt{\frac{1}{v^2} - 1} + \frac{\sin^{-1}(v)}{v^2}\right)^2\right.\right. \\
&\qquad \left.\left. - (1-2\mu)^2\right]v\right) \\
&\quad \times \left(\sqrt{\frac{1}{v^2} - (1-2\mu)^2} + \frac{1}{2}\left[\sqrt{\frac{1}{v^2} - 1} + \frac{\sin^{-1}(v)}{v^2}\right]\right)^{-1}.
\end{aligned} \tag{44}$$

Let $t = \sin^{-1}(v)$. Then, $t \in (0, \pi/2)$ and simple computation leads to

$$\begin{aligned}
&\frac{1}{v^2} - \frac{1}{4}\left(\sqrt{\frac{1}{v^2} - 1} + \frac{\sin^{-1}(v)}{v^2}\right)^2 - (1-2\mu)^2 \\
&= \frac{1 - (1/4)(\cos(t) + t/\sin(t))^2}{\sin^2(t)} - (1-2\mu)^2 \\
&= f_2(t) - (1-2\mu)^2,
\end{aligned} \tag{45}$$

where $f_2(t)$ is defined as in Lemma 4.

Therefore, Theorem 8 follows easily from (44) and (45) together with Lemma 4. □

Theorem 9. *Let $\alpha_3, \beta_3 \in [1/2, 1]$. Then the double inequality*

$$Q(\alpha_3 a + (1-\alpha_3) b, \alpha_3 b + (1-\alpha_3) a) < N_{QA}(a,b) < Q(\beta_3 a + (1-\beta_3) b, \beta_3 b + (1-\beta_3) a) \quad (46)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\beta_3 \geq 1/2 + \sqrt{3}/6 = 0.7886\ldots$ and $\alpha_3 \leq 1/2 + \sqrt{2\sqrt{2}\log(1+\sqrt{2}) + \log^2(1+\sqrt{2}) - 2}/4 = 0.7817\ldots$.

Proof. Since both the quadratic mean $Q(a,b)$ and arithmetic mean $A(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b) \in (0,1)$ and $p \in [1/2, 1]$. Then, (6) gives

$$\begin{aligned} &Q(pa + (1-p) b, pb + (1-p) a) - N_{QA}(a,b) \\ &= A(a,b)\sqrt{1 + v^2(2p-1)^2} - \frac{1}{2} A(a,b)\left[\sqrt{1+v^2} + \frac{\sinh^{-1}(v)}{v}\right] \\ &= \left(A(a,b)\left[\frac{1}{v^2} - \frac{1}{4}\left(\sqrt{\frac{1}{v^2}+1} + \frac{\sinh^{-1}(v)}{v^2}\right)^2 + (2p-1)^2\right] v\right) \\ &\quad \times \left(\sqrt{\frac{1}{v^2} + (2p-1)^2} + \frac{1}{2}\left[\sqrt{\frac{1}{v^2}+1} + \frac{\sinh^{-1}(v)}{v^2}\right]\right)^{-1}. \end{aligned} \quad (47)$$

Let $t = \sinh^{-1}(v)$. Then, $t \in (0, \log(1+\sqrt{2}))$ and elementary computations lead to

$$\begin{aligned} &\frac{1}{v^2} - \frac{1}{4}\left(\sqrt{\frac{1}{v^2}+1} + \frac{\sinh^{-1}(v)}{v^2}\right)^2 + (2p-1)^2 \\ &= \frac{1 - (1/4)\left[\cosh(t) + t/\sinh(t)\right]^2}{\sinh^2(t)} + (2p-1)^2 \\ &= f_3(t) + (2p-1)^2, \end{aligned} \quad (48)$$

where $f_3(t)$ is defined as in Lemma 5.

Therefore, Theorem 9 follows easily from (47) and (48) together with Lemma 5. □

Theorem 10. *Let $\alpha_4, \beta_4 \in [1/2, 1]$. Then the double inequality*

$$Q(\alpha_4 a + (1-\alpha_4) b, \alpha_4 b + (1-\alpha_4) a) < N_{AQ}(a,b) < Q(\beta_4 a + (1-\beta_4) b, \beta_4 b + (1-\beta_4) a) \quad (49)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_4 \leq 1/2 + \sqrt{\pi^2 + 4\pi - 12}/8 = 0.9038\ldots$ and $\beta_4 \geq 1/2 + \sqrt{6}/6 = 0.9082\ldots$.

Proof. We follow the same idea in the proof of Theorem 9. Let $a > b$, $v = (a-b)/(a+b) \in (0,1)$ and $q \in [1/2, 1]$. Then, from (7), we have

$$\begin{aligned} &Q(qa + (1-q) b, qb + (1-q) a) - N_{AQ}(a,b) \\ &= A(a,b)\sqrt{1 + v^2(2q-1)^2} - \frac{1}{2} A(a,b)\left[1 + \frac{1+v^2}{v}\tan^{-1}(v)\right] \\ &= \left(A(a,b)\left[\frac{1}{v^2} - \frac{1}{4}\left(\frac{1}{v} + \left(1 + \frac{1}{v^2}\right)\tan^{-1}(v)\right)^2 + (2q-1)^2\right] v\right) \\ &\quad \times \left(\sqrt{\frac{1}{v^2} + (2q-1)^2} + \frac{1}{2}\left[\frac{1}{v} + \left(1 + \frac{1}{v^2}\right)\tan^{-1}(v)\right]\right)^{-1}. \end{aligned} \quad (50)$$

Let $t = \tan^{-1}(v)$. Then $t \in (0, \pi/4)$ and

$$\begin{aligned} &\frac{1}{v^2} - \frac{1}{4}\left(\frac{1}{v} + \left(1 + \frac{1}{v^2}\right)\tan^{-1}(v)\right)^2 + (2q-1)^2 \\ &= \frac{3\sin^2(t)\cos^2(t) - 2t\sin(t)\cos(t) - t^2}{4\sin^4(t)} + (2q-1)^2 \\ &= \frac{3\sin^2(2t) - 4t\sin(2t) - 4t^2}{4\left[1 - \cos(2t)\right]^2} + (2q-1)^2 \\ &= \frac{1}{4} f_4(2t) + (2q-1)^2, \end{aligned} \quad (51)$$

where $f_4(t)$ is defined as in Lemma 6.

Therefore, Theorem 10 follows easily from (50) and (51) together with Lemma 6. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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