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Research Article

Noncommutative Multisoliton Solutions of a Supersymmetric Chiral Model

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Multisoliton configurations of a superextended and Moyal-type noncommutative deformed modified 2 + 1 chiral model have been constructed with the dressing method by Lechtenfeld and Popov several years ago. These configurations have no-scattering property. A two-soliton configuration with nontrivial scattering was constructed soon after that. More multisoliton solutions with general pole data of the superextended and noncommutative Ward model will be constructed in this paper. The method is the supersymmetric and noncommutative extension of Dai and Terng's in constructing soliton solutions of the Ward model.

1. Introduction

A generalization of the modified 2 + 1 chiral model [1] (also called the Ward model in [2]) with $\mathcal{N} \leq 8$ supersymmetries and a Moyal deformation of this model is introduced in [3]. Since the $\mathcal{N}$-extended and deformed chiral model can be formulated as the compatibility conditions of a linear system of differential equations, solutions of this model can be generated with the aid of the system just as in the nonsupersymmetric case [1, 2, 4–7].

A powerful method, the so-called dressing method, is employed in [3] to construct multisoliton configurations with only simple poles. By allowing for second-order poles in the dressing ansatz, a two-soliton configuration with genuine soliton-soliton interaction is constructed in [8].

In a recent paper [9], we extended the algebraic Bäcklund transformations (BTs) and the order k limiting method of Dai and Terng [2] to the noncommutative case, with which we constructed multisoliton solutions with general pole data of the noncommutative Ward model [7, 10]. In this paper, we further extend the above case to the supersymmetric one; that is, we use the supersymmetric and noncommutative extended algebraic BTs and the order k limiting method to construct multisoliton solutions with arbitrary poles and multiplicities of the supersymmetric and noncommutative Ward model.

The plan of the paper is as follows: we review the explicit definition of the $\mathcal{N}$-extended and deformed Ward model and the linear system associated to it in Section 2. In Section 3, we first briefly review the dressing construction in [3] and then give the supersymmetric algebraic BTs for the linear system. We apply these algebraic BTs and the superextended order k limiting method to construct a large family of multisoliton configurations with general pole data in Section 4.

2. Noncommutative $\mathcal{N}$-Extended Ward Model

To formulate the noncommutative $\mathcal{N}$-extended Ward model, we need to introduce the following notations [3, 8].

(1) $\mathbb{R}^{2,1} = (\mathbb{R}^3, g)$ with coordinates $(x^a) = (x, y, t)$ and the metric $g = \mathrm{diag}(1, 1, -1)$.

(2) Grassmann variables (see [11]): some mathematical objects obeying the following anticommutation rules $\chi_i\chi_j + \chi_j\chi_i = 0$ for any $1 \leq i, j \leq n$. In particular, when $i = j$, we have $\chi_i^2 = 0$. So any function K of the Grassmann variables is a finite polynomial $K(\chi_1, \chi_2, \dots, \chi_n) = \sum_{\alpha_i=0,1} a(\alpha_1, \alpha_2, \dots, \alpha_n)\chi_1^{\alpha_1}\chi_2^{\alpha_2}\cdots\chi_n^{\alpha_n}$.

(3) The antichiral superspace $\mathbb{R}^{3|\mathcal{N}}$ with coordinates (x^a, η_i^α) for $\alpha = 1, 2$ and $i = 1, \dots, (1/2)\mathcal{N} \leq 4$; here η_i^α are called fermionic coordinates, which are Grassmann variables. The tensor fields on $\mathbb{R}^{3|\mathcal{N}}$ are called superfields.

(4) The noncommutative star product on the antichiral superspace $\mathbb{R}^{3|\mathcal{N}}$ is

$$(f \star g)(x, y, t, \eta_i^\alpha) = f(x, y, t, \eta_i^\alpha) \exp\left\{\frac{i}{2}\theta\left(\overleftarrow{\partial_x}\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\overrightarrow{\partial_x}\right)\right\} g(x, y, t, \eta_i^\alpha). \quad (1)$$

Note that the time coordinate remains commutative and no derivatives with respect to the Grassmann variables η_i^α.

With the notations listed above, we can now state the definition of this model.

Definition 1 (see [3, 8]). The $U(n)$-valued superfield $\Phi(x^a, \eta_i^\alpha)$ of the noncommutative $\mathcal{N}$-extended $U(n)$ Ward model satisfies the classical field equations

$$\begin{gathered}\partial_x\left(\Phi^\dagger \star \partial_x\Phi\right) + \partial_y\left(\Phi^\dagger \star \partial_y\Phi\right) - \partial_t\left(\Phi^\dagger \star \partial_t\Phi\right) \\ + \partial_y\left(\Phi^\dagger \star \partial_t\Phi\right) - \partial_t\left(\Phi^\dagger \star \partial_y\Phi\right) = 0, \\ \partial_1^i\left(\Phi^\dagger \star \partial_x\Phi\right) - \partial_t\left(\Phi^\dagger \star \partial_2^i\Phi\right) + \partial_y\left(\Phi^\dagger \star \partial_2^i\Phi\right) = 0, \\ \partial_1^i\left(\Phi^\dagger \star \partial_t\Phi\right) + \partial_1^i\left(\Phi^\dagger \star \partial_y\Phi\right) - \partial_x\left(\Phi^\dagger \star \partial_2^i\Phi\right) = 0, \\ \partial_1^i\left(\Phi^\dagger \star \partial_2^j\Phi\right) + \partial_1^j\left(\Phi^\dagger \star \partial_2^i\Phi\right) = 0,\end{gathered} \quad (2)$$

where $\partial_\alpha^i := \partial/\partial\eta_i^\alpha$, and the unitarity condition

$$\Phi^\dagger \star \Phi = \Phi \star \Phi^\dagger = I_n. \quad (3)$$

To avoid cluttering the formulae we suppress the "$\star$" notation for the supersymmetric version of noncommutative multiplication from now on and most products between classical fields and their components are assumed to be star products until mentioned otherwise.

By [3], model (2) is the compatibility condition for the following linear system of differential equations involving a spectral parameter $\zeta \in \mathbb{C} \cup \{\infty\}$:

$$\begin{gathered}(\zeta\partial_x - \partial_u)\psi = \mathscr{A}\psi, \\ (\zeta\partial_v - \partial_x)\psi = \mathscr{B}\psi, \\ \left(\zeta\partial_1^i - \partial_2^i\right)\psi = \mathscr{C}^i\psi, \quad i = 1, \ldots, \frac{1}{2}\mathcal{N},\end{gathered} \quad (4)$$

where the $n \times n$ matrix ψ depends on $(x^a \mid \zeta, \eta_i^\alpha)$ and the $n \times n$ matrices $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}^i$ are superfields on $(x^a \mid \eta_i^\alpha) \in \mathbb{R}^{3|\mathcal{N}}$ independent of the spectral parameter ζ. Moreover, ψ is subject to the following reality condition:

$$\psi(x^a, \zeta, \eta_i^\alpha)\left[\psi\left(x^a, \bar{\zeta}, \eta_i^\alpha\right)\right]^\dagger = I_n. \quad (5)$$

Then the superfield $\Phi(x^a, \eta_i^\alpha) = \psi(x^a, 0, \eta_i^\alpha)^{-1} := \psi^{-1}$ $(\zeta = 0)$ clearly satisfies model (2) and unitary condition (3).

