

Research Article

A Note on Some Numerical Approaches to Solve a θ Neuron Networks Model

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Space time integration plays an important role in analyzing scientific and engineering models. In this paper, we consider an integrodifferential equation that comes from modeling θ neuron networks. Here, we investigate various schemes for time discretization of a theta-neuron model. We use collocation and midpoint quadrature formula for space integration and then apply various time integration schemes to get a full discrete system. We present some computational results to demonstrate the schemes.

1. Introduction

Modeling real life problems using differential and integral operators and search for numerical schemes of such models are of ongoing interest [1–9]. We consider such a nonlinear model of transmission line in neural networks with “ θ -synapses” during bursting activity [8, 10, 11]:

$$\theta_t(x, t) = \varepsilon \int_{\Omega} J^{\infty}(x - y) \theta_t(y, t) dy + f(\theta, (x, t)) \quad (1)$$

with initial function $\theta(x, 0) = \theta_0(x)$, where $x \in \Omega \subseteq \mathbb{R}$, $t \geq 0$, the angle function $\theta(x, t)$ represents the phase of the signal associated with a neuron at (x, t) , f is a smooth function that represents potential effects and external inputs, J^{∞} is a kernel function, and $\varepsilon > 0$ is the parameter of the model. When $J^{\infty} < 0$ the model (1) represents inhibitory neurons and when $J^{\infty} > 0$ the model is an excitatory one. For this paper, we consider $J^{\infty}(x) \geq 0$ and $\int_{\Omega} J^{\infty}(x) dx = 1$, which is a nonnegative normalized kernel function. Now if we consider a normalized kernel and $\Omega = \mathbb{R}$, then (1) can be written as

$$(\mathcal{L}\theta_t(\cdot, t))(x) = f(\theta, (x, t)), \quad (2)$$

where $(\mathcal{L}\psi)(x) = \int_{\Omega} J^{\infty}(x - y)(\psi(x) - \varepsilon\psi(y))dy$. In most articles $f(t, \theta) = \frac{1}{a(t)} \pm \cos(\theta)$ has been considered as

a nonlinearity. Here $\theta \rightarrow \cos^{-1}a(t)$ as $t \rightarrow \infty$ for all $0 \leq a(t) \leq 1$ which stabilizes the output, and $\theta \rightarrow \infty$ when $a(t) > 1$ which oscillates the output [12]. One may observe a saddle node bifurcation when $a(t)$ increases or decreases through the value $a(t) = 1$. In this study we consider

$$a(t) = \begin{cases} 2, & \text{if } t \leq 3 \\ 1, & \text{otherwise.} \end{cases} \quad (3)$$

It is well understood from the studies [10, 11] that $\varepsilon < 1$ can be an excitatory parameter and the Gaussian kernels are associated with its bidirectional influence. So we find an intense interest in using a normalized Gaussian kernel function:

$$J^{\infty}(x) = \sqrt{\frac{\gamma}{\pi}} \exp(-\gamma x^2), \quad (4)$$

where $\gamma > 0$, $x \in \Omega$, and $0 \leq \varepsilon < 1$. Now the problem, with a kernel of type

$$J^{\infty}(x) = \begin{cases} \sqrt{\frac{\gamma}{\pi}} \exp(-\gamma x^2), & \text{when } x \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

and $\gamma > 0$, corresponds to unidirectional connectivity. Thus (2) describes a one-dimensional chain of single neurons interacting with each other where the interaction depends on the choices of the kernel function $J^\infty(x)$. From the detailed study in [10] that the integral operator is positive semidefinite, bounded and invertible operator if $0 < \varepsilon < 1$ which has been well discussed in the next section.

Our study is motivated by [6]. In [6], the authors study numerical approximation of a nonlocal, partly nonlinear, phase transitions model. They analyze and approximate the problem using various schemes, being a finite difference method, finite element methods with collocation and the Galerkin approach (using piecewise Lagrange polynomials to form finite element basis functions), and the Legendre and Tchebychev spectral methods in space followed by implicit schemes for the time integration. The authors demonstrate some numerical solutions as well as the computational error. They also estimate the theoretical errors of finite difference approximations and finite element approximations.

