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Research Article

Hybrid Stability Checking Method for Synchronization of Chaotic Fractional-Order Systems

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A hybrid stability checking method is proposed to verify the establishment of synchronization between two hyperchaotic systems. During the design stage of a synchronization scheme for chaotic fractional-order systems, a problem is sometimes encountered. In order to ensure the stability of the error signal between two fractional-order systems, the arguments of all eigenvalues of the Jacobian matrix of the erroneous system should be within a region defined in Matignon's theorem. Sometimes, the arguments depend on the state variables of the driving system, which makes it difficult to prove the stability. We propose a new and efficient hybrid method to verify the stability in this situation. The passivity-based control scheme for synchronization of two hyperchaotic fractional-order Chen-Lee systems is provided as an example. Theoretical analysis of the proposed method is validated by numerical simulation in time domain and examined in frequency domain via electronic circuits.

1. Introduction

Nonlinear systems may exhibit dynamical chaotic behavior. The study of chaos synchronization has received increasing attention due to its predicted potentials in technological applications in recent years. In 1983, Fujisaka and Yamada [1] first described the synchronization of chaotic signals. Some years later, Pecora and Carroll [2] synchronized two identical chaotic systems with different initial conditions. Subsequently, research activities of chaos synchronization have grown continuously.

Fractional calculus has been studied in a speedy pace during the recent years. It has been implemented in various engineering fields such as control [3], modeling [4], thermal engineering [5], and bioengineering [6]. There are mainly two ways to approximate a fractional-order system: in frequency domain based on Riemann-Liouville definition or Grunwald-Letnikov definition and in time domain based on Caputo definition. In frequency domain, linear approximation may be obtained with a given discrepancy over a frequency range [7]. The discrepancy and frequency range must be carefully chosen; otherwise, huge error may occur between the approximate and true results [8]. There are other frequency domain methods using continued fraction expansions and interpolation techniques [9], FIR filters [10], and so forth. Regrettably, the time memory characteristic of fractional-order systems is not considered. Meanwhile, the approximation in time domain outperforms the frequency domain approximation. The improved $\mathrm{P(EC)}^m\mathrm{E}$ method based on the Adams-Bashforth-Moulton algorithm [11] produces pretty accurate results. The only drawback is its heavy computation load. Recently, discrete fractional calculus [12] starts drawing researchers' attention. For example, various versions of fractional discrete-time Logistic map have been proposed and studied using different approaches, such as left Caputo discrete delta difference [13], discontinuous dynamical systems [14], and discrete-time Fourier transform [15]. Meanwhile, other discrete fractional systems have also been developed

[16]. Jarad et al. investigated the stability of discrete fractional systems in detail using the Lyapunov direct method [17]. It is also interesting to develop applications such as chaos synchronization in fractional discrete systems.

In 2004, Chen and Lee [18] developed a new chaotic system based on the Euler equations for the motion of a rigid body. This system describes the chaotic behavior for the anticontrol of chaos in rigid-body motion. The system is then called the Chen-Lee system [19]. Complete synchronization [20], synchronization and antisynchronization [21], controlling chaos with multiple time-delays [22], electronic circuit implementation [23, 24], fractional-order behavior [19, 25], and so forth have been studied for this system recently. It is believed that hyperchaotic systems are demanded by practical engineering applications, such as secure communication. The hyperchaotic Chen-Lee system and its hybrid projective synchronization were proposed by Chen et al. [26]. Due to more complex structure in the differential equations of the hyperchaotic Chen-Lee system, the synchronization was achieved under a complicate controller. Furthermore, the hyperchaotic Chen-Lee system may exhibit even more complex dynamical behavior in the fractional-order domain. It is hence necessary to seek for a simple and effective controller for those systems.

The concept of passive control theory [27] has been a focus for the control of chaos and chaos synchronization. A simple state-feedback controller has been built based on the passivity to stabilize the Lorenz equation [28] and Rabinovich system [29]. Chaos synchronization in the unified chaotic system, the Rikitake attractor, and the hyperchaotic complex Chen system with unknown parameters has been achieved using the passive control technique [30–32]. Passivity synchronization has been implemented to fractional-order hyperchaotic Liu's systems [33] by using inequality relation through absolute maximum bound of its system variable. Unfortunately, it is not always that the stability can be determined by upper bounds of system variables. This happens to the case when we apply passivity theory to the hyperchaotic Chen-Lee system. Hence, we proved the control scheme based on the proposed stability checking procedure without using any bound of system's variable.

In this paper, the synchronization between two hyperchaotic Chen-Lee systems with different initial conditions was established via the passive control technique first. The influence of the controller parameters was discussed for enhancing the efficiency of synchronization. Next, the proposed controller was also applied to fractional-order hyperchaotic Chen-Lee systems. Synchronization was ensured based on the stability theorem by Matignon [34]. An efficient and easy strategy was proposed to find the range of eigenvalues when the Jacobian matrix involves system states. Finally, numerical simulation and the corresponding electronic circuits were included to show the feasibility and effectiveness of the proposed methods.

2. Preliminaries

Matignon's theorem and the passivity theory were reviewed in this section. The former was used for analyzing the stability of fractional differential equations and the latter for designing the synchronization scheme between two systems.

2.1. The Matignon Theorem. Consider an error dynamical system $e(x, y, t)$, where $x(t)$ and $y(t)$ are the driving and response systems and $J_0(x)$ the Jacobian matrix which contains $x(t)$, and q is the fractional order of the systems.

Theorem 1 (see [34]). *The trivial solution of the system $e(x, y, t)$ is asymptotically stable if and only if*

$$\left|\arg(\lambda)\right| > \frac{q\pi}{2} \quad (1)$$

is satisfied for all eigenvalues λ of Jacobian matrix J_0 of system $e(x, y, t)$ evaluated at the origin.

Corollary 2. *The response system is asymptotically synchronized with the driving system if and only if $|\arg(\lambda)| > q\pi/2$ is satisfied for all eigenvalues $\lambda(x, t)$ of $J_0(x)$ for any possible $x(t)$.*

2.2. Review of the Passivity Theory. Consider the following nonlinear system [27]:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned} \quad (2)$$

where the state variable $x \in \mathfrak{R}^m$, the input $u \in \mathfrak{R}^m$, and the output $y \in \mathfrak{R}^m$. $f(x)$ and $g(x)$ are the smooth vector fields and $f(0) = 0$. $h(x)$ is a smooth mapping. The system (2) is passive if the conditions below are all satisfied.

(1) $f(x)$ and $g(x)$ exist.

(2) For all $t \geq 0$, there is a real constant β such that the inequality holds

$$\int_0^t u^T(\tau)\, y(\tau)\, d\tau \geq \beta \quad (3)$$

or there are a real constant β and a constant $\rho > 0$ such that

$$\int_0^t u^T(\tau)\, y(\tau)\, d\tau + \beta \geq \int_0^t \rho y^T(\tau)\, y(\tau)\, d\tau. \quad (4)$$

If a system is passive, a suitable controller can stabilize asymptotically the equilibrium point $x = 0$ of system (2).