The noncommutative star product can be replaced by ordinary product via the Moyal-Weyl map [3, 6, 7]; that is, $f \star g \to \widehat{f}\widehat{g}$, where $\widehat{f}$ and $\widehat{g}$ are the corresponding operators of f and g under the Moyal-Weyl map, respectively. In later sections, we will mainly use the star-product formulation but use the operator formalism when the former does not work well.

3. $\mathcal{N}$-Extended Multisoliton Configurations with Simple Pole Data

Dressing method is employed in [3] to construct solutions to the linear system (4). This method is a recursive procedure for generating a new solution from an old one. We briefly review their construction. Rewrite (4) in the form

$$\begin{gathered}\psi(\partial_u - \zeta\partial_x)\psi^\dagger = \mathscr{A}, \\ \psi(\partial_x - \zeta\partial_v)\psi^\dagger = \mathscr{B}, \\ \psi\left(\partial_2^i - \zeta\partial_1^i\right)\psi^\dagger = \mathscr{C}^i, \quad i = 1, \ldots, \frac{1}{2}\mathcal{N},\end{gathered} \quad (6)$$

and build a multisoliton solution ψ_m with m simple poles at position $\mu_1, \ldots, \mu_m$ with $\operatorname{Im}\mu_k < 0$ by left-multiplying an $(m-1)$ simple pole solution ψ_{m-1} with a single pole factor of the form

$$I_n + \frac{\mu_m - \bar{\mu}_m}{\zeta - \mu_m} P_m(x^a, \eta_i^\alpha), \quad (7)$$

where the $n \times n$ matrix P_m is a Hermitian projection of rank r_m; that is, $P_m^\dagger = P_m$ and $P_m^2 = P_m$; this is obtained by reality condition (5), and therefore one can decompose $P_m = T_m(T_m^\dagger T_m)^{-1}T_m^\dagger$, where T_m is an $n \times r_m$ $(n \geq 2, r_m \leq n)$ matrix depending on x^a and η_i^α and the r_m columns of T_m span the image of P_m. Therefore the iteration $\psi_1 \mapsto \cdots \mapsto \psi_m$ yields the multiplicative ansatz

$$\psi_m = \prod_{l=0}^{m-1}\left(I_n + \frac{\mu_{m-l} - \bar{\mu}_{m-l}}{\zeta - \mu_{m-l}} P_{m-l}\right), \quad (8)$$

which, via a partial fraction decomposition, may be written in the additive form

$$\psi_m = I_n + \sum_{k=1}^{m} \frac{\Lambda_{mk}S_k^\dagger}{\zeta - \mu_k}, \quad (9)$$

where Λ_{mk} and S_k are some $n \times r_k$ matrices depending on x^a and η_i^α.

It was shown in [3] that ψ_m is the solution of the linear system (6) if

$$\begin{gathered}S_k = S_k\left(w_k, \eta_k^i\right), \\ T_k = \left\{\prod_{l=1}^{k-1}\left(I_n + \frac{\mu_{k-l} - \bar{\mu}_{k-l}}{\bar{\mu}_k - \mu_{k-l}} P_{k-l}\right)\right\} S_k\end{gathered} \quad (10)$$

with $w_k = x + \bar{\mu}_k u + \bar{\mu}_k^{-1} v$ and $\eta_k^i = \eta_i^1 + \bar{\mu}_k\eta_i^2$ for $k = 1, \ldots, m$. The associated superfield is

$$\Phi_m = \psi_m^{-1}(\zeta = 0) = \prod_{k=1}^{m}(I_n - \rho_k P_k) \quad \text{with } \rho_k = 1 - \frac{\mu_k}{\bar{\mu}_k}. \quad (11)$$

Thus, the solutions of (2) described by the simple pole ansatz are parametrized by the set $\{S_k\}_1^m$ of matrix-valued functions of w_k and η_k^i and by the pole position μ_k.

Example 2. One-soliton configuration

$$\psi_1 = I_n + \frac{\mu - \overline{\mu}}{\zeta - \mu} P_1, \qquad \Phi_1 = \psi_1^{-1}(\zeta = 0) = I_n - \frac{\overline{\mu} - \mu}{\overline{\mu}} P_1, \tag{12}$$

with $\mu \in \mathbb{C} \setminus \mathbb{R}$, $P_1 = T_1 (T_1^\dagger T_1)^{-1} T_1^\dagger$, and $T_1 = T_1(w, \eta^i)$, is an $n \times r$ matrix, where

$$w = x + \overline{\mu} u + \overline{\mu}^{-1} v, \qquad \eta^i = \eta_i^1 + \overline{\mu}\eta_i^2, \quad i = 1, \ldots, \frac{1}{2}\mathcal{N}. \tag{13}$$

This configuration will describe a moving soliton if the matrix T_1 depends on w rationally [3, 8].

We now change the form of the one-soliton solution ψ_1 in (12) into the following one:

$$\psi_1 = I_n + \frac{\mu - \overline{\mu}}{\zeta - \mu} P^\perp, \tag{14}$$

where $P^\perp := P_1$ and thus $P = I_n - P^\perp = I_n - P_1 = P_1^\perp$ built from the $n \times r$ matrix $T := T_{1\perp}$; then $T_1^\dagger T = T_1^\dagger T_{1\perp} = 0$. Since $T_1 = T_1(w, \eta^i)$, we know $T_1^\dagger = T_1^\dagger(\overline{w}, \overline{\eta}^i)$; hence $\partial_w T_1^\dagger = 0$ and $\partial_{\eta^i} T_1^\dagger = 0$. Thus $0 = \partial_w (T_1^\dagger T) = (\partial_w T_1^\dagger) T + T_1^\dagger (\partial_w T) = T_1^\dagger (\partial_w T)$ implies $\partial_w T = 0$. Similarly, $\partial_{\eta^i} T = 0$; hence $T = T(\overline{w}, \overline{\eta}^i) = T(x + \mu u + \mu^{-1} v, \eta_i^1 + \mu \eta_i^2)$. If we suppose

$$w = \mu x + \mu^2 u + v, \qquad \eta^i = \eta_i^1 + \mu \eta_i^2, \quad i = 1, \ldots, \frac{1}{2}\mathcal{N}. \tag{15}$$

Then $T = T(w, \eta^i)$ is an $n \times r$ matrix function of w and η^i and depends on w rationally. With these notations at hand, we can give the following supersymmetric extension of our noncommutative version of algebraic BT, which will be used to construct multisoliton solutions with only simple poles of the linear system (4).

