Research Article

Common Fixed Point Theorems Satisfying Contractive Type Conditions in Complex Valued Metric Spaces

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Some common fixed point theorems satisfying contractive conditions involving rational expressions and product for four mappings that satisfy property (E.A) along with weak compatibility of pairs are proved and further some results using (CLR)-property are obtained in complex valued metric spaces which generalize various results of ordinary metric spaces.

1. Introduction

Fixed point theory is one of the fundamental theories in nonlinear analysis which has various applications in different branches of mathematics. In this theory, to prove the existence and the uniqueness of a fixed point of operators or mappings has been a valuable research area by using the Banach contraction principle. There are many generalizations of the Banach contraction principle particularly in metric spaces. So, many researches attempted various generalizations of the concept of metric spaces such as 2-metric spaces, D-metric spaces, G-metric spaces, K-metric spaces, cone metric spaces, and probabilistic metric space.

Recently, Huang and Zhang [1] generalized the concept of metric spaces, replacing the set of real numbers by an ordered Banach space; hence they have defined the cone metric spaces. They also described the convergence of sequences and introduced the notion of completeness in cone metric spaces. They have proved some fixed point theorems of contractive mappings on complete cone metric space with the assumption of normality condition of a cone. Subsequently, various authors have generalized the results of Huang and Zhang and have studied fixed point theorems in cone metric spaces over normal and nonnormal cones.

Many results of analysis cannot be generalized to cone metric since the definition of these spaces is based on a Banach space which is not a division ring. So, in a recent time, Azam et al. [2] introduced and studied the notion of complex valued metric space and established some common fixed point theorems for mappings involving rational expressions which are not meaningful in cone metric spaces. Later, several authors have studied the problem of existence of uniqueness of a fixed point for mappings satisfying different type contractive conditions in the framework of complex valued metric spaces.

In 2002, Aamri and Moutawakil [3] introduced the property (E.A) and pointed out that this property buys containment of ranges without any continuity requirements besides minimizing the commutativity conditions of the maps to the commutativity at their points of coincidence. Further, property (E.A) allows replacing the completeness condition of the space with a natural condition of closeness of the range. Subsequently, there are numerous papers which contain fixed point results related to property (E.A) in various metric spaces. Most recently, Sintunavarat and Kumam [4] defined the notion of the (CLR)-property (or common limit in the range property) which does not impose either completeness of the whole space or any of the range spaces or continuity of maps. The importance of this property ensures that one does not require the closeness of the range of subspaces. Various fixed point theorems have been proved by using the notion of (CLR)-property (see [4–15]).

The aim of this paper is to establish common fixed point theorems for two pairs of weakly compatible self-mappings of a complex valued metric space satisfying contractive condition involving product and rational expressions using

(E.A) property. Moreover, we give some results using the property common limit in the range of one of the mappings.

2. Basic Facts and Definitions

We recall some notations and definitions which will be utilized in our subsequent discussion.

Let $\mathbf{C}$ be a set of complex numbers and $z_1, z_2 \in \mathbf{C}$. Define a partial order $\preceq$ on $\mathbf{C}$ as follows:

$$z_1 \preceq z_2 \quad \text{iff} \ \operatorname{Re}(z_1) \le \operatorname{Re}(z_2), \ \operatorname{Im}(z_1) \le \operatorname{Im}(z_2). \quad (1)$$

It follows that $z_1 \preceq z_2$ if one of the following conditions is satisfied:

   (i) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2), \ \operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,

   (ii) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2), \ \operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,

   (iii) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2), \ \operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,

   (iv) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2), \ \operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.

In (i), (ii), and (iii), we have $|z_1| < |z_2|$. In (iv), we have $|z_1| = |z_2|$. So $|z_1| \le |z_2|$. In particular, we will write $z_1 \precsim z_2$ if $z_1 \ne z_2$ and one of (i), (ii), and (iii) is satisfied. In this case $|z_1| < |z_2|$. We will write $z_1 \prec z_2$ if and only if (iii) is satisfied.

Take into account some fundamental properties of the partial order $\preceq$ on $\mathbf{C}$ as follows.

   (i) If $0 \preceq z_1 \precsim z_2$, then $|z_1| < |z_2|$.

   (ii) If $z_1 \preceq z_2$, $z_2 \prec z_3$, then $z_1 \prec z_3$.

