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Research Article

The Existence of a Global Attractor for the S-K-T Competition Model with Self-Diffusion

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This paper concerns the uniform bounds of the global existence of solutions in time for the S-K-T competition model with self-diffusion. We prove that the system has a global attractor for $n < 8$.

1. Introduction and Statement of Main Result

Shigesada et al. [1] introduced the following competition model to describe the spatial segregation of two competing species under inter- and intraspecies population pressures:

$$\begin{aligned} u_t &= \Delta\left[\left(d_1+\alpha_{11}u+\alpha_{12}v\right)u\right]+u\left(a_1-b_1u-c_1v\right), \\ &\qquad x\in\Omega\subset\mathbb{R}^n,\quad t>0, \\ v_t &= \Delta\left[\left(d_2+\alpha_{21}u+\alpha_{22}v\right)v\right]+v\left(a_2-b_2u-c_2v\right), \\ &\qquad x\in\Omega\subset\mathbb{R}^n,\quad t>0, \\ \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n}=0,\quad x\in\partial\Omega,\ t>0, \end{aligned} \tag{1}$$

where Ω is a bounded smooth region in $\mathbb{R}^n$ with n as its unit outward normal vector to the smooth boundary $\partial\Omega$. u and v are the population densities of the two competing species. The constants a_j, b_j, c_j, and d_j $(j=1,2)$ are all positive, and constants α_{ij} $(i,j=1,2)$ are nonnegative. d_1 and d_2 are the random diffusion rates, α_{11} and α_{22} are the self-diffusion rates which represent intraspecific population pressures, and α_{12} and α_{21} are the so-called cross-diffusion rates which represent the interspecific population pressures.

If $\alpha_{ij}=0$ $(i,j=1,2)$, system (1) is reduced to the classical Lotka-Volterra competition model with diffusion; it has been extensively studied in the past few decades. When initial value is nonnegative and bounded, it is easy to prove that (1) has a unique uniformly bounded global solution.

For $\alpha_{11}=0$, the global existence of solutions has been widely investigated by many authors. When $n=1$, $d_1=d_2$, $\alpha_{12}>0$, $\alpha_{21}>0$, and $\alpha_{11}=\alpha_{22}=0$ hold, Kim [2] proved the global existence of classical solutions by energy method. For $n\geq 1$, $\alpha_{11}=\alpha_{22}=0$, Deuring [3] proved the global existence of solutions if α_{12} and α_{21} are small enough depending on the $C^{2,\alpha}$ norm of initial values u_0, v_0. Choi et al. [4] improved Deuring's result and proved the global existence of solutions if the cross-diffusion coefficients are small depending only on the L^∞ norm of initial value v_0. By applying more detailed interpolated estimates, especially Gagliardo-Nirenberg inequality, Shim [5] improved Kim and Deuring's results and established the uniform bounds of the global existence of solutions in time. For $n=2$, Lou et al. [6] established the unique global existence of solutions for $\alpha_{21}=0$, $\alpha_{12}>0$, $\alpha_{11}=0$, and $\alpha_{22}\geq 0$.

For $\alpha_{11}>0$, (1) can be written as

$$\begin{aligned} u_t &= \Delta\left[\left(d_1+\alpha_{11}u+\alpha_{12}v\right)u\right]+u\left(a_1-b_1u-c_1v\right), \\ &\qquad x\in\Omega\subset\mathbb{R}^n,\quad t>0, \\ v_t &= \Delta\left[\left(d_2+\alpha_{22}v\right)v\right]+v\left(a_2-b_2u-c_2v\right), \\ &\qquad x\in\Omega\subset\mathbb{R}^n,\quad t>0, \end{aligned}$$

$$\frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0,\quad x\in\partial\Omega,\ t>0. \tag{2}$$

Equation (2) has been investigated by many authors; we state the results as follows.

For $n=2$, either $8\alpha_{11}>\alpha_{12}>0$, $8\alpha_{22}>\alpha_{21}>0$ or $\alpha_{22}=\alpha_{21}=0$, $\alpha_{11}>0$; Yagi [7] proved the global existence of solutions. For $\alpha_{11}>0$, $\alpha_{22}>0$, and $\alpha_{21}=0$, Kuiper and Dung [8] established the uniform bounds of global solutions for any n when $\|v\|_{L^\infty(\Omega)}$ and $\|u\|_{L^p(\Omega)}$ $(p>n)$ are uniformly bounded. Choi et al. [9] applied more detailed interpolated estimates and energy methods to prove the global existence of solutions for $n<6$, $\alpha_{11}>0$, and $\alpha_{22}>0$.