If the system (2) has the relative degree $\{1, \dots, 1\}$ at $x = 0$ ($L_g h(0)$ is nonsingular) and the distribution spanned by the vector field $g_1(x), \dots, g_m(x)$ is involutive, then it can be represented in the generalized form

$$\begin{aligned} \dot{z} &= f_0(z) + p(z, y)\, y, \\ \dot{y} &= b(z, y) + k(z, y)\, u, \end{aligned} \quad (5)$$

where $k(z, y)$ is nonsingular $\forall(z, y)$. By designing a suitable controller u, the system (5) may be passive. Thus, the system (5) can be asymptotically stabilized to the equilibrium point by applying the controller u. If the system (5) describes an error dynamical system, then the synchronization between two systems is achieved.

Theorem 3. *The system* (5) *is asymptotically stabilized to the equilibrium point by applying the controller as follows:*

$$u = k^{-1}(z,y)\left[-b^{T}(z,y) - \frac{\partial}{\partial z}W(z)\,p(z,y) - \alpha y + \nu\right], \tag{6}$$

where $W(z)$ is the Lyapunov function of $f_0(z)$ and $W(0) = 0$, α is a positive real constant, and ν is an external signal connected to the reference input.

3. Chaos Synchronization of Hyperchaotic Chen-Lee Systems

Chen et al. [26] introduced the hyperchaotic Chen-Lee system:

$$\begin{aligned}
\dot{x}_1 &= -x_2x_3 + ax_1,\\
\dot{x}_2 &= x_1x_3 + bx_2,\\
\dot{x}_3 &= \frac{1}{3}x_1x_2 + cx_3 + 0.2x_4,\\
\dot{x}_4 &= 0.5x_2x_3 + 0.05x_4 + dx_1,
\end{aligned} \tag{7}$$

where $a = 5$, $b = -10$, $c = -3.8$, and d are system parameters. Hyperchaotic behaviors have been observed when $0 \le d \le 1.3$ excluding $d = 0.8$ and $d = 1.1$. For the synchronization purpose, consider that the system (7) is the driving system and the response system is defined as

$$\begin{aligned}
\dot{y}_1 &= -y_2y_3 + ay_1 + u_1,\\
\dot{y}_2 &= y_1y_3 + by_2,\\
\dot{y}_3 &= \frac{1}{3}y_1y_2 + cy_3 + 0.2y_4 + u_2,\\
\dot{y}_4 &= 0.5y_2y_3 + 0.05y_4 + dy_1 + u_3,
\end{aligned} \tag{8}$$

where u_1, u_2, and u_3 are the controllers to be designed. Let $e_1 = y_1 - x_1$, $e_2 = y_2 - x_2$, $e_3 = y_3 - x_3$, and $e_4 = y_4 - x_4$. The error dynamical system is then expressed by

$$\begin{aligned}
\dot{e}_1 &= ae_1 - e_2e_3 - e_2x_3 - e_3x_2 + u_1,\\
\dot{e}_2 &= be_2 + e_1e_3 + e_1x_3 + e_3x_1,\\
\dot{e}_3 &= ce_3 + \frac{1}{3}\left(e_1e_2 + e_1x_2 + e_2x_1\right) + 0.2e_4 + u_2,\\
\dot{e}_4 &= 0.5\left(e_2e_3 + e_2x_3 + e_3x_2\right) + 0.05e_4 + de_1 + u_3.
\end{aligned} \tag{9}$$

By choosing $u_2 = -(4/3)e_2x_1 - 0.2e_4$ and $u_3 = -0.5(y_2y_3 - x_2x_3) - \sigma e_4$, where $\sigma \ge 0.05$, the system (9) can be rewritten as

$$\begin{aligned}
\dot{e}_1 &= ae_1 - e_2e_3 - e_2x_3 - e_3x_2 + u_1,\\
\dot{e}_2 &= be_2 + e_1e_3 + e_1x_3 + e_3x_1,\\
\dot{e}_3 &= ce_3 + \frac{1}{3}\left(e_1e_2 + e_1x_2 - e_2x_1\right),\\
\dot{e}_4 &= (0.05 - \sigma)e_4 + de_1.
\end{aligned} \tag{10}$$

The aim is to design the controller u_1 for stabilizing the system (10) at the origin asymptotically. Therefore, the driving and the response systems are synchronized globally asymptotically.

Let $z_1 = e_2$, $z_2 = e_3$, $z_3 = e_4$, and $y = e_1$. The system (10) can be rewritten as

$$\begin{aligned}
\dot{z}_1 &= bz_1 + x_1z_2 + (z_2 + x_3)\,y,\\
\dot{z}_2 &= cz_2 - x_1z_1 + \frac{1}{3}(z_1 + x_2)\,y,\\
\dot{z}_3 &= (0.05 - \sigma)z_3 + dy,\\
\dot{y} &= ay - z_1z_2 - z_1x_3 - z_2x_2 + u_1.
\end{aligned} \tag{11}$$

It is in the form of (5), where

$$f_0(z) = \begin{bmatrix} bz_1 + x_1z_2 \\ cz_2 - x_1z_1 \\ 0.05 - \sigma \end{bmatrix}, \qquad p(x,y) = \begin{bmatrix} z_2 + x_3 \\ \frac{1}{3}(z_1 + x_2) \\ d \end{bmatrix},$$
$$b(z,y) = ay - z_1z_2 - z_1x_3 - z_2x_2, \qquad k(z,y) = 1. \tag{12}$$

Because $b < 0$, $c < 0$, and $\sigma \ge 0.05$,

$$\frac{d}{dt}W(z) = [z_1, z_2, z_3]\begin{bmatrix} bz_1 + x_1z_2 \\ cz_2 - x_1z_1 \\ (0.05 - \sigma)z_3 \end{bmatrix} \le 0. \tag{13}$$

According to Theorem 3, the system (10) is stabilized asymptotically at the origin if we choose the controller u_1 as

$$u_1 = \frac{2}{3}e_3x_2 - \frac{1}{3}e_2e_3 - de_4 - (a + \alpha)e_1 + \nu. \tag{14}$$

This means that the driving system (7) and the response system (8) can be synchronized globally asymptotically via the designed controller.