   (iii) If $z_1 \preceq z_2$ and $\lambda \ge 0$ is a real number, then $\lambda z_1 \preceq \lambda z_2$.

Definition 1 (see [12]). The "max" function for the partial order relation "$\preceq$" is defined by the following.

   (i) $\max\{z_1, z_2\} = z_2$ if and only if $z_1 \preceq z_2$.

   (ii) If $z_1 \preceq \max\{z_2, z_3\}$, then $z_1 \preceq z_2$, or $z_1 \preceq z_3$.

   (iii) $\max\{z_1, z_2\} = z_2$ if and only if $z_1 \preceq z_2$ or $|z_1| \le |z_2|$.

Using Definition 1 one can have the following lemma.

Lemma 2 (see [12]). *Let $z_1, z_2, z_3, \ldots \in \mathbf{C}$ and the partial order relation $\preceq$ is defined on $\mathbf{C}$. Then the following statements are easy to prove.*

   (i) *If $z_1 \preceq \max\{z_2, z_3\}$, then $z_1 \preceq z_2$ if $z_3 \preceq z_2$.*

   (ii) *If $z_1 \preceq \max\{z_2, z_3, z_4\}$, then $z_1 \preceq z_2$ if $\max\{z_3, z_4\} \preceq z_2$.*

   (iii) *If $z_1 \preceq \max\{z_2, z_3, z_4, z_5\}$, then $z_1 \preceq z_2$ if $\max\{z_3, z_4, z_5\} \preceq z_2$.*

Now we give the definition of complex valued metric space which has been introduced by Azam et al. [2].

Definition 3. Let X be a nonempty set. If a mapping $d : X \times X \to \mathbf{C}$ satisfies

   (C_1) $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$,

   (C_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

   (C_3) $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$,

then d is called a complex valued metric on X and the pair (X, d) is called a complex valued metric space.

Let $\{x_n\}$ be a sequence in a complex valued metric space X and $x \in X$. If for every $\varepsilon \in \mathbf{C}$ with $\varepsilon \succ 0$ there is $N \in \mathbf{N}$ such that, for all $n > N$, $d(x_n, x) \prec \varepsilon$, then x is called the limit of $\{x_n\}$ and is written as $\lim_{n \to \infty} x_n = x$ as $n \to \infty$. If for every $\varepsilon \in \mathbf{C}$ with $\varepsilon \succ 0$ there is $N \in \mathbf{N}$ such that, for all $n, m > N$, $d(x_n, x_m) \prec \varepsilon$, then $\{x_n\}$ is called a Cauchy sequence in X. If every Cauchy sequence is convergent in X, then X is called a complete complex valued metric space.

The following lemma has been given in [2] that we utilize to prove the theorems.

Lemma 4. *Let X be a complex valued metric space and $\{x_n\}$ a sequence in X. Then*

   (i) *$\{x_n\}$ converges to $x \in X$ if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$;*

   (ii) *$\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_m)| \to 0$ as $n, m \to \infty$.*

Definition 5 (see [14]). A pair of self-mappings $S, T : X \to X$ is called weakly compatible if they commute at their coincidence point; that is, if there is a point $z \in X$ such that $Sz = Tz$, then $STz = TSz$, for each $z \in X$.

The definition of property (E.A) has been introduced by Aamri and Moutawakil in [3] and redefined by Verma and Pathak [12] in complex valued metric spaces.

Definition 6. Let $S, T : X \to X$ be two self-mappings of a complex valued metric space (X, d). The pair (S, T) is said to satisfy property (E.A), if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to \infty} d(Sx_n, u) = \lim_{n \to \infty} d(Tx_n, u) = 0, \quad (2)$$

for some $u \in X$.

Example 7. Let $X = \mathbf{C}$ be endowed with the complex valued metric $d : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ as

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|, \quad (3)$$

where $z_1 = x_1 + iy_1$ and $x_2 + iy_2$. Then $(\mathbf{C}, d)$ is a complete complex valued metric space. Define the mappings $T, S : X \to X$ as $Tz = 2z - 1$, $Sz = z^2$ for all $z \in X$ and consider the sequence $\{z_n\} = \{1 + i/2^n\}$. Thus we obtain

$$\lim_{n \to \infty} d(Tz_n, z) = \lim_{n \to \infty} d(Sz_n, z),$$

$$\lim_{n \to \infty} d\left(2\left(1 + i\frac{1}{2^n}\right) - 1, 1\right) = \lim_{n \to \infty} d\left(\left(1 + i\frac{1}{2^n}\right)^2, 1\right) = 0, \quad (4)$$

where $z = 1$ is the limit of sequence $\{z_n\}$. Hence the pair (S, T) satisfies property (E.A).