Le and his collaborators [10] have shown the existence of a global attractor for (2) in case $n\le 5$. Le and Nguyen [11] constructed a special test function to prove the global existence of solutions for any dimension n under some certain restrictions on coefficients. Tuôc [12] improved the results of Le and Nguyen by a nontrivial application of maximum principle. Recently, Tuoc [13] has established the L^4-estimate of ∇v; then by an iteration method, they show $u\in L^r$ for any $r\ge 1$ and $n<10$, which implies the global existence of solutions.

In this paper, we consider the uniform bounds of the global existence of solutions in time of system (2) for $\alpha_{21}=0$, $\alpha_{11}>0$, and $\alpha_{22}>0$. In Section 2, we show some preliminary knowledge used in this paper. In Section 3, we follow the arguments of Le et al. and improve their results. We will prove the uniform bounds of the global existence of solutions in time of system (2) for $n<8$.

The main result in this paper is as follows.

Theorem 1. *Assume $n<8$ holds; for any $p_0>n$, system* (2) *has a global attractor with finite Hausdorff dimension in the space $\mathcal{X}$ defined by*

$$\begin{aligned}\mathcal{X}=\Big\{&(u,v)\in W^{1,p_0}(\Omega)\times W^{1,p_0}(\Omega):\\ &u(x)\ge 0,\ v(x)\ge 0,\ \forall x\in\Omega\Big\}.\end{aligned} \tag{3}$$

2. Preliminary Results

System (2) can be written in the divergence form as

$$\begin{aligned}\frac{\partial u}{\partial t}&=\nabla\left[\left(d_1+2\alpha_{11}u+\alpha_{12}v\right)\nabla u+\alpha_{12}u\nabla v\right]\\ &\quad+u\left(a_1-b_1u-c_1v\right),\quad x\in\Omega,\ t>0,\\ \frac{\partial v}{\partial t}&=\nabla\left[\left(d_2+2\alpha_{22}v\right)\nabla v\right]\\ &\quad+v\left(a_2-b_2u-c_2v\right),\quad x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial n}&=\frac{\partial v}{\partial n}=0,\quad x\in\partial\Omega.\end{aligned} \tag{4}$$

Definition 2 (see [10, Definition 2.1]). Assume that there exists a solution (u,v) of system (4) defined on a subinterval I of $\mathbb{R}^+$. Let $\mathcal{O}$ be the set of function ω on I such that there exists a positive constant C_0, which may generally depend on the parameters of the system and the W^{1,p_0} norm of the initial value (u_0,v_0), such that

$$\omega(t)\le C_0,\quad \forall t\in I. \tag{5}$$

Furthermore, if $I=(0,\infty)$, one says that ω is in $\mathcal{P}$ if $\omega\in\mathcal{O}$ and there exists a positive constant C_∞ that depends only on the parameters of the system but does not depend on the initial value of (u_0,v_0) such that

$$\lim_{t\downarrow\infty}\sup\omega(t)\le C_\infty. \tag{6}$$

If $\omega\in\mathcal{P}$ and $I=(0,\infty)$, one says ω is ultimately uniformly bounded.

Lemma 3 (the uniform Gronwall inequality). *Assume that $u(t)\ge 0$, $a(t)\ge 0$, and $b(t)\ge 0$ hold and that they are integrable in $[t_0,+\infty]$ satisfying*

$$\begin{gathered}\int_t^{t+r}a(s)\,ds\le a,\qquad \int_t^{t+r}b(s)\,ds\le b,\\ \int_t^{t+r}u(s)\,ds\le C,\end{gathered} \tag{7}$$

where a, b, and C are positive constants. If $u'(t)\le a(t)u(t)+b(t)$, then one has

$$u(t+r)\le\left(\frac{C}{r}+b\right)e^a,\quad \forall t\ge t_0. \tag{8}$$

Lemma 4 (see [10, Lemmas 3.2-3.3]). *For any dimension n, one has the following estimates for the solutions of system* (4):