If the ordinary differential operators in (7) are replaced by fractional differential ones, we have the fractional-order hyperchaotic Chen-Lee system as shown as follows:

$$\begin{aligned}
D^q x_1 &= -x_2x_3 + ax_1,\\
D^q x_2 &= x_1x_3 + bx_2,\\
D^q x_3 &= \frac{1}{3}x_1x_2 + cx_3 + 0.2x_4,\\
D^q x_4 &= 0.5x_2x_3 + 0.05x_4 + dx_1.
\end{aligned} \tag{15}$$

By taking (15) as the driving system and letting

$$\begin{aligned}
D^q y_1 &= -y_2y_3 + ay_1 + u_1,\\
D^q y_2 &= y_1y_3 + by_2,\\
D^q y_3 &= \frac{1}{3}y_1y_2 + cy_3 + 0.2y_4 + u_2,\\
D^q y_4 &= 0.5y_2y_3 + 0.05y_4 + dy_1 + u_3
\end{aligned} \tag{16}$$

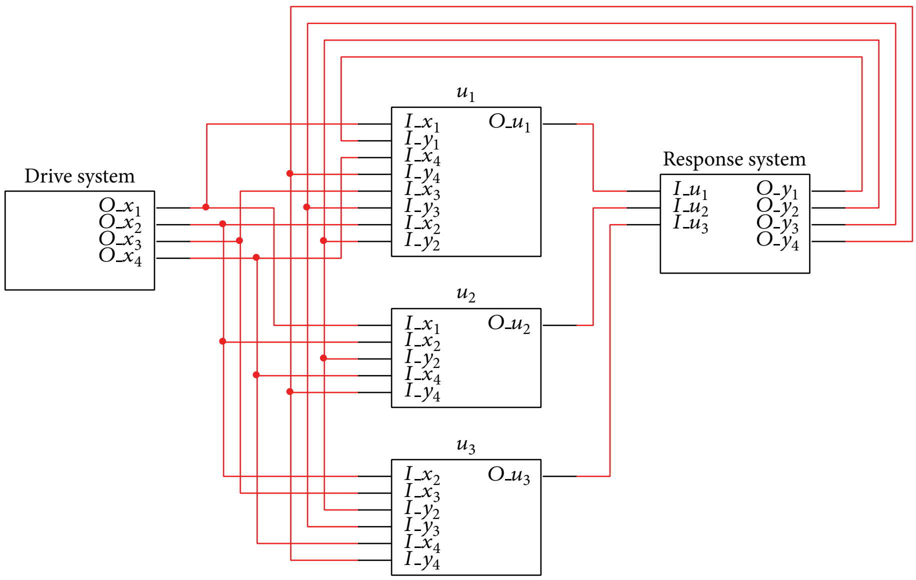

FIGURE 1: Schematic diagram of the synchronization system.

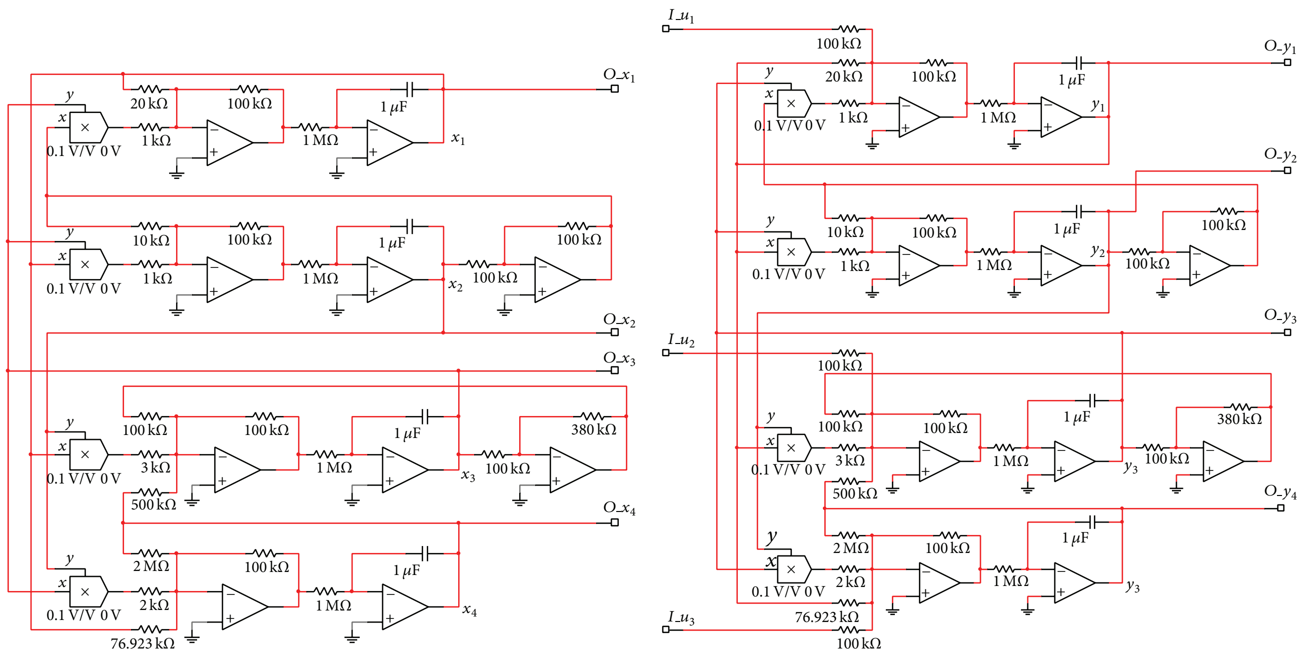

FIGURE 2: Electronic circuits of (a) the driving system and (b) the response system.

be the response system and supposing that the external source input $\nu = 0$, the synchronization error between two fractional-order hyperchaotic Chen-Lee systems with identical order is then expressed as

$$D^q e_1 = -\alpha e_1 - \frac{4}{3}e_2 e_3 - e_2 x_3 - \frac{1}{3}e_3 x_2 - de_4,$$

$$D^q e_2 = be_2 + e_1 e_3 + e_1 x_3 + e_3 x_1,$$

$$D^q e_3 = ce_3 + \frac{1}{3}\left(e_1 e_2 + e_1 x_2 - e_2 x_1\right),$$

$$D^q e_4 = (0.05 - \sigma) e_4 + de_1, \tag{17}$$

where $q \in [0.98, 1)$ is the fractional order. It has not been determined yet whether the passivity theory above can directly be applied to the synchronization for the fractional