Definition 8 (see [15]). Let S and T be two self-mappings of complex valued metric space X. S and T are said to satisfy the common limit in the range of S property if

$$\lim_{n\to\infty} d\left(Sx_n, Sx\right) = \lim_{n\to\infty} d\left(Tx_n, Sx\right) = 0, \qquad (5)$$

for some $x \in X$.

Example 9. Let X and d be given as in Example 7. Define $S, T : X \to X$ by $Tz = 5x + i5y$ and $Sz = x + 4 + iy$ for all $x, y \in \mathbf{R}$. Consider the sequence $\{z_n\} = \{x_n + iy_n\} = \{1 + i/n\}$. Then for $z = 1$, with an easy calculation, we see that

$$\lim_{n\to\infty} d\left(Tz_n, Sz\right) = \lim_{n\to\infty} d\left(Sz_n, Sz\right)$$

$$\lim_{n\to\infty} d\left(5 + i\frac{5}{n}, 5\right) = \lim_{n\to\infty} d\left(5 + i\frac{1}{n}, 5\right) = 0. \qquad (6)$$

Hence T, S satisfy the common limit in the range of S property ((CLR$_S$)-property).

3. Main Results

In this section, initially, some common fixed point results for the pairs, which are weakly compatible and satisfy property (E.A), have been proved, by reconstructing the contractive conditions given in [16].

Theorem 10. *Let (X, d) be a complex valued metric space and let $S, T, I, J : X \to X$ be four self-mappings satisfying the following:*

(i) $T(X) \subseteq I(X),\ S(X) \subseteq J(X)$;

(ii)

$$\left[d\left(Sx, Ty\right)\right]^3 \preceq \alpha d\left(Ix, Jy\right) d\left(Ix, Sx\right) d\left(Jy, Ty\right), \qquad (7)$$

for all $x, y \in X$, where $\alpha \in (0, 1)$;

(iii) *the pairs $\{S, I\}$ and $\{T, J\}$ are weakly compatible;*

(iv) *one of the pairs $\{S, I\}$ or $\{T, J\}$ satisfies property (E.A).*

If the range of one of the mappings $J(X)$ or $I(X)$ is a complete subspace of X, then the mappings $I, J, S,$ and T have a unique common fixed point in X.

Proof. Suppose that the pair $\{T, J\}$ satisfies property (E.A). Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Jx_n = z, \qquad (8)$$

for some $z \in X$. Further, since $T(X) \subseteq I(X)$, there exists a sequence $\{y_n\}$ in X such that $Tx_n = Iy_n$. Hence $\lim_{n\to\infty} Iy_n = z$. Our claim is $\lim_{n\to\infty} Sy_n = z$. Using condition (7), we have

$$\left[d\left(Sy_n, Tx_n\right)\right]^3 \preceq \alpha d\left(Iy_n, Jx_n\right) d\left(Iy_n, Sy_n\right) d\left(Jx_n, Tx_n\right)$$

$$= \alpha d\left(Tx_n, Jx_n\right) d\left(Tx_n, Sy_n\right) d\left(Tx_n, Jx_n\right). \qquad (9)$$

By dividing two sides of the above inequality with $d(Sy_n, Tx_n)$ we get

$$\left[d\left(Sy_n, Tx_n\right)\right]^2 \preceq \alpha \left[d\left(Tx_n, Jx_n\right)\right]^2. \qquad (10)$$

Thus $|d(Sy_n, Tx_n)| \leq \sqrt{\alpha}\, |d(Tx_n, Jx_n)| = 0$ and letting $n \to \infty$ we have

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tx_n = z. \qquad (11)$$

Now, suppose that $I(X)$ is complete subspace of X; then $z = Iu$ for some $u \in X$. Subsequently, we obtain