$$\|v\|_{L^\infty(\Omega)}\in\mathcal{P}, \tag{9}$$

$$\|\nabla v(\cdot,t)\|_{L^2(\Omega)}\in\mathcal{P}, \tag{10}$$

$$\|u(\cdot,t)\|_{L^1(\Omega)}\in\mathcal{P}, \tag{11}$$

$$\int_t^{t+1}\int_\Omega u^2(x,s)\,dx\,ds\in\mathcal{P}, \tag{12}$$

$$\int_t^{t+1}\int_\Omega v_t^2(x,s)\,dx\,ds\in\mathcal{P}. \tag{13}$$

Lemma 5 (see [10, Theorem 2.4]). *For the system* (4), *if*

$$\|u\|_{q,r,[t,t+1]\times\Omega}=\left(\int_t^{t+1}\|u(\cdot,s)\|_{q,\Omega}^r ds\right)^{1/r}\in\mathcal{P} \tag{14}$$

holds, with q, r satisfying

$$\frac{1}{r}+\frac{n}{2q}=1-\chi,q\in\left[\frac{n}{2(1-\chi)},\infty\right],\qquad r\in\left[\frac{1}{1-\chi},\infty\right], \tag{15}$$

where $\chi\in(0,1)$, then there exists $\gamma>1$ such that

$$\|v(\cdot,t)\|_{C^\gamma(\Omega)}\in\mathcal{P},\qquad \|u(\cdot,t)\|_{C^\gamma(\Omega)}\in\mathcal{P}. \tag{16}$$

3. Proof of Theorem 1

Lemma 6. *For any dimension n, any solution u of (4) has the following estimate:*

$$\int_t^{t+1}\int_\Omega |\nabla v|^4 dx\, ds \in \mathscr{P}. \tag{17}$$

Proof. Define

$$w = (d_2 + \alpha_{22} v)\, v; \tag{18}$$

then w satisfies the following equation:

$$w_t = (d_2 + 2\alpha_{22} v)\,\Delta w + (d_2 + 2\alpha_{22} v)\, v\,(a_2 - b_2 u - c_2 v). \tag{19}$$

Multiplying (19) by Δw and integrating with respect to x over Ω, we have

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx &= \int_\Omega (d_2 + 2\alpha_{22} v)\,|\Delta w|^2 dx \\ &\quad + \int_\Omega (d_2 + 2\alpha_{22} v)\, v\,(a_2 - b_2 u - c_2 v)\,\Delta w\, dx. \end{aligned} \tag{20}$$

Integrating (20) over $[t, t+1]$, we obtain

$$\begin{aligned} &\frac{1}{2}\|\nabla w(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla w(t+1)\|^2_{L^2(\Omega)} \\ &\quad = \int_t^{t+1}\int_\Omega (d_2 + 2\alpha_{22} v)\,|\Delta w|^2 dx\, ds \\ &\qquad + \int_t^{t+1}\int_\Omega (d_2 + 2\alpha_{22} v)\, v\,(a_2 - b_2 u - c_2 v)\,\Delta w\, dx\, ds. \end{aligned} \tag{21}$$

In virtue of (9), there exist positive constants C_1, C_2, and C_3 such that

$$\begin{aligned} C_1\int_t^{t+1}\int_\Omega |\Delta w|^2 dx\, ds &\le \frac{1}{2}\|\nabla w(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla w(t+1)\|^2_{L^2(\Omega)} \\ &\quad + \int_t^{t+1}\int_\Omega (C_2 + C_3 u)\,|\Delta w|\, dx\, ds. \end{aligned} \tag{22}$$

Here (18) implies

$$\nabla w = d_2 \nabla v + 2\alpha_{22} v \nabla v. \tag{23}$$

By (9)-(10) and (23), we have

$$\|\nabla w\|_{L^2(\Omega)} \in \mathscr{P}. \tag{24}$$

Hence (22) and Hölder's inequality imply

$$\begin{aligned} C_1\int_t^{t+1}\int_\Omega |\Delta w|^2 dx\, ds &\le C_4 + \frac{C_1}{2}\int_t^{t+1}\int_\Omega |\Delta w|^2 dx\, ds \\ &\quad + C_5\int_t^{t+1}\int_\Omega (C_2 + C_3 u)^2 dx\, ds. \end{aligned} \tag{25}$$