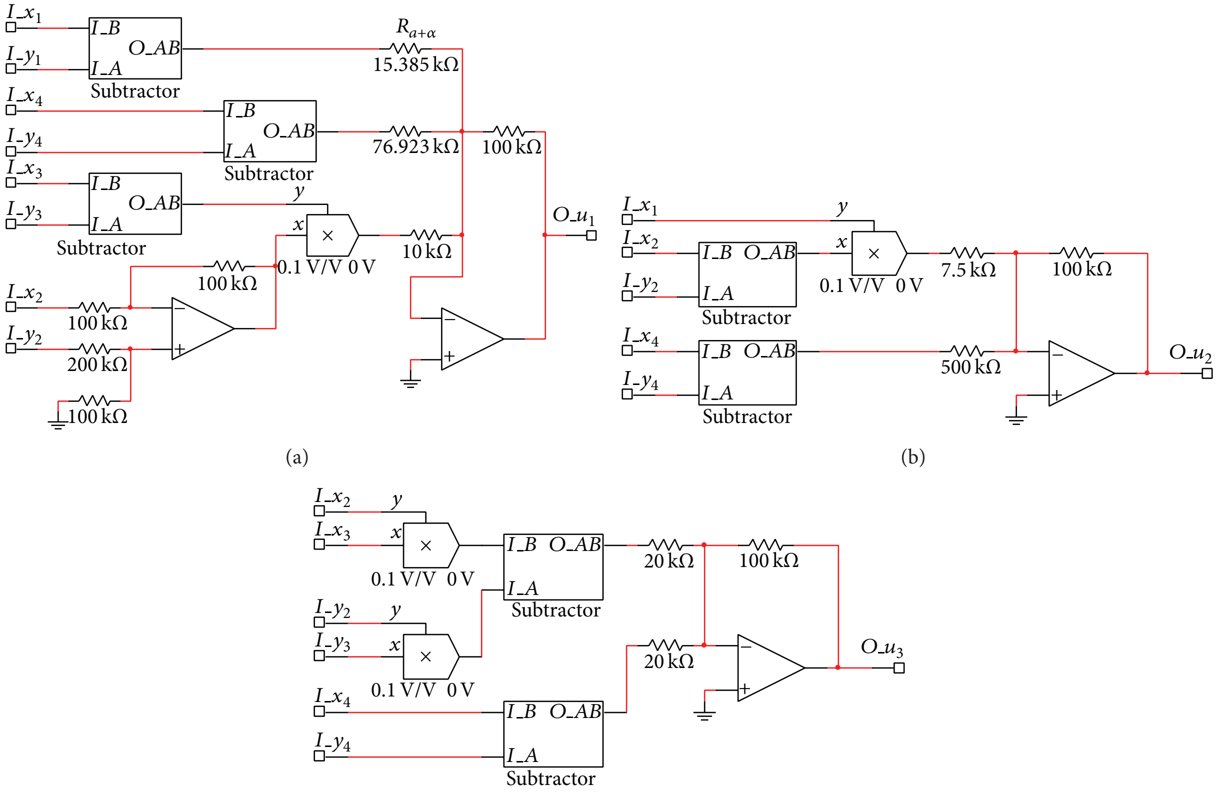

FIGURE 3: Electronic circuit of the controllers: (a) u_1; (b) u_2; (c) u_3.

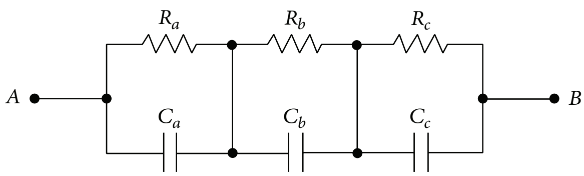

FIGURE 4: Chain fractance of $1/s^{0.98}$.

counterpart or not. The stability of (17) can be determined by the following theorem.

Theorem 4. *System* (17) *is globally asymptotically stable at the origin if* $|\arg(\lambda(t))| > q\pi/2$ *is satisfied for all* $\lambda(t)$ *in*

$$\begin{aligned}&\lambda^4 + \lambda^3(\alpha - b - c - \omega)\\&\quad + \lambda^2\left[\frac{1}{3}x_1^2 + \frac{1}{9}x_2^2 - x_3^2 + d^2 - \omega(\alpha - b - c) + \zeta\right]\\&\quad + \lambda\left[\frac{4}{9}x_1x_2x_3 + \frac{1}{3}(\alpha - \omega)x_1^2 - \frac{1}{9}(b + \omega)x_2^2 + (c + \omega)x_3^2\right.\\&\qquad\qquad \left. -d^2(b + c) - \zeta\omega + \alpha bc\right]\\&\quad - \frac{4}{9}\omega x_1x_2x_3 + \frac{1}{3}\left(d^2 - \alpha\omega\right)x_1^2 + \frac{1}{9}b\omega x_2^2\\&\quad - c\omega x_3^2 + bcd^2 - \alpha bc\omega = 0\end{aligned}\tag{18}$$

for all sets of $x(t) = [x_1(t), x_2(t), x_3(t)]^T$ *of system* (15), *where* $\omega = 0.05 - \sigma$ *and* $\zeta = bc - \alpha(b + c)$.

Proof. To gain the eigenvalues at the origin, $e_1 = e_2 = e_3 = e_4 = 0$, we let

$$\begin{aligned}&\det\left|J_0(x) - \lambda I\right|\\&\quad = \det\begin{vmatrix}-\alpha - \lambda & -x_3 & -\dfrac{1}{3}x_2 & -d\\ -x_3 & b - \lambda & x_1 & 0\\ \dfrac{1}{3}x_2 & -\dfrac{1}{3}x_1 & c - \lambda & 0\\ d & 0 & 0 & 0.05 - \sigma - \lambda\end{vmatrix} = 0\end{aligned}\tag{19}$$

and we have

$$\begin{aligned}&\lambda^4 + \lambda^3(\alpha - b - c - \omega)\\&\quad + \lambda^2\left[\frac{1}{3}x_1^2 + \frac{1}{9}x_2^2 - x_3^2 + d^2 - \omega(\alpha - b - c) + \zeta\right]\\&\quad + \lambda\left[\frac{4}{9}x_1x_2x_3 + \frac{1}{3}(\alpha - \omega)x_1^2 - \frac{1}{9}(b + \omega)x_2^2 + (c + \omega)x_3^2\right.\\&\qquad\qquad \left. -d^2(b + c) - \zeta\omega + \alpha bc\right]\end{aligned}$$