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Jx_n$$

$$= \lim_{n\to\infty} Iy_n = z = Iu. \qquad (12)$$

We claim that $Su = Iu$. To prove this, in (7)

$$\left[d\left(Su, Tx_n\right)\right]^3 \preceq \alpha d\left(Iu, Jx_n\right) d\left(Iu, Su\right) d\left(Jx_n, Tx_n\right), \qquad (13)$$

and letting $n \to \infty$ and using (12) we have

$$\left|\left[d\left(Su, z\right)\right]^3\right| \leq \alpha\, |d\left(z, z\right) d\left(z, Su\right) d\left(z, z\right)| = 0, \qquad (14)$$

and consequently $Su = z = Iu$. Thus z is a coincidence point of $\{S, I\}$. Weak compatibility of the pair $\{S, I\}$ implies that

$$SIu = ISu = Sz = Iz. \qquad (15)$$

Conversely, since $S(X) \subseteq J(X)$, there exists $v \in X$ such that $Su = Jv$. Hence $Su = Iu = Jv = z$. Now we show that v is a coincidence point of $\{T, J\}$; that is, $Tv = Jv = z$. Putting $x = u,\ y = v$ in (7), we get

$$\left[d\left(Su, Tv\right)\right]^3 \preceq \alpha d\left(Iu, Jv\right) d\left(Iu, Su\right) d\left(Jv, Tv\right),$$

$$\left|\left[d\left(z, Tv\right)\right]^3\right| \leq |\alpha d\left(z, Jv\right) d\left(z, z\right) d\left(Jv, Tv\right)|; \qquad (16)$$

thus $Tv = z$. Hence $Tv = Jv = z$ and v is a coincidence point of T and J. Weak compatibility of the pair $\{T, J\}$ implies that

$$TJv = JTv = Tz = Jz. \qquad (17)$$

Therefore, z is a common coincidence point of $S, T, I,$ and J.

In order to show that z is a common fixed point of these mappings, we write in (7)

$$\left[d\left(z, Tz\right)\right]^3$$

$$= \left[d\left(Su, Tz\right)\right]^3 \preceq \alpha d\left(Iu, Jz\right) d\left(Iu, Su\right) d\left(Jz, Tz\right) \qquad (18)$$

$$|d\left(z, Tz\right)|^3 \leq 0.$$

Thus,

$$Sz = Iz = Jz = Tz = z. \qquad (19)$$

A similar argument derives if we assume that $J(X)$ is a complete subspace of X and also using the property (E.A) of the pair $\{S, J\}$ gives us the same result.

Uniqueness. To prove that z is a unique common fixed point, let us suppose that z^* is another common fixed point of I, J, S, and T. In (7) take $x = z^*$ and $y = z$; then

$$[d(z^*, z)]^3 = [d(Sz^*, Tz)]^3$$
$$\leq \alpha d(Iz, Jz^*) d(Iz, Sz) d(Jz^*, Tz^*) \quad (20)$$
$$\left| [d(z^*, z)]^3 \right| \leq 0$$

is a contradiction. Thus $z = z^*$. Consequently, $Sz = Iz = Tz = Jz = z$ and z is the unique common fixed point of I, J, S, and T. □

Putting $J = I$ in Theorem 10 we have the following corollary.

Corollary 11. *Let S, T, and I be three self-mappings of a complex valued metric space (X, d) satisfying the inequality*

$$[d(Sx, Ty)]^3 \leq \alpha d(Ix, Iy) d(Ix, Sx) d(Iy, Ty) \quad (21)$$

for all x, y in X, where $\alpha \in (0, 1)$. Suppose that the following conditions hold:

(i) *$I(X) \supseteq S(X) \bigcup T(X)$,*

(ii) *both the pairs $\{I, S\}$ and $\{I, T\}$ are weakly compatible,*

(iii) *one of the pairs $\{I, S\}$ and $\{I, T\}$ satisfies the property (E.A).*

If $I(X)$ is complete subspace of X, then S, T, and I have a unique common fixed point in X.

In Theorem 10, if we put $S = T$ and $I = J$, we have the following.