By (12) and (25), we get

$$\int_t^{t+1}\int_\Omega |\Delta w|^2 dx\, ds \in \mathscr{P}. \tag{26}$$

Multiplying (19) by $w|\nabla w|^2$ and integrating with respect to x over Ω, we have

$$\begin{aligned} &\int_\Omega w_t w |\nabla w|^2 dx \\ &\quad = \int_\Omega \Delta w\,(d_2 + 2\alpha_{22} v)\, w|\nabla w|^2 dx + \int_\Omega f w|\nabla w|^2 dx \\ &\quad = -\int_\Omega \nabla w \nabla\left[(d_2 + 2\alpha_{22} v)\, w|\nabla w|^2\right] dx + \int_\Omega f w|\nabla w|^2 dx \\ &\quad \le -\int_\Omega (d_2 + 2\alpha_{22} v)\,|\nabla w|^4 dx \\ &\qquad + \int_\Omega 2\,(d_2 + 2\alpha_{22} v)\, w|\nabla w|^2 \left|\nabla^2 w\right| dx \\ &\qquad - \int_\Omega w|\nabla w|^2 2\alpha_{22} \nabla v \nabla w\, dx + \int_\Omega f w|\nabla w|^2 dx, \end{aligned} \tag{27}$$

with $f = (d_2 + 2\alpha_{22} v)v(a_2 - b_2 u - c_2 v)$.

By (27), we get

$$\begin{aligned} &\int_\Omega (d_2 + 2\alpha_{22} v)\,|\nabla w|^4 dx \\ &\quad \le -\int_\Omega w_t w|\nabla w|^2 dx + \int_\Omega 2\,(d_2 + 2\alpha_{22} v)\, w|\nabla w|^2 \left|\nabla^2 w\right| dx \\ &\qquad - \int_\Omega w|\nabla w|^2 2\alpha_{22}\left(\frac{\nabla w}{d_2 + 2\alpha_{22} v}\right)\nabla w\, dx \\ &\qquad + \int_\Omega f w|\nabla w|^2 dx. \end{aligned} \tag{28}$$

Recall that (9) and (18) yield

$$\|w\|_{L^\infty(\Omega)} \in \mathscr{P}. \tag{29}$$

It follows from (28) and (29) that

$$\begin{aligned} &\int_\Omega (d_2 + 4\alpha_{22} v)\,|\nabla w|^4 dx \\ &\quad \le C\left(\int_\Omega |w_t|\,|\nabla w|^2 dx + \int_\Omega |\nabla w|^2 \left|\nabla^2 w\right| dx\right) \\ &\qquad + \int_\Omega |f|\,|\nabla w|^2 dx. \end{aligned} \tag{30}$$

By Young's inequality and (30)

$$\begin{aligned} d_2 \int_\Omega |\nabla w|^4 dx &\le C\left(\frac{d_2}{3C}\int_\Omega |\nabla w|^4 dx + C_6 \int_\Omega |w_t|^2 dx + \frac{d_2}{4C}\int_\Omega |\nabla w|^4 dx \right. \\ &\quad + C_7 \int_\Omega \left|\nabla^2 w\right|^2 dx + \frac{d_2}{3C}\int_\Omega |\nabla w|^4 dx \\ &\quad \left. + C_8 \int_\Omega |f|^2 dx\right). \end{aligned} \tag{31}$$

Since

$$\int_\Omega \left|\nabla^2 w\right|^2 dx \le C_0 \int_\Omega |\Delta w|^2 dx + C_0 \int_\Omega u^2 dx, \tag{32}$$

together with (31), we see from (31) that

$$\begin{aligned} \int_t^{t+1}\int_\Omega |\nabla w|^4 dx\,ds &\le C\left(\int_t^{t+1}\int_\Omega |w_t|^2 dx\,ds + \int_t^{t+1}\int_\Omega |\Delta w|^2 dx\,ds \right. \\ &\quad \left. + \int_t^{t+1}\int_\Omega |u|^2 dx\,ds\right) \le \widetilde{C}, \end{aligned} \tag{33}$$

where $\widetilde{C}$ is independent of t.