In [8, 11], Jackiewicz et al. consider the model (1). They use the forward Euler method for time integration to form the resulting model as an integral of Fredholm type. Then the authors approximate the resulting problem using various spectral collocation methods. They present some numerical results to demonstrate their schemes. The motivation was to use global polynomials to approximate $\theta(\cdot, t)$. Solutions converge fast in such approximations if one considers smooth initial condition as well as smooth boundaries.

In [8, 11], the authors consider forward Euler scheme for time integration only. Thus we find an interest to approximate the problem using piecewise basis functions for spatial approximation and then investigate various time integration schemes.

Now if we consider a spatially one-periodic initial function $\theta(x, 0)$, then for all $x \in \mathbb{R}$ and $t \in \mathbb{R}_+$, that is, $\theta(x, t) = \theta(x + 1, t)$. Then (2) can be written as

$$(\mathcal{L}\theta_t(\cdot, t))(x) = f(\theta(x, t)), \quad (6)$$

where

$$(\mathcal{L}\psi)(x) = \int_0^1 J(x-y)(\psi(x) - \varepsilon\psi(y)) dy, \quad (7)$$

with

$$J(x) = \sum_{r=-\infty}^{\infty} J^\infty(x-r), \quad \forall x \in [0, 1]. \quad (8)$$

We are interested to consider the periodic domain $\Omega = [0, 1]$ for spatial approximations of the model.

However, it is well understood from [13] that an integrodifferential equation of type (2) defined in the infinite domain can be defined in a truncated finite domain $[A, B]$, where A and B depend on the decay of the kernel function $J^\infty(x)$. A closed form formula to find suitable A and B is well presented in [13]. Thus the analysis and the approximation we present here in a periodic spatial interval $\Omega = [0, 1]$ can also be applied to any bounded interval $[A, B]$.

In this study, we consider the integrodifferential equation (6) with a Gaussian kernel defined by (8) and $0 < \varepsilon < 1$.

The rest of the paper is organized in the following way. In Section 2, we discuss some preliminary results. We present the approximation of the problem using collocation and quadrature for space integration in Section 3. We present some time integration schemes in Section 4. We conclude this study in Section 5 presenting some numerical results and discussions.

2. Preliminaries

In this section we discuss some properties of the model operator which shows the boundedness and inevitability of the integral operator. Here

$$\begin{aligned} &(\theta_t, \mathcal{L}\theta_t) \\ &= \iint_{\Omega} J^\infty(x-y) (\theta_t^2(x, t) - \varepsilon\theta_t(x, t)\theta_t(y, t)) dy dx \\ &= \frac{1}{2} \iint_{\Omega} J^\infty(x-y) (\theta_t^2(x, t) + \theta_t^2(y, t) \\ &\quad - 2\varepsilon\theta_t(x, t)\theta_t(y, t)) dy dx, \end{aligned} \quad (9)$$

and thus $(\theta_t, \mathcal{L}\theta_t) \geq 0$. This shows the positive semidefiniteness of the operator \mathcal{L} when $\varepsilon \leq 1$. To investigate the boundedness property of the operator \mathcal{L} let us introduce a proposition. For more details see [3].

Theorem 1 (see [3]). Assume that $J(x) \in L_1(\mathbb{R})$ and the following conditions hold:

- (H1) is $J(x) \geq 0$;
- (H2) is normalized such that $\int_{-\infty}^{\infty} K(x) dx = 1$;
- (H3) is symmetric; that is, $J(x) = J(-x)$, for all $x \in \mathbb{R}$;
- (H4) is decreasing on $(0, \infty)$;
- (H5) is $\widehat{K}(\xi) > 0$.

Then (H1)–(H4) give the DFT results $0 \leq \widehat{J}(0)$ and $\widehat{J}(\xi) \leq \widehat{J}(0) \leq (2/\sqrt{2\pi}) + \widehat{J}(\xi)$ for all $\xi \in [-(\pi/h), \pi/h]$ and the CFT results $\widehat{J}(\xi) \leq \widehat{J}(0) \leq (2/\sqrt{2\pi}) + \widehat{J}(\xi)$. Further, if (H5) holds, then $\widehat{J}(\xi) \geq 0$ for all $J \in H^r$, $r > 1/2$.

The following theorem concludes with the boundedness of the operator.

Theorem 2 (see [10]). If $J(x) \geq 0$ for all $x \in \mathbb{R}$, $J \in L_2(\mathbb{R})$, and $\int_{\mathbb{R}} J(x) dx = 1$, then \mathcal{L} is bounded and $\|\mathcal{L}\| \leq 1 + \varepsilon$.