Corollary 12. *Let (X, d) be a complex valued metric space and let S and T be two self-mappings satisfying the following:*

(i) *$S(X) \subseteq I(X)$;*

(ii)

$$[d(Sx, Sy)]^3 \leq \alpha d(Ix, Iy) d(Ix, Sx) d(Iy, Sy); \quad (22)$$

(iii) *$\{I, S\}$ is a weakly compatible pair;*

(iv) *the pair $\{I, S\}$ satisfies property (E.A).*

If $I(X)$ is complete subspace of X, then I and S have the unique common fixed point in X.

Theorem 13. *Let S, I and T, J be four self-mappings of a complex valued metric space (X, d) satisfying the following:*

(i) *$T(X) \subseteq I(X)$ and $S(X) \subseteq J(X)$;*

(ii)

$$d(Sx, Ty) \leq \alpha \left(\max \left\{ [d(Ix, Jy)]^2, [d(Ix, Sx)]^2, \right.\right.$$
$$[d(Jy, Ty)]^2, \frac{1}{2}[d(Ix, Ty)]^2,$$
$$\left.\left. \frac{1}{2}[d(Jy, Sx)]^2 \right\} \right.$$
$$\left. \times \left(d(Ix, Sx) + d(Jy, Ty) \right)^{-1} \right) \quad (23)$$

if $d(Ix, Sx) + d(Jy, Ty) \neq 0$ where $\alpha \in (0, 1)$ and "max" is as in Definition 1, or

$$d(Sx, Ty) = 0 \quad \text{if } d(Ix, Sx) + d(Jy, Ty) = 0, \quad (24)$$

for all $x, y \in X$;

(iii) *the pairs $\{S, I\}$ and $\{T, J\}$ are weakly compatible;*

(iv) *one of the pairs $\{S, I\}$ or $\{T, J\}$ satisfies property (E.A).*

If the range of one of the mappings $J(X)$ or $I(X)$ is a complete subspace of X, then the mappings I, J, S, and T have a unique common fixed point in X.

Proof. Let us suppose that $d(Ix, Sx) + d(Jy, Ty) \neq 0$ so $d(Sx, Ty) \neq 0$ and the pair $\{T, J\}$ satisfies property (E.A). Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Jx_n = z, \quad (25)$$

for some $z \in X$. Since $T(X) \subseteq I(X)$, there exists a sequence $\{y_n\}$ in X such that $Tx_n = Iy_n$. Hence $\lim_{n \to \infty} Iy_n = z$. We show that $\lim_{n \to \infty} Sy_n = z$. In inequality (23), putting $x = y_n$ and $y = x_n$ we get

$$d(Sy_n, Tx_n) \leq \alpha \left(\max \left\{ [d(Iy_n, Jx_n)]^2, [d(Iy_n, Sy_n)]^2, \right.\right.$$
$$[d(Jx_n, Tx_n)]^2, \frac{1}{2}[d(Iy_n, Tx_n)]^2,$$
$$\left.\left. \frac{1}{2}[d(Jx_n, Sy_n)]^2 \right\} \right.$$
$$\left. \times \left(d(Iy_n, Sy_n) + d(Jx_n, Tx_n) \right)^{-1} \right)$$
$$= \alpha \left(\max \left\{ [d(Tx_n, Jx_n)]^2, [d(Tx_n, Sy_n)]^2, \right.\right.$$
$$\left. 0, \frac{1}{2}[d(Jx_n, Sy_n)]^2 \right\}$$
$$\left. \times \left(d(Tx_n, Sy_n) + d(Jx_n, Tx_n) \right)^{-1} \right).$$
$$\quad (26)$$

Thus,

$$
\begin{aligned}
&\left| d\left(Sy_n, Tx_n\right) \right| \\
&\leq \alpha \Big| \max \Big\{ \left[d\left(Tx_n, Jx_n\right) \right]^2, \left[d\left(Tx_n, Sy_n\right) \right]^2, \\
&\qquad 0, \frac{1}{2} \left[d\left(Jx_n, Sy_n\right) \right]^2 \Big\} \\
&\qquad \times \left(d\left(Tx_n, Sy_n\right) + d\left(Jx_n, Tx_n\right) \right)^{-1} \Big|,
\end{aligned}
\tag{27}
$$

and letting $n \to \infty$ we have

$$
(1 - \alpha) \left| d\left(Sy_n, z\right) \right| \leq 0
\tag{28}
$$

which is a contradiction since $\alpha \in (0, 1)$. Therefore,

$$
\lim_{n \to \infty} Sy_n = \lim_{n \to \infty} Tx_n = z.
\tag{29}
$$

Assuming $I(X)$ is complete subspace of X, then $z = Iu$ for some $u \in X$. Right after, we obtain

$$
\begin{aligned}
\lim_{n \to \infty} Sy_n &= \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Jx_n \\
&= \lim_{n \to \infty} Iy_n = z = Iu.
\end{aligned}
\tag{30}
$$