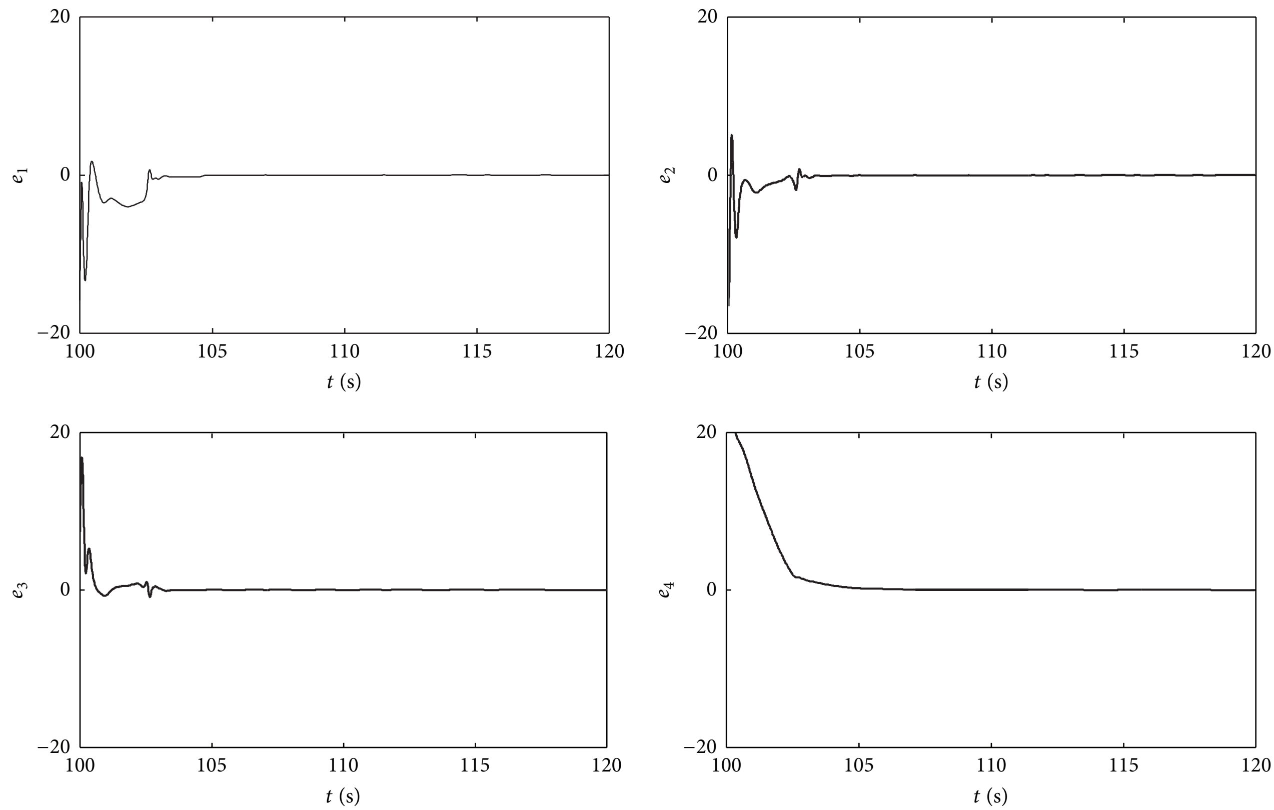

Figure 5: Time histories of the synchronization errors (10) between systems (7) and (8) with $\sigma = 0.5$.

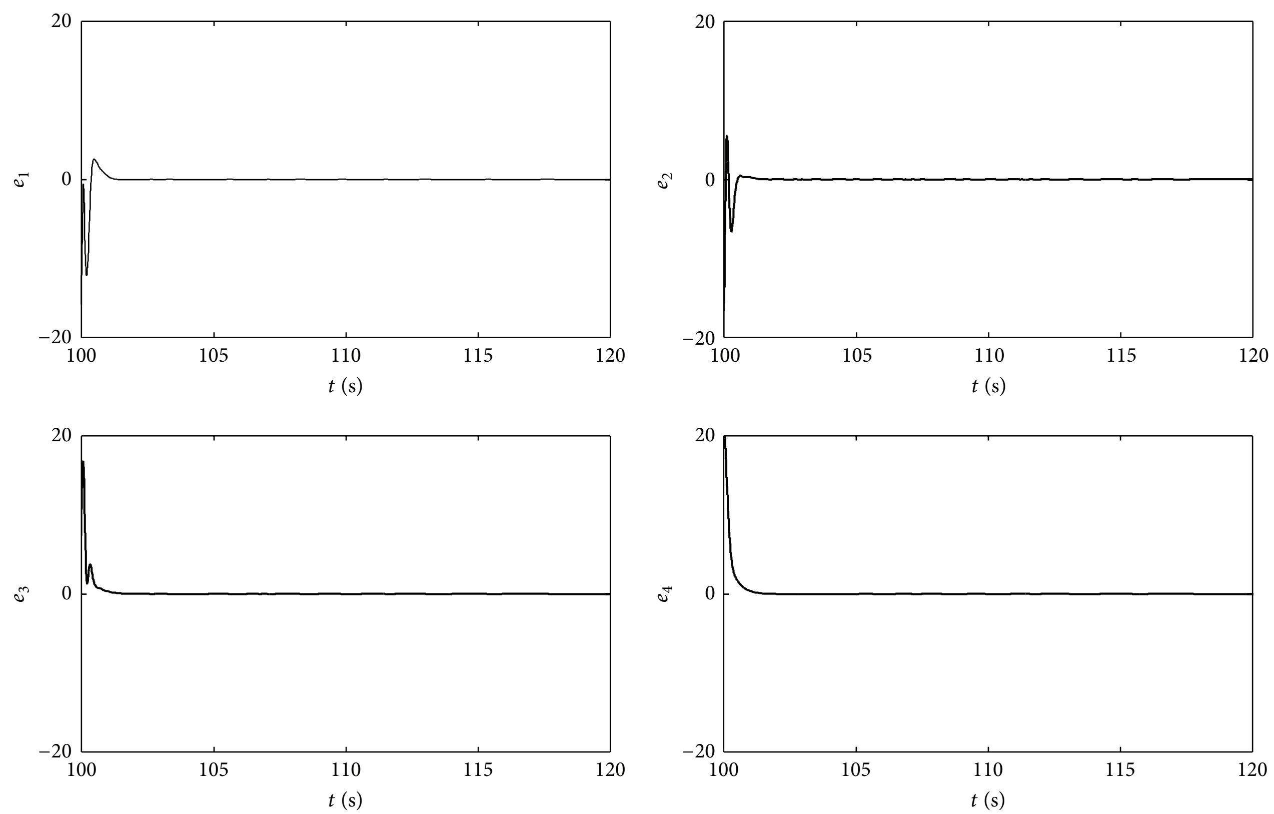

Figure 6: Time histories of the synchronization errors (10) between systems (7) and (8) with $\sigma = 5$.