The invertibility of the operator can be obtained by the following theorem.

Theorem 3 (see [10]). If the kernel function satisfies (H1)–(H5), then, for any $\vartheta(x) \in L_2(\mathbb{R})$, there exist constants $0 < C_1 < C_2$ such that $C_1(\vartheta(x), \vartheta(x)) \leq (\mathcal{L}\vartheta(x), \vartheta(x)) \leq C_2(\vartheta(x), \vartheta(x))$.

3. Numerical Approximation

We consider the periodic domain $\Omega = [0, 1]$. We subdivide Ω into N subintervals so that $\Omega = \cup \Omega_i$, where $\Omega_i = [y_i, y_{i+1}]$, $y_i = ih$, $i = 0, 1, 2, \dots, N$. Let x_i be the midpoint of Ω_i for all $i = 0, 1, 2, 3, \dots, N$, $h = 1/N$. Now we collocate (6) at x_i to get

$$\int_{\Omega} J(x_i - y) (\theta_t(x_i, t) - \varepsilon \theta_t(y, t)) dy = f(\theta_t(x_i, t)). \quad (10)$$

Now using midpoint quadrature formula in the above equation for spatial integration, one gets

$$\begin{aligned} & \int_{\Omega} J(x_i - y) \theta_t(x_i, t) dy \\ &= \theta_t(x_i, t) \int_{\Omega} J(x_i - y) dy \\ &= \theta_t(x_i, t) \Delta x \sum_j J(x_i - x_j), \end{aligned} \quad (11)$$

and we write

$$\begin{aligned} & \int_{\Omega} J(x_i - y) \theta_t(y, t) dy \\ &= \theta_t(x_i, t) \int_{\Omega} J(x_i - y) dy \\ &= \sum_j \theta_t(x_j, t) \Delta x J(x_i - x_j). \end{aligned} \quad (12)$$

Thus we get

$$\mathcal{L}\theta_t(t) = \mathcal{L}_1\theta_t(t) + \mathcal{L}_2\theta_t(t) \approx A_1\theta_t(t) - \varepsilon A_2\theta_t(t), \quad (13)$$

where

$$\begin{aligned} A_1(i, j) &= \begin{cases} \sum_j \Delta x J(x_i - x_j), & \text{if } i = j \\ 0, & \text{if } i \neq j, \end{cases} \\ A_2(i, j) &= \Delta x J(x_i - x_j), \quad \forall i, j. \end{aligned} \quad (14)$$

Now considering $\theta_t(x, t) = \theta(x, t)$, (6) can be written as

$$\theta(x_i, t) = \varepsilon \int_{\Omega} J(x_i, y) \theta(y, t) dy + f(\theta(x_i, t), t), \quad (15)$$

and so

$$\theta(x_i, t) = \varepsilon \sum_{j=1}^n \int_{\Omega_j} J(x_i, y) \theta(y, t) dy + f(\theta(x_i, t), t). \quad (16)$$

Using (6) we get

$$\theta(x_i, t) = \varepsilon h \sum_{j=1}^n J(x_i, x_j) \theta(x_j, t) + f(\theta(x_i, t), t). \quad (17)$$

The above mentioned equations can be presented in the matrix form as

$$\emptyset(t) = \varepsilon h A \emptyset(t) + f(\theta), \quad (18)$$

where

$$\begin{aligned} \emptyset(t) &= \begin{pmatrix} \emptyset(x_1, t) \\ \emptyset(x_2, t) \\ \vdots \\ \emptyset(x_n, t) \end{pmatrix}, \\ A &= \begin{pmatrix} J(x_1, y_1) & J(x_1, y_2) & \cdots & J(x_1, y_n) \\ J(x_2, y_1) & J(x_2, y_2) & \cdots & J(x_2, y_n) \\ \vdots & \vdots & \ddots & \vdots \\ J(x_n, y_1) & J(x_n, y_2) & \cdots & J(x_n, y_n) \end{pmatrix}, \\ f(\theta) &= \begin{pmatrix} f(\theta_1) \\ f(\theta_2) \\ \vdots \\ f(\theta_n) \end{pmatrix}. \end{aligned} \quad (19)$$

The above system of (18) can be arranged as $(I - \varepsilon h A)\emptyset(t) = f(\theta)$, which yields

$$(I - \varepsilon h A) \frac{d\theta}{dt} = f(\theta). \quad (20)$$

Substituting $B = (I - \varepsilon h A)$, we get a first-order time dependent system of equations:

$$B \frac{d\theta}{dt} = f(\theta), \quad \text{with initial function } \theta_0. \quad (21)$$

4. Time Integration to Solve the System of Differential Equations

Here in this section we investigate various one- and multistep schemes to approximate the system of first-order nonlinear differential equation (21).