Our aim is to prove $Su = Iu$ and for this putting $x = u$ and $y = x_n$ in (23) we get

$$
\begin{aligned}
d\left(Su, Tx_n\right) \leq \alpha \Big(\max \Big\{ &\left[d\left(Iu, Jx_n\right) \right]^2, \left[d\left(Iu, Su\right) \right]^2, \\
&\left[d\left(Jx_n, Tx_n\right) \right]^2, \\
&\frac{1}{2} \left[d\left(Iu, Tx_n\right) \right]^2, \frac{1}{2} \left[d\left(Jx_n, Su\right) \right]^2 \Big\} \\
&\times \left(d\left(Iu, Su\right) + d\left(Jx_n, Tx_n\right) \right)^{-1} \Big).
\end{aligned}
\tag{31}
$$

Letting $n \to \infty$ and using (30)

$$
\left| d\left(Su, z\right) \right| \leq \alpha \left| \frac{\max \left\{ 0, \left[d\left(z, Su\right) \right]^2, (1/2) \left[d\left(z, Su\right) \right]^2 \right\}}{d\left(z, Su\right)} \right|,
\tag{32}
$$

and hence $(1 - \alpha)|d(Su, z)| \leq 0$, and $Su = z$ since $\alpha \in (0, 1)$. Therefore, z is a coincidence point of $\{S, I\}$. Weak compatibility of the pair $\{S, I\}$ implies that $SIu = ISu = Sz = Iz$.

Otherwise, since $S(X) \subseteq J(X)$, there exists $v \in X$ such that $Su = Jv$. Hence, $Su = Iu = Jv = z$. To show

that v is a coincidence point of pair $\{T, J\}$, by using similar arguments in Theorem 10 and inequality (23) we have

$$
\begin{aligned}
d\left(Su, Tv\right) \leq \alpha \Big(\max \Big\{ &\left[d\left(Iu, Jv\right) \right]^2, \left[d\left(Iu, Su\right) \right]^2, \\
&\left[d\left(Jv, Tv\right) \right]^2, \\
&\frac{1}{2} \left[d\left(Iu, Tv\right) \right]^2, \frac{1}{2} \left[d\left(Jv, Su\right) \right]^2 \Big\} \\
&\times \left(d\left(Iu, Su\right) + d\left(Jv, Tv\right) \right)^{-1} \Big),
\end{aligned}
$$

$$
\begin{aligned}
\left| d\left(z, Tv\right) \right| &\leq \alpha \left| \frac{\max \left\{ 0, \left[d\left(z, Tv\right) \right]^2, (1/2) \left[d\left(z, Tv\right) \right]^2 \right\}}{d\left(z, Tv\right)} \right| \\
&\leq \alpha \left| d\left(z, Tv\right) \right|,
\end{aligned}
\tag{33}
$$

and then $Tv = z$ because $\alpha \in (0, 1)$. With the same assertions as in Theorem 13 one gets that z is a common coincidence point of S, T, I, and J.

Other details of Theorem 13, in which z is a unique common fixed point of the mappings I, J, S, and T, can be obtained in view of the final part of the proof of Theorem 10 with suitable modifications.

In concluding, we note that the conclusions of Theorem 13 are still valid if we replace inequality (23) with the following inequality:

$$
\begin{aligned}
&d\left(Sx, Ty\right) \\
&\leq \alpha \max \Big\{ \frac{d\left(Ix, Ty\right) \left[1 + d\left(Ix, Sx\right) + d\left(Jy, Sx\right) \right]}{2 \left[1 + d\left(Ix, Jy\right) \right]}, \\
&\qquad\qquad \frac{d\left(Jy, Sx\right) \left[1 + d\left(Ix, Ty\right) + d\left(Jy, Ty\right) \right]}{2 \left[1 + d\left(Ix, Jy\right) \right]} \Big\},
\end{aligned}
\tag{34}
$$

where $\alpha \in (0, 1)$ and the mappings S, I and T, J are defined as in Theorem 13. $\qquad\square$

Finally, at the end of the section some common fixed point theorems for weakly compatible pairs which satisfy the (CLR)-property have been proved.