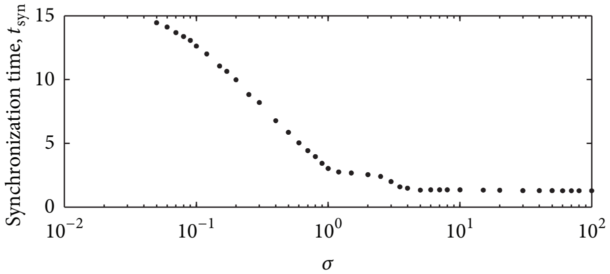

Figure 7: The influence of the controller parameter σ on the synchronization time t_{syn}.

$$- \frac{4}{9}\omega x_1 x_2 x_3 + \frac{1}{3}\left(d^2 - \alpha\omega\right) x_1^2 + \frac{1}{9} b\omega x_2^2 - c\omega x_3^2 + bcd^2 - \alpha bc\omega = 0. \tag{20}$$

Obviously, the eigenvalues depend on the system states of (15). However, it is difficult to determine the stability by just considering the bounds of the system states. In order to examine the stability, we apply Theorem 1. The procedure is described in the following. Time histories of the system states are first numerically evaluated for a sufficient long time, so that they can reflect the dynamical behavior of the chaotic fractional-order system. Next, the eigenvalues $\lambda(t)$ are evaluated for each set of the system states. The minimum values of $|\arg(\lambda(t))|$ are then collected at every time step. If all $\min|\arg(\lambda(t))| > q\pi/2$, it implies that system (17) is stabilized at the origin. This completes the proof. □

4. Electronic Circuits

An electronic circuit was constructed for implementing the proposed scheme. Figure 1 showed the synchronization system, which consisted of a driving system, a response system, and three controllers. The systems (7) and (8) were normalized by a factor of 10. The controller parameter σ was set to 5. The corresponding electronic circuits of the systems and controllers were presented in Figures 2 and 3.

The above circuit can be converted to a fractional-order one by simply replacing the capacitors on the feedback path of the integrators by chain fractance [35]. Chain fractance is constructed of a group of resistors in parallel with a capacitor in series.

To approximately realize the fractional-order operator with $q = 0.98$, the corresponding transfer function [7] with discrepancy of 0.5 dB between the actual and approximated signals over the frequency bandwidth $\omega_{max} = 100$ for a corner frequency $p_T = 0.01$ was

$$H(s) \approx \frac{1.2234s^2 + 1463.2s + 4893.2}{(s + 0.0106)(s + 3.7716)(s + 1341.4)}. \tag{21}$$

It can be realized by a chain fractance of order 3 (Figure 4). Its transfer function was described by

$$H(s) = \left(R_a // \frac{1}{sC_a}\right) + \left(R_b // \frac{1}{sC_b}\right) + \left(R_c // \frac{1}{sC_c}\right). \tag{22}$$

The values of the resistors and capacitors of the chain fractance were then determined by solving the equations

$$\begin{gathered}
\left(R_a C_a\right)^{-1} = 0.0106, \\
\left(R_b C_b\right)^{-1} = 3.7716, \\
\left(R_c C_c\right)^{-1} = 1341.4, \\
C_a^{-1} + C_b^{-1} + C_c^{-1} = 1.2234, \\
\frac{R_a C_a \left(R_b + R_c\right) + R_b C_b \left(R_a + R_c\right) + R_c C_c \left(R_a + R_b\right)}{R_a C_a R_b C_b R_c C_c} = 1463.2, \\
\frac{R_a + R_b + R_c}{R_a C_a R_b C_b R_c C_c} = 4893.2
\end{gathered} \tag{23}$$

which gave $R_a = 91.17\,\mathrm{M\Omega}$, $R_b = 32.046\,\mathrm{k\Omega}$, $R_c = 101.12\,\Omega$, $C_a = 1.0656\,\mu\mathrm{F}$, $C_b = 8.5245\,\mu\mathrm{F}$, and $C_c = 7.596\,\mu\mathrm{F}$.

5. Numerical Results

In this section, the Runge-Kutta method of order 4 was used to solve the differential equations in the systems (7) and (8), while the improved method [11] based on Caputo derivative was implemented to approximate the fractional differential equations in (15) and (16) with time step size $\Delta t = 0.001$ s. The control scheme took place at 100 s in all cases.

5.1. Synchronization in Integer-Order Systems. First of all, the value of d was chosen to be $d = 1.3$, with the fact that the Lyapunov exponents of the system (7) were $\lambda_1 = 0.677$, $\lambda_2 = 0.107$, $\lambda_3 = 0$, and $\lambda_4 = -13.412$ $\lambda_4 = -13.412$. Two positive Lyapunov exponents indicated the system (7) is hyperchaotic [26]. Next, the control parameters were chosen to be $\sigma = 0.5$, $\alpha = 1.5$, and $\nu = 0$. The initial conditions for the driving and response systems were chosen as $[2, 3, 2, 2]^T$ and $[4, -6, -6, -3]^T$, respectively. Figure 5 showed the time histories of the synchronization errors between the two systems (7) and (8). The response system was synchronized with the driving system in 5.9 s. The synchronization time was shortened by setting $\sigma = 5$. The results in Figure 6 showed that synchronization was well achieved within 1.1 s. In addition, Figure 7 represented the influence of σ on the synchronization time. It was noticed that the performance of synchronization was improved by increasing σ from a small value up to $\sigma = 5$. If $\sigma > 5$, the synchronization time only slightly decreased with increasing σ, though the influence was quite unnoticeable.

5.2. Synchronization in Fractional-Order Systems. The order of the systems was chosen to be $q = 0.98$. The phase diagrams of the system (15) were plotted in Figure 8(a). A four-scroll attractor was found in the x-y plot, which made the system (15) exhibit more complicated dynamical behavior than the system (7). The stability of the error dynamical system (17) was examined by Theorem 4. When $\alpha = 5$ and $\sigma = 5$, the time history of $\min|\arg(\lambda(t))|$ was plotted in Figure 9(a).