4.1. Euler's Methods. Let t_n be the time at the n th time-step, let θ^n be the computed solution at the n th time-step, let $\theta^n \equiv \theta(t_n)$, let Δt be the step size, and let $\Delta t = t_n - t_{n-1}$ be constant here. Now from (21)

$$B \frac{\theta^{i+1} - \theta^i}{\Delta t} = f(\theta^i), \quad (22)$$

which can be written as $\theta^{i+1} = (\theta^i) + \Delta t(B^{-1}(f(\theta^i)))$, $i = 0, 1, 2, \dots$, which is known as forward Euler scheme for initial value problems. The explicit forward/Euler method is based on a truncated Taylor series expansion. Expanding θ in the neighborhood of t_n , one gets

$$\theta^{n+1} = \theta(t_n) + \Delta t B^{-1} f(\theta^n(t_n)) + O(\Delta t^2). \quad (23)$$

Here the local truncation error of the forward Euler method is $O(\Delta t^2)$. That is to say, the forward Euler method is a first-order technique. A simple Taylor expansion can be used to show that

$$\begin{aligned} E &= \widehat{\theta}(t_n + \Delta t) - \theta(t_n + \Delta t) \\ &= \Delta t B^{-1} \left[f(\theta^n) - \theta'(t_n) \right] - \frac{1}{2} \Delta t^2 \theta''(\xi), \end{aligned} \quad (24)$$

which shows that there is an error of $O(\Delta t^2)$ in a single step of the explicit Euler method; that is, we get second-order local truncation error.

In this subsection, our goal is to find finite difference schemes which are more accurate than the simple Euler method; that is, the global error of the sought methods should be $O(\Delta t^2)$ or better. We first want to develop an intuitive understanding of how this can be done and then actually do it. An alternative to the above scheme can be to consider the midpoint of the interval to approximate

$$\theta^{i+1} = \theta^i + \Delta t B^{-1} f\left(\theta^i + \frac{\Delta t}{2} f(\theta^i)\right). \quad (25)$$

Equation (25) is referred to as the midpoint method, which is also known as a modified Euler scheme. Similar to the explicit Euler's scheme expanding θ^{i+1} one can easily show that (25) is determined from the requirement that the corresponding finite difference scheme has the global error $O(\Delta t^2)$ or, equivalently, the local truncation error $O(\Delta t^3)$.

4.2. Runge-Kutta Method. We start with the following Taylor expansion:

$$\theta(t + \Delta t) = \theta(t) + \Delta t \theta'(t) + \frac{\Delta t^2}{2} \theta''(t) + O(\Delta t^3). \quad (26)$$

The first derivative can be replaced by the right hand side if the differential equation (21) and the second derivatives is obtained by differentiating (21)

$$\begin{aligned} \theta''(t) &= B^{-1} f_t(\theta, t) + B^{-1} f_\theta(\theta, t) \theta'(t) \\ &= B^{-1} f_t(\theta, t) + (B^{-1})^2 f_\theta(\theta, t) f(\theta, t), \end{aligned} \quad (27)$$

with Jacobian f_θ . We will from now neglect the dependence of θ on t when it appears as an argument to f ; therefore the Taylor expansion and the multivariate Taylor expansion

$$\theta^{i+1} = \theta^i + \frac{\Delta t}{2} B^{-1} f(\theta^i) + \frac{\Delta t}{2} B^{-1} f\left(\theta^i + \Delta t B^{-1} f(\theta^i)\right), \quad (28)$$

which is the classical second-order Runge-Kutta method. It is also known as Heun's method or the improved Euler method.