Since

$$w = (d_2 + \alpha_{22} v) v, \quad w_t = d_2 v_t + 2\alpha_{22} v v_t, \tag{34}$$

together with (9) and (13), we have $\int_t^{t+1}\int_\Omega w_t^2(x,s)dx\,ds \in \mathscr{P}$. This fact, together with (12) and (26), implies $\int_t^{t+1}\int_\Omega |\nabla w(x,s)|^4 dx\,ds \in \mathscr{P}$. Hence, in view of $\nabla v = \nabla w/(d_2 + 2\alpha_{22} v)$ and (9), we get the desired result $\int_t^{t+1}\int_\Omega |\nabla v|^4 dx\,ds \in \mathscr{P}$. □

Lemma 7. *For any dimension n, any solution u of* (4) *satisfies the following estimates:*

$$\|u\|_{L^2(\Omega)} \in \mathscr{P}, \qquad \|u\|_{L^3(\Omega)} \in \mathscr{P}. \tag{35}$$

Proof. Multiplying the first equation of (4) by u and integrating, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx &= -\int_\Omega \left[(d_1 + 2\alpha_{11} u + \alpha_{12} v)|\nabla u|^2 - \alpha_{12} u \nabla u \cdot \nabla v\right] dx \\ &\quad + \int_\Omega u^2 (a_1 - b_1 u - c_1 v)\, dx. \end{aligned} \tag{36}$$

Young's inequality and (36) imply

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega d_1 |\nabla u|^2 dx + 2\alpha_{11}\int_\Omega u|\nabla u|^2 dx \\ &\quad + \alpha_{12}\int_\Omega v|\nabla u|^2 dx \\ &= -\alpha_{12}\int_\Omega u\nabla u\cdot\nabla v\, dx + \int_\Omega u^2 (a_1 - b_1 u - c_1 v)\, dx \\ &\le \varepsilon \int_\Omega u|\nabla u|^2 dx + C(\varepsilon)\int_\Omega u|\nabla v|^2 dx + \int_\Omega a_1 u^2 dx. \end{aligned} \tag{37}$$

Taking $\varepsilon = \alpha_{11}$ in (37), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega d_1 |\nabla u|^2 dx + \alpha_{11}\int_\Omega u|\nabla u|^2 dx \\ &\quad + \alpha_{12}\int_\Omega v|\nabla u|^2 dx \\ &\le C(\alpha_{11},\alpha_{12})\int_\Omega u|\nabla v|^2 dx + \int_\Omega a_1 u^2 dx \\ &\le C_9 \int_\Omega u^2 dx + C_{10}\int_\Omega |\nabla v|^4 dx + \int_\Omega a_1 u^2 dx. \end{aligned} \tag{38}$$

By the uniform Gronwall inequality, together with (12), (17), and (38), we obtain

$$\|u\|_{L^2(\Omega)} \in \mathscr{P}. \tag{39}$$

In virtue of (36), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega d_1 |\nabla u|^2 dx + 2\alpha_{11}\int_\Omega u|\nabla u|^2 dx \\ &\quad + \alpha_{12}\int_\Omega v|\nabla u|^2 dx + b_1 \int_\Omega u^3 dx \\ &= -\alpha_{12}\int_\Omega u\nabla u\cdot\nabla v\, dx + \int_\Omega u^2 (a_1 - c_1 v)\, dx. \end{aligned} \tag{40}$$

Integrating (40) over $[t, t+1]$, we get

$$\begin{aligned} &\frac{1}{2}\|u(t+1)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_t^{t+1}\int_\Omega d_1|\nabla u|^2 dx\,ds \\ &\quad + 2\alpha_{11}\int_t^{t+1}\int_\Omega u|\nabla u|^2 dx\,ds + \alpha_{12}\int_t^{t+1}\int_\Omega v|\nabla u|^2 dx\,ds \\ &\quad + b_1\int_t^{t+1}\int_\Omega u^3 dx\,ds \\ &\le \varepsilon\alpha_{12}\int_t^{t+1}\int_\Omega u|\nabla u|^2 dx\,ds + C\int_t^{t+1}\int_\Omega u|\nabla v|^2 dx\,ds \\ &\quad + \int_t^{t+1}\int_\Omega a_1 u^2 dx\,ds. \end{aligned} \tag{41}$$