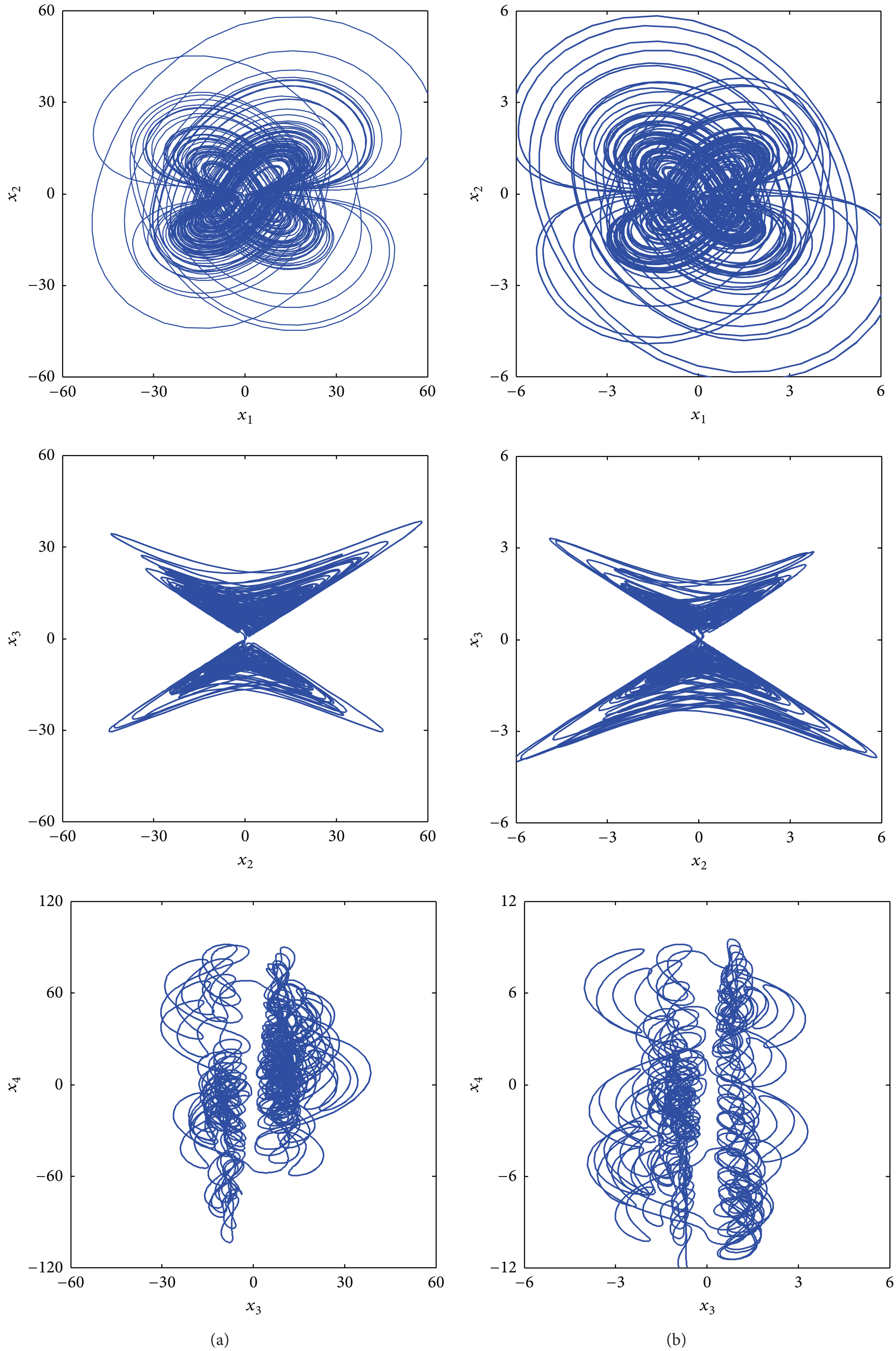

FIGURE 8: Phase diagrams of the hyperchaotic systems: (a) numerical solution in time domain; (b) circuit simulation in frequency domain.

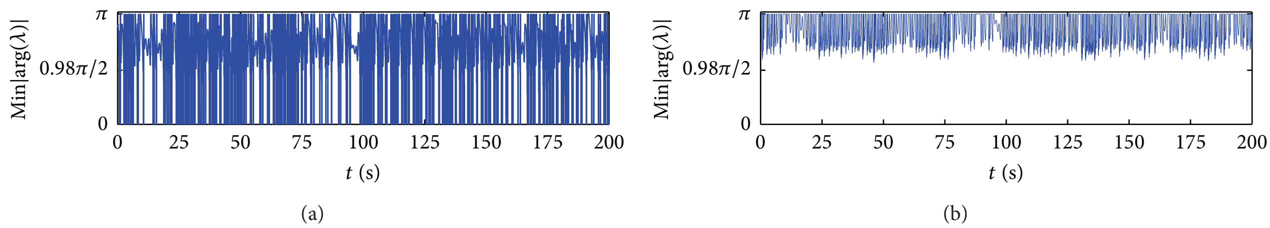

FIGURE 9: Time histories of min $|\arg(\lambda(t))|$: (a) $\alpha = 5$; (b) $\alpha = 155$.

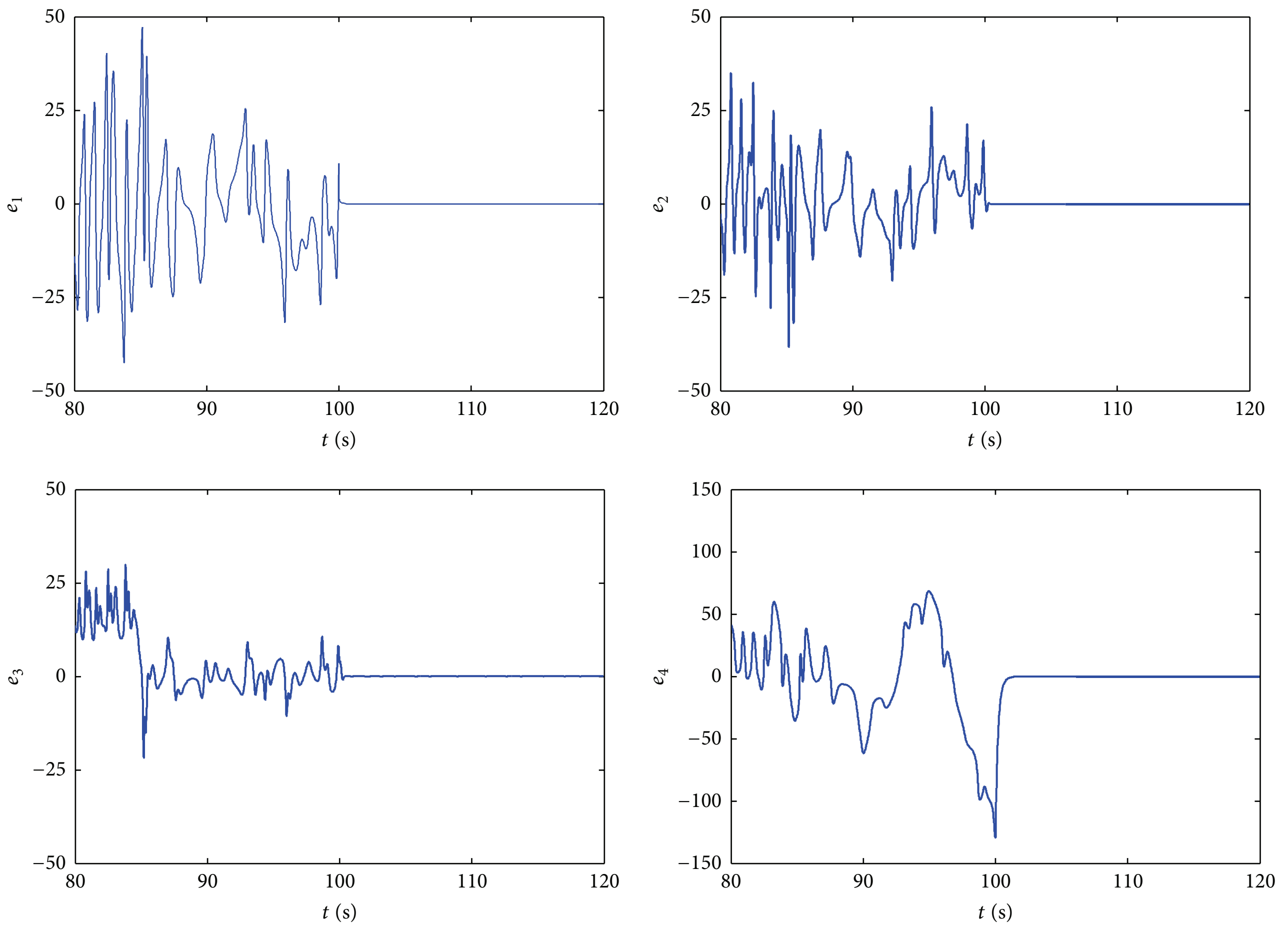

FIGURE 10: Time histories of the error dynamical system (17) when $\alpha = 155$.