Theorem 3. *Let $\psi(x^a, \zeta, \eta_i^\alpha)$ be a solution of (4) and $\Phi = \psi^{-1}(\zeta = 0)$ the associated superfield; choose $\mu \in \mathbb{C} \setminus \mathbb{R}$ such that ψ is holomorphic and nondegenerate at $\zeta = \mu$. Let*

$$\psi_{\mu,P}(\zeta) := I_n + \frac{\mu - \overline{\mu}}{\zeta - \mu} P^\perp \tag{16}$$

be a solution of (4) with a simple pole at $\zeta = \mu$, where $P = P(x^a, \eta_i^\alpha)$ is the Hermitia projection built from an $n \times r$ matrix $T = T(w, \eta^i)$ depending on w rationally. Let $\widetilde{P} = \widetilde{P}(x^a, \eta_i^\alpha)$ be the Hermitian projection built from the $n \times r$ matrix $\widetilde{T} = \psi(x^a, \mu, \eta_i^\alpha) T$. Then

(1) *$\widetilde{\psi}(x^a, \zeta, \eta_i^\alpha) = \psi_{\mu,\widetilde{P}}(\zeta) \psi(x^a, \zeta, \eta_i^\alpha) \psi_{\mu,P}(\zeta)^{-1}$ is holomorphic and nondegenerate at $\zeta = \mu, \overline{\mu}$;*

(2) *$\psi_1 = \psi_{\mu,\widetilde{P}} \psi = \widetilde{\psi} \psi_{\mu,P}$ is a new solution to the linear system (4) with some new superfields $\mathcal{A}_1$, $\mathcal{B}_1$, and $\mathcal{C}_1^i$, and the new associated superfield is $\Phi_1(x^a, \eta_i^\alpha) = \Phi(\widetilde{P} + (\mu/\overline{\mu}) \widetilde{P}^\perp)$.*

Proof. (1) The residues of the right hand side of $\widetilde{\psi}(\zeta) = \psi_{\mu,\widetilde{P}}(\zeta) \psi(\zeta) \psi_{\mu,P}(\zeta)^{-1}$ at μ and $\overline{\mu}$ are

$$(\mu - \overline{\mu}) \widetilde{P}^\perp \psi(\mu) (I_n - P^\perp), \qquad (\mu - \overline{\mu}) (I_n - \widetilde{P}^\perp) \psi(\overline{\mu}) P^\perp, \tag{17}$$

respectively, and both equal zero by the definition of $\widetilde{P}$. Thus we have two factorizations of $\psi_1 = \psi_{\mu,\widetilde{P}} \psi = \widetilde{\psi} \psi_{\mu,P}$.

(2) It suffices to show that $\mathcal{A}_1 := (\zeta \partial_x \psi_1 - \partial_u \psi_1) \psi_1^{-1}$ is independent of the spectral parameter ζ. Using $\psi_1 = \psi_{\mu,\widetilde{P}} \psi$, we have

$$\mathcal{A}_1 = \left(\zeta \partial_x \psi_{\mu,\widetilde{P}} - \partial_u \psi_{\mu,\widetilde{P}}\right) \psi_{\mu,\widetilde{P}}^{-1} + \psi_{\mu,\widetilde{P}} \left(\zeta \partial_x \psi - \partial_u \psi\right) \psi^{-1} \psi_{\mu,\widetilde{P}}^{-1}. \tag{18}$$

Since $(\zeta \partial_x \psi - \partial_u \psi) \psi^{-1}$ is constant in ζ by assumption, (18) is holomorphic at $\zeta \in \mathbb{C} \cup \{\infty\} \setminus \{\mu, \overline{\mu}\}$. On the other hand, using $\psi_1 = \widetilde{\psi} \psi_{\mu,P}$, we have

$$\mathcal{A}_1 = \left(\zeta \partial_x \widetilde{\psi} - \partial_u \widetilde{\psi}\right) \widetilde{\psi}^{-1} + \widetilde{\psi} \left(\zeta \partial_x \psi_{\mu,P} - \partial_u \psi_{\mu,P}\right) \psi_{\mu,P}^{-1} \widetilde{\psi}^{-1}. \tag{19}$$

Since $\psi_{\mu,P}$ is a solution of (4), $(\zeta \partial_x \psi_{\mu,P} - \partial_u \psi_{\mu,P}) \psi_{\mu,P}^{-1}$ is independent of ζ. Together with (1), we deduce that (19) is holomorphic at $\zeta = \mu, \overline{\mu}$. Thus $\mathcal{A}_1$ is holomorphic on $\mathbb{C} \cup \{\infty\}$ and hence independent of ζ by Liouville's theorem. Likewise $\mathcal{B}_1 := (\zeta \partial_v \psi_1 - \partial_x \psi_1) \psi_1^{-1}$ and $\mathcal{C}_1^i := (\zeta \partial_1^i \psi_1 - \partial_2^i \psi_1) \psi_1^{-1}$ are also independent of ζ.

The associated superfield is $\Phi_1 = \psi_1^{-1}(\zeta = 0) = \Phi(\widetilde{P} + (\mu/\overline{\mu}) \widetilde{P}^\perp)$, and then the proof is completed. □

Let $\psi_1 = \psi_{\mu,P} \sharp \psi$, $\Phi_1 = \psi_{\mu,P} \sharp \Phi$ denote the $\mathcal{N}$-extended noncommutative algebraic BT generated by $\psi_{\mu,P}$. If we apply these BTs repeatedly (with distinct poles) to a one-soliton solution, then we obtain a multisoliton solution of (4) with simple poles and the associated superfield. Such configurations coincide with the ones constructed in [3], and we call them multisoliton configurations with simple pole data. In contrast, we call solutions of (4) with higher-order poles and the associated fields multisoliton configurations with higher-order pole data, which will be constructed in the next section.

Example 4. Two-soliton configurations with two simple poles.

Let μ_1, μ_2 be two distinct complex numbers and $\mu_1 \neq \overline{\mu}_2$, and let $T_k(w_k, \eta_k^i)$ be an $n \times 1$ matrix function depending on w_k rationally, where

$$w_k = \mu_k x + \mu_k^2 u + v, \qquad \eta_k^i = \eta_i^1 + \mu_k \eta_i^2, \quad i = 1, \ldots, \frac{1}{2}\mathcal{N}. \tag{20}$$

Let $P_k = T_k (T_k^\dagger T_k)^{-1} T_k^\dagger$, $k = 1, 2$; then ψ_{μ_1,P_1} and ψ_{μ_2,P_2} are one-soliton solutions and $\psi_2 = \psi_{\mu_2,P_2} \sharp \psi_{\mu_1,P_1} = \psi_{\mu_2,\widetilde{P}_2} \psi_{\mu_1,P_1}$

is a two-soliton solution of (4) with two simple poles at $\zeta = \mu_1$ and $\zeta = \mu_2$, where $\widetilde{P}_2$ is the Hermitian projection built from the $n \times 1$ matrix $\widetilde{T}_2 = \psi_{\mu_1,P_1}(\mu_2)T_2 = (I_n + ((\mu_1 - \overline{\mu}_1)/(\mu_2 - \mu_1))P_1^{\perp})T_2$. The associated superfield is $\Phi_2 = (P_1 + (\mu_1/\overline{\mu}_1)P_1^{\perp})(\widetilde{P}_2 + (\mu_2/\overline{\mu}_2)\widetilde{P}_2^{\perp})$.