By Young's inequality, we have

$$\begin{aligned}\int_t^{t+1}\int_\Omega u|\nabla v|^2 dx\, ds \\ \le \frac{1}{2}\int_t^{t+1}\int_\Omega u^2 dx\, ds + \frac{1}{2}\int_t^{t+1}\int_\Omega |\nabla v|^4 dx\, ds.\end{aligned} \tag{42}$$

Taking $\varepsilon = \alpha_{11}/\alpha_{12}$ in (41) and applying Hölder's inequality, we see from (42) that

$$\begin{aligned} b_1\int_t^{t+1}\int_\Omega u^3 dx\, ds \\ \le \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(t+1)\|^2_{L^2(\Omega)} \\ + C_{11}\int_t^{t+1}\int_\Omega u^2 dx\, ds + C_{12}\int_t^{t+1}\int_\Omega |\nabla v|^4 dx\, ds.\end{aligned} \tag{43}$$

By (12), (17), and (39), we get

$$\int_t^{t+1}\int_\Omega u^3 dx\, ds \in \mathcal{P}. \tag{44}$$

Next we prove $\|u\|_{L^3(\Omega)} \in \mathcal{P}$. Multiplying (4) by u^2 and integrating with respect to x over Ω, we get

$$\begin{aligned}\frac{1}{3}\frac{d}{dt}\int_\Omega u^3 dx + 2\int_\Omega d_1 u|\nabla u|^2 dx + 4\alpha_{11}\int_\Omega u^2|\nabla u|^2 dx \\ + 2\alpha_{12}\int_\Omega uv|\nabla u|^2 dx \\ = -2\int_\Omega \alpha_{12}u^2\nabla v\cdot\nabla u\, dx + \int_\Omega u^3(a_1 - b_1 u - c_1 v)\, dx.\end{aligned} \tag{45}$$

Apply the following inequalities:

$$\begin{aligned}\int_\Omega v^2 dx \le \varepsilon\left(\int_\Omega |\nabla v|^2 dx + \|v\|^2_{L^1(\Omega)}\right) + C\varepsilon^{-n/2}\|v\|^2_{L^1(\Omega)}, \\ \int_\Omega u^2\nabla u\cdot\nabla v\, dx \le \varepsilon_1\int_\Omega u^2|\nabla u|^2 dx + C(\varepsilon_1)\int_\Omega u^2|\nabla v|^2 dx, \\ \int_\Omega u^2|\nabla v|^2 dx \le \frac{1}{2}\int_\Omega u^4 dx + \frac{1}{2}\int_\Omega |\nabla v|^4 dx.\end{aligned} \tag{46}$$

Use (46) with $v = u^2$ to get

$$\begin{aligned}\int_\Omega u^4 dx \le \varepsilon\left\{\int_\Omega \left|\nabla\left(u^2\right)\right|^2 dx + \left\|u^2\right\|^2_{L^1(\Omega)}\right\} + C\varepsilon^{-n/2}\left\|u^2\right\|^2_{L^1(\Omega)} \\ = \varepsilon\left\{4\int_\Omega u^2|\nabla u|^2 dx + \|u\|^4_{L^2(\Omega)}\right\} + C\varepsilon^{-n/2}\|u\|^4_{L^2(\Omega)}.\end{aligned} \tag{47}$$

Choosing small positive numbers ε and ε_1 in the above inequalities, we get

$$\begin{aligned}\frac{1}{3}\frac{d}{dt}\int_\Omega u^3 dx + 2\int_\Omega d_1 u|\nabla u|^2 dx + \alpha_{11}\int_\Omega u^2|\nabla u|^2 dx \\ + 2\alpha_{12}\int_\Omega uv|\nabla u|^2 dx \\ \le C\left(\int_\Omega u^2 dx\right)^2 + C\int_\Omega |\nabla v|^4 dx + \int_\Omega a_1 u^3 dx.\end{aligned} \tag{48}$$