As of the similar fashion the fourth-order Runge-Kutta method can be presented as

$$\theta^{i+1} = \theta^i + \Delta t \left(\frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} \right). \quad (29)$$

with

$$\begin{aligned} k_1 &= B^{-1} f(\theta^i), \\ k_2 &= B^{-1} f\left(\theta^i + \frac{\Delta t}{2} k_1\right), \\ k_3 &= f\left(t_i + \frac{\Delta t}{2}, \theta_i + \frac{\Delta t}{2} k_2\right), \\ k_4 &= f\left(t_i + \frac{\Delta t}{2}, \theta_i + \frac{\Delta t}{2} k_3\right). \end{aligned} \quad (30)$$

Here θ^{i+1} is the Runge-Kutta approximation of $\theta(t_{i+1})$ and next value is determined by the value θ^i plus the weighted average of four increments, where each increment is the product of the size of the interval, Δt , and an estimated slope specified by function f on the right hand side of the differential equation.

4.3. An Explicit Two-Step Scheme. Let t_n be the time at the n th time-step, let θ^n be the computed solution at the n th time-step, let $\theta^n \equiv \theta(t_n)$, let Δt be the step size, and let $\Delta t = t_n - t_{n-1}$ be constant here. Now from (21) $B((\theta^{i+1} - \theta^{i-1})/2\Delta t) = f(\theta^i)$ and so $\theta^{i+1} = \theta^{i-1} + 2\Delta t(B^{-1}(f(\theta^i)))$. By computer implementation we notice that the above mentioned scheme does not work well. So we modify the scheme by $\theta^{i+1} = \theta^{i-1} + 2\Delta t(B^{-1}(f(\theta^{i-1})))$, which converges numerically as of the other schemes discussed previously.

4.4. A Linear Two-Step Adams-Bashforth Scheme. Keeping the notations as above using a simple two-step scheme (21) can be approximated by

$$\theta^{i+2} = \theta^{i+1} + \frac{3}{2} \Delta t B^{-1} f(\theta^{i+1}) - \frac{1}{2} \Delta t B^{-1} f(\theta^i), \quad (31)$$

which needs two values θ^i and θ^{i+1} to compute the value of θ^{i+2} .

In Table 1 we present some one-order and higher order schemes with local truncation errors for the system of (21).

4.5. Implicit Schemes. Let t_n be the time at the n th time-step, let θ^n be the computed solution at the n th time-step, let $\theta^n \equiv \theta(t_n)$, let Δt be the step size, and let $\Delta t = t_n - t_{n-1}$ be constant here. Now using backward substitution (21) can be approximated by $B((\theta^{i+1} - \theta^i)/\Delta t) = f(\theta^{i+1})$, which can be written as $\theta^{i+1} = (\theta^i) + \Delta t(B^{-1}(f(\theta^{i+1})))$, $i = 0, 1, 2, \dots$, which is popularly known as implicit/backward Euler scheme. As of the explicit Euler scheme, expanding θ^{i+1} at t_i one can easily show that this scheme is determined from the requirement that the corresponding finite difference scheme have the global error $O(\Delta t)$ or equivalently, the local truncation error $O(\Delta t^2)$.

Another popular alternative to the forward Euler is

$$\theta^{i+1} = \theta^i + \frac{1}{2} \Delta t B^{-1} (f(\theta^{i+1}) + f(\theta^i)), \quad (32)$$

which is an implicit scheme. The scheme (32) is known as the Trapezoidal method. As of the explicit Euler scheme, expanding θ^{i+1} at t_i , one can easily show that (32) is determined

TABLE 1

Order	Formula	Local truncation error
1	$\theta^{i+1} = \theta^i + \Delta t B^{-1} f(\theta^i)$	$\frac{\Delta t^2}{2} \theta''(\xi)$
2	$\theta^{i+2} = \theta^{i+1} + \frac{\Delta t}{2} B^{-1} (3f(\theta^{i+1}) - f(\theta^i))$	$\frac{5\Delta t^3}{12} \theta'''(\xi)$
3	$\theta^{i+3} = \theta^{i+2} + \frac{\Delta t}{12} B^{-1} (23f(\theta^{i+2}) - 16f(\theta^{i+1}) + 5f(\theta^i))$	$\frac{3\Delta t^4}{8} \theta^{(4)}(\xi)$
4	$\theta^{i+4} = \theta^{i+3} + \frac{\Delta t}{24} B^{-1} (55f(\theta^{i+3}) - 59f(\theta^{i+2}) + 37f(\theta^{i+1}) - 9f(\theta^i))$	$\frac{251\Delta t^5}{720} \theta^{(5)}(\xi)$
5	$\theta^{i+5} = \theta^{i+4} + \frac{\Delta t}{720} B^{-1} (1901f(\theta^{i+4}) - 2774f(\theta^{i+3}) + 2616f(\theta^{i+2}) - 1274f(\theta^{i+1}) + 251f(\theta^i))$	$\frac{95\Delta t^6}{2888} \theta^{(6)}(\xi)$