Obviously min $|\arg(\lambda(t))|$ reached zero at some t. According to Theorem 1, this implied that the synchronization may not be achieved. All values of min $|\arg(\lambda(t))|$ were completely greater than $0.98\pi/2$ until α increased to 155, as depicted in Figure 9(b). Under this condition, the response system (16) was synchronized successfully with the driving system (15). The time histories of the error system (17) were plotted in Figure 10. After the synchronization scheme took place at $t = 100$ s, the error signal was asymptotically stabilized at the origin, meaning that the synchronization was well achieved.

5.3. Circuit Simulations. The simulation was done on Multisim package. The initial conditions were set to $[0.2, 0.3, 0.2, 0.2]^T$ and $[0.4, -0.6, -0.6, -0.3]^T$ for the integer-order driving and response systems, respectively. The outputs of the circuit simulation were depicted in Figure 11. The driving and response systems developed differently during the time period $0 < t < 100$ s. The response system was synchronized with the driving system after activating the controller at $t = 100$ s. It meant that the synchronization circuit with integer order was built successfully.

For simulating systems with fractional order, the capacitors were replaced by chain fractance. The value of the resistor $R_{a+\alpha} = 15.385\,\mathrm{k}\Omega$ in Figure 3(a) was changed to $650\,\Omega$ for $\alpha = 155$. The numerical circuit-simulated phase diagrams of the hyperchaotic Chen-Lee system with order $q = 0.98$ were depicted in Figure 8(b). The output

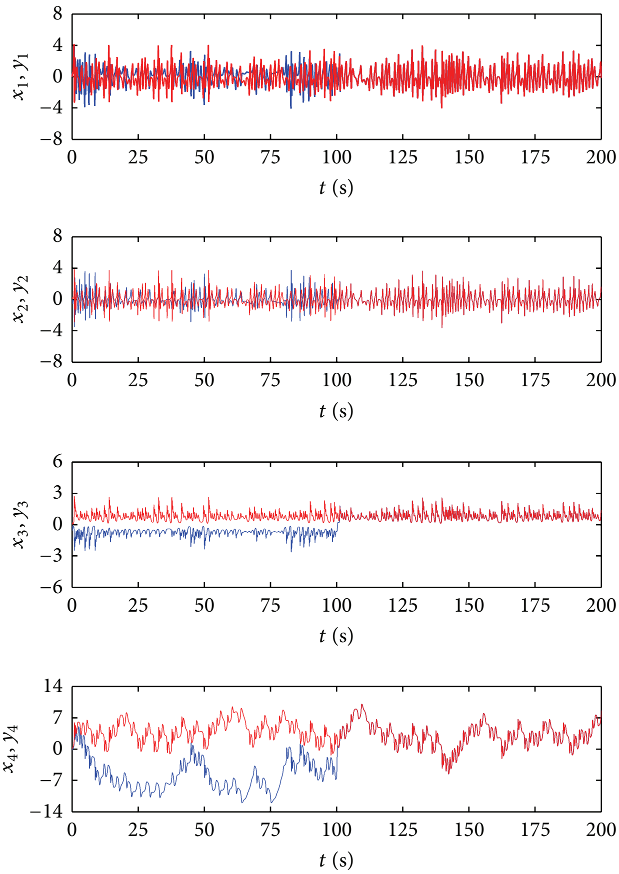

Figure 11: Time histories of the circuit outputs of the driving system (red) and the response system (blue) with integer order.

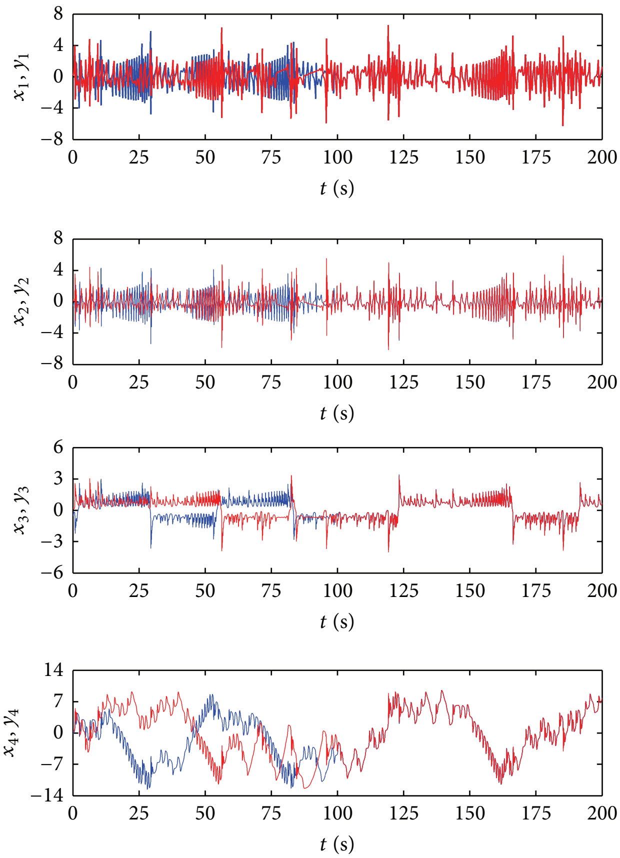

Figure 12: Time histories of the circuit outputs of the driving system (red) and the response system (blue) with fractional order.

trajectories of the circuit simulation were plotted in Figure 12. It was clear that the response system was well synchronized with the driving system after 100 s. Both circuit simulations were in good agreement with their corresponding numerical results.

6. Conclusions

A novel and efficient strategy was proposed to examine the stability of the error dynamical systems with fractional order. The chaos synchronization between two hyperchaotic Chen-Lee systems with fractional order was achieved via feedback passive control technique. The passive controller was first designed for the integer-order system. The control scheme was proved based on the stability theorem for fractional calculus. Numerical simulation was given to validate the proposed approaches. Chain fractance was designed for approximating the fractional order $q = 0.98$ and the electronic circuits were simulated to verify the scheme. It is discovered that the synchronization scheme designed for integer-order systems may not directly be valid for systems with fractional order. Fortunately, the scheme may become effective by just simply changing a single parameter.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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