4. $\mathcal{N}$-Extended Multisoliton Configurations with Higher-Order Pole Data

A solution of (6) with a double pole at $\zeta = -i$ is constructed in [8] by making a second-order pole at $\zeta = -i$ in the dressing ansatz, that is, considering the following dressing transformation:

$$\psi = I_n - \frac{2i}{\zeta + i}P \longmapsto \widetilde{\psi} = \left(I_n - \frac{2i}{\zeta + i}\widetilde{P}\right)\left(I_n - \frac{2i}{\zeta + i}P\right), \tag{21}$$

where the Hermitian projection P is known and $\widetilde{P} = \widetilde{T}(\widetilde{T}^{\dagger}\widetilde{T})^{-1}\widetilde{T}^{\dagger}$ is yet to be determined. Demanding that $\widetilde{\psi}$ is again a solution of (6) with some new superfields $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{B}}$, and $\widetilde{\mathcal{C}}^i$, which are independent of ζ, they obtained the following equations:

$$\begin{gathered}\left(I_n - \widetilde{P}\right)\left(\partial_{\overline{z}}\widetilde{T} + \left(\partial_{\overline{z}}P\right)\widetilde{T}\right) = 0,\\ \left(I_n - \widetilde{P}\right)\left(\partial_t\widetilde{T} - 2i\left(\partial_z P\right)\widetilde{T}\right) = 0,\\ \left(I_n - \widetilde{P}\right)\left(\frac{1}{2}\left(\partial_1^i + i\partial_2^i\right)\widetilde{T} + \left(\partial_1^i P\right)\widetilde{T}\right) = 0.\end{gathered} \tag{22}$$

After constructing a projection $\widetilde{P}$ via a solution $\widetilde{T}$ of (22), $\widetilde{\psi}$ and the associated superfield $\widetilde{\Phi} = \widetilde{\psi}^{-1}(\zeta = 0) = (I_n - 2P)(I_n - 2\widetilde{P})$ are derived.

Their construction of the solution $\widetilde{T}$ is inspired by the known form of $\widetilde{T}$ in the bosonic case [4, 5], that is, making the ansatz $\widetilde{T} = T + T_{\perp}(T_{\perp}^{\dagger}T_{\perp})^{-1}g$ with $T_{\perp}$ orthogonal to T.

Below we will present a supersymmetric extension of our noncommutative limiting method as in the bosonic case to construct multisoliton configurations with a higher-order pole at $\zeta = \mu$. We give an example of two-soliton configuration with a second-order pole by taking a limit at first.

Example 5. Two-soliton configuration with a second-order pole.

Let $\mu \in \mathbb{C} \setminus \mathbb{R}$, f, g be two holomorphic functions of (w, η^i) and depend on the bosonic variable w rationally:

$$T_1 = \begin{pmatrix} 1 \\ f\left(w, \eta^i\right) \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 1 \\ f\left(w_{\epsilon}, \eta_{\epsilon}^i\right) + \epsilon g\left(w_{\epsilon}, \eta_{\epsilon}^i\right) \end{pmatrix}, \tag{23}$$

where $w = \mu x + \mu^2 u + v$ and $\eta^i = \eta_i^1 + \mu\eta_i^2$, $w_{\epsilon} = (\mu + \epsilon)x + (\mu + \epsilon)^2 u + v$ and $\eta_{\epsilon}^i = \eta_i^1 + (\mu + \epsilon)\eta_i^2$, $i = 1, \ldots, (1/2)\mathcal{N}$, for any $\epsilon \in \mathbb{C}$ with $|\epsilon|$ small, $P_1 = T_1(T_1^{\dagger}T_1)^{-1}T_1^{\dagger}$, and $P_2 = T_2(T_2^{\dagger}T_2)^{-1}T_2^{\dagger}$. The Taylor expansion of $f(w_{\epsilon}, \eta_{\epsilon}^i)$ in ϵ is

$$\begin{aligned} f\left(w_{\epsilon}, \eta_{\epsilon}^i\right) &= f\left(w, \eta^i\right) + \frac{\partial f}{\partial w}\left(w_{\epsilon} - w\right) \\ &\quad + \sum_{i=1}^{(1/2)\mathcal{N}} \frac{\partial f}{\partial \eta^i}\left(\eta_{\epsilon}^i - \eta^i\right) + O\left(\epsilon^2\right) \\ &= f\left(w, \eta^i\right) + \left[\frac{\partial f}{\partial w}(x + 2\mu u) + \sum_{i=1}^{(1/2)\mathcal{N}} \frac{\partial f}{\partial \eta^i}\eta_i^2\right]\epsilon \\ &\quad + O\left(\epsilon^2\right), \end{aligned} \tag{24}$$

where $\partial f/\partial \eta^i$ is right derivative of f with respect to the Grassmann variable η^i. Note that the products in the expansion are the ordinary products and that we omit the variables (w, η^i) of $\partial f/\partial w$ and $\partial f/\partial \eta^i$ for simplicity. Clearly, $g(w_{\epsilon}, \eta_{\epsilon}^i)$ has similar expansion; therefore

$$\begin{aligned} &f\left(w_{\epsilon}, \eta_{\epsilon}^i\right) + \epsilon g\left(w_{\epsilon}, \eta_{\epsilon}^i\right) \\ &\quad = f\left(w, \eta^i\right) + \left[\frac{\partial f}{\partial w}(x + 2\mu u)\right. \\ &\qquad \left. + \sum_{i=1}^{(1/2)\mathcal{N}} \frac{\partial f}{\partial \eta^i}\eta_i^2 + g\left(w, \eta^i\right)\right]\epsilon + O\left(\epsilon^2\right). \end{aligned} \tag{25}$$

Then

$$\begin{aligned} \widetilde{T}_2 &= \psi_{\mu_1,P_1}(\mu + \epsilon)T_2 \\ &= \left(I_n + \frac{\mu_1 - \overline{\mu}_1}{\epsilon}P_1^{\perp}\right) \\ &\quad \times \begin{pmatrix} 1 \\ f + \left[\dfrac{\partial f}{\partial w}(x + 2\mu u) + \displaystyle\sum_{i=1}^{(1/2)\mathcal{N}} \frac{\partial f}{\partial \eta^i}\eta_i^2 + g\right]\epsilon + O\left(\epsilon^2\right) \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ f \end{pmatrix} + (\mu - \overline{\mu})P_1^{\perp} \\ &\quad \times \begin{pmatrix} 0 \\ \dfrac{\partial f}{\partial w}(x + 2\mu u) + \displaystyle\sum_{i=1}^{(1/2)\mathcal{N}} \frac{\partial f}{\partial \eta^i}\eta_i^2 + g \end{pmatrix} + O(\epsilon); \\ \widehat{T}_2 &= \lim_{\epsilon \to 0}\widetilde{T}_2 = \begin{pmatrix} 1 \\ f \end{pmatrix} + (\mu - \overline{\mu})P_1^{\perp} \\ &\quad \times \begin{pmatrix} 0 \\ \dfrac{\partial f}{\partial w}(x + 2\mu u) + \displaystyle\sum_{i=1}^{(1/2)\mathcal{N}} \frac{\partial f}{\partial \eta^i}\eta_i^2 + g \end{pmatrix}. \end{aligned} \tag{26}$$

Let $\widetilde{P}_2 = \widetilde{T}_2(\widetilde{T}_2^{\dagger}\widetilde{T}_2)^{-1}\widetilde{T}_2^{\dagger}$ and $\widehat{P}_2 = \widehat{T}_2(\widehat{T}_2^{\dagger}\widehat{T}_2)^{-1}\widehat{T}_2^{\dagger}$, then

$$\begin{aligned}\widehat{\psi}_2 = \psi_{\mu,\widehat{P}_2}\psi_{\mu,P_1} &= \lim_{\epsilon\to 0}\psi_{\mu+\epsilon,\widetilde{P}_2}\psi_{\mu,P_1} \\ &= \lim_{\epsilon\to 0}\psi_{\mu+\epsilon,P_2}\sharp\psi_{\mu,P_1}\end{aligned} \tag{27}$$

is a two-soliton solution of (4) with a double pole at $\zeta = \mu$, and the associated superfield is

$$\widehat{\Phi}_2 = \widehat{\psi}_2^{-1}(\zeta = 0) = \left(P_1 + \frac{\mu}{\bar{\mu}}P_1^{\perp}\right)\left(\widehat{P}_2 + \frac{\mu}{\bar{\mu}}\widehat{P}_2^{\perp}\right). \tag{28}$$

Next, we use a systematic limiting method and our extended algebraic BTs in Section 3 to construct multisoliton solutions of (4) with pole data (μ, k) for any $\mu \in \mathbb{C}\setminus\mathbb{R}$ and $k \geq 2$.