Theorem 14. *Let S, I and T, J be four self-mappings of a complex valued metric space (X, d) satisfying the following:*

(i) *$T(X) \subseteq I(X)$ and $S(X) \subseteq J(X)$;*

(ii)

$$
\begin{aligned}
&d\left(Sx, Ty\right) \\
&\leq \alpha \max \Big\{ \frac{d\left(Ix, Ty\right) \left[1 + d\left(Ix, Sx\right) + d\left(Jy, Sx\right) \right]}{2 \left[1 + d\left(Ix, Jy\right) \right]}, \\
&\qquad\qquad \frac{d\left(Jy, Sx\right) \left[1 + d\left(Ix, Ty\right) + d\left(Jy, Ty\right) \right]}{2 \left[1 + d\left(Ix, Jy\right) \right]} \Big\}
\end{aligned}
\tag{35}
$$

for all $x, y \in X$ where $\alpha \in (0, 1)$ and "max" is as in Definition 1;

(iii) $\{S, I\}$ and $\{T, J\}$ are weakly compatible pairs.

If the pair $\{S, I\}$ satisfies (CLR_S)-property, or the pair $\{T, J\}$ satisfies (CLR_T)-property, then the mappings $I, J, S,$ and T have a unique common fixed point in X.

Proof. Let us suppose that the pair $\{T, J\}$ satisfies (CLR_T)-property; then by Definition 8 there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Jx_n = Tx \tag{36}$$

for some $x \in X$. And also, since $T(X) \subseteq I(X)$, we have $Tx = Iz$ for some $z \in X$. We claim that $Sz = Iz = u$. Then putting $x = z$ and $y = x_n$ in inequality (35) we have

$$d\left(Sz, Tx_n\right)$$
$$\leq \alpha \max \left\{ \frac{d\left(Iz, Tx_n\right)\left[1 + d\left(Iz, Sz\right) + d\left(Jx_n, Sz\right)\right]}{2\left[1 + d\left(Iz, Jx_n\right)\right]}, \right.$$
$$\left. \frac{d\left(Jx_n, Sz\right)\left[1 + d\left(Iz, Tx_n\right) + d\left(Jx_n, Tx_n\right)\right]}{2\left[1 + d\left(Iz, Jx_n\right)\right]} \right\}. \tag{37}$$

If n tends to infinity and with equality (36), then

$$\left|d\left(Sz, Iz\right)\right| \leq \alpha \left| \max \left\{ 0, \frac{d\left(Iz, Sz\right)}{2} \right\} \right|,$$
$$\left(1 - \frac{\alpha}{2}\right)\left|d\left(Sz, Iz\right)\right| \leq 0, \tag{38}$$

which is possible for $Iz = Sz$ since $\alpha \in (0, 1)$. Therefore, $Sz = Iz = u$; that is, z is a coincidence point of the pair $\{S, I\}$. Also weak compatibility of the mappings S and I implies the following equality:

$$ISz = SIz = Iu = Su. \tag{39}$$

Besides, since $S(X) \subseteq J(X)$, there exist some $\omega \in X$ such that $Sz = J\omega$. We claim that $T\omega = u$. Then from (35), we have

$$d\left(Sz, T\omega\right)$$
$$\leq \alpha \max \left\{ \frac{d\left(Iz, T\omega\right)\left[1 + d\left(Iz, Sz\right) + d\left(J\omega, Sz\right)\right]}{2\left[1 + d\left(Iz, J\omega\right)\right]}, \right.$$
$$\left. \frac{d\left(J\omega, Sz\right)\left[1 + d\left(Iz, T\omega\right) + d\left(J\omega, T\omega\right)\right]}{2\left[1 + d\left(Iz, J\omega\right)\right]} \right\},$$

$$d\left(u, T\omega\right)$$
$$\leq \alpha \max \left\{ \frac{d\left(u, T\omega\right)\left[1 + d\left(u, u\right) + d\left(Sz, Sz\right)\right]}{2\left[1 + d\left(u, u\right)\right]}, \right.$$
$$\left. \frac{d\left(Sz, Sz\right)\left[1 + d\left(u, T\omega\right) + d\left(u, T\omega\right)\right]}{2\left[1 + d\left(u, u\right)\right]} \right\}. \tag{40}$$