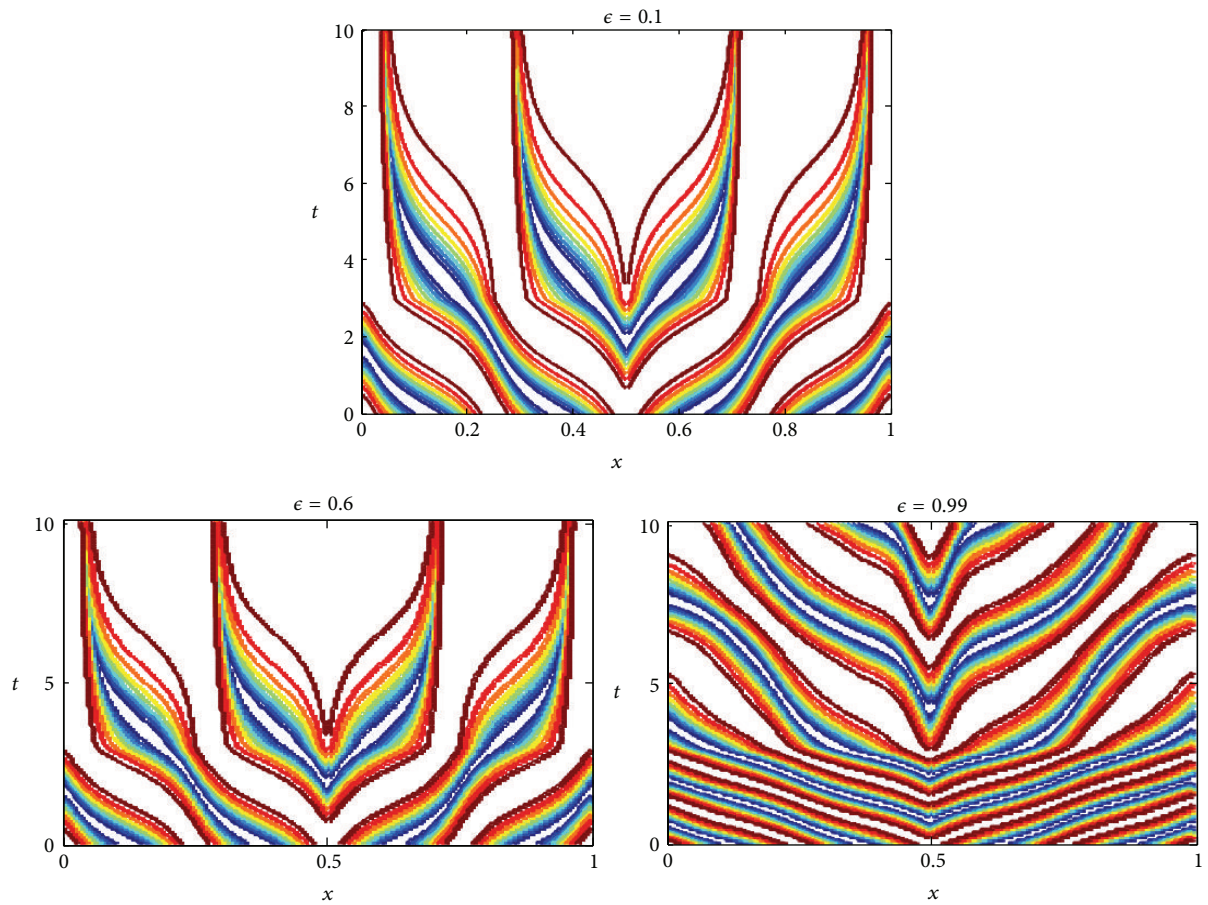


FIGURE 1: Here we present contour of $\cos(\theta_h(x, t))$ for $0 \leq x \leq 1$, $0 \leq t \leq 10$. Here we use $N = 128$ space nodes to compute the spatial integrals, then we use $\Delta t = 0.1$ for time integration, and we use modified Euler for time integration.

from the requirement that the corresponding finite difference scheme has the global error $O(\Delta t^2)$ or, equivalently, the local truncation error $O(\Delta t^3)$. It is to note that both the implicit solvers give us system of nonlinear equations in terms of θ^{i+1} which needs to be solved by using Newton or some Newton type solvers. We keep ourselves restricted with the schemes discussed above. However there are many other one-order

and higher order schemes that are also available for time integration.