Let $\mu \in \mathbb{C}\setminus\mathbb{R}$ be a constant and $\{a_j(w,\eta^i)\}_{j=0}^{\infty}$ a sequence of $n\times 1$ matrix functions depending on w rationally. For any $\epsilon \in \mathbb{C}$ with $|\epsilon|$ small, let $w = \mu x + \mu^2 u + v$, $\eta^i = \eta_i^1 + \mu\eta_i^2$, $w_\epsilon = (\mu+\epsilon)x + (\mu+\epsilon)^2 u + v$, and $\eta_\epsilon^i = \eta_i^1 + (\mu+\epsilon)\eta_i^2$, for $i = 1,\ldots,(1/2)\mathcal{N}$. Here we only consider the case of $\mathcal{N} = 8$ since it is easily truncated to any smaller even number $\mathcal{N}$ of supersymmetries (see [8]); thus

$$\begin{aligned} w_\epsilon - w &= (x + 2\mu u)\epsilon + u\epsilon^2, \\ \eta_\epsilon^i - \eta^i &= \eta_i^2\epsilon, \quad i = 1,\ldots,4. \end{aligned} \tag{29}$$

Let $F_{k,\epsilon} = \sum_{j=0}^{k} a_j(w_\epsilon,\eta_\epsilon^i)\epsilon^j$ for $k \geq 1$. The Taylor expansion of $a_j(w_\epsilon,\eta_\epsilon^i)$ at (w,η^i) is

$$\begin{aligned} a_j(w_\epsilon,\eta_\epsilon^i) &= a_j(w,\eta^i) + \frac{\partial a_j}{\partial w}(w_\epsilon - w) + \sum_{i=1}^{4}\frac{\partial a_j}{\partial\eta^i}(\eta_\epsilon^i - \eta^i) \\ &+ \frac{1}{2!}\left[\frac{\partial^2 a_j}{\partial w^2}(w_\epsilon - w)^2 + 2\sum_{i=1}^{4}\frac{\partial^2 a_j}{\partial w\partial\eta^i}(w_\epsilon - w)(\eta_\epsilon^i - \eta^i) + \sum_{i,k=1}^{4}\frac{\partial^2 a_j}{\partial\eta^i\partial\eta^k}(\eta_\epsilon^k - \eta^k)(\eta_\epsilon^i - \eta^i)\right] \\ &+ \frac{1}{3!}\left[\frac{\partial^3 a_j}{\partial w^3}(w_\epsilon - w)^3 + 3\sum_{i=1}^{4}\frac{\partial^3 a_j}{\partial w^2\partial\eta^i}(w_\epsilon - w)^2(\eta_\epsilon^i - \eta^i) + 3\sum_{i,k=1}^{4}\frac{\partial^3 a_j}{\partial w\partial\eta^i\partial\eta^k}\cdot(w_\epsilon - w)(\eta_\epsilon^k - \eta^k)(\eta_\epsilon^i - \eta^i) \right. \\ &\left. + \sum_{i,k,l=1}^{4}\frac{\partial^3 a_j}{\partial\eta^i\partial\eta^k\partial\eta^l}(\eta_\epsilon^l - \eta^l)\times(\eta_\epsilon^k - \eta^k)(\eta_\epsilon^i - \eta^i)\right] \\ &+ \frac{1}{4!}\left[\frac{\partial^4 a_j}{\partial w^4}(w_\epsilon - w)^4 + 4\sum_{i=1}^{4}\frac{\partial^4 a_j}{\partial w^3\partial\eta^i}(w_\epsilon - w)^3(\eta_\epsilon^i - \eta^i) + 6\sum_{i,k=1}^{4}\frac{\partial^4 a_j}{\partial w^2\partial\eta^i\partial\eta^k}(w_\epsilon - w)^2\times(\eta_\epsilon^k - \eta^k)(\eta_\epsilon^i - \eta^i) \right. \\ &+ 4\sum_{i,k,l=1}^{4}\frac{\partial^4 a_j}{\partial w\partial\eta^i\partial\eta^k\partial\eta^l}(w_\epsilon - w)\times(\eta_\epsilon^l - \eta^l)(\eta_\epsilon^k - \eta^k)(\eta_\epsilon^i - \eta^i) \\ &\left. + \sum_{i,k,l,m=1}^{4}\frac{\partial^4 a_j}{\partial\eta^i\partial\eta^k\partial\eta^l\partial\eta^m}\times(\eta_\epsilon^m - \eta^m)(\eta_\epsilon^l - \eta^l)\times(\eta_\epsilon^k - \eta^k)(\eta_\epsilon^i - \eta^i)\right] + \cdots, \end{aligned} \tag{30}$$

where the sums are taken over different indices when there are more than one index and the partial derivatives of $a_j(w,\eta^i)$ with respect to the Grassmann variables are given by the following remark.

Remark 6. We can write

$$\begin{aligned} a_j(w,\eta^i) &= a_{j,0}(w) + \sum_{i=1}^{4}a_{j,i}(w)\eta^i + \sum_{k,i=1}^{4}a_{j,ki}(w)\eta^k\eta^i \\ &+ \sum_{l,k,i=1}^{4}a_{j,lki}(w)\eta^l\eta^k\eta^i + \sum_{m,l,k,i=1}^{4}a_{j,mlki}(w)\eta^m\eta^l\eta^k\eta^i; \end{aligned} \tag{31}$$

hence the right derivatives of $a_j(w,\eta^i)$ with respect to one or several of the Grassmann variables are computed as follows:

$$\begin{aligned} \frac{\partial a_j}{\partial\eta^i} &= a_{j,i} + \sum_{\substack{k=1\\k\neq i}}^{4}a_{j,ki}\eta^k + \sum_{\substack{l,k=1\\l,k\neq i}}^{4}a_{j,lki}\eta^l\eta^k \\ &+ \sum_{\substack{m,l,k=1\\m,l,k\neq i}}^{4}a_{j,mlki}\eta^m\eta^l\eta^k, \end{aligned}$$

$$\frac{\partial^2 a_j}{\partial\eta^i\partial\eta^k} = a_{j,ki} - a_{j,ik} + \sum_{\substack{l=1\\ l\neq i,k}}^{4} \left(a_{j,lki} - a_{j,lik}\right)\eta^l + \sum_{\substack{m,l=1\\ m,l\neq i,k}}^{4} \left(a_{j,mlki} - a_{j,mlik}\right)\eta^m\eta^l,$$

$$\frac{\partial^3 a_j}{\partial\eta^i\partial\eta^k\partial\eta^l} = \sum_{\sigma} \operatorname{sign}\sigma a_{j,\sigma(l)\sigma(k)\sigma(i)} + \sum_{\substack{m=1\\ m\neq i,k,l}}^{4} \sum_{\sigma} \operatorname{sign}\sigma a_{j,m\sigma(l)\sigma(k)\sigma(i)}\eta^m,$$

$$\frac{\partial^4 a_j}{\partial\eta^i\partial\eta^k\partial\eta^l\partial\eta^m} = \sum_{\tau} \operatorname{sign}\tau a_{j,\tau(m)\tau(l)\tau(k)\tau(i)}, \tag{32}$$

where $\sum_\sigma$, $\sum_\tau$ represent that the sums are taken over the permutation group of $\{l,k,i\}$ and $\{m,l,k,i\}$, respectively.