By (17), (39), (44), (48), and uniform Gronwall's inequality, we get the desired result

$$\|u\|_{L^3(\Omega)} \in \mathcal{P}. \tag{49}$$

□

Proof of Theorem 1. It follows from (48) that

$$\begin{aligned}\alpha_{11}\int_t^{t+1}\int_\Omega u^2|\nabla u|^2 dx\, ds \\ \le \frac{1}{3}\|u(t)\|^3_{L^3(\Omega)} - \frac{1}{3}\|u(t+1)\|^3_{L^3(\Omega)} \\ + C\int_t^{t+1}\left(\int_\Omega u^2 dx\right)^2 ds + C\int_t^{t+1}\int_\Omega |\nabla v|^4 dx\, ds \\ + \int_t^{t+1}\int_\Omega a_1 u^3 dx\, ds.\end{aligned} \tag{50}$$

In virtue of (17), (35), (44), and (50), we obtain

$$\int_t^{t+1}\int_\Omega u^2|\nabla u|^2 dx\, ds \in \mathcal{P}. \tag{51}$$

For $l = 2$, $v = u^l$, we see $\int_t^{t+1}\int_\Omega |\nabla v|^2 dx\, ds = 4\int_t^{t+1}\int_\Omega u^2|\nabla u|^2 dx\, ds \in \mathcal{P}$.

Let $w = v - \int_\Omega v\, dx$; then Gagliardo-Nirenberg inequality gives

$$\|w\|_{L^{2^*}(\Omega)} \le C\|\nabla w\|_{L^2(\Omega)}, \tag{52}$$

which implies

$$\|v\|_{L^{2^*}(\Omega)} \le C\left(\|\nabla v\|_{L^2(\Omega)} + \|v\|_{L^1(\Omega)}\right), \tag{53}$$

with $2^* = 2n/(n-2)$.

For $r = 2l$, $q = 2^* l$, in virtue of (53), we have

$$\begin{aligned}\int_t^{t+1}\|u\|^r_{L^q(\Omega)} ds = \int_t^{t+1}\|v\|^2_{L^{2^*}(\Omega)} ds \\ \le C\left(\int_t^{t+1}\|\nabla v\|^2_{L^2(\Omega)} ds + \sup_{[t,t+1]}\|v\|^2_{L^1(\Omega)}\right).\end{aligned} \tag{54}$$

Note

$$\|v\|^2_{L^1(\Omega)} = \|u(\cdot,t)\|^l_{L^l(\Omega)} = \|u(\cdot,t)\|^2_{L^2(\Omega)} \in \mathcal{P}; \tag{55}$$

thus $\int_t^{t+1} \|u\|_{L^q(\Omega)}^{\gamma} ds \in \mathscr{P}$, with q, r satisfying

$$1 - \chi := \frac{1}{r} + \frac{n}{2q} = \frac{1}{l}\left(\frac{1}{2} + \frac{n}{2 \cdot 2^*}\right) = \frac{n}{4l}. \tag{56}$$

Let

$$A = q - \frac{n}{2(1-\chi)} = q - 2l, \qquad B = r - \frac{1}{1-\chi} = 2l - \frac{4l}{n}, \tag{57}$$

when $l = 2$ holds; in order to satisfy (15) in Lemma 5, we need to check $\chi \in (0,1)$, $A \geq 0$, and $B \geq 0$. By (56), we have the following results:

$$n = 3 \qquad \chi = \frac{5}{8} \qquad A = 8 \qquad B = \frac{4}{3}, \tag{58}$$

$$n = 4 \qquad \chi = \frac{1}{2} \qquad A = 4 \qquad B = 2, \tag{59}$$

$$n = 5 \qquad \chi = \frac{3}{8} \qquad A = \frac{8}{3} \qquad B = \frac{12}{5}, \tag{60}$$

$$n = 6 \qquad \chi = \frac{1}{4} \qquad A = 2 \qquad B = \frac{8}{3}, \tag{61}$$

$$n = 7 \qquad \chi = \frac{1}{8} \qquad A = \frac{8}{5} \qquad B = \frac{20}{7}. \tag{62}$$

Since $C^{\upsilon} \times C^{\upsilon}$ $(\upsilon > 1)$ is compact in $\mathscr{X}$, by the attractor theory in [14], we complete the proof of Theorem 1. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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