Thus,

$$\left[d\left(u, T\omega\right)\right] \leq \alpha \max \left\{ \frac{1}{2}d\left(u, T\omega\right), 0 \right\},$$
$$\left(1 - \frac{\alpha}{2}\right)\left|d\left(u, T\omega\right)\right| \leq 0, \tag{41}$$

which implies that $T\omega = u$, since $\alpha \in (0, 1)$. Hence,

$$Iz = Sz = u = T\omega = J\omega, \tag{42}$$

and this shows that ω is a coincidence point of the pair $\{T, J\}$. Weak compatibility of the pair $\{T, J\}$ yields that $TJ\omega = JT\omega = Tu = Ju$. In conclusion we show that u is a common fixed point of $I, J, S,$ and T. Using (35), we get

$$d\left(u, Tu\right)$$
$$= d\left(Sz, Tu\right)$$
$$\leq \alpha \max \left\{ \frac{d\left(Iz, Tu\right)\left[1 + d\left(Iz, Sz\right) + d\left(Ju, Sz\right)\right]}{2\left[1 + d\left(Iz, Ju\right)\right]}, \right.$$
$$\left. \frac{d\left(Ju, Sz\right)\left[1 + d\left(Iz, Tu\right) + d\left(Ju, Tu\right)\right]}{2\left[1 + d\left(Iz, Ju\right)\right]} \right\},$$

$$d\left(u, Tu\right)$$
$$= d\left(Sz, Tu\right)$$
$$\leq \alpha \max \left\{ \frac{d\left(u, Tu\right)\left[1 + d\left(u, u\right) + d\left(Tu, u\right)\right]}{2\left[1 + d\left(u, Tu\right)\right]}, \right.$$
$$\left. \frac{d\left(Tu, u\right)\left[1 + d\left(u, Tu\right) + d\left(Tu, Tu\right)\right]}{2\left[1 + d\left(u, Tu\right)\right]} \right\},$$

$$\left(1 - \frac{\alpha}{2}\right)\left|d\left(u, Tu\right)\right| \leq 0, \tag{43}$$

and hence $Tu = u$ which is the desired result. The uniqueness of common fixed point u follows easily. The details of the proof of this theorem can be obtained by using the argument that the pair $\{S, I\}$ satisfies (CLR_S)-property with suitable modifications. This completes the proof. $\square$

Theorem 14 is still true if we replace condition (35) with the following condition:

$$d\left(Sx, Ty\right)$$
$$\leq \alpha \left(\max \left\{ \left[d\left(Ix, Jy\right)\right]^2, \left[d\left(Ix, Sx\right)\right]^2, \right. \right.$$
$$\left[d\left(Jy, Ty\right)\right]^2,$$
$$\left. \frac{1}{2}\left[d\left(Ix, Ty\right)\right]^2, \frac{1}{2}\left[d\left(Jy, Sx\right)\right]^2 \right\}$$
$$\left. \times \left(d\left(Ix, Sx\right) + d\left(Jy, Ty\right)\right)^{-1} \right), \tag{44}$$

where $\alpha \in (0, 1)$ and the mappings S, I and T, J are defined as in Theorem 14.

Theorem 15. *Let (X, d) be a complex valued metric space and let $S, T, I, J : X \to X$ be four self-mappings satisfying the following:*

(i) *$T(X) \subseteq I(X)$, $S(X) \subseteq J(X)$;*

(ii) *$[d(Sx, Ty)]^3 \leq \alpha d(Ix, Jy) d(Ix, Sx) d(Jy, Ty)$, for all $x, y \in X$, where $\alpha \in (0, 1)$;*

(iii) *the pairs $\{S, I\}$ and $\{T, J\}$ are weakly compatible.*

If the pair $\{S, I\}$ satisfies (CLR_S)-property, or the pair $\{T, J\}$ satisfies (CLR_T)-property, then the mappings $I, J, S,$ and T have a unique common fixed point in X.

Proof. This theorem can be obtained by using a similar technique as in the above theorem. So we omit it. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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