Substituting (29) into the Taylor expansion of $a_j(w_\epsilon,\eta^i_\epsilon)$, we get the Taylor expansion of $a_j(w_\epsilon,\eta^i_\epsilon)$ in ϵ. Assume that $a_j(w_\epsilon,\eta^i_\epsilon) = \sum_{p=0}^{\infty} b_{j,p}\epsilon^p$, then

$$\begin{aligned}
b_{j,0} &= a_j\left(w,\eta^i\right),\\
b_{j,1} &= \frac{\partial a_j}{\partial w}(x+2\mu u) + \sum_{i=1}^{4}\frac{\partial a_j}{\partial\eta^i}\eta_i^2,\\
b_{j,2} &= \frac{\partial a_j}{\partial w}u + \frac{1}{2}\frac{\partial^2 a_j}{\partial w^2}(x+2\mu u)^2 + \sum_{i=1}^{4}\frac{\partial^2 a_j}{\partial w\partial\eta^i}(x+2\mu u)\eta_i^2 + \frac{1}{2}\sum_{i,k=1}^{4}\frac{\partial^2 a_j}{\partial\eta^i\partial\eta^k}\eta_k^2\eta_i^2,\\
b_{j,3} &= \frac{\partial^2 a_j}{\partial w^2}(x+2\mu u)u + \sum_{i=1}^{4}\frac{\partial^2 a_j}{\partial w\partial\eta^i}u\eta_i^2 + \frac{1}{6}\frac{\partial^3 a_j}{\partial w^3}(x+2\mu u)^3 + \frac{1}{2}\sum_{i=1}^{4}\frac{\partial^3 a_j}{\partial w^2\partial\eta^i}(x+2\mu u)^2\eta_i^2\\
&\quad + \frac{1}{2}\sum_{i,k=1}^{4}\frac{\partial^3 a_j}{\partial w\partial\eta^i\partial\eta^k}(x+2\mu u)\eta_k^2\eta_i^2 + \frac{1}{6}\sum_{i,k,l=1}^{4}\frac{\partial^3 a_j}{\partial\eta^i\partial\eta^k\partial\eta^l}\eta_l^2\eta_k^2\eta_i^2,\\
b_{j,4} &= \frac{1}{2}\frac{\partial^2 a_j}{\partial w^2}u^2 + \frac{1}{2}\frac{\partial^3 a_j}{\partial w^3}(x+2\mu u)^2 u + \sum_{i=1}^{4}\frac{\partial^3 a_j}{\partial w^2\partial\eta^i}(x+2\mu u)^2 u\eta_i^2 + \frac{1}{2}\sum_{i,k=1}^{4}\frac{\partial^3 a_j}{\partial w\partial\eta^i\partial\eta^k}u\eta_k^2\eta_i^2\\
&\quad + \frac{1}{24}\frac{\partial^4 a_j}{\partial w^4}(x+2\mu u)^4 + \frac{1}{6}\sum_{i=1}^{4}\frac{\partial^4 a_j}{\partial w^3\partial\eta^i}(x+2\mu u)^3\eta_i^2 + \frac{1}{4}\sum_{i,k=1}^{4}\frac{\partial^4 a_j}{\partial w^2\partial\eta^i\partial\eta^k}(x+2\mu u)^2\eta_k^2\eta_i^2\\
&\quad + \frac{1}{6}\sum_{i,k,l=1}^{4}\frac{\partial^4 a_j}{\partial w\partial\eta^i\partial\eta^k\partial\eta^l} \times (x+2\mu u)\eta_l^2\eta_k^2\eta_i^2 + \frac{1}{24}\sum_{i,k,l,m=1}^{4}\frac{\partial^4 a_j}{\partial\eta^i\partial\eta^k\partial\eta^l\partial\eta^m}\cdot\eta_m^2\eta_l^2\eta_k^2\eta_i^2;\\
&\;\;\vdots
\end{aligned} \tag{33}$$

Expanding $F_{k,\epsilon}$ in ϵ, we obtain

$$\begin{aligned}
F_{k,\epsilon} &= \sum_{j=0}^{k} a_j\left(w_\epsilon,\eta^i_\epsilon\right)\epsilon^j\\
&= \sum_{j=0}^{k}\sum_{p=0}^{\infty} b_{j,p}\epsilon^{j+p}\\
&= c_0 + c_1\epsilon + \cdots + c_k\epsilon^k + O\left(\epsilon^{k+1}\right),
\end{aligned} \tag{34}$$

where $c_p = \sum_{j=0}^{p} b_{j,p-j}$ are given as follows:

$$c_0 = a_0\left(w,\eta^i\right), \tag{35}$$

$$c_1 = \frac{\partial a_0}{\partial w}(x+2\mu u) + \sum_{i=1}^{4}\frac{\partial a_0}{\partial\eta^i}\eta_i^2 + a_1\left(w,\eta^i\right), \tag{36}$$

$$c_2 = \frac{\partial a_0}{\partial w}u + \frac{1}{2}\frac{\partial^2 a_0}{\partial w^2}(x+2\mu u)^2 + \sum_{i=1}^{4}\frac{\partial^2 a_0}{\partial w\partial\eta^i}(x+2\mu u)\eta_i^2$$

$$+\frac{1}{2}\sum_{i,k=1}^{4}\frac{\partial^2 a_0}{\partial\eta^i\partial\eta^k}\eta_k^2\eta_i^2+\frac{\partial a_1}{\partial w}(x+2\mu u)$$
$$+\sum_{i=1}^{4}\frac{\partial a_1}{\partial\eta^i}\eta_i^2+a_2(w,\eta^i),\tag{37}$$

$$c_3=\frac{\partial^2 a_0}{\partial w^2}(x+2\mu u)u$$
$$+\sum_{i=1}^{4}\frac{\partial^2 a_0}{\partial w\partial\eta^i}u\eta_i^2+\frac{1}{6}\frac{\partial^3 a_0}{\partial w^3}(x+2\mu u)^3$$
$$+\frac{1}{2}\sum_{i=1}^{4}\frac{\partial^3 a_0}{\partial w^2\partial\eta^i}(x+2\mu u)^2\eta_i^2$$
$$+\frac{1}{2}\sum_{i,k=1}^{4}\frac{\partial^3 a_0}{\partial w\partial\eta^i\partial\eta^k}(x+2\mu u)\eta_k^2\eta_i^2$$
$$+\frac{1}{6}\sum_{i,k,l=1}^{4}\frac{\partial^3 a_0}{\partial\eta^i\partial\eta^k\partial\eta^l}\eta_l^2\eta_k^2\eta_i^2$$
$$+\frac{\partial a_1}{\partial w}u+\frac{1}{2}\frac{\partial^2 a_1}{\partial w^2}(x+2\mu u)^2$$
$$+\sum_{i=1}^{4}\frac{\partial^2 a_1}{\partial w\partial\eta^i}(x+2\mu u)\eta_i^2$$
$$+\frac{1}{2}\sum_{i,k=1}^{4}\frac{\partial^2 a_1}{\partial\eta^i\partial\eta^k}\eta_k^2\eta_i^2+\frac{\partial a_2}{\partial w}(x+2\mu u)$$
$$+\sum_{i=1}^{4}\frac{\partial a_2}{\partial\eta^i}\eta_i^2+a_3(w,\eta^i),$$
$$\vdots\tag{38}$$

From the expressions, we know that c_j's are $n\times 1$ matrix functions of (w,η^i) and depend on w rationally.