5. Numerical Experiments and Discussions

In this section we discuss computer implementation of the schemes. For all the computations we consider N subintervals

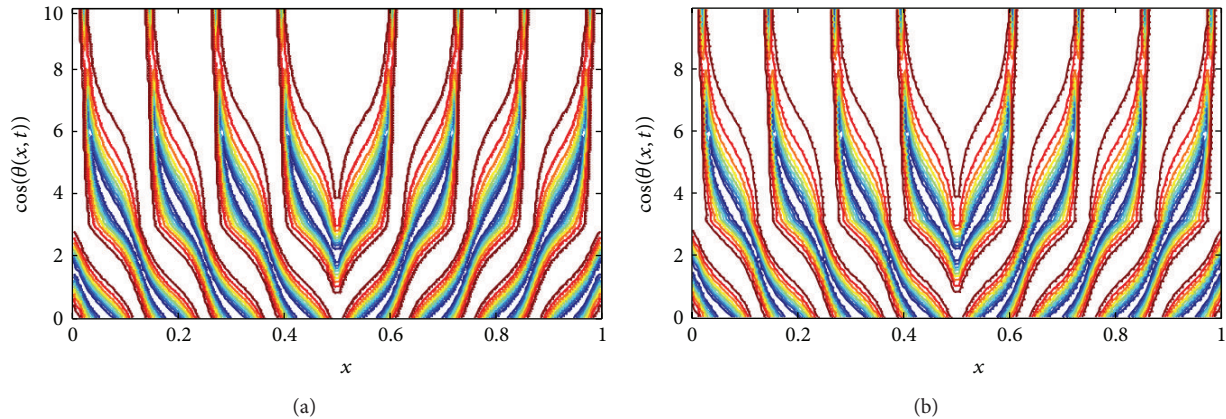


FIGURE 2: Here we present contour of $\cos(\theta_h(x, t))$ for $0 \leq x \leq 1$, $0 \leq t \leq 10$. Here we use $N = 128$ space nodes to compute the spatial integrals and then use various choices of Δt for time integration. Here (a) backward Euler, (b) Trapezoidal method.

$[x_i, x_{i+1}]$, $i = 0, 1, 2, \dots, N-1$, for the spatial domain $[0, 1]$, and consider the midpoints of the interval for collocation to derive the semidiscrete time dependent system of differential equation (21). Then depending on the choice of the solver for the system (21) we choose different time stepping Δt as required. For all the computations we consider the Gaussian kernel defined in (8) with $\gamma = 100$, and $0 < \epsilon < 1$.

In Figure 1 we present solutions for (21) for different choices of the parameter ϵ . In all cases we consider $\theta(x, 0) = 8\pi x$, if $0 \leq x < 0.5$, and $\theta(x, 0) = 8\pi(1 - x)$, if $0.5 \leq x < 1$. Here we observe different patterns for different choices of the parameter values. The bifurcation of the solutions at $t = 3$ are also visible from all the computational results. The other explicit one and multistep solvers also produce same results.

In Figure 2 we present solutions for different choices time schemes for (21). In both cases we consider that $\epsilon = 0.5$, and $\theta(x, 0) = 16\pi x$, if $0 \leq x < 0.5$, and $\theta(x, 0) = 16\pi(1 - x)$, if $0.5 \leq x < 1$. From this computations we notice a bifurcation of solutions at the transition point $t = 3$ of the nonlinearity $f(\theta)$. Here from the computations we observe that the implicit solvers work well for large Δt , and as a result they need less computational time, though the explicit solvers are easy to implement. In terms of computational costs and stability issues we recommend the implicit and multistep schemes for this type of nonlinear integrodifferential equations.

Here we restrict ourselves with piecewise constant approximations for space integrations, and we study one space dimensional model only. Thus the multidimensional version of the model with higher order quadratures and higher order schemes for time integration leaves as future studies.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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