For $k\geq 1$, let $T_{k,\epsilon}=F_{k-1,\epsilon}$, $P_{k,\epsilon}=T_{k,\epsilon}(T_{k,\epsilon}^{\dagger}T_{k,\epsilon})^{-1}T_{k,\epsilon}^{\dagger}$. Define $\psi_{k,\epsilon}$ and $\widehat{\psi}_k$ by induction as follows:

$$\psi_{1,\epsilon}=\psi_{\mu+\epsilon,P_{1,\epsilon}},\qquad \widehat{\psi}_1=\lim_{\epsilon\to 0}\psi_{1,\epsilon},$$
$$\psi_{k,\epsilon}=\psi_{\mu+\epsilon,P_{k,\epsilon}}\sharp\widehat{\psi}_{k-1},\qquad \widehat{\psi}_k=\lim_{\epsilon\to 0}\psi_{k,\epsilon}.\tag{39}$$

Let $\widetilde{T}_{k,\epsilon}=\widehat{\psi}_{k-1}(\mu+\epsilon)T_{k,\epsilon}$, $\widetilde{P}_{k,\epsilon}=\widetilde{T}_{k,\epsilon}(\widetilde{T}_{k,\epsilon}^{\dagger}\widetilde{T}_{k,\epsilon})^{-1}\widetilde{T}_{k,\epsilon}^{\dagger}$; then $\psi_{k,\epsilon}=\psi_{\mu+\epsilon,\widetilde{P}_{k,\epsilon}}\widehat{\psi}_{k-1}$, where implicit "$\star$" products are still assumed between classical fields and their components.

Theorem 7. *Let $a_0,a_1,\ldots$ be a sequence of $n\times 1$ matrix functions of w and η^i and depend on w rationally. Let $T_{k,\epsilon}$, $P_{k,\epsilon}$, $\psi_{k,\epsilon}$, $\widehat{\psi}_k$, and $\widetilde{T}_{k,\epsilon}$ be defined as above. Then one has*

(1) *$\widetilde{T}_{k,\epsilon}=\widehat{T}_k+\epsilon S_{k,1}+\epsilon^2 S_{k,2}+\cdots$, where $\widehat{T}_k=c_0+\sum_{j=1}^{k-1}(\mu-\overline{\mu})^j P_{k-1,j}c_j, P_{l,j}=\sum_{l\geq i_1>\cdots>i_j\geq 1}\widehat{P}_{i_1}^{\perp}\cdots\widehat{P}_{i_j}^{\perp}$, and $\widehat{P}_i=\widehat{T}_i(\widehat{T}_i^{\dagger}\widehat{T}_i)^{-1}\widehat{T}_i^{\dagger}$;*

(2) *$\widehat{\psi}_k=\psi_{\mu,\widehat{P}_k}\cdots\psi_{\mu,\widehat{P}_1}$ is a multisoliton solution of (4) with only a pole at $\zeta=\mu$ of order k;*

(3) *the associated superfield is $\widehat{\Phi}_k=(\widehat{P}_1+(\mu/\overline{\mu})\widehat{P}_1^{\perp})\cdots(\widehat{P}_k+(\mu/\overline{\mu})\widehat{P}_k^{\perp})$.*

Proof. We prove the theorem by induction on k. For $k=1$, the theorem is clearly true. Suppose the theorem is true for k, we will prove that (1)–(3) hold for $k+1$.

(1) By Theorem 3 and induction hypothesis,

$$\widetilde{T}_{k+1,\epsilon}=\widehat{\psi}_k(\mu+\epsilon)T_{k+1,\epsilon}$$
$$=\left(I_n+\frac{\mu-\overline{\mu}}{\epsilon}\widehat{P}_k^{\perp}\right)\widehat{\psi}_{k-1}(\mu+\epsilon)F_{k,\epsilon}$$
$$=\left(I_n+\frac{\mu-\overline{\mu}}{\epsilon}\widehat{P}_k^{\perp}\right)\widehat{\psi}_{k-1}(\mu+\epsilon)$$
$$\times\left(F_{k-1,\epsilon}+a_k(w_\epsilon,\eta_\epsilon^i)\epsilon^k\right)$$
$$=\left(I_n+\frac{\mu-\overline{\mu}}{\epsilon}\widehat{P}_k^{\perp}\right)$$
$$\times\left(\widehat{\psi}_{k-1}(\mu+\epsilon)T_{k,\epsilon}+\widehat{\psi}_{k-1}(\mu+\epsilon)a_k(w_\epsilon,\eta_\epsilon^i)\epsilon^k\right)$$
$$=\left(I_n+\frac{\mu-\overline{\mu}}{\epsilon}\widehat{P}_k^{\perp}\right)\left(\widetilde{T}_{k,\epsilon}+\left(I_n+\frac{\mu-\overline{\mu}}{\epsilon}\widehat{P}_{k-1}^{\perp}\right)\cdots\left(I_n+\frac{\mu-\overline{\mu}}{\epsilon}\widehat{P}_1^{\perp}\right)\times a_k(w_\epsilon,\eta_\epsilon^i)\epsilon^k\right)$$
$$=\left(I_n+\frac{\mu-\overline{\mu}}{\epsilon}\widehat{P}_k^{\perp}\right)\left(\widehat{T}_k+\epsilon S_{k,1}+\epsilon(\mu-\overline{\mu})^{k-1}\times P_{k-1,k-1}a_k(w,\eta^i)+O(\epsilon^2)\right)$$
$$=\widehat{T}_k+(\mu-\overline{\mu})\widehat{P}_k^{\perp}\left(S_{k,1}+(\mu-\overline{\mu})^{k-1}\times P_{k-1,k-1}a_k(w,\eta^i)\right)+O(\epsilon).\tag{40}$$

In the last step we have used $\widehat{P}_k^{\perp}\widehat{T}_k=0$. Thus all terms of negative powers of ϵ vanish in the Laurent series expansion of $\widetilde{T}_{k+1,\epsilon}$ in ϵ; hence

$$\widetilde{T}_{k+1,\epsilon}=\widehat{\psi}_k(\mu+\epsilon)F_{k,\epsilon}$$
$$=\left(I_n+\frac{\mu-\overline{\mu}}{\epsilon}\widehat{P}_k^{\perp}\right)\cdots\left(I_n+\frac{\mu-\overline{\mu}}{\epsilon}\widehat{P}_1^{\perp}\right)$$
$$\times\left(c_0+c_1\epsilon+\cdots+c_k\epsilon^k+O(\epsilon^{k+1})\right)$$
$$=\left(I_n+\frac{\mu-\overline{\mu}}{\epsilon}P_{k,1}+\cdots+\frac{(\mu-\overline{\mu})^k}{\epsilon^k}P_{k,k}\right)$$
$$\times\left(c_0+c_1\epsilon+\cdots+c_k\epsilon^k+O(\epsilon^{k+1})\right)$$

$$= c_0 + (\mu - \overline{\mu}) P_{k,1} c_1 + \cdots + (\mu - \overline{\mu})^k P_{k,k} c_k + O(\epsilon). \tag{41}$$

Therefore we obtain

$$\widehat{T}_{k+1} = \lim_{\epsilon \to 0} \widetilde{T}_{k+1,\epsilon} = c_0 + \sum_{j=1}^{k} (\mu - \overline{\mu})^j P_{k,j} c_j. \tag{42}$$

(2) By the inductive definition before this theorem,

$$\begin{aligned}
\widehat{\psi}_{k+1} &= \lim_{\epsilon \to 0} \psi_{\mu+\epsilon, P_{k+1,\epsilon}} \sharp \widehat{\psi}_k = \lim_{\epsilon \to 0} \psi_{\mu+\epsilon, \widetilde{P}_{k+1,\epsilon}} \widehat{\psi}_k \\
&= \psi_{\mu, \widehat{P}_{k+1}} \widehat{\psi}_k = \psi_{\mu, \widehat{P}_{k+1}} \psi_{\mu, \widehat{P}_k} \cdots \psi_{\mu, \widehat{P}_1} \\
&= \left(I_n + \frac{\mu - \overline{\mu}}{\zeta - \mu} \widehat{P}_{k+1}^{\perp} \right) \cdots \left(I_n + \frac{\mu - \overline{\mu}}{\zeta - \mu} \widehat{P}_1^{\perp} \right) \\
&= I_n + \sum_{j=1}^{k+1} \frac{(\mu - \overline{\mu})^j}{(\zeta - \mu)^j} P_{k+1,j}.
\end{aligned} \tag{43}$$

By Theorem 3, $\psi_{\mu+\epsilon, P_{k+1,\epsilon}} \sharp \widehat{\psi}_k$ is a solution of (4) for small $|\epsilon| > 0$. By continuity, so is $\widehat{\psi}_{k+1}$. The coefficient of $(\zeta - \mu)^{-k-1}$ of $\widehat{\psi}_{k+1}$ is $(\mu - \overline{\mu})^{k+1} P_{k+1,k+1}$. To show that $\widehat{\psi}_{k+1}$ has a pole at $\zeta = \mu$ of multiplicity $k+1$, it suffices to show that

$$P_{k+1,k+1} = \widehat{P}_{k+1}^{\perp} \widehat{P}_k^{\perp} \cdots \widehat{P}_1^{\perp} \neq 0. \tag{44}$$

As in the nonsupersymmetric scenario, we choose the operator formalism to prove (44). So now the Hermitian projections $\widehat{P}_i$, $i = 1, \ldots, k+1$,

$$\begin{aligned}
P_{k+1,k+1} &= \widehat{P}_{k+1}^{\perp} \widehat{P}_k^{\perp} \cdots \widehat{P}_1^{\perp}, \\
P_{l,j} &= \sum_{l \geq i_1 > \cdots > i_j \geq 1} \widehat{P}_{i_1}^{\perp} \cdots \widehat{P}_{i_j}^{\perp}
\end{aligned} \tag{45}$$

(under the Moyal-Weyl map) are $U(n)$-valued matrices, and the products in (45) are the ordinary products instead of star products. We write $P_{k,j} = P_{k-1,j} + \widehat{P}_k^{\perp} P_{k-1,j-1}$. So (42) for $k+1$ can be written as

$$\begin{aligned}
\widehat{T}_{k+1} &= c_0 + \sum_{j=1}^{k-1} (\mu - \overline{\mu})^j P_{k-1,j} c_j \\
&\quad + \widehat{P}_k^{\perp} \sum_{j=0}^{k-1} (\mu - \overline{\mu})^{j+1} P_{k-1,j} c_{j+1} \\
&= \widehat{T}_k + (\mu - \overline{\mu}) \widehat{P}_k^{\perp} \sum_{j=0}^{k-1} (\mu - \overline{\mu})^j P_{k-1,j} c_{j+1}.
\end{aligned} \tag{46}$$

By the induction hypothesis, $\widehat{T}_k \neq 0$, then the above formula implies that

$$\operatorname{Im} \widehat{P}_{k+1} \cap \operatorname{Im} \widehat{P}_k^{\perp} = 0. \tag{47}$$

If (44) does not hold, that is, $\widehat{P}_{k+1}^{\perp} \widehat{P}_k^{\perp} \cdots \widehat{P}_1^{\perp} = 0$, then $\widehat{P}_k^{\perp} \cdots \widehat{P}_1^{\perp} \subset \operatorname{Im} \widehat{P}_{k+1}$, but $\widehat{P}_k^{\perp} \cdots \widehat{P}_1^{\perp} \neq 0$ by induction hypothesis, and clearly $\widehat{P}_k^{\perp} \cdots \widehat{P}_1^{\perp} \subset \operatorname{Im} \widehat{P}_k^{\perp}$, so we have

$$0 \neq \widehat{P}_k^{\perp} \cdots \widehat{P}_1^{\perp} \subset \operatorname{Im} \widehat{P}_{k+1} \cap \operatorname{Im} \widehat{P}_k^{\perp}, \tag{48}$$

which contradicts (47). Therefore (2) holds for $k+1$.

(3) The expression for the associated superfield $\widehat{\Phi}_{k+1}$ is straightforward; that is

$$\begin{aligned}
\widehat{\Phi}_{k+1} &= \widehat{\psi}_{k+1}^{-1} (\zeta = 0) \\
&= \left(\widehat{P}_1 + \frac{\mu}{\overline{\mu}} \widehat{P}_1^{\perp} \right) \cdots \left(\widehat{P}_k + \frac{\mu}{\overline{\mu}} \widehat{P}_k^{\perp} \right) \left(\widehat{P}_{k+1} + \frac{\mu}{\overline{\mu}} \widehat{P}_{k+1}^{\perp} \right).
\end{aligned} \tag{49}$$

The proof is completed. □

Note that all $\widehat{P}_j$'s are of rank one in the above construction. But the same limiting method also produces multisoliton solutions of the form $\psi_{\mu, \widehat{P}_k} \cdots \psi_{\mu, \widehat{P}_1}$ with $\operatorname{rank}(\widehat{P}_1) \geq \cdots \geq \operatorname{rank}(\widehat{P}_k)$. Similar computations give the construction of multisoliton solutions with pole data (μ, k) and arbitrary rank data $(n_1, \ldots, n_k)$.

Like the bosonic case, we can also associate to each multisoliton solution of (4) with pole data (μ, k) a generalized algebraic BT. Using these generalized BTs, we can construct multisoliton solutions with general pole data $(\mu_1, \ldots, \mu_k, n_1, \ldots, n_k)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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